Journal of Computational Mathematics, Vol.24, No.1, 2006, 9–18.

SPLITTING EXTRAPOLATIONS FOR SOLVING BOUNDARY INTEGRAL EQUATIONS OF LINEAR ELASTICITY DIRICHLET PROBLEMS ON POLYGONS BY MECHANICAL QUADRATURE METHODS *1)

Jin Huang
(*University of electronic & science technology of China, Chengdu 610054; Mathematical College, Sichuan University, Chengdu 610064, China*)

Tao Lu
(*Mathematical College, Sichuan University, Chengdu 610064, China*)

Abstract

Taking h_m as the mesh width of a curved edge Γ_m $(m = 1, ..., d\ )$ of polygons and using quadrature rules for weakly singular integrals, this paper presents mechanical quadrature methods for solving BIES of the first kind of plane elasticity Dirichlet problems on curved polygons, which possess high accuracy $O(h_0^3)$ and low computing complexities. Since multivariate asymptotic expansions of approximate errors with power h_i^3 $(i = 1, 2, ..., d)$ are shown, by means of the splitting extrapolations high precision approximations and a posteriori estimate are obtained.

Mathematics subject classification: 65N38, 65R20, 75C35
Key words: Splitting extrapolation; Linear elasticity Dirichlet problem; Boundary integral equation of the first kind; Mechanical quadrature method

1. Introduction

Let $\Omega \subset R^2$ be curved polygons with the edges $\cup_{m=1}^{d}\Gamma_m = \Gamma$. Consider plane linear elasticity Dirichlet problems:

$$\begin{cases} Au \equiv \mu\Delta u + (\mu+\lambda)\text{graddiv}u = 0, \text{ in } \Omega, \\ u = u^0, \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2)$ is the displacement field, and μ and λ are Lame constants. By using the single layer potential theory, (1.1) can be converted into the following boundary integral equation system (BIES) of the first kind[2,3,17,18]

$$\int_\Gamma u_{ij}^*(y,x)p_j(x)ds_x = \alpha_{ij}(y)u_j^0(y) + \int_\Gamma p_{ij}^*(y,x)u_j^0(x)ds_x,\ i = 1, 2,\ \forall y \in \Gamma, \tag{1.2}$$

where $y = (y_1,y_2)$, $r = |y-x| = [(y_1-x_1)^2 + (y_2-x_2)^2]^{1/2}$; $\alpha_{ij}(y)$ is a constant dependent on $y \in \Gamma$;

$$\begin{cases} u_{ij}^* = [-(3-4\nu)(\ln r)\delta_{ij} + r_{,.i}r_{,j}]/[8\pi\mu(1-\nu)], \\ p_{ij}^* = -[\partial r/\partial n((1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}) + (1-2\nu)(n_i r_{,j} - n_j r_{,i})]/[4\pi(1-\nu)r] \end{cases} \tag{1.3}$$

are Kelvin's fundamental solutions[2,3]; $\nu = \lambda/[2(\lambda+\mu)]$ is Poisson$'$s ratio; $n = (n_1, n_2)$ is the unit outward normal vector on Γ; $r_{,i} = \partial r/\partial x_i$ and repeated subscript means a summation from 1 to 2. Obviously, the equation (1.2) is a weakly singular boundary integral equation system

* Received January 14, 2004; Final revised June 6, 2005 .
1) Supported by Natioal Science Foundation of China (10171073).

of the first kind. If the traction $p=(p_1,p_2)^T$ is solved by (1.2), then the displacement vectors and stress tensor components can be calculated by

$$u_i(y)=\int_\Gamma u^*_{ij}(y,x)p_j(x)ds_x-\int_\Gamma p^*_{ij}(y,x)u^0_j(x)ds_x,\forall y\in\Omega, \tag{1.4}$$

$$\sigma_{ij}(y)=\int_\Gamma u^*_{ijk}(y,x)p_k(x)ds_x-\int_\Gamma p^*_{ijk}(y,x)u^0_k(x)ds_x,\forall y\in\Omega, \tag{1.5}$$

where

$$\left\{\begin{array}{l} u^*_{ijk}=[(1-2\nu)(r_{,j}\delta_{ki}+r_{,i}\delta_{kj}-r_{,k}\delta_{ij})+2r_{,i}r_{,j}r_{,k})]/[4\pi\mu(1-\nu)r].\\ p^*_{ijk}=\mu/[2\pi(1-\nu)r^2]\{2\partial r/\partial n[(1-2\nu)\delta_{ij}r_{,k}+\nu(\delta_{ik}r_{,j}+\delta_{jk}r_{,i})-4r_{,i}r_{,j}r_{,k}]\\ +2\nu(n_ir_{,j}r_{,k}+n_jr_{,i}r_{,k})+(1-2\nu)(2n_kr_{,i}r_{,j}+\delta_{ik}n_j+\delta_{jk}n_i)-(1-4\nu)\delta_{ij}n_k\}. \end{array}\right. \tag{1.6}$$

Unfortunately, the homogeneous equations corresponding to (1.2) might admit non-trivial solutions[7,23]. For simplicity, in the paper we assume

$$d(\Omega)=\max_{x,y\in\Gamma}|x-y|<1, \tag{1.7}$$

which can ensure that the solution of (1.2) is unique (see Remark 1).

So far the numerical methods for solving (1.2) are Galerkin methods[7,22] and collocation methods [24] based on the projective theory, which have been applied to many engineering computations and application software. However there exist the following disadvantages: (1) Since the discrete matrix is full, the generating each element has to calculate an improper integral for the collocation method or a double improper integral for the Galerkin method, which implies that the work calculating discrete matrix is so large as greatly to exceed to solve the discrete equations. (2) The order of accuracy is lower, especially, for concave domain problems[22,24]. Obviously, using mechanical quadrature methods for solving (1.2) can save a lot of computations generating the discrete matrix. However, the convergent proof of the mechanical quadrature methods appears to be some difficulties without the projective theory as a mathematical tool. So far there are only a few papers to discuss the mechanical quadrature methods of the second-kind BIE[10]. In the paper, we propose a high accuracy mechanical quadrature method for solving the first-kind BIES of plane elasticity Dirichlet problems on curved polygons, which is based on the quadrature rules of the weakly singular periodic functions and the periodical transformations. Using the methods, we not only get the convergence rate of approximations, but also prove that the errors of the approximations possess the multivariate asymptotic expansions with power h_i^3 $(i=1,2,...,d)$ given mesh widths. Thus as soon as some discrete equations with respect to some coarse mesh partitions are solved in parallel, the accuracy order of approximations will be improved by splitting extrapolation methods (SEM). Moreover, a posteriori asymptotic error estimate as adaptive algorithms is derived.

