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# ON THE CONVERGENCE OF THE NONNESTED V-CYCLE MULTIGRID METHOD FOR NONSYMMETRIC AND INDEFINITE SECOND-ORDER ELLIPTIC PROBLEMS *

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**Abstract**

This paper provides a proof for the uniform convergence rate (independently of the number of mesh levels) for the nonnested V-cycle multigrid method for nonsymmetric and indefinite second-order elliptic problems.

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*Key words*: Nonnested V-cycle multigrid method, Second-order elliptic problems.

## 1. Introduction

In this paper we study the convergence of the nonnested V-cycle multigrid method, cf. [2,3]. The nonnestedness is usually caused either by the nature of a specific element (e.g., nonconforming finite elements) or by the nonnested mesh refinement. Due to the varieties of the elements and the triangulations for various problems, nonnestedness is universal, cf. [2], [5], [3].

It is well-known that a general proof of the uniform convergence of the nonnested V-cycle multigrid method had been open for many years, although there have been numerous numerical experiments showing that a uniform convergence rate does exist, see [7], [14], [5], [18], [9] and references cited. Among others, the analysis of the V-cycle for the non-conforming finite element method for the second-order elliptic problem has been and is still an active research subject. Let us mention some works in this aspect. The authors of [14][21] proposed a so-called Galerkin V-cycle nested multigrid method and obtained a uniform convergence rate. Since the iterated intergrid transfer operator is employed and different discrete equations on different levels are solved, when dealing with anisotropic problems, the computational work is huge for this Galerkin V-cycle. Recently, the author of [23] gave a proof under a less regularity requirement for the nonconforming V-cycle of the symmetric and positive definite second-order elliptic problem. Nevertheless, it is not clear if the analysis therein could be carried over to other cases where the nonnestedness may be caused by bubble functions (the bubbles are either artificial or come from cubic and above finite elements) or by unstructured mesh refinements, due to its lengthy analysis and its long list of assumptions. Other related works may be referred to [8], [5], [9], [2], [24], [25], [27], [28], [29], [26]. More importantly, however, *up to now there is no a general convergence proof for the nonnested V-cycle for nonsymmetric and indefinite second-order elliptic problems.*

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In this paper, inspired by an argument developed in [4], we give a general convergence proof for the nonnested V-cycle for nonsymmetric and indefinite second-order elliptic problems. Our proof covers all existing nonnested V-cycle where Assumptions A1) and A2) hold (see Section 3 of this paper), including the non-conforming V-cycle with nested meshes [6,10,15], conforming V-cycle with nonnested meshes [9] and Mortar element V-cycle [26, 29, 28, 20]. We obtain a uniform convergence rate (independently of the number of mesh levels), under the condition that the number of pre and post-smoothing steps are sufficiently large and that the coarsest mesh-size is sufficiently small (see Theorem 3.1 and Remark 3.1 of this paper). We point out that, for all existing nonnested V-cycle methods, Assumptions A1) and A2) are valid, see the comments on various nonnested V-cycles in Remark 3.2 of this paper. The key Assumption A1) is the usual regularity-approximation property as in [4,2,3,1], whose verification here requires the full elliptic regularity assumption. The Assumption A2) concerns the approximation property of the coarse-to-fine intergrid transfer operator, which is usually either the interpolation or the $L^2$ projection operator. Also, the assumption A2) holds for all existing nonnested V-cycles, see Remark 3.2 of this paper. We would like also to remark that it is not clear if our approach could be applied to the case of less elliptic regularity, see related works [27] for nonconforming W-cycle and [23] for nonconforming V-cycle for symmetric and positive definite second-order elliptic problems.

The outline of this paper is as follows. In section 2, we review the nonsymmetric and indefinite second-order elliptic problem and the V-cycle multigrid method as well as some notations. In section 3, we obtain the convergence rate for the nonnested V-cycle multigrid method for nonsymmetric and indefinite second-order elliptic problems.

## 2. Preliminaries

### 2.1. Nonsymmetric and indefinite second-order elliptic problem

Let $\Omega$ be a bounded, connected domain in $\mathbb{R}^n$, $(n = 2, 3)$, with Lipschitz continuous boundary $\partial\Omega$. We will use Sobolev spaces $H^k(\Omega)$, with norm $||\cdot||_{H^k(\Omega)}$ and seminorm $|\cdot|_{H^k(\Omega)}$, and $H_0^1(\Omega) = \{v \in H^1(\Omega); v|_{\partial\Omega} = 0\}$. We denote by $(\cdot,\cdot)$ the inner product of $L^2(\Omega)(\equiv H^0(\Omega))$ or $(L^2(\Omega))^n$.

We consider the nonsymmetric and indefinite second-order elliptic problem:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + d(x)u = f, \quad \text{in } \Omega, \quad u_{|\partial\Omega} = 0, \tag{2.1}$$

where $\mathcal{A}(x) := (a_{ij}(x)) \in \mathbb{R}^{n\times n}$ is bounded symmetric and uniformly positive definite in the usual sense, and $a_{ij}, b_i \in C^1(\bar{\Omega})$ and $d \in C^0(\bar{\Omega})$. The variational problem of (2.1) reads as follows: Find $u \in U := H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in U, \tag{2.2}$$

where

$$a(u, v) := \tilde{a}(u, v) + b(u, v), \tag{2.3}$$

$$\tilde{a}(u, v) := (\mathcal{A}\bigtriangledown u, \bigtriangledown v) + (u, v), \tag{2.4}$$

$$b(u, v) := (\boldsymbol{b}\cdot\bigtriangledown u, v) + ((d-1)u, v), \tag{2.5}$$

with $\boldsymbol{b} := (b_1, \cdots, b_n)^T$ and $\bigtriangledown u = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)^T$.

