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A LQP BASED INTERIOR PREDICTION-CORRECTION METHOD FOR NONLINEAR COMPLEMENTARITY PROBLEMS *[1)]

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Abstract

To solve nonlinear complementarity problems (NCP), at each iteration, the classical proximal point algorithm solves a well-conditioned sub-NCP while the Logarithmic-Quadratic Proximal (LQP) method solves a system of nonlinear equations (*LQP system*). This paper presents a practical LQP method-based prediction-correction method for NCP. The predictor is obtained via solving the *LQP system* approximately under significantly relaxed restriction, and the new iterate (the corrector) is computed directly by an explicit formula derived from the original LQP method. The implementations are very easy to be carried out. Global convergence of the method is proved under the same mild assumptions as the original LQP method. Finally, numerical results for traffic equilibrium problems are provided to verify that the method is effective for some practical problems.

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1. Introduction

The nonlinear complementarity problem (NCP) is to determine a vector $x \in R^n$ such that

$$x \geq 0, \quad F(x) \geq 0 \quad \text{and} \quad x^T F(x) = 0, \tag{1.1}$$

where F is a nonlinear mapping from R^n into itself. NCP has received a lot of attention due to its various applications in operations research, economic equilibrium, engineering design, and others, e.g., [7, 8].

A classical method for solving NCP is the Proximal Point Algorithm (PPA) proposed first by Martinet [12] and then developed by many researchers, e.g., [6, 9, 15, 16]. For given $x^k \in R^n_+$ and $\beta_k > 0$, the new iterate x^{k+1} generated by PPA is the unique solution of the following auxiliary NCP: Find $x \in R^n$ such that

$$x \geq 0, \quad \beta_k F(x) + (x - x^k) \geq 0 \quad \text{and} \quad x^T(\beta_k F(x) + (x - x^k)) = 0. \tag{1.2}$$

Recently, a number of articles have concentrated on the generalization of PPA by replacing the linear term $x - x^k$ with some nonlinear functions $r(x, x^k)$. As a result, some "interior point" proximal methods for variational inequality problems have been developed by introducing entropic proximal terms arising from appropriately formulated Bregman functions [1, 4, 5, 6]

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and entropic φ-divergence [16]. For given $x^k \in R^n_{++} := \text{int}R^n_+$ and $\beta_k > 0$, the Logarithmic-Quadratic Proximal (LQP) method presented by Auslender, *et al.* in [2] takes the unique solution of the following auxiliary NCP as the new iterate:

$$x \geq 0, \quad \beta_k F(x) + \nabla_x D(x, x^k) \geq 0 \quad \text{and} \quad x^T(\beta_k F(x) + \nabla_x D(x, x^k)) = 0, \tag{1.3}$$

where

$$\nabla_x D(x, x^k) = (x - x^k) + \mu(x^k - X_k^2 x^{-1}), \tag{1.4}$$

μ is a parameter in $(0, 1)$, $X_k = \text{diag}(x_1^k, x_2^k, \ldots, x_n^k)$ and x^{-1} is an n-vector whose j-th element is $1/x_j$. Note that the integral function of $\nabla_x D(x, x^k)$ satisfying $D(x^k, x^k) = 0$ is

$$D(x, x^k) = \begin{cases} \frac{1}{2}\|x - x^k\|^2 + \mu \sum_{j=1}^n \Big((x_j^k)^2 \log \frac{x_j^k}{x_j} + x_j x_j^k - (x_j^k)^2\Big), & \text{if } x \in \mathcal{R}^n_{++}, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.5}$$

Since $D(x, x^k)$ includes logarithmic and quadratic terms, the method is called *Logarithmic-Quadratic Proximal method*. The first term of $\nabla_x D(x, x^k)$ is to avoid that the new iterate is too far away from x^k; and the second term is to guarantee that the new iterate lies in R^n_{++}. Therefore, at the k-th iteration, solving NCP by the LQP method is equivalent to finding the positive solution of the following system of nonlinear equations

$$\beta_k F(x) + x - (1 - \mu)x^k - \mu X_k^2 x^{-1} = 0. \tag{1.6}$$

Throughout this paper, we call (1.6) the *LQP system of nonlinear equations* (abbreviated as *LQP system*). Generally speaking, solving the *LQP system* is much easier than solving the auxiliary NCP (1.2). Thus the LQP method is attractive for solving NCP. In general, however, it is not trivial to obtain the exact positive solution of the *LQP system*. An inexact LQP method solving (1.6) approximately was also presented in [2].

In this paper, inspired by the LQP method, we present a prediction-correction method [11] for NCP. Both the predictor and corrector are computed via an explicit formula derived from (1.6) (For details, see (2.1) and (2.3)). Similar to the LQP method, all the iterative points generated by the method lie in R^n_{++} whenever the initial point does. Thus the method inherits theoretical properties of the original LQP method. Based on these observations, we call the method a LQP based interior prediction-correction method.

The rest of this paper is organized as follows. In Section 2, the new method is presented and some remarks are also provided. In Section 3, we prove the contractive properties of the proposed method. These properties play important roles in the convergence analysis. Convergence of the new method is discussed in Section 4. In Section 5, some implementation details of the proposed method are addressed. In addition, numerical results for problems in traffic equilibrium are also reported. Finally, some conclusions are drawn in Section 6.

Throughout this paper we make the following standard assumptions:

A1. $F(x)$ is continuous and monotone mappings with respect to R^n_+, i.e.,

$$(x - \tilde{x})^T(F(x) - F(\tilde{x})) \geq 0, \quad \forall x, \tilde{x} \in R^n_+. \tag{1.7}$$

A2. The solution set of the NCP, denoted by $\mathcal{X}^*$, is nonempty.

2. The Proposed Method

At the k-th iteration, the LQP method solves the *LQP system* (1.6) exactly or approximately. We now present a LQP based interior prediction-correction method for NCP.

