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AN INVERSE EIGENVALUE PROBLEM FOR JACOBI MATRICES *

Haixia Liang and Erxiong Jiang
(*Department of Mathematics, Shanghai University, Shanghai 200444, China*
Email: lianghaixia1980@163.com, ejiang@fudan.edu.cn)

Abstract

In this paper, we discuss an inverse eigenvalue problem for constructing a $2n \times 2n$ Jacobi matrix T such that its $2n$ eigenvalues are given distinct real values and its leading principal submatrix of order n is a given Jacobi matrix. A new sufficient and necessary condition for the solvability of the above problem is given in this paper. Furthermore, we present a new algorithm and give some numerical results.

Mathematics subject classification: 65L09.
Key words: Symmetric tridiagonal matrix, Jacobi matrix, Eigenvalue problem, Inverse eigenvalue problem.

1. Introduction

A real symmetric tridiagonal matrix $T_{1,n}$ of the form

$$T_{1,n} = \begin{pmatrix} \alpha_1 & \beta_1 & & 0 \\ \beta_1 & \ddots & & \\ & \ddots & & \\ 0 & & \beta_{n-1} & \alpha_n \end{pmatrix}$$

with $\beta_i > 0$ is called a Jacobi matrix.

In 1979, Hochstand [1] put forward the inverse eigenvalue problem (I): Given a Jacobi matrix T_n and real values: $\lambda_1 < \lambda_2 < \cdots < \lambda_{2n}$, construct an irreducible symmetric tridiagonal matrix $T_{1,2n}$ whose eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$ and the leading principal submatrix $T_{1,n}$ is the given T_n.

Hochstand also proved that the solution is unique if it exists. In 1987, Boley and Golub [2] proposed a numerical method for solving Problem (I), but this method needs to compute all the eigenvalues and eigenvectors of $T_{1,n}$, which seems expensive in computational time. Dai [3] gave a sufficient and necessary condition for solving this problem, which was further improved by Xu [4]. But both algorithms need to compute $2n + 1$ determinants of matrices of order $2n$. Furthermore, in the process of constructing $T_{1,2n}$, we find that $T_{1,n}$ is reconstructed, which may make $T_{1,n}$ different from the given one due to the computing error. In this paper, the inverse problem is solved by an idea completely different from the previous ones. In fact, since $T_{1,n}$ is given , we may only take measures to obtain $T_{n+1,2n}$ and β_n.

In this paper, we present a new algorithm based on the following (k) Jacobi inverse eigenvalue problem [5]: Given real number sets $S_1 = \{\mu_1, \cdots, \mu_{k-1}\}$, $S_2 = \{\mu_{k+1}, \cdots, \mu_n\}$ *and* $S_3 =$

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$\{\lambda_1, \cdots, \lambda_n\}$, we construct $T_{1,n}$ of the form

$$T_{1,n} = \begin{pmatrix} T_{1,k-1} & \beta_{k-1} & \\ \beta_{k-1} & \alpha_k & \beta_k \\ & \beta_k & T_{k+1,n} \end{pmatrix},$$

where the eigenvalue sets of $T_{1,k-1}, T_{k+1,n}$ and $T_{1,n}$ are S_1, S_2 and S_3, respectively.

Let

$$T_{1,2n} = \begin{pmatrix} \alpha_1 & \beta_1 & & & & & 0 \\ \beta_1 & \ddots & \ddots & & & & \\ & \ddots & & \beta_{n-1} & & & \\ & & \beta_{n-1} & \alpha_n & \beta_n & & \\ & & & \beta_n & \ddots & \ddots & \\ & & & & \ddots & \ddots & \beta_{2n-1} \\ 0 & & & & & \beta_{2n-1} & \alpha_{2n} \end{pmatrix}$$

be a $2n \times 2n$ irreducible symmetric tridiagonal matrix, and denote its submatrix $T_{p,q}$ $(p < q)$ by

$$T_{p,q} = \begin{pmatrix} \alpha_p & \beta_p & & & 0 \\ \beta_p & \alpha_{p+1} & \beta_{p+1} & & \\ & \beta_{p+1} & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{q-1} \\ 0 & & & \beta_{q-1} & \alpha_q \end{pmatrix}.$$

Here, we assume that $T_{1,2n}$ and $T_{p,q}$ are Jacobi matrices. Rewrite

$$T_{1,2n} = \begin{pmatrix} T_{1,n-1} & \beta_{n-1} & \\ \beta_{n-1} & \alpha_n & \beta_n \\ & \beta_n & T_{n+1,2n} \end{pmatrix},$$

where β_n and $T_{n+1,2n}$ in problem (I) need to be obtained from the given values $\{\lambda_i\}_{i=1}^{2n}$ and the matrix $T_{1,n}$.