SEM[11,12,13] based on the multivariate asymptotic expansion of the error is a new extrapolation technique to solve large problems in parallel, which possesses a high order of accuracy, a high degree of parallelism and an almost optimal computational complexity. Since Lin and Lü published the first paper [12] in 1983, SEM has been applied to many multidimensional problems,e.g., the multidimensional numerical integrals[11,15], finite differential methods[11] and finite element methods[11]. Using Galerkin methods, Rüde and Zhou[19] gave SEM for solving the second kind BIE of Laplace's equation on polygonal domains. In the paper, SEM is first applied to solve the first kind BIES of the elasticity problems on curved polygons.

This paper is organized as follows: in Section 2 we derive a new integral equation system of the first kind with weakly singular kernels under the periodical transformations; in Section 3 the quadrature method and its convergent proof are given; in Section 4 the multivariate asymptotic expansions of the errors are derived, and SEM and a posteriori error estimate are got; in Section 5 some numerical examples are shown.

Remark 1. If (1.7) is satisfied, then it is easily verified that the matrix

$$\mathfrak{S}=ı8\pi\mu(1-\nu)\{-(3-4\nu)\ln r\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}r_{,1}r_{,1}&r_{,1}r_{,2}\\r_{,1}r_{,2}&r_{,2}r_{,2}\end{bmatrix}\}.$$

is uniformly positive definite and $\mathfrak{G} \geq (3-4\nu)\ln d(\Omega)/[8\pi\mu(1-\nu)]\mathrm{diag}(1,1) = \kappa\mathrm{diag}(1,1)$. Hence under (1.7) we have

$$\int_\Gamma\int_\Gamma u^*_{ij}(y,x)p_j(x)p_i(y)ds_x ds_y \geq \kappa\sum_{j=1}^{2}\int_\Gamma p_j^2(x)ds_x$$

which implies that the solution of (1.2) is unique.

Remark 2. If (1.7) is not satisfied, then replacing (1.2), we consider the following BIES: find $p_j(x)$, γ_j $(j=1,2)$ and β satisfying

$$\int_\Gamma u^*_{ij}(y,x)p_j(x)ds_x = \alpha_{ij}(y)u^0_j(y)$$
$$+\int_\Gamma p^*_{ij}(y,x)u^0_j(x)ds_x + \gamma_i + \beta z_i,\ \ i=1,2,\ \ \forall y\in\Gamma, \tag{1.8a}$$

and

$$\begin{cases} \int_\Gamma p_j(x)ds_x = 0, i=1,2, \\ \int_\Gamma (p_1 y - p_2 x)ds_x = 0, \end{cases} \tag{1.8b}$$

where

$$z_1 = y,\ \text{and}\ z_2 = x.$$

It has been shown by [8] that the solution of the first kind BIES (1.8) is unique. The assumption (1.7) is not essential, so our method can be applied in general case.

2. Weakly Singular BIES of the First Kind

Define the boundary integral operators on Γ_m as follows

$$(K^{qm}_{ij}v)(y) = \int_{\Gamma_m} p^*_{ij}(y,x)v(x)ds_x, y\in\Gamma_q,\ m,q=1,...,d;\ i,j=1.2, \tag{2.1a}$$

$$(H^{qm}_{ij}v)(y) = \int_{\Gamma_m} u^*_{ij}(y,x)v(x)ds_x, y\in\Gamma_q,\ m,q=1,...,d;\ i,j=1.2, \tag{2.1b}$$

then (1.2) can be converted into an operator equation system

$$\sum_{m=1}^{d}\begin{pmatrix} H^{qm}_{11} & H^{qm}_{12} \\ H^{qm}_{21} & H^{qm}_{22}\end{pmatrix}\begin{pmatrix} p^m_1 \\ p^m_2\end{pmatrix} = \begin{pmatrix} f^q_1 \\ f^q_2\end{pmatrix}, q=1,...,d, \tag{2.2a}$$

where

$$f^q_j = \sum_{i=1}^{2}\alpha^q_{ji}u^{0q}_i + \sum_{i=1}^{2}\sum_{m=1}^{d}K^{qm}_{ji}u^{0m}_i, j=1,2. \tag{2.2b}$$

Let T_m denote the measurable length of Γ_m and Γ_m can be described by the parameter mapping $x_m(s) = (x_{m1}(s), x_{m2}(s)) : [0,T_m] \to \Gamma_m$ with $|x'_m(s)|^2 = |x'_{m1}(s)|^2 + |x'_{m2}(s)|^2 > 0$. Let α be a positive integer and

$$\phi_\alpha(t) = \vartheta_\alpha(t)/\vartheta_\alpha(1), \vartheta_\alpha(t) = \int_0^t (\sin\pi t)^\alpha ds \tag{2.3a}$$

be the periodical transformation functions[21]. Then under the periodical transformation

$$s = T_m\phi_\alpha(t) : [0,1] \to [0,T_m], \alpha\in N, \tag{2.3b}$$

(2.2) is an integral equation system on [0,1]. Define the following integral operators on [0,1] transformed by Γ_m

$$(A^m_{00}\bar{p}^m_i)(t) = -c_0\int_0^1 \ln|2e^{-1/2}\sin\pi(t-\tau)|\bar{p}^m_i(\tau)d\tau; \tag{2.4a}$$

$$(B_{00}^{qm}\bar{p}_i^m)(t)=\left\{\begin{array}{l} -c_0\int_0^1 \ln|\frac{\bar{x}_m(t)-\bar{x}_m(\tau)}{2e^{-1/2}\sin\pi(t-\tau)}|\bar{p}_i^m(\tau)d\tau, q=m, \\ -c_0\int_0^1 \ln|\bar{x}_q(t)-\bar{x}_m(\tau)|\bar{p}_i^m(\tau)d\tau, q\neq m; \end{array}\right] \tag{2.4b}$$

$$(B_{ij}^{qm}\bar{p}_j^m)(t)=c_1\int_0^1 \frac{(\bar{x}_{qi}(t)-\bar{x}_{mi}(\tau))(\bar{x}_{qj}(t)-\bar{x}_{mj}(\tau))}{|\bar{x}_m(t)-\bar{x}_q(\tau)|^2}\bar{p}_j^m(\tau)d\tau,\ i,j=1,2 \tag{2.4c}$$

