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ON THE DIVIDED DIFFERENCE FORM OF FAÀ DI BRUNO'S FORMULA II*

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Abstract

In this paper, we consider the higher divided difference of a composite function $f(g(t))$ in which $g(t)$ is an s-dimensional vector. By exploiting some properties from mixed partial divided differences and multivariate Newton interpolation, we generalize the divided difference form of Faà di Bruno's formula with a scalar argument. Moreover, a generalized Faà di Bruno's formula with a vector argument is derived.

Mathematics subject classification: 65D05, 05A10, 41A05.
Key words: Bell polynomial, Faà di Bruno's formula, Mixed partial divided difference, Multivariate Newton interpolation.

1. Introduction

The well-known formula of Faà di Bruno [5] for higher derivative of a composite function plays an important role in combinatorial algebra. The formula together with its representative tools, the Bell polynomials and cycle indicator polynomials of symmetric groups, have wide applications in numerical analysis. Recently, applying Faà di Bruno's formula [10], the coefficients in Lagrangian numerical differentiation formulas and asymptotic expansions of the remainders on local approximation have been derived explicitly. It is proved in [14] that the solutions of a system of equations of algebraic sum of equal powers can be converted to all roots of two univariate algebraic equations whose degree sum equals to the number of unknowns of the underlying system of equations. In [15], we proposed the best quadrature for the function which belongs to any order Sobolev class $KW^r[a,b]$ with the L_∞ norm. In [11], the authors applied the formula again and proved that, for complex polynomials, all extraneous fixed points for any iteration of Halley iterative family (König's algorithms) are repelling. Then six years earlier than [1] they solved the problem left by [9]. Moreover, using the Bell polynomials and cycle indicator polynomials of symmetric groups, a family of parallel and interval iterations for finding all roots of a polynomial simultaneously is established, see, e.g., [16, 17]. These results became the main part of the monograph [8].

Several generalizations of Faà di Bruno's formula for multivariate composite functions have been given in [3, 4, 7]. However, it seems that the details of the proofs and expressions are so cumbersome that they are difficult to be used for practical computations. In a recent paper [13], we established the divided difference form of Faà di Bruno's formula which is of simple form. It happened that Floater [6] also derived a similar result independently. It is the purpose of this paper to continue the above work and to give the high order divided difference of h which is represented by the divided differences of f and g. The functions f and g satisfy $f: V \to \mathbb{F}$, $g: U \to V$, and the function $h: U \to \mathbb{F}$ is the composite function $f \circ g$, which is denoted

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by $t \mapsto f(g(t))$. Here $\mathbb{F}$ denotes the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$. Moreover, $U \subset \mathbb{F}$, $V \subset \mathbb{F}^s$ and s is a positive integer. We follow the definition of the divided difference of g from [2]. So we can obtain the following generalized Faà di Bruno's formula,

$$\frac{1}{n!}h^{(n)}(t) = \sum_{1\le|\alpha|\le n} \frac{1}{|\alpha|!} f^{(\alpha)}(g(t)) \hat{B}_{n,\alpha}\left(\frac{g'(t)}{1!}, \frac{g''(t)}{2!}, \cdots, \frac{g^{(n)}(t)}{n!}\right), \tag{1.1}$$

where

$$\hat{B}_{n,\alpha}(x_1, x_2, \cdots, x_n) := \sum_{e_{i_1}+\cdots+e_{i_{|\alpha|}}=\alpha} \sum_{\substack{k_1+\cdots+k_{|\alpha|}=n \\ k_1,\ \ldots\ ,\ k_{|\alpha|}\ge 1}} \prod_{j=1}^{|\alpha|} x_{k_j}^{e_{i_j}} \tag{1.2}$$

is the ordinary partial Bell polynomial with several vector variables $x_1, x_2, \ldots, x_n$. We will introduce all notations in the next section. However, in order to understand formula (1.1) well, here we present an example for the generalized Faà di Bruno's formula. Let $h(t) = f(x(t), y(t))$. Then we have