The paper is organized as follows. In Section 2, a sufficient and necessary condition for solving Problem (I) is given in two cases: $T_{1,n-1}$ and $T_{1,2n}$ have or do not have common eigenvalues. We also prove that if the solution exists, then it is unique. In Section 3, we present the corresponding algorithm and give two numerical examples.

2. The Basic Theorems

2.1. Some basic lemmas

In this section, we first give some preliminary results which play a fundamental role in this paper. The proofs are omitted here; they can be found in the corresponding references.

Lemma 2.1. [6] *Let the eigenvalues of T_n be $\theta_1 < \theta_2 < \cdots < \theta_n$, with corresponding unit eigenvectors S_1, $S_2, \cdots, S_n$. Denote the i-th component of S_j $(j = 1, 2, \cdots, n)$ by S_{ij}. Then, for $\mu \le \nu$,*

$$\chi'_{1,n}(\theta_j) S_{\mu j} S_{\nu j} = \chi_{1,\mu-1}(\theta_j)\beta_\mu \cdots \beta_{\nu-1}\chi_{\nu+1,n}(\theta_j),$$

where $\chi_{p,q} = det(\lambda I - T_{p,q})$.

For

$$T_n = \begin{pmatrix} T_{1,k-1} & \beta_{k-1} & \\ \beta_{k-1} & \alpha_k & \beta_k \\ & \beta_k & T_{k+1,n} \end{pmatrix},$$

let the eigenvalue sets of $T_{1,k-1}$ and $T_{k+1,n}$ be $\{\mu_i\}_{i=1}^{k-1}$ and $\{\mu_i\}_{i=k+1}^{2n}$ respectively. Then we have

$$(S_{k-1,i}^{(1)})^2 = \frac{\chi_{1,k-2}(\mu_i)}{\chi_{1,k-1}'(\mu_i)}, \quad i = 1, \cdots, k-1 \tag{2.1}$$

and

$$(S_{1,i}^{(2)})^2 = \frac{\chi_{k+2,n}(\mu_i)}{\chi_{k+1,n}'(\mu_i)}, \quad i = k+1, \cdots, n, \tag{2.2}$$

where $S_{k-1,i}^{(1)}$ is the $(k-1)$-th element of the i-th eigenvector of $T_{1,k-1}$ and $S_{1,i}^{(2)}$ is the first element of the i-th eigenvector of $T_{k+1,n}$.

Lemma 2.2. [5] *If* $T_{1,k-1}$ *and* $T_{k+1,n}$ *have no common eigenvalues, then any root of the equation*

$$F(\lambda) = \lambda - \alpha_k - \sum_{i=1}^{k-1} \frac{(\beta_{k-1} S_{k-1,i}^{(1)})^2}{\lambda - \mu_i} - \sum_{i=k+1}^{n} \frac{(\beta_k S_{1,i}^{(2)})^2}{\lambda - \mu_i} = 0 \tag{2.3}$$

is an eigenvalue of T_n*. On the other hand, any eigenvalue of* T_n *is a root of Eq.* (2.3)*. If* $T_{1,k-1}$ *and* $T_{k+1,n}$ *have common eigenvalues, each common eigenvalue is an eigenvalue of* T_n*, and other eigenvalues of* T_n *are roots of Eq.* (2.3)*. Similarly, in this case, any root of Eq.* (2.3) *is an eigenvalue of* T_n*.*

Lemma 2.3. [5] *Let* $\lambda_1 < \mu_{j_1} < \lambda_2 < \mu_{j_2} < \cdots < \mu_{j_{n-1}} < \lambda_n$*. Then the following linear algebraic equations system:*

$$\sum_{j=1}^{n-1} \frac{x_j}{\lambda_i - \mu_j} = \lambda_i - \alpha_k \qquad i = 1, \cdots, n,$$

has unique solution $x = (x_1, x_2, \cdots, x_{n-1})$ *and*

$$x_j = -\frac{\prod_{i=1}^{n} (\lambda_i - \mu_j)}{\prod_{\substack{i=1 \\ i \neq j}}^{n-1} (\mu_i - \mu_j)} > 0, \quad j = 1, \cdots, n-1. \tag{2.4}$$

Lemma 2.4. [5] *If there is no common number between* $\{\mu_i\}_{i=1}^{k-1}$ *and* $\{\mu_i\}_{i=k+1}^{n}$*, then the necessary and sufficient condition of the (k) problem having a solution is*

$$\lambda_1 < \mu_{i_1} < \lambda_2 < \cdots < \mu_{i_{n-1}} < \lambda_n.$$

Furthermore, if a given (*k*) *problem has a solution, then the solution is unique.*