Let $\mu, \eta \in (0, 1)$. For given $x^k > 0$ and $\beta_k > 0$, the new iterate x^{k+1} is generated by the following steps:

Prediction step: Find an approximate solution $\tilde{x}^k$ of (1.6), called predictor, such that

$$0 \approx \beta_k F(\tilde{x}^k) + \tilde{x}^k - (1 - \mu)x^k - \mu X_k^2 (\tilde{x}^k)^{-1} =: \xi^k, \tag{2.1}$$

where ξ^k satisfies

$$\|\xi^k\| \le \eta\sqrt{1-\mu^2}\|x^k - \tilde{x}^k\|. \tag{2.2}$$

Correction step: Take the positive solution of the following system of equations, called corrector, as the new iterate $x^{k+1}(\tau)$

$$\tau\beta_k F(\tilde{x}^k) + x - (1-\mu)x^k - \mu X_k^2 x^{-1} = 0, \tag{2.3}$$

where τ is a positive scalar. How to choose the parameter τ will be discussed in Section 3.

Remark 1. In general, the prediction step is implementable. Sometimes we can get the approximate solution of (2.1) directly via choosing a suitable small $\beta_k > 0$. For example, if F is Lipschitz continuous with constant L on R^n_+, i.e.,

$$\|F(x) - F(y)\| \le L\|x - y\|, \qquad \forall x, y \in R^n_+; \tag{2.4}$$

and $0 < \beta \le \beta_k \le \eta\sqrt{1-\mu^2}/L$, then we can take the positive solution of the following equation

$$\beta_k F(x^k) + x - (1-\mu)x^k - \mu X_k^2 x^{-1} = 0 \tag{2.5}$$

as $\tilde{x}^k$. Note that in this special case

$$\xi^k = \beta_k(F(\tilde{x}^k) - F(x^k)) \tag{2.6}$$

and condition (2.2) is satisfied. In addition, the components of the positive solution of (2.5) can be computed directly by

$$\tilde{x}^k_j = \frac{(1-\mu)x^k_j - \beta_k F_j(x^k) + \sqrt{[(1-\mu)x^k_j - \beta_k F_j(x^k)]^2 + 4\mu(x^k_j)^2}}{2}. \tag{2.7}$$

Remark 2. The predictor $\tilde{x}^k$ is computed by solving (2.1) under significantly relaxed inexact criterion (2.2) due to that it will be corrected further by the correction step (2.3). Note that the coefficient of the inexact criterion (2.2) can be fixed as $\eta\sqrt{1-\mu^2}$. Thus (2.2) is much more relaxed than the criteria of the inexact LQP method in [2] which require their coefficients to converge to 0.

Remark 3. Note that in the correction step the components of the positive solution of (2.3) can be computed directly by

$$x^{k+1}_j = \frac{(1-\mu)x^k_j - \tau\beta_k F_j(\tilde{x}^k) + \sqrt{[(1-\mu)x^k_j - \tau\beta_k F_j(\tilde{x}^k)]^2 + 4\mu(x^k_j)^2}}{2}. \tag{2.8}$$

It is easy to verify that $\tilde{x}^k > 0$ and $x^{k+1} > 0$ whenever $x^k > 0$. Thus all the iterative points generated by the proposed method lie in R^n_{++} whenever the initial point does. Therefore, the proposed method inherits theoretical properties of the original LQP method.

Remark 4. At each iteration, the main task of the LQP method is to solve the *LQP system* (1.6) (exactly or approximately). The proposed method sometimes obtains both the predictor $\tilde{x}^k$ and the corrector x^{k+1} quite easily by an explicit formula, i.e., $\tilde{x}^k$ and x^{k+1} can be computed directly by (2.7) and (2.8), respectively. Therefore, the proposed method is easily implementable.

3. Contractive Property of the Generated Sequence

The following proposition is similar to Lemma 2 in [2]. For completeness, a proof is provided.

Lemma 3.1 *For given $x^k > 0$ and $q \in R^n$, the equation*

$$q + x - (1-\mu)x^k - \mu X_k^2 x^{-1} = 0 \tag{3.1}$$

has a positive solution x whose j-th element x_j is given by

$$x_j = \frac{(1-\mu)x^k_j - q_j + \sqrt{[(1-\mu)x^k_j - q_j]^2 + 4\mu(x^k_j)^2}}{2}. \tag{3.2}$$

Moreover, for this positive solution $x > 0$ and any $y \ge 0$, we have

$$(x-y)^T(-q) \ge \tfrac{1+\mu}{2}\Big(\|x-y\|^2 - \|x^k - y\|^2\Big) + \tfrac{1-\mu}{2}\|x^k - x\|^2. \tag{3.3}$$

Proof. The first assertion is clear and we only prove the second one. Since $x > 0$, $x^k > 0$ and $y \geq 0$, we have

$$y_i(x_i^k)^2/x_i \geq y_i(2x_i^k - x_i), \quad i = 1, \ldots, n. \tag{3.4}$$

It follows from (3.1) that for $i = 1, \ldots, n$,

$$\begin{aligned}(x_i - y_i)(-q_i) &= (x_i - y_i)\big(x_i - (1-\mu)x_i^k - \mu(x_i^k)^2/x_i\big)\\ &\geq (x_i)^2 - (1-\mu)x_i x_i^k - \mu(x_i^k)^2 - x_i y_i + (1-\mu)x_i^k y_i + \mu y_i(2x_i^k - x_i)\\ &= (x_i)^2 - (1-\mu)x_i x_i^k - \mu(x_i^k)^2 - (1+\mu)x_i y_i + (1+\mu)x_i^k y_i\\ &= \tfrac{1+\mu}{2}\Big((x_i - y_i)^2 - (x_i^k - y_i)^2\Big) + \tfrac{1-\mu}{2}(x_i^k - x_i)^2.\end{aligned}$$

Hence, (3.3) holds and the proof is completed.

Note that the corrector $x^{k+1}(\tau)$ obtained by solving (2.3) is dependent on the parameter τ. How to choose values of τ to ensure that $x^{k+1}(\tau)$ is closer to the solution set than x^k deserves further investigation. For this purpose, we define

$$\Theta(\tau) := \|x^k - x^*\|^2 - \|x^{k+1}(\tau) - x^*\|^2, \tag{3.5}$$

where $x^* \in \mathcal{X}^*$ is any solution point of NCP. Clearly, $\Theta(\tau)$ measures the progress made by the new iterate $x^{k+1}(\tau)$ at the k-th iteration.