Let $\mathcal{J}_k$, $k \geq 1$, be a sequence of quasi-uniform triangulations of $\Omega$ into elements $K$, with mesh parameter $h_k = \sup_{K\in\mathcal{J}_k} h_K$, here $h_K$ is the diameter of $K$, cf. [13]. For each $\mathcal{J}_k$, let $U_k$ be a finite element space (not necessarily a subspace of $U$ and not necessarily $U_{k-1} \subset U_k$), and let

$$a_k(\cdot,\cdot) = \tilde{a}_k(\cdot,\cdot) + b_k(\cdot,\cdot) \tag{2.6}$$

be a discrete bilinear form, where, for example, without loss of generality, for conforming and non-conforming elements, we may often have (without numerical integration)

$$\tilde{a}_k(u,v) = \sum_{K\in\mathcal{J}_k} (\mathcal{A}\bigtriangledown u, \bigtriangledown v)_{0,K} + (u,v), \tag{2.7}$$

$$b_k(u,v) = \sum_{K\in\mathcal{J}_k} (\boldsymbol{b}\cdot\bigtriangledown u, v)_{0,K} + ((d-1)u,v). \tag{2.8}$$

Note that we may encounter other forms, instead of (2.7) and (2.8), e.g., numerical integration forms. Anyway, as usual, we require that $a_k(\cdot,\cdot)$ is defined on $U_{k-1}+U_k+U$, and such that $a_k(\cdot,\cdot)$ reduces to $a(\cdot,\cdot)$ on $U$ and $a_{k-1}(\cdot,\cdot)$ on $U_{k-1}$. We further require that $\tilde{a}_k(\cdot,\cdot)$ is a symmetric, positive definite, bounded bilinear form. These two requirements are reasonable for most cases, e.g., conforming elements, Crouzeix-Raviart nonconforming elements with nested triangulations [22], and so forth. We furthermore require that

$$|b_k(u,v)| \leq C\,|||u|||_{1,k}\,||v||_{L^2(\Omega)} \quad \forall u,v\in U_k, \tag{2.9}$$

$$|b_k(u,v)| \leq C\,||u||_{L^2(\Omega)}\,|||v|||_{1,k} \quad \forall u,v\in U_k. \tag{2.10}$$

Note that (2.9) generally holds trivially. For conforming elements, (2.10) can be shown with the use of Green's formula of integration by parts. For nonconforming elements which satisfy

$$\int_F [v] = 0 \quad \forall v\in U_k, \text{for all edges } F, \tag{2.11}$$

where $[v]$ denotes the jump of $v$ across $F$, applying the nonconforming estimation [13,12,22], we can easily obtain (2.10). Note that (2.11) is often true for nonconforming elements [22,17] and Mortar elements [26,29,20,28].

We now state the finite element problem for (2.2): Find $u_k \in U_k$ such that

$$a_k(u_k,v) = (f,v) \quad \forall v\in U_k. \tag{2.12}$$

There are numerous solution algorithms for (2.12). Among others, the multigrid algorithm is most efficient, because of its uniform convergence rate with respect to mesh levels and its optimal computational complexity in the sense that the number of operations scales linearly with the number of unknowns, see [1,3]. Before describing this algorithm, we need some more notations for the analysis in Section 3.

Let $(\cdot,\cdot)_{0,k}$ be an inner product over $U_k$. Associated with $U_k$ and $\tilde{a}_k$, we introduce eigenvalues $\lambda_{k,i}$ and eigenvectors $\psi_{k,i}$, $1\leq i\leq N_k$, which satisfy

$$\tilde{a}_k(\psi_{k,i},v) = \lambda_{k,i}(\psi_{k,i},v)_{0,k} \quad \forall v\in U_k, \tag{2.13}$$

$$0 < \lambda_{k,1} \leq \lambda_{k,2} \leq \cdots \leq \lambda_{k,N_k}, \tag{2.14}$$

$$(\psi_{k,i},\psi_{k,j})_{0,k} = \delta_{ij}, \quad \tilde{a}_k(\psi_{k,i},\psi_{k,j}) = \lambda_{k,i}\delta_{ij}, \tag{2.15}$$

where $\delta_{ij}$ is the Kronecker symbol, $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$. For any $v\in U_k$, expanded in terms of eigenvectors as $v=\sum_{i=1}^{N_k} c_i\,\psi_{k,i}$, we define mesh-dependent norms $|||\cdot|||_{s,k}$

by

$$|||v|||_{s,k} := \sqrt{\sum_{i=1}^{N_k} \lambda_{k,i}^s \, c_i^2} \quad \forall s \in \mathbb{R}. \tag{2.16}$$

We would see that

$$|||v|||_{1,k} = \sqrt{\tilde{a}_k(v,v)}, \quad |||v|||_{0,k} = \sqrt{(v,v)_{0,k}}, \tag{2.17}$$

$$|\tilde{a}_k(u,v)| \le |||u|||_{1+t,k} \, |||v|||_{1-t,k} \quad \forall u,v \in U_k, \forall t \in \mathbb{R}. \tag{2.18}$$

With respect to $(\cdot,\cdot)_{0,k}$, we define an operator $\tilde{A}_k : U_k \to U_k$ of $\tilde{a}_k(\cdot,\cdot)$ by

$$(\tilde{A}_k u, v)_{0,k} = \tilde{a}_k(u,v) \quad \forall u,v \in U_k. \tag{2.19}$$

Noting that $\tilde{A}_k$ is symmetric, positive definite with respect to $(\cdot,\cdot)_{0,k}$, we have

$$|||v|||_{s,k} = \sqrt{(\tilde{A}_k^s v, v)_{0,k}} \quad \forall v \in U_k. \tag{2.20}$$

For all $k$, we always assume that the following inverse inequalities hold([13]):

$$|||v|||_{t,k} \le C\, h_k^{s-t} \, |||v|||_{s,k} \quad \forall s \le t, \quad \forall v \in U_k \tag{2.21}$$

and that there holds the following equivalence between $||\cdot||_{L^2(\Omega)}$ and $|||\cdot|||_{0,k}$:

$$C^{-1} \, ||v||_{L^2(\Omega)} \le |||v|||_{0,k} \le C \, ||v||_{L^2(\Omega)} \quad \forall v \in U_k. \tag{2.22}$$

It follows that

$$\lambda_{k,N_k} \le \Lambda_k := C\, h_k^{-2}. \tag{2.23}$$

### 2.2. The Nonnested V-cycle Multigrid method

In this subsection, we review the nonnested V-cycle multigrid method of the $k$th-level iteration for the nonsymmetric and indefinite second-order elliptic problem [16,2].