Lemma 2.5. [5] *Given three real number sets* $S_1 = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, $S_2 = \{\mu_1, \mu_2, \cdots, \mu_{k-1}\}$ *and* $S_3 = \{\mu_{k+1}, \mu_{k+2}, \cdots, \mu_n\}$ *so that each set has different elements and*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < \lambda_n, \qquad \mu_{j_1} \le \mu_{j_2} \le \cdots \le \mu_{j_{n-2}} \le \mu_{j_{n-1}},$$

where $(j_1, j_2, \cdots, j_{n-1})$ *is a permutation of* $(1, 2, \cdots, k-1, k+1, \cdots, n)$. *If* $\mu_{j_q} = \mu_{j_{q+1}}$, *the sufficient and necessary condition for the* (k) *problem to have a solution is that the following strict separation*

$$\lambda_1 < \mu_{j_1} < \lambda_2 < \cdots < \mu_{j_{n-1}} < \lambda_n,$$

holds except $\mu_{j_q} = \lambda_{q+1} = \mu_{j_{q+1}}$ *instead of above* $\mu_{j_q} < \lambda_{q+1} < \mu_{j_{q+1}}$. *Furthermore, if the* (k) *problem has a solution, then there are infinitely many solutions.*

2.2. Basic theorems

Now, we consider Problem (I). Denote the eigenvalues of $T_{1,n-1}$ by $\mu_1, \mu_2, \cdots, \mu_{n-1}$ whose corresponding unit eigenvectors are $S_1^{(1)}, S_2^{(1)}, \cdots, S_{n-1}^{(1)}$. Denote the $(n-1)$-th component of $S_j^{(1)}$ by $S_{n-1,j}^{(1)}$ $(j = 1, \cdots, n-1)$. We discuss Problem (I) in two cases similar to [5]:

- (1) There is no common number between $\{\mu_i\}_{i=1}^{n-1}$ and $\{\lambda_i\}_{i=1}^{2n}$.
- (2) There are common numbers between $\{\mu_i\}_{i=1}^{n-1}$ and $\{\lambda_i\}_{i=1}^{2n}$.

In Case (1), we define

$$\varphi_{n+1,2n}(\mu) = \mu^n + \sum_{i=1}^{n} C_{n-i}\mu^{n-i}, \tag{2.5}$$

where

$$C_{n-1} = -\sum_{i=1}^{2n} \lambda_i + \sum_{i=1}^{n} \alpha_i, \tag{2.6a}$$

$$\varphi_{n+1,2n}(\mu_j) = \frac{(-1)^{n+1} \prod_{i=1}^{2n} (\lambda_i - \mu_j)}{(\beta_{n-1} S_{n-1,j}^{(1)})^2 \prod_{\substack{i=1 \\ i \ne j}}^{n-1} (\mu_i - \mu_j)}, \qquad j = 1, \cdots, n-1 . \tag{2.6b}$$

Here, $(C_0, C_1, \cdots, C_{n-2})^T$ is the solution of the following system of equations

$$\begin{pmatrix} 1 & \mu_1 & \cdots & \mu_1^{n-2} \\ 1 & \mu_2 & \cdots & \mu_2^{n-2} \\ & \cdots & \cdots & \\ 1 & \mu_{n-1} & \cdots & \mu_{n-1}^{n-2} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-2} \end{pmatrix} = \begin{pmatrix} \rho(\mu_1) \\ \rho(\mu_2) \\ \vdots \\ \rho(\mu_{n-1}) \end{pmatrix}, \tag{2.7}$$

where $\rho(\mu) = \varphi_{n+1,2n}(\mu) - \mu^n - C_{n-1}\mu^{n-1}$. It is known that $\varphi_{n+1,2n}(\mu)$ is uniquely determined.

In Case (2), for simplicity, we assume that there is only one common element between $\{\mu_i\}_{i=1}^{n-1}$ and $\{\lambda_i\}_{i=1}^{2n}$, say $\mu_1 = \lambda_q$. If there are two or more common elements, the analysis is analogous. Define

$$\varphi_{n+1,2n}(\mu) = (\mu - \mu_1)\psi_{n-1}(\mu), \tag{2.8}$$

where

$$\psi_{n-1}(\mu) = \mu^{n-1} + \sum_{i=2}^{n} C_{n-i}\mu^{n-i}, \tag{2.9}$$

$$C_{n-2} = -\sum_{i=1}^{2n}\lambda_i + \sum_{i=1}^{n}\alpha_i + \lambda_q,$$

and

$$\psi_{n-1}(\mu_j) = \frac{(-1)^n \prod\limits_{\substack{i=1\\ i\neq q}}^{2n}(\lambda_i - \mu_j)}{(\beta_{n-1}S^{(1)}_{n-1,j})^2 \prod\limits_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i - \mu_j)}, \qquad j = 2, \cdots, n-1.$$