$$\begin{aligned}
\frac{h^{(4)}(t)}{4!} =& \frac{\partial f}{\partial x}\frac{x^{(4)}(t)}{4!} + \frac{\partial f}{\partial y}\frac{y^{(4)}(t)}{4!} \\
&+ \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\left(\frac{1}{3}x'(t)x'''(t) + \frac{1}{3}x''(t)^2\right) + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}\left(\frac{1}{3}y'(t)y'''(t) + \frac{1}{4}y''(t)^2\right) \\
&+ \frac{1}{2!}\frac{\partial^2 f}{\partial x \partial y}\left(\frac{1}{3}x'(t)y'''(t) + \frac{1}{2}x''(t)y''(t) + \frac{1}{3}x'''(t)y'(t)\right) \\
&+ \frac{1}{3!}\frac{\partial^3 f}{\partial x^3}\left(\frac{3}{2}x'(t)^2x''(t)\right) + \frac{1}{3!}\frac{\partial^3 f}{\partial x^2 \partial y}\left(\frac{3}{2}x'(t)^2y''(t) + 3x'(t)x''(t)y'(t)\right) \\
&+ \frac{1}{3!}\frac{\partial^3 f}{\partial x \partial y^2}\left(3x'(t)y'(t)y''(t) + \frac{3}{2}x''(t)y'(t)^2\right) + \frac{1}{3!}\frac{\partial^3 f}{\partial y^3}\left(\frac{3}{2}y'(t)^2y''(t)\right) \\
&+ \frac{1}{4!}\frac{\partial^4 f}{\partial x^4}x'(t)^4 + \frac{1}{4!}\frac{\partial^4 f}{\partial x^3 \partial y}\left(4x'(t)^3y'(t)\right) + \frac{1}{4!}\frac{\partial^4 f}{\partial x^2 \partial y^2}\left(6x'(t)^2y'(t)^2\right) \\
&+ \frac{1}{4!}\frac{\partial^4 f}{\partial x \partial y^3}\left(4x'(t)y'(t)^3\right) + \frac{1}{4!}\frac{\partial^4 f}{\partial y^4}y'(t)^4.
\end{aligned}$$

2. Preliminaries

In order to simplify expressions, it is convenient to recall some multivariate notations. We denote by $\mathbb{Z}_+$ the set of nonnegative integers and by $\mathbb{Z}_+^s$ the set of multi-integers. $e_i \in \mathbb{Z}_+^s$ is a unit vector whose jth component is δ_{ij} where

$$\delta_{ij} := \begin{cases} 0, & j \ne i; \\ 1, & j = i. \end{cases}$$

Let $\mathbb{F}^s$ be the s-dimensional Euclidean space and $x \in \mathbb{F}^s$ be an s-dimensional vector whose ith component is denoted by x^{e_i}. $\alpha \in \mathbb{Z}_+^s$ denotes a multi-index. $|\alpha| := \sum_{i=1}^s \alpha^{e_i}$ is its length. More generally,

$$(\cdot)^\alpha : \mathbb{F}^s \to \mathbb{F} : \quad x \mapsto x^\alpha := \prod_{i=1}^s (x^{e_i})^{\alpha^{e_i}},$$

and $0^0 := 1$. Let

$$f^{(\alpha)}(x) := \left(\frac{\partial}{\partial x^{e_1}}\right)^{\alpha^{e_1}} \left(\frac{\partial}{\partial x^{e_2}}\right)^{\alpha^{e_2}} \cdots \left(\frac{\partial}{\partial x^{e_s}}\right)^{\alpha^{e_s}} f(x).$$

For $x_0, x_1, \ldots, x_n \in \mathbb{F}^s$, we also denote the convex hull of $X := \{x_0, x_1, \ldots, x_n\} \subset \mathbb{F}^s$ by

$$\bar{X} := \left\{ \left(1 - \sum_{i=1}^{n} t_i\right) x_0 + t_1 x_1 + \cdots + t_n x_n \mid t_i \geq 0, \sum_{i=1}^{n} t_i \leq 1 \right\}.$$

According to the Hermite-Genocchi formula for univariate divided difference, we give the definition of the mixed partial divided difference of order $|\alpha|$ at $x_0, x_1, \ldots, x_{|\alpha|}$.