Denote by $I_k : U_{k-1} \to U_k$ the coarse-to-fine intergrid transfer operator and by $m \ge 1$ an integer and by $U_k'$ the dual of $U_k$. Given $g \in U_k'$ and an initial guess $z_0 \in U_k$, we obtain an approximate solution $MG(k, z_0, g)$ to the general problem (cf. also (2.12))

$$\text{Find } z \in U_k \text{ such that } a_k(z,v) = g(v) \quad \forall v \in U_k. \tag{2.24}$$

For $k = 1$, $MG(1, z_0, g)$ is the solution to (2.24) obtained from a direct method.

For $k > 1$, $MG(k, z_0, g)$ is obtained from three steps as follows.

*Presmoothing step.* Let $z_i \in U_k$, $1 \le i \le m$, be defined recursively by

$$(z_i - z_{i-1}, v)_{0,k} = \Lambda_k^{-1}\{g(v) - a_k(z_{i-1}, v)\} \quad \forall v \in U_k. \tag{2.25}$$

*Correction step.* Let $\bar{g} \in U_{k-1}'$ be defined by $\bar{g}(v) := g(I_k v) - a_k(z_m, I_k v)$ for all $v \in U_{k-1}$, we obtain $q_1 \in U_{k-1}$ from

$$q_1 = MG(k-1, 0, \bar{g}). \tag{2.26}$$

Then set

$$z_{m+1} := z_m + I_k q_1. \tag{2.27}$$

*Postsmoothing step.* Let $z_i \in U_k$, $m+2 \le i \le 2m+1$ be defined recursively by

$$(z_i - z_{i-1}, v)_{0,k} = \Lambda_k^{-1}\{g(v) - a_k(z_{i-1}, v)\} \quad \forall v \in U_k. \tag{2.28}$$

Set

$$MG(k, z_0, g) := z_{2m+1}. \tag{2.29}$$

Note that pre and post-smoothings are well-known Richardson iterations.

## 3. The Convergence Analysis

In this section, we give a general convergence rate for the V-cycle multigrid method described in Subsection 2.2.

We define two operators $A_k, B_k : U_k \to U_k$ by

$$(A_k u, v)_{0,k} = a_k(u, v) \quad \forall u, v \in U_k, \tag{3.1}$$

$$(B_k u, v)_{0,k} = b_k(u, v) \quad \forall u, v \in U_k. \tag{3.2}$$

The Richardson iterative operator may be written as

$$R_k = Id_k - \Lambda_k^{-1} A_k = \tilde{R}_k - \Lambda_k^{-1} B_k, \tag{3.3}$$

where $Id_k$ is the identity operator on $U_k$ and

$$\tilde{R}_k = Id_k - \Lambda_k^{-1} \tilde{A}_k. \tag{3.4}$$

The eigenvalues for $\tilde{R}_k$ are

$$\mu_{k,i} = 1 - \frac{\lambda_{k,i}}{\Lambda_k} \geq 0, \quad 1 \leq i \leq N_k, \tag{3.5}$$

with its eigenvectors being the same as those of $\tilde{A}_k$. For the convergence analysis, we also need the operators $P_{k-1} : u \in U_k \to P_{k-1}u \in U_{k-1}$ defined by

$$a_{k-1}(P_{k-1}u, v) = a_k(u, I_k v) \quad \forall v \in U_{k-1} \tag{3.6}$$

and $Q_k : U_k \to U_k$ defined by

$$Q_k := Id_k - I_k P_{k-1}. \tag{3.7}$$

Let the iterative operator $e_{k,2m+1} : U_k \to U_k$ for the V-cycle algorithm be defined by

$$z - z_{2m+1} = e_{k,2m+1}(z - z_0). \tag{3.8}$$

From (2.24)-(2.29) we have

$$e_{k,2m+1}(z - z_0) = R_k^m Q_k R_k^m (z - z_0) + R_k^m I_k e_{k-1,2m+1} P_{k-1} R_k^m \, (z - z_0). \tag{3.9}$$

We now list the assumptions which will be used for the convergence analysis.

**Assumption A1)** (regularity-approximation assumption)

$$|||Q_k v|||_{0,k} \leq C \, h_k^2 \, |||v|||_{2,k} \quad \forall v \in U_k. \tag{3.10}$$

**Assumption A2)**

$$||I_k v - v||_{L^2(\Omega)} \leq C \, h_k \, |||v|||_{1,k-1} \quad \forall v \in U_{k-1}, \tag{3.11}$$

$$|||P_{k-1} v|||_{1,k-1} \leq C \, |||v|||_{1,k} \quad \forall v \in U_k. \tag{3.12}$$

We are now in a position to state the main theorem of this paper.

**Theorem 3.1.** *Let Assumptions A1) and A2) hold. In addition, we assume that*

$$\delta := \frac{C}{2m - C} + C \sum_{j=1}^{\infty} (m^2 \, h_j^2 \, (1 + h_j)^{2m-2} + m \, h_j \, (1 + h_j)^{m-1}) < 1 \tag{3.13}$$

*holds. Then, for all $k \geq 2$ we have*

$$\varepsilon_k = \frac{C}{2m - C} + C \sum_{j=2}^{k} (m^2 h_j^2 \, (1 + h_j)^{2m-2} + m h_j \, (1 + h_j)^{m-1}) < \delta < 1, \tag{3.14}$$

*such that*

$$|||e_{k,2m+1}v|||_{1,k} \le \varepsilon_k\, |||v|||_{1,k} \quad \forall v \in U_k. \tag{3.15}$$