Lemma 3.2 *For given predictor $\tilde{x}^k$ and any $\tau > 0$, we have*

$$\Theta(\tau) \geq \tfrac{1-\mu}{1+\mu}\|x^k - x^{k+1}(\tau)\|^2 + 2\tau\big(x^{k+1}(\tau) - \tilde{x}^k\big)^T d^k - \tfrac{2\tau\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2, \tag{3.6}$$

where

$$d^k = (x^k - \tilde{x}^k) + \tfrac{1}{1+\mu}\xi^k. \tag{3.7}$$

Proof. The proof consists of applying Lemma 3.1 to both the prediction and correction steps. First, we apply Lemma 3.1 to the prediction step (2.1). By setting $q = \beta_k F(\tilde{x}^k) - \xi^k$ in (3.1) and $y = x^{k+1}(\tau)$ in (3.3), it follows from (2.1) that

$$\begin{aligned}(x^{k+1}(\tau) - \tilde{x}^k)^T(\tfrac{1}{1+\mu}(\xi^k - \beta_k F(\tilde{x}^k)) &\leq \frac{1}{2}\Big(\|x^k - x^{k+1}(\tau)\|^2 - \|\tilde{x}^k - x^{k+1}(\tau)\|^2\Big)\\ &\quad - \tfrac{1-\mu}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2.\end{aligned} \tag{3.8}$$

Notice the following identity

$$(x^{k+1}(\tau) - \tilde{x}^k)^T(x^k - \tilde{x}^k) = \frac{1}{2}\big(\|\tilde{x}^k - x^{k+1}(\tau)\|^2 - \|x^k - x^{k+1}(\tau)\|^2\big) + \frac{1}{2}\|x^k - \tilde{x}^k\|^2 \tag{3.9}$$

Adding (3.8) and (3.9) we obtain

$$(x^{k+1}(\tau) - \tilde{x}^k)^T\big\{(x^k - \tilde{x}^k) + \tfrac{1}{1+\mu}(\xi^k - \beta_k F(\tilde{x}^k))\big\} \leq \tfrac{\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2, \tag{3.10}$$

which implies

$$0 \geq 2\tau(x^{k+1}(\tau) - \tilde{x}^k)^T\big\{(x^k - \tilde{x}^k) + \tfrac{1}{1+\mu}(\xi^k - \beta_k F(\tilde{x}^k))\big\} - \tfrac{2\tau\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2. \tag{3.11}$$

Then, we apply Lemma 3.1 to the correction step (2.3). Setting $q = \tau\beta_k F(\tilde{x}^k)$ in (3.1) and $y = x^*$ in (3.3), it follow from (2.3) that

$$(x^{k+1}(\tau) - x^*)^T(-\tau\beta_k F(\tilde{x}^k)) \geq \tfrac{1+\mu}{2}(\|x^{k+1}(\tau) - x^*\|^2 - \|x^k - x^*\|^2) + \tfrac{1-\mu}{2}\|x^k - x^{k+1}(\tau)\|^2.$$

Thus we get

$$\Theta(\tau) \geq \tfrac{1-\mu}{1+\mu}\|x^k - x^{k+1}(\tau)\|^2 + 2\tau\tfrac{\beta_k}{1+\mu}(x^{k+1}(\tau) - x^*)^T F(\tilde{x}^k). \tag{3.12}$$

Since $\tilde{x}^k \in R^n_+$ and x^* is a solution of NCP, using the monotonicity of F, we have

$$(\tilde{x}^k - x^*)^T F(\tilde{x}^k) \geq (\tilde{x}^k - x^*)^T F(x^*) = (\tilde{x}^k)^T F(x^*) \geq 0, \tag{3.13}$$

and consequently

$$\Theta(\tau) \geq \tfrac{1-\mu}{1+\mu}\|x^k - x^{k+1}(\tau)\|^2 + 2\tau\tfrac{\beta_k}{1+\mu}(x^{k+1}(\tau) - \tilde{x}^k)^T F(\tilde{x}^k). \tag{3.14}$$

Adding (3.11) and (3.14) and using the notation of d^k in (3.7), the assertion is proved.

Based on Lemma 3.2, we discuss the contractive property of the proposed method by setting $\tau = \frac{1-\mu}{1+\mu}\alpha$ in the correction step (2.3).

Theorem 3.1 *Let $\Theta(\tau)$ be defined in (3.5) and d^k be defined in (3.7). If we take $\tau = \frac{1-\mu}{1+\mu}\alpha$ in (2.3), then for any $x^* \in \mathcal{X}^*$ and $\alpha > 0$, we have*

$$\Theta(\tfrac{1-\mu}{1+\mu}\alpha) \geq \tfrac{1-\mu}{1+\mu}\Phi(\alpha), \tag{3.15}$$

where

$$\Phi(\alpha) := 2\alpha\varphi_k - \alpha^2\|d^k\|^2 \tag{3.16}$$

and

$$\varphi_k := \tfrac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \tfrac{1}{1+\mu}(x^k - \tilde{x}^k)^T\xi^k. \tag{3.17}$$

Proof. Recall that

$$\Theta(\tfrac{1-\mu}{1+\mu}\alpha) = \|x^k - x^*\|^2 - \|x^{k+1}(\tfrac{1-\mu}{1+\mu}\alpha) - x^*\|^2. \tag{3.18}$$

It follows from (3.6) that

$$\begin{aligned}
\tfrac{1+\mu}{1-\mu}\Theta(\tfrac{1-\mu}{1+\mu}\alpha) \quad &\geq \quad 2\alpha\{(x^{k+1}(\tfrac{1-\mu}{1+\mu}\alpha) - x^k) + (x^k - \tilde{x}^k)\}^T d^k - 2\tfrac{\alpha\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 \\
&\quad\quad + \|x^k - x^{k+1}(\tfrac{1-\mu}{1+\mu}\alpha)\|^2 \\
&= \quad 2\alpha(x^k - \tilde{x}^k)^T d^k - 2\tfrac{\alpha\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 - \alpha^2\|d^k\|^2 \\
&\quad\quad + \|(x^k - x^{k+1}(\tfrac{1-\mu}{1+\mu}\alpha)) - \alpha d^k\|^2 \\
&\geq \quad 2\alpha\big\{(x^k - \tilde{x}^k)^T d^k - \tfrac{\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2\big\} - \alpha^2\|d^k\|^2 \\
&\overset{(3.7)}{=} \quad 2\alpha\big\{\tfrac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \tfrac{1}{1+\mu}(x^k - \tilde{x}^k)^T\xi^k\big\} - \alpha^2\|d^k\|^2 \\
&\overset{(3.17)}{=} \quad 2\alpha\varphi_k - \alpha^2\|d^k\|^2.
\end{aligned} \tag{3.19}$$

The assertion follows from (3.18) and (3.19) directly.