Definition 2.1 ([2]) *For $\alpha \in \mathbb{Z}_+^s$, the mixed partial divided difference of order $|\alpha|$ of f is defined by*

$$[x_0]_0 f := f(x_0),$$

and

$$\begin{aligned} &[x_0, x_1, \cdots, x_{|\alpha|}]_\alpha f \\ := &\int_0^1 \int_0^{u_1} \cdots \int_0^{u_{|\alpha|-1}} f^{(\alpha)}\big(x_0 + u_1 \Delta x_0 + \cdots + u_{|\alpha|} \Delta x_{|\alpha|-1}\big) du_{|\alpha|} \cdots du_1, \end{aligned}$$

where

$$\Delta x_j := x_{j+1} - x_j, \quad j = 0, 1, \cdots, |\alpha| - 1, \quad |\alpha| \geq 1.$$

The above expressions may be regarded as a generalization of univariate divided difference to high dimensions.

Next, we give the definition of multivariate Newton interpolation of total degree n.

Definition 2.2. *Let $X := \{x_0, x_1, \cdots, x_n\} \subset \mathbb{F}^s$ and $f \in \mathcal{C}^n(\bar{X})$. Then the multivariate Newton interpolation of f at the nodes $X \subset \mathbb{F}^s$ is defined by*

$$N(x; f, X) = \sum_{0 \leq |\alpha| \leq n} [x_0, x_1, \cdots, x_{|\alpha|}]_\alpha f \cdot \omega_\alpha(x; X),$$

where

$$\omega_\alpha(x; X) = \begin{cases} 1, & |\alpha| = 0, \\ \displaystyle\sum_{e_{i_1} + e_{i_2} + \cdots + e_{i_{|\alpha|}} = \alpha} \prod_{j=1}^{|\alpha|} (x - x_{j-1})^{e_{i_j}}, & |\alpha| > 0. \end{cases}$$

We call $\omega_\alpha(x; X)$ the Newton fundamental functions for all $\alpha \in \mathbb{Z}_+^s$. For more properties of the multivariate Newton polynomials, see [12] and references therein.

In the rest of this section, we state some lemmas which will be needed in the proofs of our theorems.

Lemma 2.1 ([12]) *If $f \in \mathcal{C}^{|\alpha|}(\bar{X})$ and $X := \{x_0, x_1, \cdots, x_{|\alpha|}\} \subset \mathbb{F}^s$, then there exists an $\xi \in \bar{X}$ such that*

$$[x_0, x_1, \cdots, x_{|\alpha|}]_\alpha f = \frac{1}{|\alpha|!} f^{(\alpha)}(\xi).$$

Lemma 2.2 ([13]) *Assume that $\{t_i\}_{i=0}^n$ is a sequence of $n+1$ distinct points. Let ϕ_i satisfy*

$$\phi_i(t_i) = 0, \quad i = 0, 1, \cdots, k-1,$$

and let $\Phi(t) = \prod_{i=0}^{k-1} \phi_i(t)$. Then for all $n \geq k$, we have

$$\Phi[t_0, t_1, \cdots, t_n] = \sum_{\nu_0 \leq \nu_1 \leq \cdots \leq \nu_k} \prod_{i=1}^{k} \phi_{i-1}[t_{i-1}, t_{\nu_{i-1}}, t_{\nu_{i-1}+1}, \cdots, t_{\nu_i}],$$

where $\nu_0 := k,\ \nu_k := n$.

3. Main Results

In this section, we suppose that $t_0, t_1 \ldots, t_n \in \mathbb{F}$, $T := \{t_0, t_1, \cdots, t_n\}$ and $\bar{T}$ is a convex hull of T. Let $g : \bar{T} \to \mathbb{F}^s$, $g \in \mathcal{C}^n(\bar{T})$ and $g^{e_i} := [g(\cdot)]^{e_i}$. We will consider the divided difference of a composite function $h := f \circ g$.