**Remark 3.1.** This theorem says that the convergence rate is $\delta$ which is less than 1 and is uniform with respect to mesh levels. We consider nested triangulations into triangles. In spite of the nestedness of meshes, the use of Crouzeix-Raviart nonconforming elements or conforming elements with bubbles still makes the V-cycle is nonnested. The nested triangulations are obtained as follows. Suppose $\mathcal{J}_1$ is given and $\mathcal{J}_k$, $k \ge 2$, is obtained from $\mathcal{J}_{k-1}$ via a bi-subdivision: edge midpoints in each triangle $\mathcal{J}_{k-1}$ are connected. Obviously, we have $h_k = h_{k-1}/2 = h_1/2^{k-1}$. We can require that $h_k \le 1$ for all $k$. We then have

$$\delta < \frac{C}{2\,m - C} + C\left(\frac{4}{3}\,m^2\,4^{m-1}\,h_1^2 + 2\,m\,2^{m-1}\,h_1\right),$$

from which we first choose $m > C$ such that

$$\delta_1 := \frac{C}{2\,m - C} < 1, \tag{3.16}$$

and then choose $h_1$ such that

$$\frac{4}{3}\,m^2\,4^{m-1}\,h_1^2 + 2\,m\,2^{m-1}\,h_1 < \frac{1-\delta_1}{C}.$$

That $h_j = h_1/2^{j-1}$ is sufficiently small because of $h_1$ being sufficiently small is an essential requirement in the finite element discretization of nonsymmetric and indefinite second-order elliptic problems, see [12, page 139]. On the other hand, we point out that for symmetric and positive definite second-order elliptic problems the convergence rate is $\delta_1$. It would be interesting comparing $\delta_1$ with the convergence rate $C/(2\,m + C)$ for the nested V-cycle [4,2].

In the sequent, we shall prove Theorem 3.1.

To that goal, we first need to thoroughly investigate some properties of the Richardson iteration operator $R_k$. They are closely related to a quantity $\rho(w)$, characterizing the smooth effect of the Richardson iteration, which was first introduced in [4] for the convergence analysis for the nested V-cycle. Here we generalize that quantity to the case of nonnested V-cycle. For $w \in U_k$, we define

$$\rho_k(w) = \tilde{a}_k(w, \tilde{R}_k w)/\tilde{a}_k(w,w) = |||\tilde{R}_k^{1/2} w|||_{1,k}^2 / |||w|||_{1,k}^2 \tag{3.17}$$

if $w \ne 0$; otherwise, we define $\rho_k(w) = 0$ if $w = 0$. Expanding $w = \sum_{i=1}^{N_k} c_i \psi_{k,i} \ne 0$, then $\rho_k(w) = \sum\limits_{i=1}^{N_k} \lambda_{k,i}\mu_{k,i} c_i^2 / \sum\limits_{i=1}^{N_k} \lambda_{k,i} c_i^2$, which, together with $\mu_{k,i} \le 1$, leads to $0 \le \rho_k(w) \le 1$.

The following three Propositions mainly give some properties associated with $R_k$.

**Proposition 3.1.** *For all $w \in U_k$ and for all $m \ge 1$, we have*

$$|||\tilde{R}_k^m w|||_{1,k} \le \rho_k^m(\tilde{R}_k^m w)\, |||w|||_{1,k}, \tag{3.18}$$

$$|||w|||_{2,k} \le C\, h_k^{-1} \sqrt{1 - \rho_k(w)}\, |||w|||_{1,k}. \tag{3.19}$$

*Proof.* (3.18) and (3.19) can be easily shown by the same argument as in [4,Lemma 4.3, Lemma 4.4].

**Proposition 3.2.** *For all $v \in U_k$ and for all $m \ge 1$, we have*

$$|||R_k^m v|||_{i,k} \le |||\tilde{R}_k^m v|||_{i,k} + C\, m\, h_k^{2-i}\,(1 + h_k)^{m-1}\, |||v|||_{1,k}, \quad i = 1, 2, \tag{3.20}$$

$$|||R_k^m v|||_{1,k}^2 \le |||\tilde{R}_k^m v|||_{1,k}^2 + C(m^2 h_k^2 (1+h_k)^{2m-2} + m h_k (1+h_k)^{m-1}) |||v|||_{1,k}^2, \tag{3.21}$$

$$|||R_k^m v|||_{2,k}^2 \le |||\tilde{R}_k^m v|||_{2,k}^2 + C(m^2 (1+h_k)^{2m-2} + m^{\frac{1}{2}} h_k^{-1} (1+h_k)^{m-1}) |||v|||_{1,k}^2, \tag{3.22}$$

$$\begin{aligned} |||R_k^m v|||_{1,k}^2 + C\, h_k^2\, |||R_k^m v|||_{2,k}^2 \quad &\le \quad \rho_k^{2m}(\tilde{R}_k^m v)\{1 + C[1-\rho_k(\tilde{R}_k^m v)]\}\, |||v|||_{1,k}^2 \\ &+ \quad C(m^2 h_k^2 (1+h_k)^{2m-2} + m h_k (1+h_k)^{m-1})\, |||v|||_{1,k}^2, \end{aligned} \tag{3.23}$$

$$|\tilde{a}_k(R_k^m u, v) - \tilde{a}_k(u, R_k^m v)| \le C\, m\, h_k (1+h_k)^{m-1} |||u|||_{1,k} |||v|||_{1,k}, \quad \forall u, v \in U_k. \tag{3.24}$$

*Proof.* With the application of Proposition 3.1, (3.23) follows from (3.21) and (3.22), while (3.21) and (3.22) follow from (3.20) and

$$|||\tilde{R}_k^m v|||_{1,k} \le |||v|||_{1,k}, \quad |||\tilde{R}_k^m v|||_{2,k} \le C\,(h_k \sqrt{m})^{-1}\, |||v|||_{1,k}. \tag{3.25}$$