Note that $\Theta(\frac{1-\mu}{1+\mu}\alpha)$ can be regarded as the progress made by $x^{k+1}(\frac{1-\mu}{1+\mu}\alpha)$ at the $k-th$ iteration. Therefore, it motivates us to choose such an α that reaches the maximum of $\Phi(\alpha)$. Since $\Phi(\alpha)$ is a quadratic function of α, it reaches its maximum at

$$\alpha_k^* = \frac{\varphi_k}{\|d^k\|^2} \tag{3.20}$$

with

$$\Phi(\alpha_k^*) = \alpha_k^*\varphi_k. \tag{3.21}$$

Under the condition (2.2) we have

$$\begin{aligned}
2\varphi_k \quad &\overset{(3.17)}{=} \quad \tfrac{2}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \tfrac{2}{1+\mu}(x^k - \tilde{x}^k)^T\xi^k \\
&= \quad \|x^k - \tilde{x}^k\|^2 + \tfrac{2}{1+\mu}(x^k - \tilde{x}^k)^T\xi^k + \tfrac{1-\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 \\
&\overset{(2.2)}{\geq} \quad \|x^k - \tilde{x}^k\|^2 + \tfrac{2}{1+\mu}(x^k - \tilde{x}^k)^T\xi^k + \tfrac{1}{(1+\mu)^2}\|\xi^k\|^2 + \tfrac{1-\mu}{1+\mu}(1-\eta^2)\|x^k - \tilde{x}^k\|^2 \\
&\overset{(3.7)}{=} \quad \|d^k\|^2 + \tfrac{1-\mu}{1+\mu}(1-\eta^2)\|x^k - \tilde{x}^k\|^2.
\end{aligned} \tag{3.22}$$

Therefore, it follows from (3.20) and (3.22) that

$$\alpha_k^* > \frac{1}{2}. \tag{3.23}$$

In addition, we have

$$\begin{aligned}
\varphi_k \quad &\overset{(3.17)}{\geq} \quad \tfrac{1}{1+\mu}\Big(\|x^k - \tilde{x}^k\|^2 - \|x^k - \tilde{x}^k\| \cdot \|\xi^k\|\Big) \\
&\overset{(C.S.I)}{\geq} \quad \tfrac{1}{1+\mu}\Big(\|x^k - \tilde{x}^k\|^2 - \tfrac{1-\mu^2}{2}\|x^k - \tilde{x}^k\|^2 - \tfrac{1}{2(1-\mu^2)}\|\xi^k\|^2\Big) \\
&\overset{(2.2)}{\geq} \quad \tfrac{1-\eta^2+\mu^2}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2,
\end{aligned} \tag{3.24}$$

where the second inequality comes from the Cauchy-Schwartz inequality.

Consequently, from (3.21), (3.23) and (3.24) we obtain

$$\Phi_k(\alpha_k^*) \geq \frac{1-\eta^2+\mu^2}{4(1+\mu)}\|x^k-\tilde{x}^k\|^2. \tag{3.25}$$

Based on numerical experiments, we prefer multiplying the 'optimal' value α^* by a relaxation factor $\gamma \in (1,2)$ (better when close to 2). Thus the correction step (2.3) with 'optimal' α of the proposed method is to find the positive solution $x^{k+1}(\frac{1-\mu}{1+\mu}\gamma\alpha^*)$ of the following system of equations

$$(\text{LQP P-C}_{\gamma\tau_k^*}) \qquad \gamma\tau_k^*\beta_k F(\tilde{x}^k) + x - (1-\mu)x^k - \mu X_k^2 x^{-1} = 0, \tag{3.26}$$

where (see (3.20))

$$\tau_k^* = \frac{1-\mu}{1+\mu}\alpha_k^* \qquad \text{and} \qquad \alpha_k^* = \frac{\|x^k-\tilde{x}^k\|^2 + (x^k-\tilde{x}^k)^T\xi^k}{(1+\mu)\|d^k\|^2}. \tag{3.27}$$

Theorem 3.2 *Let $x^{k+1}(\frac{1-\mu}{1+\mu}\gamma\alpha^*)$ be the positive solution of (3.26). Then for any $x^* \in \mathcal{X}^*$ and $\gamma \in (1,2)$, we have*

$$\|x^{k+1}(\tfrac{1-\mu}{1+\mu}\gamma\alpha^*) - x^*\|^2 \leq \|x^k-x^*\|^2 - \frac{\gamma(2-\gamma)(1-\eta^2+\mu^2)(1-\mu)}{4(1+\mu)^2}\|x^k-\tilde{x}^k\|^2. \tag{3.28}$$

Proof. Note that for $\gamma \in (1,2)$, by a simple manipulations we obtain

$$\begin{aligned}
\Phi(\gamma\alpha_k^*) &\overset{(3.16)}{=} 2\gamma\alpha_k^*\varphi_k - (\gamma^2\alpha_k^*)(\alpha_k^*\|d^k\|^2) \\
&\overset{(3.20)}{=} (2\gamma\alpha_k^* - \gamma^2\alpha_k^*)\varphi_k \\
&\overset{(3.21)}{=} \gamma(2-\gamma)\Phi(\alpha_k^*).
\end{aligned} \tag{3.29}$$

It follows from Theorem 3.1 and (3.29) that

$$\begin{aligned}
\Theta(\tfrac{1-\mu}{1+\mu}\gamma\alpha^*) &= \|x^k-x^*\|^2 - \|x^{k+1}(\tfrac{1-\mu}{1+\mu}\gamma\alpha^*) - x^*\|^2 \\
&\geq \tfrac{1-\mu}{1+\mu}\Phi(\gamma\alpha^*) \\
&= \gamma(2-\gamma)\tfrac{1-\mu}{1+\mu}\Phi(\alpha_k^*).
\end{aligned} \tag{3.30}$$

Then the assertion follows from (3.25) immediately.