where $\bar{x}_m(t)=x_m(T_m\phi_\alpha(t))$, $\bar{p}_j^m(t)=p_j^m(\bar{x}_m(t))|\bar{x}_m'(t)|T_m\phi_\alpha'(t)$, $m,\ q=1,...,d,\ i;\ j=1,\ 2,$ $c_0=(3-4\nu)/[8\pi(1-\nu)\mu]$, and $c_1=1/[8\pi(1-\nu)\mu]$, so that (2.2) can be expressed by

$$(A+B)\bar{p}=F, \tag{2.5}$$

where $\bar{p}=(\bar{p}^1(t),...,\bar{p}^d(t))^T$, $\bar{p}^m(t)=(\bar{p}_1^m(t),\bar{p}_2^m(t))^T$; $F=(F^1(t),...,F^d(t))^T$, $F^m(t)=(F_1^m(t),$ $F_2^m(t))^T$, $F_j^m(t)=f_j^m(\bar{x}_m(t))$, and $A=$diag$(A_{11},...,A_{dd})$, $B=(B_{qm})_{q,m=1}^d$ are matrix operators with the block constructions

$$A_{mm}=\begin{pmatrix} A_{00}^m & 0 \\ 0 & A_{00}^m \end{pmatrix}, B_{qm}=\begin{pmatrix} B_{00}^{qm}+B_{11}^{qm} & B_{12}^{qm} \\ B_{21}^{qm} & B_{00}^{qm}+B_{22}^{qm} \end{pmatrix}. \tag{2.6}$$

Remark 3. In general the solution $p_j^m(\bar{x}(t))$ $(j=1,2;\ m=1,\cdots,d)$ of (1.2) may have the singularity at a corner $t=0$ or 1 of polygons, however as $\alpha\geq 3$, $\bar{p}_j^m(t)$ is sufficiently smooth. The reason is because under the periodical transformation (2.3) $\phi_\alpha'(t)$ has the order α of zeros at $t=0,1$ (see [21]).

3. Quadrature Method

Let $h_m=1/n_m$, $m=1,...,d$ be mesh widths and $t_r=\tau_r=(r+1/2)h_m, r=0,...,\ n_m-1$ be nodes. Since the kernel $b_{ij}^{qm}(t,\tau)$ of the operator B_{ij}^{qm}, $i,\ j=0,1,2;\ q,\ m=1,...,d$ is a smooth periodic function (see Lemma 2 and Lemma 3), using the repeated midpoint formula we construct Nyström approximation[6] of (2.4) as follows

$$(\hat{B}_{ijh}^{qm}\bar{p}_j^m)(t)=\tilde{c}h_m\sum_{r=0}^{n_m-1} b_{ij}^{qm}(t,\tau_r)\bar{p}_j^m(\tau_r), r=0,\cdots,n_m-1 \tag{3.1}$$

where

$$\tilde{c}=\left\{\begin{array}{l} -c_0 \text{ for } i,j=0 \\ c_1 \text{ for } i,j=1,2. \end{array}\right.$$

Moreover,we have the error estimate[6]

$$(B_{ij}^{qm}\bar{p}_j^m)(t)-(\hat{B}_{ijh}^{qm}\bar{p}_j^m)(t)=O(h_m^{2l}),\ \text{for } b_{ij}^{qm}(t,\tau)\in C^{2l}[0,1]^2. \tag{3.2}$$

If $\bar{p}_j^m\in C^{2l}(0,1)$, applying the quadrature formula[16,20] to the weakly singular integral operator A_{00}^m, we get its Fredholm approximation[4]

$$\begin{aligned}(A_{00}^{h_m}\bar{p}_j^m)(t_k)=&-c_0h_m\{\sum_{r=0,r\neq k}^{n_m-1}\ln|2e^{-1/2}\sin\pi(t_k-\tau_r)|\bar{p}_j^m(\tau_r)\\ &+\ln|2\pi e^{-1/2}h_m/(2\pi)|\bar{p}_j^m(t_k)\}, k=0,\cdots,n_m-1,\end{aligned} \tag{3.3}$$

where $A_{00}^{h_m}$ is a matrix. From [20] the error has an asymptotic expansion

$$(A_{00}^m\bar{p}_j^m)(t_k)-(A_{00}^{h_m}\bar{p}_j^m)(t_k)=-2c_0\sum_{\mu=1}^{l-1}\frac{\varsigma'(-2\mu)}{(2\mu)!}(\bar{p}_j^m)^{(2\mu)}(t_k)h_m^{2\mu+1}+O(h_m^{2l}), \tag{3.4}$$

where $\varsigma'(t)$ is the derivative of Riemann zeta function. Taking $t=t_\beta=(\beta+1/2)h_q$, $\beta=0,...,n_q-1$, and using the midpoint rule, the approximation of the right hand term F_j^q , $j=1,2;\ q=1,\cdots,d$, of (2.5) is

$$F_{jh}^q(t_\beta)=\sum_{i=1}^2\alpha_{ji}^q\bar{u}_i^q(t_\beta)+\sum_{i=1}^2\sum_{m=1}^d\sum_{r=0}^{n_m-1}h_m\bar{k}_{ji}^{qm}(t_\beta,\tau_r)\bar{u}_i^m(\tau_r), \tag{3.5}$$

where $\bar{u}_i^m(t) = u_j^{0m}(\bar{x}_m(t))|x_m^{'}(T_m\phi_\alpha(t))|T_m\phi_\alpha^{'}(t)$, and $\bar{k}_{ji}^{qm}(t,\tau) = k_{ji}^{qm}(\bar{x}_q(t), \bar{x}_m(\tau))$, $i,j = 1,2$; $q,m = 1,...,d$. Thus we get the approximately linear equation of (2.5)

$$(A^h + B^h)\bar{p}^h = F^h, \tag{3.6}$$

where $\bar{p}^h = (\bar{p}^{1h_1}, \ ... \ ,\bar{p}^{dh_d})^T$, $\bar{p}^{mh_m} = (\bar{p}_1^{mh_m}, \ \bar{p}_2^{mh_m})^T$; $\bar{p}_j^{mh_m} = (\bar{p}_j^m(t_0), ..., \bar{p}_j^m(t_{n_m-1}))^T$, $F^h = (F^{1h_1}, ..., F^{dh_d})^T$, $F^{mh_m} = (F_1^{mh_m}, \ F_2^{mh_m})^T$, and $F_j^{mh_m} = (F_{jh}^m(t_0), \ ..., \ F_{jh}^m(t_{n_m-1}))^T$, $j = 1,2$, are the vectors; $B^h = [B_{qm}^h]_{q,m=1}^d$ and A^h =diag$(A_{11}^{h_1}, ..., A_{dd}^{h_d})$ are the matrixes with the block constructions