Here $(C_0, C_1, \cdots, C_{n-3})^T$ is the solution of the system

$$\begin{pmatrix} 1 & \mu_2 & \cdots & \mu_2^{n-3} \\ 1 & \mu_3 & \cdots & \mu_3^{n-3} \\ & \cdots & \cdots & \\ 1 & \mu_{n-1} & \cdots & \mu_{n-1}^{n-3} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-3} \end{pmatrix} = \begin{pmatrix} h(\mu_2) \\ h(\mu_3) \\ \vdots \\ h(\mu_{n-1}) \end{pmatrix}, \tag{2.10}$$

where $h(\mu) = \psi_{n-1}(\mu) - \mu^{n-1} - C_{n-2}\mu^{n-2}$. Obviously, $\varphi_{n+1,2n}(\mu)$ is also unique.

For $\varphi_{n+1,2n}(\mu)$ given in the above two cases, we have the following result.

Theorem 2.1. *If Problem* (I) *has a solution, then*

$$\varphi_{n+1,2n}(\mu) = det(\mu I - T_{n+1,2n}).$$

Proof. Rewrite

$$T_{1,2n} = \begin{pmatrix} T_{1,n-1} & \beta_{n-1} & \\ \beta_{n-1} & \alpha_n & \beta_n \\ & \beta_n & T_{n+1,2n} \end{pmatrix},$$

and denote the eigenvalues of $T_{n+1,2n}$ by $\mu_{n+1}, \cdots, \mu_{2n}$. First, in Case (1), if $T_{1,2n}$ exists, it follows from Lemma 2.2 that

$$F(\lambda_p) = \lambda_p - \alpha_n - \sum_{i=1}^{n-1}\frac{(\beta_{n-1}S^{(1)}_{n-1,i})^2}{\lambda_p - \mu_i} - \sum_{i=n+1}^{2n}\frac{(\beta_n S^{(2)}_{1,i})^2}{\lambda_p - \mu_i} = 0, \quad 1 \le p \le 2n.$$

Notice that β_{n-1} is given and $S^{(1)}_{n-1,j}(j = 1, \cdots, n-1)$ can be obtained by (2.1). Correspondingly, $(\beta_{n-1}S^{(1)}_{n-1,j})^2$ $(j = 1, \cdots, n-1)$ is known. Furthermore, by (2.4),

$$(\beta_{n-1}S^{(1)}_{n-1,j})^2 = x_j = -\frac{\prod\limits_{i=1}^{2n}(\lambda_i - \mu_j)}{\prod\limits_{\substack{i=1\\ i\neq j\\ i\neq n}}^{2n}(\mu_i - \mu_j)}, \quad j = 1, \cdots, n-1. \tag{2.11}$$

So, for $j = 1, \cdots, n-1$, we rewrite (2.11) as

$$(\beta_{n-1}S^{(1)}_{n-1,j})^2 = -\frac{\prod\limits_{i=1}^{2n}(\lambda_i - \mu_j)}{\left[\prod\limits_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i - \mu_j)\right]\prod\limits_{i=1}^{n}(\mu_{n+i} - \mu_j)}.$$

Denote

$$\phi_{n+1,2n}(\mu) = (-1)^n \prod_{i=n+1}^{2n} (\mu_i - \mu) = \prod_{i=n+1}^{2n} (\mu - \mu_i).$$

We get, for $1 \le j \le n-1$,

$$(\beta_{n-1} S_{n-1,j}^{(1)})^2 = -\frac{\prod_{i=1}^{2n} (\lambda_i - \mu_j)}{\left[\prod_{\substack{i=1\\ i\neq j}}^{n-1} (\mu_i - \mu_j)\right] \cdot (-1)^n \phi_{n+1,2n}(\mu_j)},$$

which gives

$$\phi_{n+1,2n}(\mu_j) = \frac{(-1)^{n+1} \prod_{i=1}^{2n} (\lambda_i - \mu_j)}{(\beta_{n-1} S_{n-1,j}^{(1)})^2 \prod_{\substack{i=1\\ i\neq j}}^{n-1} (\mu_i - \mu_j)}. \tag{2.12}$$

By (2.6) and (2.12),

$$\varphi_{n+1,2n}(\mu_j) = \phi_{n+1,2n}(\mu_j), \qquad j = 1, \cdots, n-1. \tag{2.13}$$

Define

$$\phi_{n+1,2n}(\mu) = \mu^n + \sum_{i=1}^{n} C'_{n-i} \mu^{n-i},$$

and notice that

$$C'_{n-1} = -\sum_{i=1}^{n} \mu_{n+i} = -\left(\sum_{i=1}^{2n} \lambda_i - \sum_{i=1}^{n} \alpha_i\right) = C_{n-1}.$$