It follows from Theorem 3.2 that there exists a constant $c > 0$ such that

$$\|x^{k+1}-x^*\|^2 \leq \|x^k-x^*\|^2 - c\|x^k-\tilde{x}^k\|^2, \quad \forall x^* \in \mathcal{X}^*, \tag{3.31}$$

whenever the new iterate x^{k+1} is obtained by LQP P-C$_1$ or LQP P-C$_{\gamma\alpha_k^*}$. Since (3.31) holds for any $x^* \in \mathcal{X}^*$, we have

$$[\text{dist}(x^{k+1},\mathcal{X}^*)]^2 \leq [\text{dist}(x^k,\mathcal{X}^*)]^2 - c\|x^k-\tilde{x}^k\|^2, \tag{3.32}$$

where

$$\text{dist}(x,\mathcal{X}^*) = \inf\{\|x-x^*\| \mid x^* \in \mathcal{X}^*\}. \tag{3.33}$$

Both LQP P-C$_1$ and LQP P-C$_{\gamma\alpha_k^*}$ belong to contractive methods because the new iterate x^{k+1} generated by either of them is closer to the solution set $\mathcal{X}^*$ than x^k.

4. Convergence of the Proposed Method

The following lemma plays an important role in the convergence analysis of the proposed method.

Lemma 4.1 *For given $x^k > 0$ and $\beta_k > 0$, let $\tilde{x}^k$ be obtained by the prediction step (2.1), then for each $x \geq 0$ we have*

$$(x-\tilde{x}^k)^T(\beta_k F(\tilde{x}^k) - \xi^k) \geq (x^k-\tilde{x}^k)^T\big\{(1+\mu)x - (\mu x^k + \tilde{x}^k)\big\}. \tag{4.1}$$

Proof. By setting $q = \beta_k F(\tilde{x}^k) - \xi^k$ in (3.1), $x = \tilde{x}^k$ and $y = x$ in (3.3), it follows from (3.3) that

$$(x - \tilde{x}^k)^T(\beta_k F(\tilde{x}^k) - \xi^k) \geq \tfrac{1+\mu}{2}\big(\|\tilde{x}^k - x\|^2 - \|x^k - x\|^2\big) + \tfrac{1-\mu}{2}\|x^k - \tilde{x}^k\|^2. \tag{4.2}$$

By a simple manipulation, we have

$$\begin{aligned} &\tfrac{1+\mu}{2}\big(\|\tilde{x}^k - x\|^2 - \|x^k - x\|^2\big) + \tfrac{1-\mu}{2}\|x^k - \tilde{x}^k\|^2 \\ = \;& (1+\mu)x^T x^k - (1+\mu)x^T\tilde{x}^k - (1-\mu)(\tilde{x}^k)^T x^k - \mu\|x^k\|^2 + \|\tilde{x}^k\|^2 \\ = \;& (1+\mu)x^T(x^k - \tilde{x}^k) - (x^k - \tilde{x}^k)^T(\mu x^k + \tilde{x}^k) \\ = \;& (x^k - \tilde{x}^k)^T\big\{(1+\mu)x - \big(\mu x^k + \tilde{x}^k\big)\big\}. \end{aligned}$$

Then the proof is completed.

Now, we are ready to prove convergence of the proposed method.

Theorem 4.1 *If $\inf_{k=0}^{\infty}\beta_k := \beta > 0$, then the sequence $\{x^k\}$ generated by the proposed method converges to some x^∞ which is a solution of NCP.*

Proof. It follows from (3.31) that $\{x^k\}$ is a bounded sequence and

$$\lim_{k\to\infty}\|x^k - \tilde{x}^k\| = 0. \tag{4.3}$$

Consequently, $\{\tilde{x}^k\}$ is also bounded. Since $\lim_{k\to\infty}\|x^k - \tilde{x}^k\| = 0$, $\|\xi^k\| < \|x^k - \tilde{x}^k\|$ and $\beta_k \geq \beta > 0$, it follows from (4.1) that

$$\lim_{k\to\infty}(x - \tilde{x}^k)^T F(\tilde{x}^k) \geq 0, \qquad \forall x \in R^n_+. \tag{4.4}$$

Because $\{\tilde{x}^k\}$ is bounded, it has at least one cluster point. Let x^∞ be a cluster point of $\{\tilde{x}^k\}$ and the subsequence $\{\tilde{x}^{k_j}\}$ converges to x^∞. It follows that

$$\lim_{j\to\infty}(x - \tilde{x}^{k_j})^T F(\tilde{x}^{k_j}) \geq 0, \qquad \forall x \in R^n_+ \tag{4.5}$$

and consequently

$$(x - x^\infty)^T F(x^\infty) \geq 0, \qquad \forall x \in R^n_+. \tag{4.6}$$

This means that x^∞ is a solution of NCP. Note that the inequality (3.31) is true for all solution point of NCP, hence we have

$$\|x^{k+1} - x^\infty\|^2 \leq \|x^k - x^\infty\|^2, \quad \forall k \geq 0. \tag{4.7}$$

Since $\tilde{x}^{k_j} \to x^\infty$ $(j \to \infty)$ and $x^k - \tilde{x}^k \to 0$ $(k \to \infty)$, for any given $\varepsilon > 0$, there exists an $l > 0$ such that

$$\|\tilde{x}^{k_l} - x^\infty\| < \varepsilon/2 \quad \text{and} \quad \|x^{k_l} - \tilde{x}^{k_l}\| < \varepsilon/2. \tag{4.8}$$

Therefore, for any $k \geq k_l$, it follows from (4.7) and (4.8) that

$$\|x^k - x^\infty\| \leq \|x^{k_l} - x^\infty\| \leq \|x^{k_l} - \tilde{x}^{k_l}\| + \|\tilde{x}^{k_l} - x^\infty\| \leq \varepsilon. \tag{4.9}$$

This implies that the sequence $\{x^k\}$ converges to x^∞ which is a solution of NCP.