We thus only need to show (3.20) and (3.24). Let us first show (3.20). Note that

$$R_k^m v = (\tilde{R}_k - \Lambda_k^{-1} B_k)^m v, \tag{3.26}$$

$$|||B_k v|||_{0,k} \le C\, |||v|||_{1,k} \quad \forall v \in U_k \quad \text{(by (2.9))}, \tag{3.27}$$

using the inverse inequality (2.21) we have

$$|||B_k v|||_{i,k} \le C\, h_k^{-i}\, |||v|||_{1,k}, \quad i = 1, 2. \tag{3.28}$$

By a simple but tedious calculation, from (3.26)-(3.28) and $\Lambda_k^{-1} = C h_k^2$, we have

$$|||R_k^m v|||_{i,k} \le |||\tilde{R}_k^m v|||_{i,k} + C\, m\, h_k^{2-i} (1+h_k)^{m-1}\, |||v|||_{1,k}, \quad i = 1, 2. \tag{3.29}$$

We now show (3.24). Note that

$$R_k^m u = \tilde{R}_k^m u + \Phi(\Lambda_k^{-1}, \tilde{R}_k, B_k) u \quad \forall u \in U_k, \tag{3.30}$$

where, $\Phi(\cdot,\cdot,\cdot)$ is a linear operator, for which, similar to (3.29), we have

$$|||\Phi(\Lambda_k^{-1}, \tilde{R}_k, B_k) u|||_{1,k} \le C\, m\, h_k (1+h_k)^{m-1}\, |||u|||_{1,k}, \tag{3.31}$$

we then have

$$\begin{aligned} |\tilde{a}_k(R_k^m u, v) - \tilde{a}_k(u, R_k^m v)| \quad &= \quad |\tilde{a}_k(\Phi(\Lambda_k^{-1}, \tilde{R}_k, B_k) u, v) - \tilde{a}_k(u, \Phi(\Lambda_k^{-1}, \tilde{R}_k, B_k) v)| \\ &\le \quad C\, m\, h_k (1+h_k)^{m-1}\, |||u|||_{1,k}\, |||v|||_{1,k} \end{aligned} \tag{3.32}$$

because of $\tilde{a}_k(\tilde{R}_k^m u, v) = \tilde{a}_k(u, \tilde{R}_k^m v)$.

**Proposition 3.3.** *Let Assumptions A1) and A2) hold. Then, for all $v \in U_k$ and for all $m \ge 1$, we have*

$$\begin{aligned} |||P_{k-1} R_k^m v|||_{1,k-1}^2 \quad \le \quad & \rho_k^{2m}(\tilde{R}_k^m v) \left\{1 + C\left[1 - \rho_k(\tilde{R}_k^m v)\right]\right\} |||v|||_{1,k}^2 \\ & + C(m^2 h_k^2 (1+h_k)^{2m-2} + m h_k (1+h_k)^{m-1}) |||v|||_{1,k}^2. \end{aligned} \tag{3.33}$$

*Proof.* We first have

$$\begin{aligned} |||P_{k-1} R_k^m v|||_{1,k-1}^2 \quad &= \quad \tilde{a}_{k-1}(P_{k-1} R_k^m v, P_{k-1} R_k^m v) \\ &= \quad a_{k-1}(P_{k-1} R_k^m v, P_{k-1} R_k^m v) - b_{k-1}(P_{k-1} R_k^m v, P_{k-1} R_k^m v) \\ &= \quad a_k(R_k^m v, I_k P_{k-1} R_k^m v) - b_{k-1}(P_{k-1} R_k^m v, P_{k-1} R_k^m v) \\ &= \quad a_k(R_k^m v, R_k^m v) - a_k(R_k^m v, Q_k R_k^m v) - b_{k-1}(P_{k-1} R_k^m v, P_{k-1} R_k^m v) \\ &= \quad \tilde{a}_k(R_k^m v, R_k^m v) - \tilde{a}_k(R_k^m v, Q_k R_k^m v) - b_k(R_k^m v, Q_k R_k^m v) \\ & \qquad + b_k(R_k^m v, R_k^m v) - b_{k-1}(P_{k-1} R_k^m v, P_{k-1} R_k^m v) \\ &= \quad \tilde{a}_k(R_k^m v, R_k^m v) - \tilde{a}_k(R_k^m v, Q_k R_k^m v) \\ & \qquad + b_k(I_k P_{k-1} R_k^m v - P_{k-1} R_k^m v, P_{k-1} R_k^m v) \\ & \qquad + b_k(Q_k R_k^m v, P_{k-1} R_k^m v) + b_k(R_k^m v, I_k P_{k-1} R_k^m v - P_{k-1} R_k^m v). \end{aligned} \tag{3.34}$$

where, using (2.10), (3.10) and (3.12) of Assumptions A1) and A2), (3.21), we have

$$\begin{aligned}
&|b_k(Q_k R_k^m v, P_{k-1} R_k^m v)| \le C\,|||Q_k R_k^m v|||_{0,k}\,|||P_{k-1} R_k^m v|||_{1,k-1} \le C\,h_k^2\,|||R_k^m v|||_{2,k}\,|||R_k^m v|||_{1,k}\\
&\le C\,h_k\,\Big\{\rho_k^{2m}(\tilde{R}_k^m v)\Big[1 + C\Big(1-\rho_k(\tilde{R}_k^m v)\Big)\Big]\\
&\qquad + C(m^2 h_k^2\,(1+h_k)^{2m-2} + m h_k\,(1+h_k)^{m-1})\Big\}\,|||v|||_{1,k}^2\\
&\le C\,h_k\,\{\max_{0\le\rho\le 1}\rho^{2m}[1+C(1-\rho)] + C(m^2 h_k^2\,(1+h_k)^{2m-2} + m h_k\,(1+h_k)^{m-1})\}\,|||v|||_{1,k}^2\\
&\le C\,h_k\,\left\{\max\left(1, \frac{C+1}{2m+1}\right) + C(m^2 h_k^2\,(1+h_k)^{2m-2} + m h_k\,(1+h_k)^{m-1})\right\}\,|||v|||_{1,k}^2\\
&\le C(m^2 h_k^2\,(1+h_k)^{2m-2} + m h_k\,(1+h_k)^{m-1})\,|||v|||_{1,k}^2.
\end{aligned} \tag{3.35}$$