Hence, $\varphi_{n+1,2n}(\mu) - \phi_{n+1,2n}(\mu)$ is a polynomial of degree $n-2$. This, together with (2.13), shows that

$$\varphi_{n+1,2n}(\mu) = \phi_{n+1,2n}(\mu) = det(\mu I - T_{n+1,2n}).$$

Second, for Case (2), we assume that $\mu_1 = \lambda_q$. If $T_{1,2n}$ exists, then there must exist one eigenvalue of $T_{n+1,2n}$ such that $\mu_k = \lambda_q = \mu_1$ $(n+1 \le k \le 2n)$. By Lemma 2.2, λ_p $(p \neq q)$ can be substituted into Eq. (2.3):

$$F(\lambda_p) = \lambda_p - \alpha_n - \sum_{i=2}^{n-1} \frac{(\beta_{n-1} S_{n-1,i}^{(1)})^2}{\lambda_p - \mu_i} - \sum_{\substack{i=n+1\\ i\neq k}}^{2n} \frac{(\beta_n S_{1,i}^{(2)})^2}{\lambda_p - \mu_i} - \frac{(\beta_{n-1} S_{n-1,1}^{(1)})^2 + (\beta_n S_{1,k}^{(2)})^2}{\lambda_p - \mu_1} = 0.$$

For $j = 2, \cdots, n-1$,

$$(\beta_{n-1} S_{n-1,j}^{(1)})^2 = -\frac{\prod_{\substack{i=1\\ i\neq q}}^{2n} (\lambda_i - \mu_j)}{\prod_{\substack{i=1\\ i\neq j,n,k}}^{2n} (\mu_i - \mu_j)} = -\frac{\prod_{\substack{i=1\\ i\neq q}}^{2n} (\lambda_i - \mu_j)}{\left[\prod_{\substack{i=1\\ i\neq j}}^{n-1} (\mu_i - \mu_j)\right] \prod_{\substack{i=1\\ i\neq k}}^{n} (\mu_{n+i} - \mu_j)}, \tag{2.14}$$

$$x_1 = (\beta_{n-1} S_{n-1,1}^{(1)})^2 + (\beta_n S_{1,k}^{(2)})^2 = -\frac{\prod_{\substack{i=1\\ i\neq q}}^{2n} (\lambda_i - \mu_1)}{\prod_{\substack{i=2\\ i\neq n\\ i\neq k}}^{2n} (\mu_i - \mu_1)}. \tag{2.15}$$

Denote

$$\sigma_{n-1}(\mu) = (-1)^{n-1} \prod_{\substack{i=n+1\\ i\neq k}}^{2n} (\mu_i - \mu) = \prod_{\substack{i=n+1\\ i\neq k}}^{2n} (\mu - \mu_i).$$

Then we have

$$\sigma_{n-1}(\mu_j) = \frac{(-1)^n \prod\limits_{\substack{i=1\\ i\neq q}}^{2n} (\lambda_i - \mu_j)}{(\beta_{n-1} S^{(1)}_{n-1,j})^2 \prod\limits_{\substack{i=1\\ i\neq j}}^{n-1} (\mu_i - \mu_j)}, \qquad j = 2, \cdots, n-1.$$

Let

$$\sigma_{n-1}(\mu) = \mu^{n-1} + \sum_{i=2}^{n} C'_{n-i}\mu^{n-i},$$

where

$$C'_{n-2} = -\sum_{\substack{i=1\\ i\neq k}}^{n} \mu_{n+i} = -\sum_{i=1}^{2n} \lambda_i + \sum_{i=1}^{n} \alpha_i + \lambda_q = C_{n-2}.$$

Similarly, we can prove that $\psi_{n-1}(\mu) = \sigma_{n-1}(\mu)$. Therefore,

$$\varphi_{n+1,2n}(\mu) = (\mu - \mu_1)\psi_{n-1}(\mu) = (\mu - \mu_1)\sigma_{n-1}(\mu) = det(\mu I - T_{n+1,2n}).$$

According to Lemma 2.4, Lemma 2.5 and above discussions, we obtain

Theorem 2.2. *The sufficient and necessary condition of Problem* (I) *having a solution is:*