Theorem 3.1. *If $g \in \mathcal{C}^n(\bar{T})$, $f \in \mathcal{C}^n(g(\bar{T}))$ and $h := f \circ g$, then we have*

$$[t_0, t_1, \cdots, t_n]h = \sum_{1 \leq |\alpha| \leq n} [g(t_0), g(t_1), \cdots, g(t_{|\alpha|})]_\alpha f$$

$$\times \sum_{e_{i_1} + \cdots + e_{i_{|\alpha|}} = \alpha} \sum_{\nu_0 \leq \nu_1 \leq \cdots \leq \nu_{|\alpha|}} \prod_{j=1}^{|\alpha|} [t_{j-1}, t_{\nu_{j-1}}, t_{\nu_{j-1}+1}, \cdots, t_{\nu_j}] g^{e_{i_j}}, \tag{3.1}$$

where $\nu_0 := |\alpha|,\ \nu_{|\alpha|} := n$. In particular, when $t_1 = \cdots = t_n = t_0$, the generalized formula of Faà di Bruno is as follows

$$\frac{1}{n!} h^{(n)}(t_0) = \sum_{1 \leq |\alpha| \leq n} \frac{f^{(\alpha)}(g(t_0))}{|\alpha|!} \times \sum_{e_{i_1} + \cdots + e_{i_{|\alpha|}} = \alpha} \sum_{\nu_0 \leq \nu_1 \leq \cdots \leq \nu_{|\alpha|}} \prod_{j=1}^{|\alpha|} \frac{\left(g^{(\nu_j - \nu_{j-1}+1)}(t_0)\right)^{e_{i_j}}}{(\nu_j - \nu_{j-1}+1)!}, \tag{3.2}$$

where $\nu_0 := |\alpha|,\ \nu_{|\alpha|} := n$.

Proof. Assume that $\{t_i\}_{i=0}^n$ is a sequence of $n+1$ distinct points. Then we can obtain the following Newton polynomial of $h(t)$,

$$N(t; h, T) = h(t_0) + \sum_{l=1}^{n} [t_0, t_1, \cdots, t_l]h \cdot \omega_l(t; T), \tag{3.3}$$

where

$$\omega_l(t; T) := \prod_{i=0}^{l-1} (t - t_i).$$

For $x_0, x_1, \cdots, x_n \in g(\bar{T})$, let $X := \{x_0, x_1, \cdots, x_n\}$. Then for $x \in g(\bar{T})$ the multivariate Newton interpolation of f at the nodes $X \subset \mathbb{F}^s$ is

$$N(x; f, X) = f(x_0) + \sum_{1 \leq |\alpha| \leq n} [x_0, x_1, \cdots, x_{|\alpha|}]_\alpha f \cdot \omega_\alpha(x; X),$$

where

$$\omega_\alpha(x;X)=\sum_{e_{i_1}+\cdots+e_{i_{|\alpha|}}=\alpha}\prod_{j=1}^{|\alpha|}(x-x_{j-1})^{e_{i_j}}.$$

Let $x=g(t)$ and $X=g(T)$. Then we have

$$N(g(t);f,g(T))=f(g(t_0))+\sum_{1\le|\alpha|\le n}[g(t_0),g(t_1),\cdots,g(t_{|\alpha|})]_\alpha f\cdot\omega_\alpha(g(t);g(T)), \tag{3.4}$$

where

$$\omega_\alpha(g(t);g(T))=\sum_{e_{i_1}+\cdots+e_{i_{|\alpha|}}=\alpha}\prod_{j=1}^{|\alpha|}(g(t)-g(t_{j-1}))^{e_{i_j}}=:\Omega_\alpha(t).$$

Write

$$G_{i_1,\ldots,i_{|\alpha|}}(t):=\prod_{j=1}^{|\alpha|}(g(t)-g(t_{j-1}))^{e_{i_j}}.$$

Then we have

$$\Omega_\alpha(t)=\sum_{e_{i_1}+\cdots+e_{i_{|\alpha|}}=\alpha}G_{i_1,\ldots,i_{|\alpha|}}(t).$$

For $|\alpha|>l$ we have

$$[t_0,t_1,\cdots,t_l]G_{i_1,\ldots,i_{|\alpha|}}=0,$$

which yields $[t_0,t_1,\cdots,t_l]\Omega_\alpha=0$. So the Newton interpolation of $\Omega_\alpha(t)$ at the nodes T is

$$N(t;\Omega_\alpha,T)=\sum_{l=|\alpha|}^{n}[t_0,t_1,\cdots,t_l]\Omega_\alpha\cdot\omega_l(t;T). \tag{3.5}$$