Similarly,

$$|b_k(I_k P_{k-1} R_k^m v - P_{k-1} R_k^m v, P_{k-1} R_k^m v)| \le C(m^2 h_k^2\,(1+h_k)^{2m-2} + m h_k\,(1+h_k)^{m-1})\,|||v|||_{1,k}^2, \tag{3.36}$$

$$|b_k(R_k^m v, I_k P_{k-1} R_k^m v - P_{k-1} R_k^m v)| \le C(m^2 h_k^2\,(1+h_k)^{2m-2} + m h_k\,(1+h_k)^{m-1})\,|||v|||_{1,k}^2, \tag{3.37}$$

$$\begin{aligned}
&|\tilde{a}_k(R_k^m v, R_k^m v) - \tilde{a}_k(R_k^m v, Q_k R_k^m v)| \le \rho_k^{2m}(\tilde{R}_k^m v)\{1 + C[1-\rho_k(\tilde{R}_k^m v)]\}\,|||v|||_{1,k}^2\\
&+C(m^2 h_k^2\,(1+h_k)^{2m-2} + m h_k\,(1+h_k)^{m-1})\,|||v|||_{1,k}^2.
\end{aligned} \tag{3.38}$$

Summarizing (3.34)-(3.38) to get (3.33).

We next give two Lemmas from which we shall prove Theorem 3.1. In proving the first lemma, we used the previous Propositions in several places.

**Lemma 3.1.** *Let Assumptions A1) and A2) hold. If*

$$|||e_{k-1,2m+1} v|||_{1,k-1} \le \varepsilon_{k-1}\,|||v|||_{1,k-1} \quad \forall v \in U_{k-1}, \tag{3.39}$$

*where* $\varepsilon_{k-1} < 1$*. Then, we have*

$$|||e_{k,2m+1} v|||_{1,k} \le \varepsilon_k\,|||v|||_{1,k} \quad \forall v \in U_k \tag{3.40}$$

*with*

$$\varepsilon_k := \max_{0\le\rho\le 1}\rho^{2m}\{(1+\varepsilon_{k-1})C(1-\rho) + \varepsilon_{k-1}\} + C\,(m^2 h_k^2\,(1+h_k)^{2m-2} + m h_k\,(1+h_k)^{m-1}). \tag{3.41}$$

*Proof.* Note that

$$|||e_{k,2m+1} v|||_{1,k} = \sup_{w\in U_k}\frac{\tilde{a}_k(e_{k,2m+1} v, w)}{|||w|||_{1,k}}, \tag{3.42}$$

$$\begin{aligned}
&\tilde{a}_k(e_{k,2m+1} v, w) = \tilde{a}_k(R_k^m Q_k R_k^m v, w) + \tilde{a}_k(R_k^m I_k e_{k-1,2m+1} P_{k-1} R_k^m v, w)\\
&= \tilde{a}_k(R_k^m Q_k R_k^m v, w) - \tilde{a}_k(Q_k R_k^m v, R_k^m w)\\
&+\tilde{a}_k(R_k^m I_k e_{k-1,2m+1} P_{k-1} R_k^m v, w) - \tilde{a}_k(I_k e_{k-1,2m+1} P_{k-1} R_k^m v, R_k^m w)\\
&+\tilde{a}_k(Q_k R_k^m v, R_k^m w) + \tilde{a}_k(I_k e_{k-1,2m+1} P_{k-1} R_k^m v, R_k^m w),
\end{aligned} \tag{3.43}$$

$$\begin{aligned}
&\tilde{a}_k(Q_k R_k^m v, R_k^m w) + \tilde{a}_k(I_k e_{k-1,2m+1} P_{k-1} R_k^m v, R_k^m w)\\
&= \tilde{a}_k(Q_k R_k^m v, R_k^m w) + \tilde{a}_{k-1}(e_{k-1,2m+1} P_{k-1} R_k^m v, P_{k-1} R_k^m w)\\
&+b_{k-1}(P_{k-1} R_k^m w, e_{k-1,2m+1} P_{k-1} R_k^m v) - b_k(R_k^m w, I_k e_{k-1,2m+1} P_{k-1} R_k^m v).
\end{aligned} \tag{3.44}$$

In view of (3.24) and (3.20) with $i = 1$ (see Proposition 3.2), we have

$$\begin{aligned}
&|\tilde{a}_k(R_k^m Q_k R_k^m v, w) - \tilde{a}_k(Q_k R_k^m v, R_k^m w)| \le C\,m\,h_k\,(1+h_k)^{m-1}\,|||Q_k R_k^m v|||_{1,k}\,|||w|||_{1,k}\\
&\le C\,m\,h_k\,(1+h_k)^{m-1}\,|||R_k^m v|||_{1,k}\,|||w|||_{1,k}\\
&\le C\,m\,h_k\,(1+h_k)^{m-1}\,(1 + C m h_k\,(1+h_k)^{m-1})\,|||v|||_{1,k}\,|||w|||_{1,k}\\
&\le C\,(m^2 h_k^2\,(1+h_k)^{2m-2} + m\,h_k\,(1+h_k)^{m-1})\,|||v|||_{1,k}\,|||w|||_{1,k}.
\end{aligned} \tag{3.45}$$