- (L_1). *$\mu_{n+1}, \cdots, \mu_{2n}$, the roots of $\varphi_{n+1,2n}(\lambda) = 0$, are distinct real values.*
- (L_2) (l_1). *Separation $\lambda_1 < \mu_{i_1} < \lambda_2 < \cdots < \mu_{i_{2n-1}} < \lambda_{2n}$ holds, where $(i_1, i_2, \cdots, i_{2n-1})$ is a permutation of $(1, 2, \cdots, n-1, n+1, \cdots, 2n)$, or*
- $(L_2)(l_2)$. *If $\mu_{i_{q_1}} = \lambda_{q_1}, \cdots, \mu_{i_{q_l}} = \lambda_{q_l}$ between $\{\mu_i\}_1^{n-1}$ and $\{\lambda_i\}_1^{2n}$, above strict separation holds except $\mu_{i_{(q_s-1)}} = \lambda_{q_s} = \mu_{i_{q_s}}$ instead of above $\mu_{i_{(q_s-1)}} < \lambda_{q_s} < \mu_{i_{q_s}}$ ($s = 1, \cdots, l$ and $n+1 \le i_{(q_s-1)} \le 2n$), and for $j = q_1, \cdots, q_l$, $x_{i_j} - (\beta_{n-1} S^{(1)}_{n-1,i_j})^2 > 0$, where*

$$x_j = -\frac{\eta(\mu_j)}{\xi(\mu_j)}, \qquad j = 1, \cdots, n-1, n+1, \cdots, 2n, \tag{2.16}$$

where

$$\eta(\mu) = \frac{\prod\limits_{p=1}^{2n} (\lambda_p - \mu)}{\prod\limits_{s=1}^{l} (\lambda_{q_s} - \mu)}, \qquad \xi(\mu_j) = \prod_{\substack{p=1\\ p\neq j, n, i_{q_1-1}, \cdots, i_{q_l-1}}}^{2n} (\mu_p - \mu_j).$$

Furthermore, if Problem (I) *has a solution, then the solution is unique.*

Proof. By Lemma 2.4, Lemma 2.5 and Theorem 2.1, the necessity of (L_1) and (L_2) (l_1) is obvious. Moreover, if there are common numbers between $\{\mu_i\}_{i=1}^{n-1}$ and $\{\lambda_i\}_{i=1}^{2n}$, just as in (2.15), we have

$$x_{i_j} = (\beta_{n-1} S^{(1)}_{n-1,i_j})^2 + (\beta_n S^{(2)}_{1,i_{j-1}})^2.$$

Hence

$$x_{i_j} - (\beta_{n-1} S^{(1)}_{n-1,i_j})^2 > 0, \qquad j = q_1, \cdots, q_l,$$

and by the two lemmas, the necessity (L_2) (l_2) is easy to verify.

Now we prove that (L_1) and (L_2) are also sufficient. In Case (1), by (L_1) and $(L_2)(l_1)$,

$$x_j = -\frac{\prod_{i=1}^{2n}(\lambda_i - \mu_j)}{\prod_{\substack{i=1\\ i\neq j\\ i\neq n}}^{2n}(\mu_i - \mu_j)} > 0, \quad j = n+1, \cdots, 2n. \tag{2.17}$$

Let

$$X_j = (\beta_n S^{(2)}_{1,j})^2 = x_j, \tag{2.18}$$

where

$$\beta_n = \sqrt{\sum_{j=n+1}^{2n} X_j}\ ; \quad S^{(2)}_{1,j} = \sqrt{X_j}/\beta_n, \quad j = n+1, \cdots, 2n. \tag{2.19}$$

Let g_1 be an $n \times 1$ vector whose j-th element is $S^{(2)}_{1,n+j}$ $(j = 1, \cdots, n)$. Then it is well-known that $T_{n+1,2n}$ can be constructed uniquely [7] with $\mu_{n+1}, \mu_{n+2}, \cdots, \mu_{2n}$ and g_1. From the process of constructing (2.11) and (2.17), we know that $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$ are the roots of Eq. (2.3). So, by Lemma 2.2, $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$ are exactly the eigenvalues of $T_{1,2n}$ that we construct.

Now we prove the sufficiency of (L_1) and (L_2) (l_2). For simplicity and without loss of generality, we assume that there is only one common eigenvalue, $\mu_1 = \lambda_q = \mu_k$ $(n+1 \le k \le 2n)$. Consider the system of equations

$$\sum_{\substack{j=1\\ j\neq n,k}}^{2n} \frac{x_j}{\lambda_i - \mu_j} = \lambda_i - \alpha_n, \quad i \neq q,$$

whose unique solution is given by $x = (x_1, \cdots, x_{n-1}, x_{n+1}, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{2n})$, where

$$x_j = -\frac{\prod_{\substack{i=1\\ i\neq q}}^{2n}(\lambda_i - \mu_j)}{\prod_{\substack{i=1\\ i\neq n,k,j}}^{2n}(\mu_i - \mu_j)} > 0, \quad j \neq n, k.$$