From (3.4) and (3.5), we derive

$$\begin{aligned}N(t;f\circ g,T)=&f(g(t_0))+\sum_{1\le|\alpha|\le n}[g(t_0),g(t_1),\cdots,g(t_{|\alpha|})]_\alpha f\\&\times\sum_{l=|\alpha|}^{n}[t_0,t_1,\cdots,t_l]\Omega_\alpha\cdot\omega_l(t;T)\\=&f(g(t_0))+\sum_{l=1}^{n}\sum_{1\le|\alpha|\le l}[g(t_0),g(t_1),\cdots,g(t_{|\alpha|})]_\alpha f\\&\times[t_0,t_1,\cdots,t_l]\Omega_\alpha\cdot\omega_l(t;T).\end{aligned}$$

Therefore, by comparing the above expression with (3.3) and in view of the uniqueness of interpolation, we have $h(t_0)=f(g(t_0))$ and

$$[t_0,t_1,\cdots,t_n]h=\sum_{1\le|\alpha|\le n}[g(t_0),g(t_1),\cdots,g(t_{|\alpha|})]_\alpha f\cdot[t_0,t_1,\cdots,t_n]\Omega_\alpha.$$

According to Lemma 2.2, we can obtain

$$[t_0,t_1,\cdots,t_n]\Omega_\alpha=\sum_{e_{i_1}+\cdots+e_{i_{|\alpha|}}=\alpha}\sum_{\nu_0\le\nu_1\le\cdots\le\nu_{|\alpha|}}\prod_{j=1}^{|\alpha|}[t_{j-1},t_{\nu_{j-1}},t_{\nu_{j-1}+1},\cdots,t_{\nu_j}]g^{e_{i_j}},$$

where $\nu_0 := |\alpha|$, $\nu_{|\alpha|} := n$. Therefore we obtain (3.1). For general nodes, (3.1) holds automatically because $g \in \mathcal{C}^n(\bar{T})$. In particular, when $t_1 = \cdots = t_n = t_0$, we get (3.2) from Lemma 2.1 and (3.1). $\square$

In Theorem 3.1, for $j = 1, 2, \cdots, |\alpha|$, we take k_j instead of $\nu_j - \nu_{j-1} + 1$. Then we will derive an alternative form.

Theorem 3.2. *Suppose all assumptions in Theorem* 3.1 *hold. Then we have*

$$[t_0, t_1, \cdots, t_n]h = \sum_{1 \le |\alpha| \le n} [g(t_0), g(t_1), \cdots, g(t_{|\alpha|})]_\alpha f$$
$$\times \sum_{e_{i_1} + \cdots + e_{i_{|\alpha|}} = \alpha} \sum_{\substack{k_1 + \cdots + k_{|\alpha|} = n \\ k_1, \, \dots \, , k_{|\alpha|} \ge 1}} \prod_{j=1}^{|\alpha|} [t_{j-1}, t_{\nu_{j-1}}, t_{\nu_{j-1}+1}, \cdots, t_{\nu_j}] g^{e_{i_j}},$$

where

$$\nu_0 := |\alpha|; \;\; \nu_j := k_1 + k_2 + \cdots + k_j + |\alpha| - j, \quad j = 1, 2, \cdots, |\alpha| - 1; \quad \nu_{|\alpha|} := n.$$

In particular, when $t_1 = \cdots = t_n = t_0$, the above expression becomes the generalized Faà di Bruno's formula defined by

$$\frac{1}{n!} h^{(n)}(t_0) = \sum_{1 \le |\alpha| \le n} \frac{1}{|\alpha|!} f^{(\alpha)}(g(t_0)) \times \sum_{e_{i_1} + \cdots + e_{i_{|\alpha|}} = |\alpha|} \sum_{\substack{k_1 + \cdots + k_{|\alpha|} = n \\ k_1, \, \dots \, , \, k_{|\alpha|} \ge 1}} \prod_{j=1}^{|\alpha|} \frac{\left(g^{(k_j)}(t_0)\right)^{e_{i_j}}}{k_j!}$$
$$= \sum_{1 \le |\alpha| \le n} \frac{1}{|\alpha|!} f^{(\alpha)}(g(t_0)) \hat{B}_{n,\alpha} \left(\frac{g'(t_0)}{1!}, \frac{g''(t_0)}{2!}, \cdots, \frac{g^{(n)}(t_0)}{n!} \right). \tag{3.6}$$

