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NATURAL SUPERCONVERGENT POINTS OF EQUILATERAL TRIANGULAR FINITE ELEMENTS – A NUMERICAL EXAMPLE *1)

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Abstract

A numerical test case demonstrates that the Lobatto and the Gauss points are not natural superconvergent points of the cubic and the quartic finite elements under equilateral triangular mesh for the Poisson equation.

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1. Introduction

Natural superconvergent points are those points where the rate of convergence exceeds the best possible global rate without post-processing. Research in this area started in the early 70s, and may even trace back to the late 60s (see [5, 6, 7, 8, 9, 12, 14] and references therein). Consider the C^0 finite element approximation for second-order elliptic equations. It is well known that the Lobatto points and the Gauss points are superconvergent points for function values and derivatives, respectively. This result is valid for the one dimensional case, as well as for the tenser-product space in higher dimensional settings. As for triangular elements, the situation is much more complicated. Earlier researches focused only on lower-order elements, namely, the linear and the quadratic elements under strongly regular meshes.

In the mid-90s, two systematic methods to find superconvergent points were developed. One was the symmetry theory due to Schatz-Sloan-Wahlbin [11, 12]. By this theory, superconvergence occurs at mesh symmetry centers. One advantage of the symmetry theory is its generality, since it is applicable in all dimensions. Nevertheless, this theory does not say that there is no other superconvergence points, and therefore it is not conclusive. Almost at the same time, another approach was proposed by Babuška-Strouboulis *et al*[2, 3]. They established a theoretical framework which narrows the task of locating superconvergence points to finding intersections of some polynomial contours on a "master cell". The actual procedure was carried out by a computer algorithm without explicitly constructing those polynomials, and the computed superconvergent points were reported up to 10 digits. This approach is called the "computer-based proof". Using that computer algorithm, they predicted all derivative superconvergence points for the Poisson equation, the Laplace equation, and the linear elasticity equation under four different triangular mesh patterns for polynomial finite element spaces of degrees up to 7. The advantage of this approach is that it is conclusive.

Along the line of the computer-based proof, Zhang proposed an analytic approach which constructs explicitly the needed polynomials through an orthogonal decomposition [13]. Using

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this approach, Lin-Zhang studied superconvergent points for triangular elements [10]. Their results verify that the computed data for triangular elements in [3] have 9 digits of accuracy except for one pair (with 8-7 accurate digits). In addition, they reported superconvergent points for function values, which are not discussed in the computer-based proof. The Lin-Zhang approach has the following advantages:

1. The superconvergent points are computed with high accuracy. With polynomials explicitly given, one can easily verify how accurate those points are.

2. It can be readily verified. With paper and pencil, an interested reader may check if those polynomials are indeed the periodic finite element solutions on master cells. As for the superconvergent points, when polynomials are explicitly given, a root finding can be done analytically for lower-order cases (which are the most interesting cases anyway).

3. It provides more insight. An orthogonal polynomial basis functions on triangular meshes is constructed in a systematic way, which reveals the structure of the periodic finite element solution.

4. It can be generalized to 3D cases, where superconvergence results are relatively scarce.

As a related work, Chen investigated superconvergent points in the triangular elements of the regular pattern using the orthogonal decomposition and element analysis technique [5]. As an example of his approach, superconvergent points in the cubic element are calculated for the Poisson equation as well as for the Laplace equation. These points are consistent with the symmetry theory and the computer-based proof.

According to the above review, investigation of natural superconvergent points for triangular elements seems to be over except for one special case, the equilateral triangulation. Yet, this is the most important case, since automatic mesh generators based on the Delaunay triangulation produce nearly equilateral triangles for most part of the mesh. There was an interesting result due to Blum-Lin-Rannacher in 1986 [4], which revealed that the convergent rate for linear element at element vertices is of $O(h^4)$ for equilateral mesh. This rate is two orders higher than that of the regular mesh pattern. Therefore, it is natural to ask if there are other special properties associated with equilateral triangular mesh for higher-order finite elements. Motivated with the fact that the Lobatto (Gauss) points in the quadratic element are supercovergent points of function (derivative) values along element edges (this is true for both regular and equilateral triangulation [1, 14]), one may wonder if it is also true for higher-order elements. The current paper provides a negative answer to this question. With a non-trivial numerical example, we demonstrate that neither the Lobatto nor the Gauss points are natural superconvergent points for the cubic and the quartic elements in the equilateral meshes. The convergence rate is the same for the regular and the equilateral triangulations with only one exception, the linear element.