5. Implementation Details and Numerical Experiments

In this section, we provide some illustrations of the proposed method from the practical point of view. Then the algorithm with some implementation details are addressed. Finally some numerical results are reported to verify the theoretical results.

5.1 Implementation details

In the prediction step, the main task of the proposed method is to find an approximate solution $\tilde{x}^k$ of (1.6) such that (2.1)-(2.2) are satisfied. Such $\tilde{x}^k$ can be obtained by choosing a suitably small $\beta_k > 0$ in (2.7) (see Remark 1 in Section 2) such that

$$\|\xi^k\| \leq \eta\sqrt{1-\mu^2}\|x^k - \tilde{x}^k\|, \tag{5.1}$$

where

$$\xi^k = \beta_k(F(\tilde{x}^k) - F(x^k)). \tag{5.2}$$

Too small values of β_k, however, usually lead to extremely slow convergence according to our numerical experiments. Thus it is necessary to avoid this situation. To do so, a strategy of enlarging the values of β_k whenever they are too small is proposed in the following algorithm.

The implementation details of the proposed method

Step 0. Let $\beta_0 = 1$, $\eta(:= 0.95) < 1$, $\mu = 0.01$, $\gamma = 1.8$, $\varepsilon = 10^{-8}$, $k = 0$ and $x^0 > 0$.

Step 1. If $\|\min\{x, F(x)\}\|_\infty \le \varepsilon$, then stop. Otherwise, go to Step 2.

Step 2. (Prediction step)

$$s := (1-\mu)x^k - \beta_k F(x^k), \qquad \tilde{x}_i^k := \Big(s_i + \sqrt{(s_i)^2 + 4\mu(x_i^k)^2}\,\Big)/2,$$

$$\xi := \beta_k(F(\tilde{x}^k) - F(x^k)), \qquad r := \|\xi\|/(\sqrt{1-\mu^2}\,\|x^k - \tilde{x}^k\|).$$

while $(r > \eta)$

$$\beta_k := \beta_k * 0.8/r,$$

$$s := (1-\mu)x^k - \beta_k F(x^k), \qquad \tilde{x}_i^k := \Big(s_i + \sqrt{(s_i)^2 + 4\mu(x_i^k)^2}\,\Big)/2,$$

$$\xi := \beta_k(F(\tilde{x}^k) - F(x^k)), \qquad r := \|\xi\|/(\sqrt{1-\mu^2}\,\|x^k - \tilde{x}^k\|).$$

end while

Step 3. (Correction step)

$$\varphi := \tfrac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \tfrac{1}{1+\mu}(x^k - \tilde{x}^k)^T\xi,$$

$$d = (x^k - \tilde{x}^k) + \tfrac{1}{1+\mu}\xi, \quad \alpha^* = \tfrac{\varphi}{\|d\|^2},$$

$$s := (1-\mu)x^k - \tfrac{1-\mu}{1+\mu}\beta_k\gamma\alpha^* F(\tilde{x}^k),$$

$$x_i^{k+1} := \Big(s_i + \sqrt{(s_i)^2 + 4\mu(x_i^k)^2}\,\Big)/2,$$

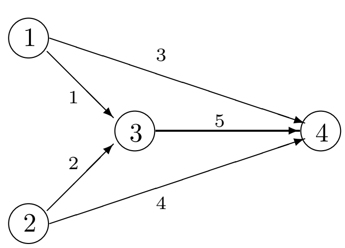
Step 4. $\beta_{k+1} := \begin{cases} \beta_k * 0.7/r, & \text{if } r \le 0.5, \\ \beta_k, & \text{otherwise.} \end{cases}$

Step 5. $k := k+1$; go to Step 1.

From the implementation details, we see that the entire computational cost of the proposed method is very small, thus it is applicable in practice. The number of evaluations of the mapping F per iteration is dependent on the trial steps in the prediction step, yet our numerical experiments show that the number of the total trial steps is small.

5.2 Numerical experiments for network equilibrium problems

As an application we use the examples in the traffic equilibrium problems [17]. Consider a network $[N, L]$ of nodes N and directed links L, which consists of a finite sequence of connecting links with a certain orientation. Let a, b, etc., denote the links, and let p, q, etc., denote the paths. We let ω denote an origin/destination (O/D) pair of nodes of the network and P_ω denote the set of all paths connecting the O/D pair ω. An illustrative example is depicted in Fig. 5.1.



O/D Pairs	Path No. & the link on the path
ω_1 : ① → ④	$p_1 = \{3\}$ $p_2 = \{1, 5\}$
ω_2 : ② → ④	$p_3 = \{4\}$ $p_4 = \{2, 5\}$

Fig. 5.1. An example of given directed network and the O/D pairs

Let A and B denote the path-arc and the path-O/D pair incidence matrices, respectively. For the given example in Fig. 5.1, A and B have the following forms:

$$A = \begin{array}{c} \text{No. link} \quad 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \leftarrow p_1 \\ \leftarrow p_2 \\ \leftarrow p_3 \\ \leftarrow p_4 \end{array}, \qquad B = \begin{array}{c} \text{No. O/D pair} \quad \omega_1 \;\; \omega_2 \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \leftarrow p_1 \\ \leftarrow p_2 \\ \leftarrow p_3 \\ \leftarrow p_4 \end{array}.$$

Let x_p represent the traffic flow on path p, f_a denote the link load on link a and d_ω denote the traffic amount between the O/D pair ω. Thus the arc-flow vector f is given by

$$f = A^T x \tag{5.3}$$

and the O/D pair-traffic amount vector d is given by

$$d = B^T x. \tag{5.4}$$

Let $t(f) = \{t_a, a \in L\}$ be the vector of link travel costs, which is a function of the link flow. A user travelling on path p incurs a (path) travel cost θ_p. For given link travel cost vector t, the path travel cost vector θ is given by

$$\theta = At(f) \qquad \text{and thus} \qquad \theta(x) = At(A^T x). \tag{5.5}$$