$$A_{mm}^{h_m} = \begin{pmatrix} A_{00}^{h_m} & 0 \\ 0 & A_{00}^{h_m} \end{pmatrix}, B_{qm}^h = \begin{pmatrix} B_{00h}^{qm} + B_{11h}^{qm} & B_{12h}^{qm} \\ B_{21h}^{qm} & B_{00h}^{qm} + B_{22h}^{qm} \end{pmatrix}, \tag{3.7}$$

in which $B_{ijh}^{qm} = \tilde{c}h_m[b_{ij}^{qm}(t_k, t_l)]_{k,l=0}^{n_q-1,n_m-1}$. As soon as $\bar{p}^h$ is solved by (3.6), the approximations of the displacement vectors and the stress tensor can be computed by

$$u_i^h(y) = \sum_{j=1}^{2}\sum_{m=1}^{d}\sum_{r=0}^{n_m-1} h_m[u_{ij}^*(y, \bar{x}_m(\tau_r))\bar{p}_j^{mh_m} - p_{ij}^*(y, \bar{x}_m(\tau_r))\bar{u}_j^m(\tau_r)], y \in \Omega \tag{3.8}$$

and

$$\sigma_{ij}^h(y) = \sum_{k=1}^{2}\sum_{m=1}^{d}\sum_{r=0}^{n_m-1} h_m[u_{ijk}^*(y, \bar{x}_m(\tau_r))\bar{p}_j^{mh_m} - p_{ijk}^*(y, \bar{x}_m(\tau_r))\bar{u}_k^m(\tau_r)], \ y \in \Omega. \tag{3.9}$$

It is easily verified that the matrix defined by (3.3) is a circulant matrix[5]

$$A_{00}^{h_m} = c_0\bar{A}_{00}^{h_m}$$
$$= -c_0h_m\text{circular}(\ln(e^{-1/2}h_m), \ln(2e^{-1/2}\sin(\pi h_m)), ..., \ln(2e^{-1/2}\sin((n_m-1)\pi h_m))).$$

The following Lemma1 is known.

Lemma 1[14]**.** *The eigenvalue $\lambda_k, k = 0, ..., n_m - 1$ of $\bar{A}_{00}^{h_m}$ is positive, and there exists a positive constant c such that $\lambda_k > c$ if $n_m < 4$,or $\lambda_k > 1/(2n_m)$ if $n_m \geq 4$.*

From Lemma 1 we get the following corollary.

Corollary 1. *(1) The conditional number of $A_{00}^{h_m}$ is $O(n_m)$.*

(2) $A_{00}^{h_m}$ is invertible and the spectral norm estimate $||(A_{00}^{h_m})^{-1}|| = O(n_m)$.

(3) The conditional number of A^h is $O(n)$, where $n = \max(n_1, ..., n_d)$.

(4) A^h is invertible and the spectral norm estimate $||(A^h)^{-1}|| = O(n)$.

By corollary 1, (3.6) is equivalent to

$$(I + (A^h)^{-1}B^h)\bar{p}^h = (A^h)^{-1}F^h \tag{3.10}$$

Let S^{h_m} =span$\{e_j(t), j = 0, \cdots, n_{m-1}\} \subset C[0,1]$ be a piecewise linear function subspace with base points $\{t_i\}_{i=0}^{n_m-1}$, where $e_j(t)$, $j = 0, \ ..., \ n_m - 1$ is the base function satisfying $e_j(t_i) = \delta_{ji}$. Define a prolongation operator $I^{h_m} : \Re^{n_m} \to S^{h_m}$ satisfying

$$I^{h_m}z = \sum_{j=0}^{n_m-1} z_je_j(t), \forall z = (z_0, ..., z_{n_m-1}) \in \Re^{n_m}, \tag{3.11}$$

and a restricted operator $R^{h_m} : C[0,1] \to \Re^{n_m}$ satisfying

$$R^{h_m}u = (u(t_0), ..., u(t_{n_m-1})) \in \Re^{n_m}, \forall u \in C[0,1], \tag{3.12}$$

Corollary 2. *The operator sequence $\{I^{h_m}(A_{00}^{h_m})^{-1}R^{h_m}A_{00}^m : C^3[0,1] \to C[0,1] \}$ is uniformly bounded and convergent to the embedding operator I.*

Proof. For $\forall \ w \in C^3[0,1]$, we construct an operator equation

$$A_{00}^m w = f, \tag{3.13}$$

and its approximate equation: find $w^h \in \Re^{n_m}$ satisfying

$$(A_{00}^{h_m}w^h)(t_i) = f(t_i), \ i = 0, ..., n_m - 1.$$

Let $e \in \Re^{n_m}$ with $e(t_j) = w^h(t_j) - w(t_j)$, $j = 0, ..., n_m - 1$, then it satisfies the linear equation

$$A_{00}^{h_m}e = \varepsilon,$$

where $e^T = (e(t_0), ..., e(t_{n_m-1})), \varepsilon^T = (\varepsilon(t_0), ..., \varepsilon(t_{n_m-1}))$. From (3.4), we get

$$\varepsilon(t_i) = \int_0^1 \Lambda(t_i,\tau)w(\tau)d\tau - [\sum_{j=0,j\neq i}^{n_m-1} h_m\Lambda(t_i,t_j)w(t_j) + c_0h_m\ln(e^{-1/2}/n_m)w(t_i)] = O(h_m^3),$$

where $\Lambda(t,\tau)$ is the kernel of A_{00}^m. It implies $\|\varepsilon\| = O(h_m^3)$ and

$$\begin{aligned}||e|| &= ||(A_{00}^{h_m})^{-1}R^{h_m}\varepsilon|| = ||R^{h_m}(A_{00}^m)^{-1}f - (A_{00}^{h_m})^{-1}R^{h_m}f|| \\ &= ||R^{h_m}w - (A_{00}^{h_m})^{-1}R^{h_m}A_{00}^m w|| = O(h_m^2). \end{aligned} \tag{3.14}$$

Since $I^{h_m}R^{h_m} \to I$, the proof of Corollary 2 is completed.