2. A Numerical Example

Define an equilateral triangular domain Ω enclosed by the three straight lines $y = 0$, $y = \sqrt{3}x$, and $y = \sqrt{3}(1 - x)$. Consider the boundary value problem

$$-\Delta u = f \quad \text{in} \ \ \Omega, \qquad u = 0 \quad \text{on} \ \ \partial\Omega. \tag{2.1}$$

We choose f such that $u = y(y - \sqrt{3}x)(y + \sqrt{3}x - \sqrt{3})e^{x+y}$, and then solve (2.1) by the linear, the quadratic, the cubic, and the quartic finite elements under uniform triangulation of Ω by dividing each side of Ω to n sub-intervals. We set $h^{-1} = n = 8$ (see Figure 1) for the initial mesh, and use the regular refinement with bi-section strategy. Using this kind of meshes, we will solve (2.1) with MATLAB 6.5.1 and FEMLAB 2.3. Earlier versions of MATLAB may lead to different results as it uses Gaussian elimination to solve unrecognized sparse linear systems. MATLAB 6.5.1 has a new solver that supports more types of sparse linear systems including the systems arising in solving (2.1).

We calculate the error $e_h = u - u_h$ at the Lobatto points and $\partial_t e_h = \vec{t} \cdot \nabla(u - u_h)$ at the Gauss points. Here $\vec{t}$ is the tangential unit vector. Table 1 lists the Lobatto and the Gauss points on $[-1, 1]$, and a suitable linear mapping is used to compute the Lobatto and the Gauss points on element edges. For each given $h = 1/n$, we denote $L(h)$, the set of all the Lobatto points on element edges; and $G(h)$, the set of all the Gauss points on element edges. Define the sub-domain

$$\Omega_H = \{(x, y) \in \Omega : dist((x, y), \partial\Omega) \geq H)\}$$

where $H > 0$ is a fixed constant (see Figure 2), and $H = 1/8$ in this particular example. Let ε_h denote the set of element edges in the mesh, and we define the following measures

$$E_h(e_h, \Omega_H) = \max_{\boldsymbol{z} \in L(h) \cap \Omega_H} |(e_h)(\boldsymbol{z})|, \quad E_h(\partial_t e_h, \Omega_H) = \max_{\boldsymbol{z} \in G(h) \cap \Omega_H} |\partial_t(e_h)(\boldsymbol{z})|;$$

$$\bar{E}_h(e_h, \Omega_H) = \frac{1}{|L(h)|} \sum_{\boldsymbol{z} \in L(h) \cap \Omega_H} |(e_h)(\boldsymbol{z})|, \quad \bar{E}_h(\partial_t e_h, \Omega_H) = \frac{1}{|G(h)|} \sum_{\boldsymbol{z} \in G(h) \cap \Omega_H} |\partial_t(e_h)(\boldsymbol{z})|;$$

$$\tilde{E}_h(e_h, \Omega_H) = \sqrt{\sum_{\ell \in \varepsilon_h \cap \Omega_H} \int_\ell e_h^2 ds}, \quad \tilde{E}_h(\partial_t e_h, \Omega_H) = \sqrt{\sum_{\ell \in \varepsilon_h \cap \Omega_H} \int_\ell (\partial_t e_h)^2 ds}.$$

Here $|G(h)|$ is the cardinal value, i.e., the number of points in $G(h)$. The integrals in $\tilde{E}_h(e_h, \Omega_H)$ and $\tilde{E}_h(\partial_t e_h, \Omega_H)$ are computed using the Gauss-Lobatto quadrature and the Gauss-Legendre quadrature, respectively. Table 2 and Table 3 collect all computed error values for different $L(h)$ and $G(h)$, respectively, where the relative graphs (in log-log scale) are plotted in Figures 3-6. All measures of e_h and $\partial_t e_h$ lead to the same conclusion, and hence we focus on $E_h(e_h, \Omega_H)$ and $E_h(\partial_t e_h, \Omega_H)$. Based on the computed data, we summarize our results in Table 4 and draw the following conclusions:

1. For the cubic and the quartic finite element approximation of the Poisson equation, the Lobatto points are not superconvergent points for $u - u_h$, and the Gauss points are not superconvergent points for $\partial_t(u - u_h)$.