We conclude this section by considering an example. Let $s = 2$ and $h(t) = f(x(t), y(t))$. The functions $x(t)$ and $y(t)$ are two components of $g(t)$, respectively. Then it follows from (3.1) that

$$\begin{aligned}
&[t_0, t_1, t_2, t_3]h \\
=&[g(t_0), g(t_1)]_{(1,0)} f \cdot [t_0, t_1, t_2, t_3]x + [g(t_0), g(t_1)]_{(0,1)} f \cdot [t_0, t_1, t_2, t_3]y \\
&+ [g(t_0), g(t_1), g(t_2)]_{(2,0)} f \cdot ([t_0, t_2]x[t_1, t_2, t_3]x + [t_0, t_2, t_3]x[t_1, t_3]x) \\
&+ [g(t_0), g(t_1), g(t_2)]_{(1,1)} f \cdot \big([t_0, t_2]x[t_1, t_2, t_3]y + [t_0, t_2, t_3]x[t_1, t_3]y \\
&\qquad + [t_0, t_2]y[t_1, t_2, t_3]x + [t_0, t_2, t_3]y[t_1, t_3]x\big) \\
&+ [g(t_0), g(t_1), g(t_2)]_{(0,2)} f \cdot ([t_0, t_2]y[t_1, t_2, t_3]y + [t_0, t_2, t_3]y[t_1, t_3]y) \\
&+ [g(t_0), g(t_1), g(t_2), g(t_3)]_{(3,0)} f \cdot [t_0, t_3]x[t_1, t_3]x[t_2, t_3]x \\
&+ [g(t_0), g(t_1), g(t_2), g(t_3)]_{(2,1)} f \cdot \big([t_0, t_3]x[t_1, t_3]x[t_2, t_3]y \\
&\qquad + [t_0, t_3]x[t_1, t_3]y[t_2, t_3]x + [t_0, t_3]y[t_1, t_3]x[t_2, t_3]x\big) \\
&+ [g(t_0), g(t_1), g(t_2), g(t_3)]_{(1,2)} f \cdot \big([t_0, t_3]x[t_1, t_3]x[t_2, t_3]y \\
&\qquad + [t_0, t_3]y[t_1, t_3]x[t_2, t_3]y + [t_0, t_3]y[t_1, t_3]y[t_2, t_3]x\big) \\
&+ [g(t_0), g(t_1), g(t_2), g(t_3)]_{(0,3)} f \cdot [t_0, t_3]y[t_1, t_3]y[t_2, t_3]y.
\end{aligned}$$

In particular, when $t_0 = t_1 = t_2 = t_3 = t$, the above expression becomes

$$\begin{aligned}\frac{h'''(t)}{3!} =& \frac{\partial f}{\partial x}\frac{x'''(t)}{3!} + \frac{\partial f}{\partial y}\frac{y'''(t)}{3!} \\ &+ \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}x'(t)x''(t) + \frac{1}{2!}\frac{\partial^2 f}{\partial x \partial y}\left(x'(t)y''(t) + x''(t)y'(t)\right) + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}y'(t)y''(t) \\ &+ \frac{1}{3!}\frac{\partial^3 f}{\partial x^3}x'(t)^3 + \frac{1}{3!}\frac{\partial^3 f}{\partial x^2 \partial y}\left(3x'(t)^2 y'(t)\right) \\ &+ \frac{1}{3!}\frac{\partial^3 f}{\partial x \partial y^2}\left(3x'(t)y'(t)^2\right) + \frac{1}{3!}\frac{\partial^3 f}{\partial y^3}y'(t)^3.\end{aligned}$$