Noting that

$$|||I_k\, v|||_{1,k} \le C\, |||v|||_{1,k} \quad \text{(by ((3.11) in Assumption A2) and (2.21))},$$

in view of (3.11) (see Assumption A2)), (3.24) and (3.20) with $i = 1$ (see Proposition 3.2) and (3.39), we have

$$\begin{aligned}
&|\tilde{a}_k(R_k^m I_k e_{k-1,2m+1} P_{k-1} R_k^m v, w) - \tilde{a}_k(I_k e_{k-1,2m+1} P_{k-1} R_k^m v, R_k^m w)| \\
&\le C\, m\, h_k\, (1+h_k)^{m-1}\, |||I_k e_{k-1,2m+1} P_{k-1} R_k^m v|||_{1,k}\, |||w|||_{1,k} \\
&\le C\, m\, h_k\, (1+h_k)^{m-1}\, \varepsilon_{k-1}\, |||R_k^m v|||_{1,k}\, |||w|||_{1,k} \\
&\le C\, (m^2 h_k^2\, (1+h_k)^{2m-2} + m\, h_k\, (1+h_k)^{m-1})\, |||v|||_{1,k}\, |||w|||_{1,k}.
\end{aligned} \tag{3.46}$$

Noting that

$$\begin{aligned}
&b_{k-1}(P_{k-1} R_k^m w, e_{k-1,2m+1} P_{k-1} R_k^m v) - b_k(R_k^m w, I_k e_{k-1,2m+1} P_{k-1} R_k^m v) \\
&= b_k(P_{k-1} R_k^m w - I_k P_{k-1} R_k^m w, e_{k-1,2m+1} P_{k-1} R_k^m v) \\
&+ b_k(R_k^m w, e_{k-1,2m+1} P_{k-1} R_k^m v - I_k e_{k-1,2m+1} P_{k-1} R_k^m v) \\
&- b_k(Q_k R_k^m w, e_{k-1,2m+1} P_{k-1} R_k^m v),
\end{aligned} \tag{3.47}$$

similarly, we have

$$\begin{aligned}
&|b_k(P_{k-1} R_k^m w - I_k P_{k-1} R_k^m w, e_{k-1,2m+1} P_{k-1} R_k^m v)| \le \\
&C\, h_k\, |||P_{k-1} R_k^m w|||_{1,k-1}\, |||e_{k-1,2m+1} P_{k-1} R_k^m v|||_{1,k-1} \\
&\le C\, h_k\, \varepsilon_{k-1}\, |||P_{k-1} R_k^m v|||_{1,k-1}\, |||P_{k-1} R_k^m w|||_{1,k-1} \\
&\le C\, (m^2 h_k^2\, (1+h_k)^{2m-2} + m\, h_k\, (1+h_k)^{m-1})\, |||v|||_{1,k}\, |||w|||_{1,k},
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
&|b_k(R_k^m w, e_{k-1,2m+1} P_{k-1} R_k^m v - I_k e_{k-1,2m+1} P_{k-1} R_k^m v)| \\
&\le C\, (m^2 h_k^2\, (1+h_k)^{2m-2} + m\, h_k\, (1+h_k)^{m-1})\, |||v|||_{1,k}\, |||w|||_{1,k},
\end{aligned} \tag{3.49}$$

$$\begin{aligned}
&|-b_k(Q_k R_k^m w, e_{k-1,2m+1} P_{k-1} R_k^m v)| \\
&\le C\, (m^2 h_k^2\, (1+h_k)^{2m-2} + m\, h_k\, (1+h_k)^{m-1})\, |||v|||_{1,k}\, |||w|||_{1,k}.
\end{aligned} \tag{3.50}$$

$$\begin{aligned}
&|\tilde{a}_k(Q_k R_k^m v, R_k^m w)| \le C\, h_k^2\, |||R_k^m v|||_{2,k}\, |||R_k^m w|||_{2,k} \\
&\le C \sqrt{\rho_k^{2m}(\tilde{R}_k^m v) C[1-\rho_k(\tilde{R}_k^m v)] + C(m^2 h_k^2\, (1+h_k)^{2m-2} + m h_k\, (1+h_k)^{m-1})} \\
&\times C \sqrt{\rho_k^{2m}(\tilde{R}_k^m w) C[1-\rho_k(\tilde{R}_k^m w)] + C(m^2 h_k^2\, (1+h_k)^{2m-2} + m h_k\, (1+h_k)^{m-1})} \\
&\times |||v|||_{1,k}\, |||w|||_{1,k},
\end{aligned} \tag{3.51}$$

$$\begin{aligned}
&|\tilde{a}_{k-1}(e_{k-1,2m+1} P_{k-1} R_k^m v, P_{k-1} R_k^m w)| \\
&\le \varepsilon_{k-1}\, |||P_{k-1} R_k^m v|||_{1,k-1}\, |||P_{k-1} R_k^m w|||_{1,k-1} \\
&\le \varepsilon_{k-1} \sqrt{\rho_k^{2m}(\tilde{R}_k^m v) \left[1 + C(1-\rho_k(\tilde{R}_k^m v))\right] + C(m^2 h_k^2\, (1+h_k)^{2m-2} + m h_k\, (1+h_k)^{m-1})} \\
&\times \sqrt{\rho_k^{2m}(\tilde{R}_k^m w) \left[1 + C(1-\rho_k(\tilde{R}_k^m w))\right] + C(m^2 h_k^2\, (1+h_k)^{2m-2} + m h_k\, (1+h_k)^{m-1})} \\
&\times |||v|||_{1,k}\, |||w|||_{1,k}.
\end{aligned} \tag{3.52}$$

For convenience, set

$$\begin{aligned}
\gamma_k &:= m^2 h_k^2\, (1+h_k)^{2m-2} + m h_k\, (1+h_k)^{m-1}, \\
\eta_k(v) &:= \rho_k^{2m}(\tilde{R}_k^m v) C[1-\rho_k(\tilde{R}_k^m v)], \\
\xi_k(v) &:= \rho_k^{2m}(\tilde{R}_k^m v) \left[1 + C(1-\rho_k(\tilde{R}_k^m v))\right].
\end{aligned}$$