Associated with every O/D pair ω, there is a travel disutility $\lambda_\omega(d)$. Since both the path costs and the travel disutilities are functions of the flow pattern x, the traffic network equilibrium problem is to seek the path flow pattern x^*:

$$x^* \geq 0, \qquad (x - x^*)^T F(x^*) \geq 0, \qquad \forall x \geq 0 \tag{5.6}$$

where

$$F_p(x) = \theta_p(x) - \lambda_\omega(d(x)), \qquad \forall\, \omega,\ \ p \in P_\omega. \tag{5.7}$$

Using matrices A and B, a compact form of mapping is $F(x) = At(A^T x) - B\lambda(B^T x)$. The problem is a NCP. We take some test examples from [14] in which the disutility function $\lambda_\omega(d)$ is given by

$$\lambda_\omega(d) = -m_\omega d_\omega + q_\omega, \quad \forall \omega. \tag{5.8}$$

Example 1. *The first traffic equilibrium problem (example 7.4 in [14]) is consisted of 20 nodes, 28 links and 8 O/D pairs. Its network is depicted in Figure 5.2. The user cost of traversing link a is given in Table 5.1. The O/D pairs, the coefficients m_ω and q_ω in the disutility function (5.8) and numbers of paths of each O/D pair for this problem are given in Table 5.2. Since there are together 49 paths for the 8 given O/D pairs, the dimension of the variable x is 49, and the path-arc incidence matrix A and the path-O/D pair incidence matrix B is a* 49×28 *matrix and a* 49×8 *matrix, respectively.*

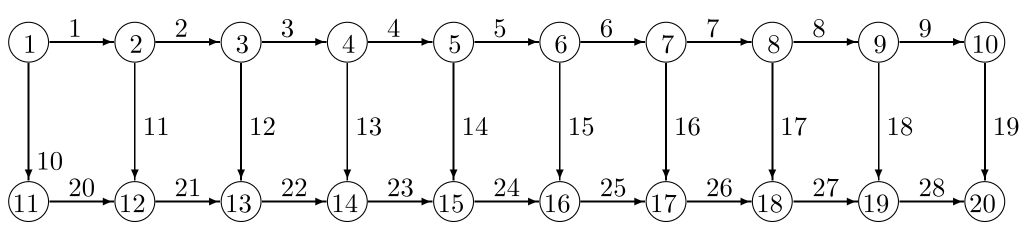


Fig. 5.2. A directed network with 20 nodes and 28 links in Example 1

Table 5.1. The link traversing cost functions $t_a(f)$ in Example 1

$t_1(f) = 5 \cdot 10^{-6} f_1^4 + 0.5f_1 + 0.2f_2 + 50$	$t_{15}(f) = 3 \cdot 10^{-6} f_{15}^4 + 0.9f_{15} + 0.2f_{14} + 20$
$t_2(f) = 3 \cdot 10^{-6} f_2^4 + 0.4f_2 + 0.4f_1 + 20$	$t_{16}(f) = 0.8f_{16} + 0.5f_{12} + 30$
$t_3(f) = 5 \cdot 10^{-6} f_3^4 + 0.3f_3 + 0.1f_4 + 35$	$t_{17}(f) = 3 \cdot 10^{-6} f_{17}^4 + 0.7f_{17} + 0.2f_{15} + 45$
$t_4(f) = 3 \cdot 10^{-6} f_4^4 + 0.6f_4 + 0.3f_5 + 40$	$t_{18}(f) = 0.5f_{18} + 0.1f_{16} + 30$
$t_5(f) = 6 \cdot 10^{-6} f_5^4 + 0.6f_5 + 0.4f_6 + 60$	$t_{19}(f) = 0.8f_{19} + 0.3f_{17} + 60$
$t_6(f) = 0.7f_6 + 0.3f_7 + 50$	$t_{20}(f) = 3 \cdot 10^{-6} f_{20}^4 + 0.6f_{20} + 0.1f_{21} + 30$
$t_7(f) = 8 \cdot 10^{-6} f_7^4 + 0.8f_7 + 0.2f_8 + 40$	$t_{21}(f) = 4 \cdot 10^{-6} f_{21}^4 + 0.4f_{21} + 0.1f_{22} + 40$
$t_8(f) = 4 \cdot 10^{-6} f_8^4 + 0.5f_8 + 0.2f_9 + 65$	$t_{22}(f) = 2 \cdot 10^{-6} f_{22}^4 + 0.6f_{22} + 0.1f_{23} + 50$
$t_9(f) = 10^{-6} f_9^4 + 0.6f_9 + 0.2f_{10} + 70$	$t_{23}(f) = 3 \cdot 10^{-6} f_{23}^4 + 0.9f_{23} + 0.2f_{24} + 35$
$t_{10}(f) = 0.4f_{10} + 0.1f_{12} + 80$	$t_{24}(f) = 2 \cdot 10^{-6} f_{24}^4 + 0.8f_{24} + 0.1f_{25} + 40$
$t_{11}(f) = 7 \cdot 10^{-6} f_{11}^4 + 0.7f_{11} + 0.4f_{12} + 65$	$t_{25}(f) = 3 \cdot 10^{-6} f_{25}^4 + 0.9f_{25} + 0.3f_{26} + 45$
$t_{12}(f) = 0.8f_{12} + 0.2f_{13} + 70$	$t_{26}(f) = 6 \cdot 10^{-6} f_{26}^4 + 0.7f_{26} + 0.8f_{27} + 30$
$t_{13}(f) = 10^{-6} f_{13}^4 + 0.7f_{13} + 0.3f_{18} + 60$	$t_{27}(f) = 3 \cdot 10^{-6} f_{27}^4 + 0.8f_{27} + 0.3f_{28} + 50$
$t_{14}(f) = 0.8f_{14} + 0.3f_{15} + 50$	$t_{28}(f) = 3 \cdot 10^{-6} f_{28}^4 + 0.7f_{28} + 65$