Lemma 2. *If $\Gamma_q \cap \Gamma_m = \emptyset$ or $\Gamma_q \cap \Gamma_m \neq \emptyset$, then the Nyström approximate $\hat{B}_{ijh}^{mq}$ defined by (3.1) holds*

$$I^{h_m}(A_{00}^{h_m})^{-1}R^{h_m}\hat{B}_{ijh}^{mq} \stackrel{c.c}{\to} (A_{00}^m)^{-1}\hat{B}_{ij}^{mq}, \text{ in } C[0,1] \to C[0,1], \ i,j = 1,2. \tag{3.15}$$

Proof. If $\Gamma_q \cap \Gamma_m = \emptyset$ or $\Gamma_q \cap \Gamma_m \neq \emptyset$, $i,j = 1,2$, obviously the kernel $b_{ij}^{qm}(t,\tau)$ of the operator B_{ij}^{qm} and its derivative of higher order are continuous[21,24]. By

$$I^{h_m}(A_{00}^{h_m})^{-1}R^{h_m}\hat{B}_{ijh}^{mq} = (I^{h_m}(A_{00}^{h_m})^{-1}R^{h_m}A_{00}^m)((A_{00}^m)^{-1}\hat{B}_{ijh}^{mq}),$$

we have

$$||I^{h_m}(A_{00}^{h_m})^{-1}R^{h_m}\hat{B}_{ijh}^{mq}||_{0,0} \leq ||I^{h_m}(A_{00}^{h_m})^{-1}R^{h_m}A_{00}^m||_{0,3}||(A_{00}^m)^{-1}\hat{B}_{ijh}^{mq}||_{3,0},$$

where $||\cdot||_{m_2,m_1}$ is the norm of the linear bounded operator space $\mathfrak{L}(C^{m_1}[0,1], C^{m_2}[0,1])$. By Corollary 2 and $(A_{00}^m)^{-1}\hat{B}_{ijh}^{mq} \in \mathfrak{L}(C[0,1], C^3[0,1])$, there exists a constant c such that

$$\begin{aligned}||(A_{00}^m)^{-1}\hat{B}_{ijh}^{mq}||_{3,0} &\leq c, \\ ||I^{h_m}(A_{00}^{h_m})^{-1}R^{h_m}A_{00}^m||_{0,3} &\leq c. \end{aligned} \tag{3.16}$$

Using the results of [1,4], the smooth operator sequence $\{(A_{00}^m)^{-1}\hat{B}_{ijh}^{mq} : C[0,1] \to C^3[0,1]\ \}$ must be collectively compact convergent to $(A_{00}^m)^{-1}B_{ij}^{qm}$, which gets the proof of (3.15).

Define the subspace

$$C_0[0,1] = \{v(t) \in C[0,1] : v(t)/\sin^3(\pi t) \in C[0,1]\}$$

of the space $C[0,1]$ with the norm $||v||^* = \max_{0\leq t\leq 1}|v(t)/\sin^3(\pi t)|$.

Lemma 3. *If $\Gamma_q \cap \Gamma_m \neq \emptyset$, $i,j = 0$ and the interior angle θ_q, $q = 1,...,d$, of the curved polygon Ω satisfy $\theta_q \in (0,\pi)\cup(\pi,2\pi)$, $q = 1,...,d$, then the Nyström approximate $\tilde{B}_{00h}^{mq}$ of the integral operator $\tilde{B}_{00}^{mq}$ holds*

$$I^{h_m}(A_{00}^{h_m})^{-1}R^{h_m}\tilde{B}_{00h}^{mq} \stackrel{c.c}{\to} (A_{00}^m)^{-1}\tilde{B}_{00}^{mq}, \text{in } C[0,1] \to C[0,1], \tag{3.17}$$

where the kernel $\tilde{b}_{00}^{mq}(t,\tau)$ of the integral operator $\tilde{B}_{00}^{mq}$ is $\sin^\alpha(\pi t)b_{00}^{mq}(t,\tau)$, and $\tilde{B}_{00h}^{mq}$ is a Nyström approximation of $\tilde{B}_{00}^{mq}$ by using the midpoint rule[6].

Proof. Assume that the origin of coordinates $(0,0) = \Gamma_m \cap \Gamma_q$ is a vertex with interior angle θ_q, then from (2.4b) we make use of the cosine theorem and get

$$\begin{aligned}\tilde{b}_{00}^{mq}(t,\tau) &= -\frac{1}{2}\sin^\alpha(\pi t)\ln[a_0^2(t) + a_1^2(\tau) - 2a_0(t)a_1(\tau)\cos\theta_q] \\ &= -\frac{1}{2}\sin^\alpha(\pi t)\ln(a_0^2(t) + a_1^2(\tau)) \\ &\quad -\frac{1}{2}\sin^\alpha(\pi t)\ln[1 - 2a_0(t)a_1(\tau)\cos\theta_q/(a_0^2(t) + a_1^2(\tau))] \end{aligned} \tag{3.18a}$$

as a new kernel of an integral operator, where $a_0(t) = |x_m(T_m\phi_\alpha(t))|$, $a_1(t) = |x_q(T_q\phi_\alpha(t))|$. Obviously if $\tilde{b}_{00}^{mq}(t,\tau)$ is smooth, then (3.17) holds. Without loss of generality, we assume $a_0(0) = a_1(0) = 0$. Since

$$|2a_0(t)a_1(\tau)\cos\theta_q/(a_0^2(t) + a_1^2(\tau))| \leq |\cos\theta_q| < 1, \tag{3.18b}$$

if we can prove that $a(t,\tau)=\sin^{\alpha}(\pi t)\ln(a_0^2(t)+a_1^2(\tau))$ is a bounded function on $[0,1]^2$, then $\tilde{b}_{00}^{mq}(t,\tau)$ is continuous. In fact, from $\phi_{\alpha}^{(j)}(t)|_{t=0,1}=0$, $j=1,...,\alpha$, we easily get $a_i^{(j)}(t)|_{t=0,t=1}=0$, $i=0,1$, $j=1,...,\alpha$. Thus we only require to prove that for an arbitrary real number $\varepsilon>0$, $a(t,\tau)$ is bounded on $[\varepsilon/2,\varepsilon]^2$. For $(t,\tau)\in[\varepsilon/2,\varepsilon]^2$, it always holds that

$$|a(t,\tau)|=O(\varepsilon^{\alpha}|\ln\varepsilon|)\to 0,\ \text{as}\ \varepsilon\to 0,$$

which means that $a(t,\tau)$ is bounded. Secondly, we can prove that $\frac{\partial}{\partial t}a(t,\tau)$ is continuous on $[0,1]^2$, because for $\forall(t,\tau)\in[\varepsilon/2,\varepsilon]^2$, it holds that

$$|\frac{\partial}{\partial\tau}a(t,\tau)|\leq|\sin^{\alpha}(\pi t)\frac{2a_1(\tau)x_q^{'}(\tau)\phi_{\alpha}^{'}(\tau)}{(a_0^2(t)+a_1^2(\tau))}|=O(\varepsilon^{\alpha}),\ \text{as}\ \varepsilon\to 0.$$

Similarly, from

$$|\frac{\partial^2}{\partial t^2}a(t,\tau)|=O(\varepsilon^{\alpha-1}),\ |\frac{\partial^3}{\partial t^3}a(t,\tau)|=O(\varepsilon^{\alpha-2}),$$

it also holds that $\frac{\partial^2}{\partial t^2}a(t,\tau)$ and $\frac{\partial^3}{\partial t^3}a(t,\tau)$ are continuous on $[0,1]^2$ for $\alpha\geq 3$. Therefor, as $\alpha\geq 3$, $\tilde{b}_{00}^{mq}(t,\tau)$ is smooth, and by Lemma 2 we get the proof of (3.17).