If we only need to calculate the high order derivative of h but not the high order divided difference of h, we can do it by the generalized ordinary partial Bell polynomial (1.2) directly. For example, we find the generalized ordinary partial Bell polynomials of order 4 as follows (we omit the polynomials which can be derived by symmetries):

$$\begin{aligned}&\hat{B}_{4,(1,0)} = x_4^{e_1}, \quad \hat{B}_{4,(2,0)} = 2x_1^{e_1}x_3^{e_1} + (x_2^{e_1})^2, \\ &\hat{B}_{4,(1,1)} = 2(x_1^{e_1}x_3^{e_2} + x_2^{e_1}x_2^{e_2} + x_3^{e_1}x_1^{e_2}), \quad \hat{B}_{4,(3,0)} = 3(x_1^{e_1})^2 x_2^{e_1}, \\ &\hat{B}_{4,(2,1)} = 3(x_1^{e_1})^2 x_2^{e_2} + 6x_1^{e_1}x_2^{e_1}x_1^{e_2}, \quad \hat{B}_{4,(4,0)} = (x_1^{e_1})^4, \\ &\hat{B}_{4,(3,1)} = 4(x_1^{e_1})^3 x_1^{e_2}, \quad \hat{B}_{4,(2,2)} = 6(x_1^{e_1})^2(x_1^{e_2})^2.\end{aligned}$$

Then from these formulas, we derive the results for the example in Section 1 according to the second conclusion of Theorem 3.2, *i.e.*, formula (1.1).

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References

[1] X. Buff and C. Henriksen, On König's root-finding algorithmes, *Nonlinearity*, **16** (2003), 989-1015.

[2] A. Cavaretta, C. Micchelli and A. Sharma, Multivariate interpolation and the Radon transform, *Math. Z.*, **174A** (1980), 263-279.

[3] G.M. Constantine and T.H. Savits, A multivariate Faà di Bruno formula with applications, *Trans. Amer. Math. Soc.*, **348** (1996), 503-520.

[4] L.H. Encinas and J.M. Masque, A short proof of the generalized Faà di Bruno's formula, *Appl. Math. Lett.*, **16** (2003), 975-979.

[5] C.F. Faà di Bruno, Note sur nouvelle formule de calcul differentiel, *Q. J. Pure Appl. Math.*, **1** (1857), 359-360.

[6] M.S. Floater and T. Lyche, Two chain rules for divided differences and Faà di Bruno's formula, *Math. Comput.*, **76** (2007), 867-877.

[7] R. Mishkov, Generalization of the formula of Faà di Bruno for a composite function with a vector argument, *Int. J. Math. Sci.*, **24**:7 (2000), 23-32.

[8] M.C. Petkovic, Iterative methods for simultaneous inclusion of polynomial zeros, *Lecture Notes in Math.*, **1387** (1989), 263.

[9] E.R. Vrscay and W.J. Gilbert, Extraneous fixed points basin boundaries and chaotic dynamics for Schröder and König rational iteration functions, *Numer. Math.*, **52** (1988), 1-16.

[10] H. Wang, F. Cui and X. Wang, Explicit representations for local Lagrangian numerical differentiation, *Acta. Math. Sinica*, **23** (2007), 365-372.

[11] X. Wang and D. Han, On fixed points and Julia sets for iterations for two families, *Math. Numer. Sin.*, **19**:2 (1997), 219-224; *Chinese J. Numer. Math. Appl.*, **19**:3 (1997), 94-100.

[12] X. Wang and M.J. Lai, On multivariate Newtonian interpolation, *Scientia Sinica*, **29** (1986), 23-32.

[13] X. Wang and H. Wang, On the divided difference form of Faà di Bruno's formula, *J. Comput. Math.*, **24** (2006), 553-560.

[14] X. Wang and S. Yang, On solving equations of algebraic sum of equal powers. *Science in China (Ser. A)*, **49** (2006), 1153-1157.

[15] X. Wang and S. Yang, The best quadrature for the Sobolev class $KW^r[a,b]$ based on given Hermite information, *Science in China (Ser. A)*, **49** (2006), 1146-1152.

[16] X. Wang and S. Zheng, A family of parallel and interval iterval iterations for finding all roots of a polynomial simultaneously with rapid convergence, *J. Comput. Math.*, **2**:1 (1984), 70-76.

[17] X. Wang, S. Zheng and G. Shen, Bell's disk polynomials and parallel disk iteration, *Numer. Math. J. Chinese Univ*, **9**:4 (1987), 328-346; *Freiburger Intervall Berichte*, **2** (1986), 37-65.