Table 5.2. The O/D pairs and the parameters in (5.8) of Example 1

No. of the Pair	1	2	3	4	5	6	7	8
(O, D)	$(1, 20)$	$(1, 19)$	$(2, 17)$	$(4, 20)$	$(6, 19)$	$(2, 20)$	$(2, 13)$	$(3, 14)$
m_ω	0.5	0.6	0.1	0.6	1	1	0.5	0.4
q_ω	100	200	500	100	500	200	100	200
No. of the Paths	10	9	6	7	4	9	2	2

Example 2. *The second example (Example 7.5 in [14]) is consisted of 25 nodes, 37 links and 6 O/D pairs. The network is depicted in Figure 5.3. The user cost of traversing link a is given in Table 5.3. The O/D pairs, the coefficients m_ω and q_ω in the disutility function (5.8) and numbers of paths of each O/D pair for this problem are given in Table 5.4. Since there are together 55 paths for the 6 given O/D pairs, the dimension of the variable x is 55, and the path-arc incidence matrix A and the path-O/D pair incidence matrix B is a 55×37 matrix and a 55×6 matrix, respectively.*

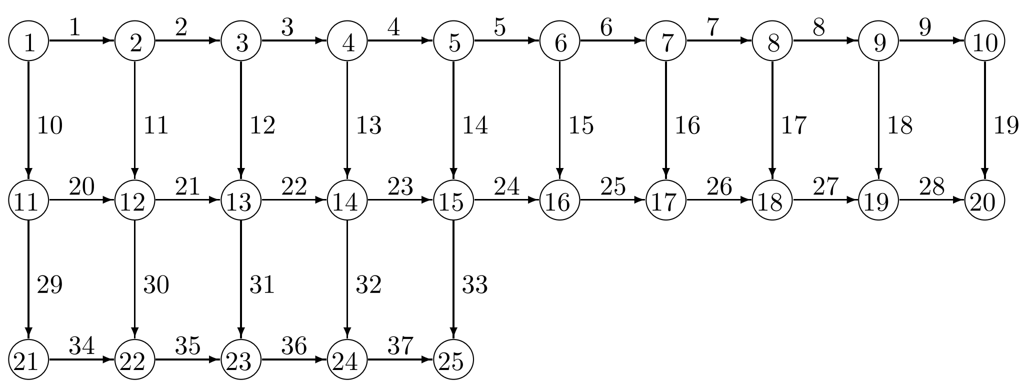


Fig. 5.3. A directed network with 25 nodes and 37 links in Example 2

Table 5.3. The link traversing cost functions in Example 2.

$t_1(f), t_2(f), \ldots, t_{28}(f)$ are given as in Table 5.3.	
$t_{29}(f) = 3 \cdot 10^{-6} f_{29}^4 + 0.3f_{29} + 0.1f_{30} + 45$	$t_{34}(f) = 6 \cdot 10^{-6} f_{34}^4 + 0.7f_{34} + 0.3f_{30} + 55$
$t_{30}(f) = 4 \cdot 10^{-6} f_{30}^4 + 0.7f_{30} + 0.2f_{31} + 60$	$t_{35}(f) = 3 \cdot 10^{-6} f_{35}^4 + 0.8f_{35} + 0.3f_{32} + 60$
$t_{31}(f) = 3 \cdot 10^{-6} f_{31}^4 + 0.8f_{31} + 0.1f_{32} + 75$	$t_{36}(f) = 2 \cdot 10^{-6} f_{36}^4 + 0.8f_{36} + 0.4f_{31} + 75$
$t_{32}(f) = 6 \cdot 10^{-6} f_{32}^4 + 0.8f_{32} + 0.3f_{33} + 65$	$t_{37}(f) = 6 \cdot 10^{-6} f_{37}^4 + 0.5f_{37} + 0.1f_{36} + 35$
$t_{33}(f) = 4 \cdot 10^{-6} f_{33}^4 + 0.9f_{33} + 0.2f_{31} + 75$	

Table 5.4. The O/D pairs and the parameters in (5.8) of Example 2

No. of the Pair	1	2	3	4	5	6
(O, D)	$(1, 20)$	$(1, 25)$	$(2, 20)$	$(3, 25)$	$(1, 24)$	$(11, 25)$
m_ω	0.1	0.6	1	0.5	0.7	0.9
q_ω	100	80	200	600	800	700
No. of the Paths	10	15	9	6	10	5

The problems are solved by the proposed method. We take $x^0 = (1, 1, \ldots, 1)^T$ as starting point and stop criterion is

$$\frac{\| \min\{x, F(x)\}\|_\infty}{\| \min\{x^0, F(x^0)\}\|_\infty} \leq \varepsilon. \tag{5.9}$$

The number of iteration, the mapping evaluations, and the CPU time on a Notebook Computer IBM T40 for different ε are reported in Table 5.5.

Table 5.5. Numerical results for different ε.

Examples	No. of iterations			No. of F evaluations			CPU-time
	10^{-6}	10^{-7}	10^{-8}	10^{-6}	10^{-7}	10^{-8}	Average
Example 1	342	419	496	770	944	1117	0.14 Sec.
Example 2	352	436	516	790	979	1159	0.15 Sec.

The preliminary numerical experiments tell us that solutions are obtained in a moderate number of iterations. Theoretically, the number of evaluations of the mapping F per iteration is at least 2. From Table 5.5 we see that it is approximately equal to 2.2 in our test examples. This means, in order to satisfy the conditions in the prediction step, the number of trial steps is insignificant.

6. Conclusion

Based on the Logarithmic-Quadratic Proximal (LQP) method, we present an interior prediction-correction method for nonlinear complementarity problems (NCP). By solving the *LQP system* approximately, the predictor can be obtained easily. Then the corrector is computed directly via an explicit formula derived from the original LQP method. The totally computational cost of the proposed method is very tiny, thus the method is applicable in practice. In addition, we prove that it is meaningful to find the optimal value of α in the correction step in both theoretical and practical senses. Preliminary numerical results show that the proposed method is a practical method for large scale NCP. How to design some LQP method-based algorithms for variational inequalities and maximal monotone operator may be interesting research topics in the future.

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