By Lemma 2 and Lemma 3, the following Corollary 3 is easily proved.

Corollary 3. $\{I+I^{h_m}(A_{00}^{h_m})^{-1}R^{h_m}B_{ijh}^{qm}\}$ *is invertible and its inverse operator is uniformly bounded.*

Below construct matrix operators:I_{h_m} =diag(I^{h_m},I^{h_m}) and R_{h_m} =diag(R^{h_m},R^{h_m}); $I^h=$ diag$(I_{h_1},...,I_{h_d})$ and R^h =diag$(R_{h_1},...,R_{h_d})$; $\hat{L}_h=(\hat{A}^h)^{-1}\hat{B}^h=I^h(A^h)^{-1}R^hB^h:(C_0[0.1]\times(C_0[0,1])^d\to\Pi_{m=1}^dS^{h_m}\times S^{h_m}$, where I^{h_m} and R^{h_m} are defined by (3.11) and (3.12). Consider the operator equation

$$(I+\hat{L}_h)\hat{P}^h=I^h\hat{F}^h \tag{3.19}$$

where $\hat{F}^h=(A^h)^{-1}F^h$ and $\hat{F}=A^{-1}F$. Obviously, if $\hat{P}^h$ is the solution of (3.19), then $R^h\hat{P}^h$ must be the solution of (3.10); conversely, if $\bar{p}^h$ is the solution of (3.10), then $I^h\bar{p}^h$ must be the solution of (3.19). In order to prove that $\hat{P}^h$ converge to $\bar{p}$, we first prove the following Theorem 1.

Theorem 1. *The operator sequence*$\{(\hat{A}^h)^{-1}\hat{B}^h\}$ *is collectively compact convergent to* $A^{-1}B$ *in* $V=(C_0[0.1])^d\times(C_0[0,1])^d$, *i.e.*

$$(\hat{A}^h)^{-1}\hat{B}^h\overset{c.c}{\to}A^{-1}B. \tag{3.20}$$

Proof. Let $\Theta=\{v:||v||\leq 1,v\in V\}$ be a unit ball and $H=\{H^{(1)},H^{(2)},...\}$ be a mesh sequence, where $H^{(n)}=\{h_1^{(n)},...,h_d^{(n)}\}$ denotes a multi-parameter step size with $\max_{1\leq m\leq d}h_m^{(n)}\to 0$ as $n_m\to\infty$. Take an arbitrary sequence $\{Z_h,h\in H\}\subset\Theta$, where $Z_h=\{Z_{1h_1},...,Z_{dh_d}\}$, $Z_{mh_m}^T=(z_{1h_m},z_{2h_m})$ with

$$\max_{1\leq m\leq d,0\leq t\leq 1}|z_{ih_m}(t)/\sin^3(\pi t)|\leq 1,i=1,2.$$

Under the above assumptions we assure that there exists a convergent subsequence in $\{(\hat{A}^h)^{-1}\hat{B}^hZ_h\}$. Considering the first component of $(\hat{A}^h)^{-1}\hat{B}^hZ_h$, we have

$$\sum_{q=1}^d I^{h_1}(A_{00}^{h_1})^{-1}R^{h_q}((B_{00h}^{1q}+B_{11h}^{1q})R^{h_q}z_{1h_q}+B_{12h}^{1q}R^{h_q}z_{2h_q}). \tag{3.21}$$

If $q=2$, $i=j=0$, and $\theta_q\in(0,\pi)\cup(\pi,2\pi)$,then

$$\begin{aligned}||I^{h_1}(A_{00}^{h_1})^{-1}R^{h_q}B_{00h}^{12}||_0&=||I^{h_1}(A_{00}^{h_1})^{-1}R^{h_q}\tilde{B}_{00h}^{12}(z_{1h_2})/\sin^3(\pi t)||_0\\&\leq||I^{h_1}(A_{00}^{h_1})^{-1}R^{h_q}A_{00}^1||_{0,3}||A_{00}^{-1}\tilde{B}_{00h}^{12}||_{3,0}||z_{1h_2}||^*.\end{aligned} \tag{3.22}$$

By Lemma 3 there exists a convergent subsequence in $\{I^{h_1}(A_{00}^{h_1})^{-1}R^{h_q}B_{ijh}^{12}R^{h_q}z_{ih_q}\}$. If $q\neq 2$, or $q=2$ and $i,j=1,2$, then by the lemma 2 we have

$$I^{h_1}(A_{00}^{h_1})^{-1}R^{h_q}B_{ijh}^{1q}\overset{c.c}{\to}(A_{00}^1)^{-1}B_{ij}^{1q},\ \text{in}\ \mathfrak{L}(C[0,1],C[0,1]). \tag{3.23}$$

However $C_0[0,1] \subset C[0,1]$, on the base of [1,4,9,14] we can find a infinite subsequence $H_1 \subset H$ such that (3.21) converge as $h \to 0$, $h \in H_1$. Continuing the above methods, we can find a infinite subsequence $H_d \subset H_1 \subset H$ such that $\{(\hat{A}^h)^{-1}\hat{B}^h Z_h,\ h \in H_d\}$ is a convergent sequence in $V = (C_0[0.1])^d \times (C_0[0,1])^d$. Obviously it means

$$(\hat{A}^h)^{-1}\hat{B}^h \xrightarrow{p} A^{-1}B = L, \tag{3.24}$$

where the notation $\xrightarrow{p}$ denotes the pointwise convergence. By [1,4], Theorem 1 is proved.

From Theorem 1 we have the following corollary[4,9].