By condition $(L_2)(l_2)$, $x_1 - (\beta_{n-1} S^{(1)}_{n-1,1})^2 > 0$. Let

$$X_k = x_1 - (\beta_{n-1} S^{(1)}_{n-1,1})^2 = (\beta_n S^{(2)}_{1,k})^2, \tag{2.20}$$

$$X_j = x_j = (\beta_n S^{(2)}_{1,j})^2, \quad j = n+1, \cdots, k-1, k+1, \cdots, 2n, \tag{2.21}$$

$$\beta_n = \sqrt{\sum_{j=n+1}^{2n} X_j}\ ; \quad S^{(2)}_{1,j} = \sqrt{X_j}/\beta_n, \quad j = n+1, \cdots, 2n. \tag{2.22}$$

With μ_j, $S^{(2)}_{1,j}$ $(j = n+1, \cdots, 2n)$, $T_{n+1,2n}$ can be uniquely constructed [7]. Similar to Case (1), $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$ are also exactly the eigenvalues of $T_{1,2n}$ that we construct. Moreover, it is easy to verify that the solution of Problem (I) is unique.

3. Algorithm and Numerical Examples

Summarizing the above discussions, we can give the following algorithm for solving Problem (I):

Step 1: Compute $\{\mu_i\}_{i=1}^{n-1}$, the eigenvalues of $T_{1,n-1}$ which is the $(n-1)\times(n-1)$ leading principal submatrix of the given matrix T_n.

Step 2: Compute $(S_{n-1,i}^{(1)})^2, i=1,\cdots,n-1$ by (2.1).

Step 3: Find $\varphi_{n+1,2n}(\mu)$ by (2.5) and (2.7) in Case (1), and by (2.8)-(2.10) in Case (2), and then obtain $\mu_{n+1},\mu_{n+2},\cdots,\mu_{2n}$ by solving $\varphi_{n+1,2n}(\mu)=0$. If they satisfy conditions (L_1) and (L_2) of Theorem 2.2, continue. Otherwise, Problem (I) has no solution.

Step 4: Compute x_{i_j} by (2.17) in Case (1) and by (2.16) in Case (2). If in Case (2), $x_{i_j}-(\beta_{n-1}S_{n-1,i_j}^{(1)})^2\le 0$, then the problem (I) has no solution. Otherwise, compute β_n by (2.18)-(2.19) or (2.20)-(2.22).

Step 5: Compute $S_{1,j}^{(2)}$ $(j=n+1,\cdots,2n)$ by (2.19) or (2.22).

Step 6: Construct $T_{n+1,2n}$ from $S_{1,j}^{(2)}$, $(j=n+1,\cdots,2n)$ and $\mu_{n+1},\mu_{n+2},\cdots,\mu_{2n}$ by Lanczos Process or Givens Orthogonal Reduction Process [2,7].

Example 1. Consider

$$T_{1,8}=\begin{pmatrix} 1&1&0&0&0&0&0&0\\ 1&2&1&0&0&0&0&0\\ 0&1&3&1&0&0&0&0\\ 0&0&1&4&1&0&0&0\\ 0&0&0&1&5&1&0&0\\ 0&0&0&0&1&6&1&0\\ 0&0&0&0&0&1&7&1\\ 0&0&0&0&0&0&1&8 \end{pmatrix}.$$

Its eigenvalues are

$\lambda_1=0.25380581740172,\quad \lambda_2=1.78932147067715,\quad \lambda_3=2.96106654125555,$
$\lambda_4=3.99627320510481,\quad \lambda_5=5.00372679490000,\quad \lambda_6=6.03893345873617,$
$\lambda_7=7.21067852932706,\quad \lambda_8=8.74619418259755.$

Now we reconstruct a Jacobi matrix with these eigenvalues and $T_{1,4}$ according to the above algorithm.

Step 1: Pick $T_{1,3}$. Its eigenvalues (μ_1,μ_2,μ_3) are

$$(0.26794919243112,\ 2.00000000000000,\ 3.73205080756887).$$

Step 2: Compute $\left((S_{3,1}^{(1)})^2,(S_{3,2}^{(1)})^2,(S_{3,3}^{(1)})^2\right)$ as follows:

$$(0.04465819873852,\ 0.33333333333333,\ 0.62200846792815).$$

Step 3: Find

$$\varphi_{5,8}(\mu)=\mu^4-26\mu^3+C_2\mu^2+C_1\mu^1+C_0,$$

where (C_0, C_1, C_2) are given by

$$(1.55500000000003 \times 10^3, \ -1.02699999999998 \times 10^3, \ 0.24799999999998 \times 10^3).$$

It can be verified that the roots of $\varphi_{5,8}(\mu) = 0$ are

$$\begin{bmatrix} \mu_5 & \mu_6 \\ \mu_7 & \mu_8 \end{bmatrix} = \begin{bmatrix} 4.25471875981424 & 5.82271708097246 \\ 7.17728291897388 & 8.74528124023941 \end{bmatrix}.$$

Step 4: Find $\beta_4 = 0.99999999998965$.