Combining (3.43)-(3.52), we get

$$\begin{aligned}
|\tilde{a}_k(e_{k,2m+1}\, v, w)| \quad \le \quad & C\, \gamma_k\, |||v|||_{1,k}\, |||w|||_{1,k} \\
& + C \sqrt{\eta_k(v) + C\, \gamma_k} \sqrt{\eta_k(w) + C\, \gamma_k}\, |||v|||_{1,k}\, |||w|||_{1,k} \\
& + \varepsilon_{k-1} \sqrt{\xi_k(v) + C\, \gamma_k} \sqrt{\xi_k(w) + C\, \gamma_k}\, |||v|||_{1,k}\, |||w|||_{1,k},
\end{aligned} \tag{3.53}$$

where, using Cauchy-inequality we have

$$\begin{aligned} & C\,\sqrt{\eta_k(v)+C\,\gamma_k}\,\sqrt{\eta_k(w)+C\,\gamma_k}+\varepsilon_{k-1}\,\sqrt{\xi_k(v)+C\,\gamma_k}\,\sqrt{\xi_k(w)+C\,\gamma_k}\le \\ & \{C\eta_k(v)+C\gamma_k+\varepsilon_{k-1}\,[\xi_k(v)+C\,\gamma_k]\}^{1/2} \\ & \times\{C\eta_k(w)+C\gamma_k+\varepsilon_{k-1}\,[\xi_k(w)+C\,\gamma_k]\}^{1/2} \\ & \le \max_{0\le\rho\le1}\,\rho^{2m}\,\{C\,(1-\rho)+\varepsilon_{k-1}\,[1+C(1-\rho)]\}+C\,\gamma_k. \end{aligned} \tag{3.54}$$

Therefore, (3.41) follows from (3.53), (3.54) and (3.42).

**Lemma 3.2.** *Let $\delta_1$ be given as (3.16). Then, for any $\theta$ satisfying*

$$\delta_1\le\theta<1, \tag{3.55}$$

*we have*

$$\max_{0\le\rho\le1}\,\rho^{2m}\,\{C\,(1-\rho)+\theta\,[1+C(1-\rho)]\}\le\theta. \tag{3.56}$$

*Proof.* Let

$$\kappa_{\theta,m}(\rho):=\rho^{2m}\,\{C(1-\rho)+\theta\,[1+C(1-\rho)]\}. \tag{3.57}$$

We see that $\kappa_{\theta,m}(\rho)$ has a nonzero stationary point

$$\rho_*(m)=\frac{2mC+2m\theta C+2m\theta}{2mC+2m\theta C+C+\theta C}. \tag{3.58}$$

Clearly, in view of (3.55), we have $\rho_*(m)\ge1$. That is to say, $\kappa_{\theta,m}(\rho)$ has not any stationary point in the interval $(0,1)$. Hence, $\kappa_{\theta,m}(\rho)$ attains its maximum $\theta$ at 1 (in fact, with (3.55) $\kappa_{\theta,m}(\rho)$ is nondecreasing over the interval $[0,1]$).

*Proof of Theorem 3.1* Reasoning by mathematical induction. For $k=1$, there is nothing to show. We assume that for $k-1$ we have

$$\varepsilon_{k-1}=\frac{C}{2m-C}+C\sum_{j=2}^{k-1}(m^2h_j^2\,(1+h_j)^{2m-2}+mh_j\,(1+h_j)^{m-1})<\delta<1. \tag{3.59}$$

In what follows, we consider $k$. Since

$$1>\varepsilon_{k-1}>\delta_1, \tag{3.60}$$

from Lemma 3.2 we have

$$\max_{0\le\rho\le1}\rho^{2m}\{(1+\varepsilon_{k-1})C(1-\rho)+\varepsilon_{k-1}\}\le\varepsilon_{k-1}. \tag{3.62}$$

It follows from Lemma 3.1 that $\varepsilon_k=\varepsilon_{k-1}+C(m^2h_k^2\,(1+h_k)^{2m-2}+mh_k\,(1+h_k)^{m-1})$. $\square$

**Remark 3.2.** Let us make some comments on Assumptions A1) and A2) for applications. The Assumption A1) plays the most important role in the convergence analysis of multigrid methods, whose validity needs the full elliptic regularity. Although it was proven for symmetric positive definite second-order elliptic problems, there are not any essential difficulties to adapt the argument to the case of nonsymmetric and indefinite second-order elliptic problems.

Let us mention some realistic applications where the verification of Assumption A1) can be found. For $\mathcal{P}_1$ and Wilson's nonconforming elements, see [6], [10], [17]. For $C^0$(continuous) elements with nonnested triangulations, see [9]. For other nonnested $C^0$ elements such as bubble-enriching element and composited element, see [11]. For nonconforming elements such as $\mathcal{Q}_1^{\rm rot}$ element and discretely divergence-free $\mathcal{P}_1$ element, see [15], [19], [8]. For Mortar element, see [26], [29], [28]. Note that there are different assumptions on the triangulations for different nonnested cases. Readers may refer to the cited references for details.

Assumption A2) is also true generally. For nonconforming elements, $I_k$ is usually a local average operator or local $L^2$ projection operator, cf. [6], [10], [18] ,[20]; for nonnested conforming elements, $I_k$ is the usual interpolation operator, cf. [9]. Under some appropriate conditions on the triangulations, Assumption A2) could be shown, see [6], [9], [10], [11], [15], [17], [18], [19], [20], [25], [26], [27], [28], [29], [2].

Finally, we remark that (3.10) and (3.11) imply (3.12), if $h_k$ is small enough. In fact, since

$$|||P_{k-1}v|||_{1,k-1} = \sup_{w \in U_{k-1}} \frac{\tilde{a}_{k-1}(P_{k-1}v, w)}{|||w|||_{1,k-1}},$$

$$\begin{aligned} \tilde{a}_{k-1}(P_{k-1}v, w) &= a_k(v, I_k w - w) + a_k(v, w) \\ &+ b_k(I_k P_{k-1}v - P_{k-1}v, w) + b_k(Q_k v, w) - b_k(v, w) \\ &\le C\, h_k \, |||P_{k-1}v|||_{1,k-1} \, |||w|||_{1,k-1} + C \, |||v|||_{1,k} \, |||w|||_{1,k-1}, \end{aligned}$$

we conclude that (3.12) holds, with a sufficiently small $h_k$.

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