Corollary 4. *If $h_0 = \max_{1\le m\le d} h_m$ is sufficiently small, then there exits the unique solution $\hat{P}^h$ of the approximate equation (3.19), which has the following error estimate under the norm of V*

$$||\hat{P}^h - \bar{p}||_0 \le ||(I+L)^{-1}|| \frac{||(\hat{L}_h - L)\hat{F}|| + ||(\hat{L}_h - L)\hat{L}_h\bar{p}||}{1 - ||(I+\hat{L}_h)^{-1}(\hat{L}_h - L)\hat{L}_h||}. \tag{3.25}$$

4. Multivariate Asymptotic Expansions and Splitting Extrapolations

Theorem 2. *If u_m^0 is piecewise smooth on $\Gamma_m \in C^4[0,T_m]$, then there exists the vector function $W = (W_1,...,W_d)^T \in (C_0[0,1])^{2d}$, and $W_m = (W_{m1}, W_{m2})^T$, $m = 1,...,d$, independent of $h = (h_1,...,h_d)^T$, such that the solution of (2.5) has the multi-parameter asymptotic expansion*

$$\bar{p} - \hat{P}^h = diag(h_1^3,...,h_d^3)W + O(h_0^4) \tag{4.1}$$

at the mesh points.

Proof. By (3.2), (3.4), (3.6) and (2.5), we obtain

$$\begin{aligned}(\hat{A}^h + \hat{B}^h)(\bar{p} - \hat{P}^h) &= I^h R^h (A+B)\bar{p} - (\hat{A}^h + \hat{B}^h)\bar{p} \\ &= \mathrm{diag}(h_1^3,...,h_d^3) I^h R^h \varpi + O(h_0^4),\end{aligned} \tag{4.2}$$

or

$$(I + \hat{L}_h)(\bar{p} - \hat{P}^h) = \mathrm{diag}(h_1^3,...,h_d^3)(\hat{A}^h)^{-1} I^h R^h \varpi + O(h_0^4), \tag{4.3}$$

where $\varpi^T = (\varpi_1,...,\varpi_d), \varpi_m = -c_0\xi^{'}(-2)(\bar{p}^m)^{''}$. Now construct the following auxiliary equation

$$(I+L)W = A^{-1}\varpi; \tag{4.4a}$$

and its approximate equation

$$(I+\hat{L}_h)W^h = (\hat{A}^h)^{-1} I^h R^h \varpi. \tag{4.4b}$$

Substituting (4.4) into (4.3), we get

$$(I+\hat{L}_h)(\bar{p} - \hat{P}^h - \mathrm{diag}(h_1^3,...,h_d^3)W^h) = O(h_0^4). \tag{4.5}$$

But $(I+\hat{L}_h)^{-1}$ is uniformly bounded from Theorem 1, we obtain

$$\bar{p} - \hat{P}^h - \mathrm{diag}(h_1^3,...,h_d^3)W^h = O(h_0^4). \tag{4.6}$$

Replacing W^h by W in (4.6) and applying the estimate (3.25), we get the proof of (4.1).

The multi-parameter asymptotic expansion (4.1) means that SEM as a parallel algorithm can be applied to solve BIES, that is, a higher order accuracy $O(h^4)$ at the coarse grid points can be obtained by solving some discrete equations in parallel. The process of SEM is as follows[11,12].

Step 1. Take $h^{(0)} = (h_1,...,h_d)$ and $h^{(m)} = (h_1,...,h_m/2,...,h_d)$, then solve the problem (3.6) according to the mesh parameter $h^{(m)}$, $m = 1,\ ...,\ d$, in parallel. Let $\bar{p}^{h^{(m)}}(t_r)$, $r = 0,\ ..,\ n_m - 1$, $m = 1,\ ...,\ d$, be their solutions.

Step 2. Implement the following SEM on the coarse grid points

$$\hat{P}^*(t_r) = 8/7[\sum_{m=1}^{d} \bar{p}^{h^{(m)}}(t_r) - (d - 7/8)\bar{p}^{h^{(0)}}(t_r)]. \tag{4.7}$$

Note that from (4.1), it holds that

$$\hat{P}^*(t_r) = \bar{p}(t_r) + O(h_0^4).$$

Step 3. Using $\hat{P}^*(t_r)$, $r = 0, .., n_m - 1$, $m = 1, ..., d$, we can compute the approximations of $u_i(y)$ and $\sigma_{ij}(y)$ by (3.8) and (3.9).

Moreover by the inequality

$$|\bar{p}(t_r) - \frac{1}{d}\sum_{m=1}^{d} \bar{p}^{h^{(m)}}(t_r)| \le |\bar{p}(t_r) - \frac{8}{7}[\sum_{m=1}^{d} \bar{p}^{h^{(m)}}(t_r) - (d - \frac{7}{8})\bar{p}^{h^{(0)}}(t_r)]| +$$

$$(\frac{8}{7}d - 1)|\frac{1}{d}\sum_{m=1}^{d} \bar{p}^{h^{(m)}}(t_r) - \bar{p}^{h^{(0)}}(t_r)| \le (\frac{8}{7}d - 1)|\frac{1}{d}\sum_{m=1}^{d} \bar{p}^{h^{(m)}}(t_r) - \bar{p}^{h^{(0)}}(t_r)| + O(h_0^4). \quad (4.8)$$

We get a posteriori error estimate of the average $\sum_{m=1}^{d} \bar{p}^{h^{(m)}}(t_r)/d$, which can be applied to construct adaptive algorithms.

5. Numerical Examples

Example. *Consider a plane elasticity Dirichlet problem (1.1), where* $\Omega = [0, 1/2]^2$, $\lambda = \mu = 1$, $\nu = 1/4$, *and the exact solution is*

$$\begin{cases} u_1(x_1, x_2) = 1/2(x_1 - x_1^2 - \mu(x_2 - 1/4)); \\ u_2(x_1, x_2) = \mu(1/2 - x_1)(x_2 - 1/4); \textit{ in } \Omega. \end{cases} \quad (5.1)$$

Using $\phi_3(t)$ *as a periodical transformation in (2.3), Table 1 and Table 2 display the errors of the displacement vectors and stress components computed by the quadrature method, average, a posteriori estimate, and SEM, in which* $e_i = |u_i(0.3, 0.3) - u_i^h(0.3, 0.3)|$, $r_i = |u_i - u_i^h|/|u_i - u_i^{2h}|$ *and* $\hat{e}_i = |p_i(0.5, 0.25) - p_i^h(0.5, 0.25)|$, $\hat{r}_i = |p_i - p_i^h|/|p_i - p_i^{2h}|$, $i = 1, 2$, *where* p_i *shows the outer normal stress at (0.25,0.25) and* p_i^h *is its approximation.*