2. For the quadratic, the cubic, and the quartic elements, the convergence behavior of equilateral mesh is the same as that of the regular mesh.

3. The only special case for equilateral mesh is the linear element, when the convergence rate is $O(h^4)$ at element vertices.

4. Convergence behavior of average error is the same as that of point-wise error.

5. Mesh symmetry points in Ω_H are element vertices and mid-edges, but the lobatto points for the linear element are only the element vertices. That explains why the convergence rate of $E_h(e_h, \Omega_H)$ drops from 4 to 2 when we switch from the Lobatto points to symmetry points in the linear element as shown in Table 4.

6. If we replace Ω_H with Ω in all the previous error measures, the superconvergence is lost in two cases: superconvergence for e_h at mesh symmetry points in the quartic element and superconvergence for $\partial_t e_h$ at mesh symmetry points in the cubic element.

Table 1: Lobatto and Gauss points on $[-1, 1]$ for elements of order 1 through 4

Element order	Labatto points	Gaussian points
1	± 1	0
2	$0, \pm 1$	$\pm \frac{1}{\sqrt{3}}$
3	$\pm \frac{1}{\sqrt{5}}, \pm 1$	$0, \pm\sqrt{\frac{3}{5}}$
4	$0, \pm\sqrt{\frac{3}{7}}, \pm 1$	$\pm\sqrt{\frac{3}{7} \pm \frac{4}{7}\sqrt{\frac{3}{10}}}$

Table 2: Various error measures for e_h in Ω_H

n	Cubic element			Quartic element		
	$\tilde{E}_h(e_h,\Omega_H)$	$\bar{E}_h(e_h,\Omega_H)$	$E_h(e_h,\Omega_H)$	$\tilde{E}_h(e_h,\Omega_H)$	$\bar{E}_h(e_h,\Omega_H)$	$E_h(e_h,\Omega_H)$
8	9.1561E-007	1.3541E-006	4.3632E-006	2.9158E-008	2.0899E-008	5.6626E-008
16	1.2569E-007	7.2703E-008	3.2206E-007	2.0769E-009	6.6178E-010	2.2385E-009
32	1.2833E-008	4.3698E-009	2.1923E-008	1.1056E-010	2.0852E-011	7.8289E-011
64	1.1545E-009	2.6768E-010	1.3691E-009	4.8401E-012	6.5100E-013	2.4815E-012

Table 3: Various error measures for $\partial_t e_h$ in Ω_H

n	Cubic element			Quartic element		
	$\tilde{E}_h(\partial_t e_h,\Omega_H)$	$\bar{E}_h(\partial_t e_h,\Omega_H)$	$E_h(\partial_t e_h,\Omega_H)$	$\tilde{E}_h(\partial_t e_h,\Omega_H)$	$\bar{E}_h(\partial_t e_h,\Omega_H)$	$E_h(\partial_t e_h,\Omega_H)$
8	1.0191E-004	8.6471E-005	1.6244E-004	1.5243E-006	1.3290E-006	3.3746E-006
16	2.8295E-005	1.0704E-005	2.4089E-005	2.1569E-007	8.6482E-008	2.5139E-007
32	5.9710E-006	1.3384E-006	3.2692E-006	2.2936E-008	5.5223E-009	1.7008E-008
64	1.0371E-006	1.6634E-007	4.0779E-007	2.0076E-009	3.4686E-010	1.0613E-009

Table 4: Convergence rates in Ω_H

Element order	Convergence rate for $E_h(e_h,\Omega_H)$		Convergence rate for $E_h(\partial_t e_h,\Omega_H)$	
	Lobatto Points	Symmetry points	Gaussian Points	Symmetry points
1	4 (superconvergence)	2 (superconvergence)	2 (no superconvergence)	2 (no superconvergence)
2	4 (superconvergence)	4 (superconvergence)	3 (superconvergence)	2 (no superconvergence)
3	4 (no superconvergence)	4 (no superconvergence)	3 (no superconvergence)	4 (superconvergence)
4	5 (no superconvergence)	6 (superconvergence)	4 (no superconvergence)	4 (no superconvergence)