Step 5: Compute $S_{1,j}^{(2)}, j = 5, 6, 7, 8$

$$\begin{bmatrix} S_{1,5}^{(2)} & S_{1,6}^{(2)} \\ S_{1,7}^{(2)} & S_{1,8}^{(2)} \end{bmatrix} = \begin{bmatrix} 0.77795054674394 & 0.55327107602016 \\ 0.29117378247848 & 0.06246512352891 \end{bmatrix}.$$

Step 6: Construct $T_{5,8}$ from $\mu_i, i = 5, 6, 7, 8$ and $S_{1,j}^{(2)}, j = 5, 6, 7, 8$

$\alpha_5 = 4.99999999967063$, $\beta_5 = 0.99999999885204$, $\alpha_6 = 5.99999999169626$,
$\beta_6 = 0.99999998932013$, $\alpha_7 = 6.99999997518876$, $\beta_7 = 0.99999999063476$,
$\alpha_8 = 8.00000003344434$.

The reconstruction of $T_{1,8}$ is then obtained.

Example 2. Given

$$T_{1,4} = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

and the eigenvalues of $T_{1,8}$ are

$\lambda_1 = 2.31949546297742$, $\lambda_2 = 3.15418996943928$, $\lambda_3 = 4.00000000000000$,
$\lambda_4 = 4.51656171330962$, $\lambda_5 = 5.14362819027225$, $\lambda_6 = 5.59203832346487$,
$\lambda_7 = 6.16629426322943$, $\lambda_8 = 7.10779207697774$.

Step 1: The eigenvalues (μ_1, μ_2, μ_3) of $T_{1,3}$ are

$$(2.58578643762690, \ 4.00000000000000, \ 5.41421356237310).$$

There is one common eigenvalue between $T_{1,8}$ and $T_{1,3}$.

Step 2: Compute $\left((S_{3,1}^{(1)})^2, (S_{3,2}^{(1)})^2, (S_{3,3}^{(1)})^2\right)$ as follows:

$$(0.25000000000000, \ 0.50000000000000, \ 0.25000000000000).$$

Step 3: Find $\psi_3(\mu) = \mu^3 - 24\mu^2 + C_1\mu + C_0$ with

$$C_0 = -2.09999999980931 \times 10^2, \qquad C_1 = 1.06999999996435 \times 10^2.$$

The roots (μ_6, μ_7, μ_8) for ψ_3 are

$$(4.99999999937818, \ 5.99999999767866, \ 7.00000000294316),$$

which are eigenvalues of $T_{5,8}$ together with $\mu_5 = 4.00000000000000$.

Step 4: Compute $x_2 = 0.75000000000014$. Then

$$(\beta_3 S_{3,2}^{(1)})^2 = 0.50000000000000 < x_2, \quad (\beta_4 S_{1,5}^{(2)})^2 = x_2 - (\beta_3 S_{3,2}^{(1)})^2 = 0.25000000000014.$$

Moreover, we have

$$\begin{bmatrix} (\beta_4 S_{1,6}^{(2)})^2 & (\beta_4 S_{1,7}^{(2)})^2 \\ (\beta_4 S_{1,8}^{(2)})^2 & \beta_4 \end{bmatrix} = \begin{bmatrix} 0.25000000190807 & 0.25000000140841 \\ 0.24999999311876 & 0.99999999821769 \end{bmatrix}.$$

Step 5: We obtain

$$\begin{bmatrix} S_{1,5}^{(2)} & S_{1,6}^{(2)} \\ S_{1,7}^{(2)} & S_{1,8}^{(2)} \end{bmatrix} = \begin{bmatrix} 0.50000000089130 & 0.50000000279922 \\ 0.50000000229956 & 0.49999999400991 \end{bmatrix}.$$

Step 6: Construct $T_{5,8}$ from $\mu_i, i = 5,6,7,8$ and $S_{1,j}^{(2)}, j = 5,6,7,8$,

$\alpha_5 = 5.49999998942810$, $\beta_5 = 1.11803398498655$, $\alpha_6 = 5.50000000604590$,
$\beta_6 = 0.89442719636495$, $\alpha_7 = 5.50000000583800$, $\beta_7 = 0.67082039502598$,
$\alpha_8 = 5.49999999868799$.

Finally, we give the eigenvalues of the constructed matrix to compare them with the given ones. The eigenvalue set is

$$S = \{\ 2.31949546304336,\ \ 3.15418996957387,\ \ 3.99999999976149,\ \ 4.51656171398453,\ \ 5.14362818931108,\ \ 5.59203832427922,\ \ 6.16629426303990,\ \ 7.10779207700656\}.$$

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