Table 1. The displacement errors at (0.3,0.3)

$(n_1, ..., n_4)$	e_1	e_2	$(n_1, ..., n_4)$	e_1	r_1	e_2	r_2
(4, 4, 4, 4)	5.98e-2	3.69e-2	(8,8,8,8)	6.97e-3	8.6	4.65e-3	7.9
(8,4,4,4)	2.42e-2	9.62e-3	(16,8,8,8)	2.76e-3	8.8	9.87e-4	9.8
(4,8,4,4)	4.32e-2	4.86e-2	(8,16,8,8)	5.12e-3	8.4	6.02e-3	8.1
(4,4,8,4)	6.78e-2	2.39e-2	(8,8,16,8)	7.82e-3	8.7	2.86e-3	8.3
(4,4,4,8)	4.85e-2	3.67e-2	(8,8,8,16)	6.29e-3	7.7	4.88e-3	7.5
average-error	4.59e-2	2.97e-2	average-error	5.50e-3	8.4	3.68e-3	8.1
a-post-error	4.95e-2	2.57e-2	a-post-error	5.25e-3		3.44e-3	
SEM-error	3.62e-3	4.01e-3	SEM-error	2.50e-4	19.4	2.46e-4	16.2

Table 2. The outer normal stress errors at (0.5,0.25)

$(n_1, ...n_4)$	$\hat{e}_1$	$\hat{e}_2$	$(n_1, ..., n_4)$	$\hat{e}_1$	$\hat{r}_1$	$\hat{e}_2$	$\hat{r}_2$
(2, 2, 2, 2)	7.04e-1	5.75e-1	(4, 4, 4, 4)	8.61e-2	8.2	7.14e-2	8.1
(4, 2, 2, 2)	3.14e-1	4.50e-1	(8, 4, 4, 4)	3.91e-2	8.1	5.70e-2	7.9
(2, 4, 2, 2)	7.72e-1	3.48e-1	(4,8,4,4)	9.50e-2	8.1	4.47e-2	7.8
(2,2,4,2)	6.09e-1	6.24e-1	(4,4,8,4)	7.99e-2	7.6	7.68e-2	8.1
(2,2,2,4)	4.26e-1	3.54e-1	(4,4,4,8)	5.02e-2	8.5	4.34e-2	8.2
average-error	5.30e-1	4.44e-1	average-error	6.61e-2	7.9	5.55e-2	8.1
a-post-error	6.21e-1	4.67e-1	a-post-error	6.65e-2		5.67e-2	
SEM-error	9.11e-2	2.31e-2	SEM-error	5.58e-3	16.3	1.27e-3	18.1

Table 1 and Table 2 show that our quadrature methods not only possess high accuracy, but also splitting extrapolations and the a posteriori error estimate are very effective. Since the discrete matrix of BIE is full, by using SEM the larger the scale of problems are, the more effective the methods are.

References

[1] P.M. Anselone, Collectively Compact Operator Approximation Theory, Prentice-Hall, Englewood Cliffs, New Jersey, 1971.

[2] E. Alarcon, A. Martin, F. Paris, Boundary Elements in Potential and Elasticity Theory, Computers and Structures, **10**(1979).

[3] C.A., Brebbia, S. Walker, Boundary Elements Techniques in Engineering, Butterworth and Co., Boston, 1980.

[4] F. Chatelin, Spectral Approximation of Linear Operator, Academic Press, 1983.

[5] P. Davis, Circulant Matrices, Joln Wiley & Sones, New York, 1979.

[6] P. Davis, Methods of Numerical Integration, Second, Academic Press, New York, 1984.

[7] G. C. Hsiao, P. Kopp, W. L. Wendland, Some applications of a Galerkin-collocation method for boundary integral equations of the first kind, *Math. Meth. in the Appl.* Sci **6** (1984), 280-325.

[8] W. He, H. Ding, H. Hu, Non-equivalence of the conventional boundary integral formulation and its elimination for plane elasticity problems, *Computers & Structures*, **59** (1996), 1059-1062.

[9] R. Kress, Linear Integral Equations, Springer-Verlag, 1989.

[10] R. Kress, A Nyström method for boundary integral equation in domains with corners, *Numer. Math.*, **58** (1990), 145-161.

[11] C.B.Lin, T. Lü, T.M.Shih, The Splitting Extrapolation Method, World Scientific, Singapore, 1995.

[12] Q. Lin, T, Lü, Splitting extrapolation for multidimensional problem, *J. Comp. Math.*, **1** (1983), 376-383.

[13] T. Lü, J. Huang, Quadrature methods and their extrapolations for solving boundary integral equations of plane elasticity Dirichlet and Neumann problems on the smooth region, *Math. Num. Sinica.*, **4** (2001), 491-509(in Chinese).

[14] T.Lü, J. Huang, Quadrature methods with high accuracy and extrapolation for solving boundary integral equations of first-kind, *Math Num. Sinica.*, **1** (2000), 59-72(in Chinese).

[15] T. Lü, J. Lu, Splitting extrapolation for solving the second order elliptic system with curved boundary in R^d by using d-quadratic isoparametric finite element, *Appl. Numer. Math.* **40** (2002), 467-481.

[16] J.L. Lyness, B.W. Ninham, Numerical quadrature and asymptotic expansions, *Math. Comp.*, **21** (1967), 162-178.

[17] N.I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity, Groningen, Noordhoff, 1953.

[18] F. Paris and J. Cans, Boundary Element Method, Oxford University press, 1997.

[19] U. Rüde and A. Zhou, Multi-parameter extrapolation methods for boundary integral equations, *Adv. Comput. Math.*, **9** (1998), 173-190.

[20] A. Sidi, M. Israrli, Quadrature methods for periodic singular Fredholm integral equation, *J. of Sci Comp.*, **3** (1988), 201-231.

[21] A. Sidi, A new variable transformation for numerical integration, in Numerical Integration IV (edited by H.Brass and G. Hämmerlin), Birkhäuser Verlag, 1993, 359-374.

[22] I. H. Sloan, A. Spence, The Galerkin method for integral equations of the first kind with logarithmic kernel, *IMA J.Numer. Anal*, V(1988), 105-122.

[23] Willian McLean, Strongly Elliptic Systems and Boundary Integrral Equations, Cambridge University Press, 2000.

[24] Y. Yan, The collocation method for first-kind boundary integral equations on polygonal regions, *Math Comp.* **54** (1990), 139-154.