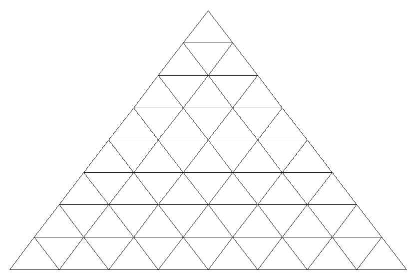

Figure 1: Initial mesh when $n = 8$

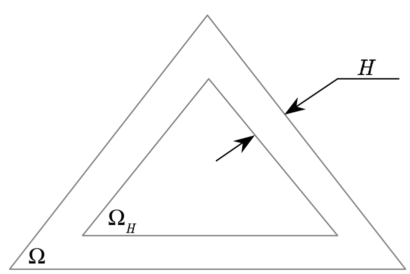

Figure 2: The subdomain Ω_H

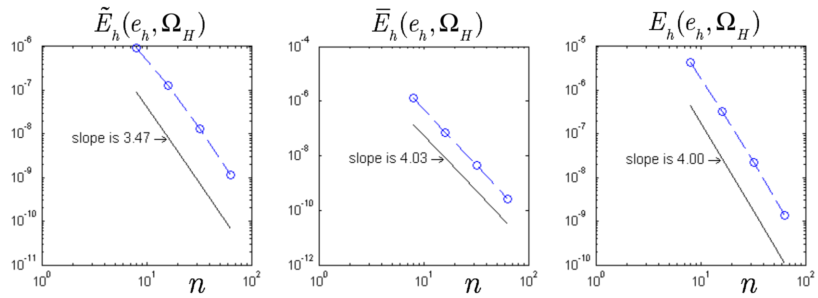

Figure 3: Various error measures for e_h in Ω_H - cubic element

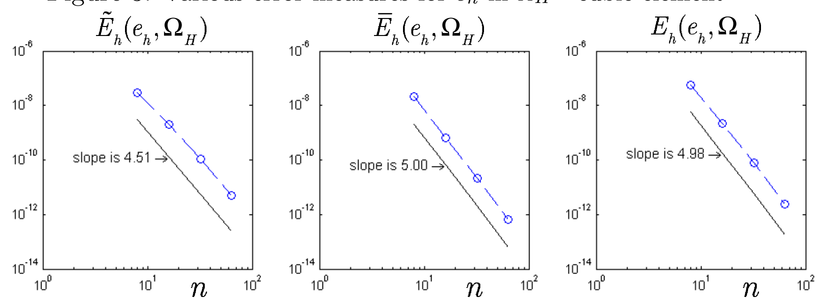

Figure 4: Various error measures for e_h in Ω_H - quartic element

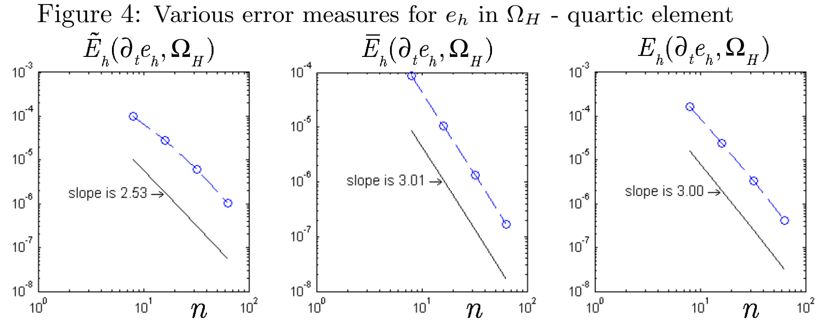

Figure 5: Various error measures for $\partial_t e_h$ in Ω_H - cubic element

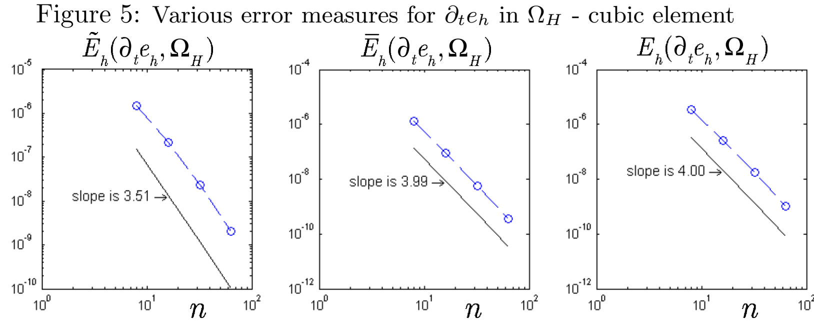

Figure 6: Various error measures for $\partial_t e_h$ in Ω_H - quartic element

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