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INCREMENTAL UNKNOWNS FOR THE HEAT EQUATION WITH TIME-DEPENDENT COEFFICIENTS: SEMI-IMPLICIT θ-SCHEMES AND THEIR STABILITY *

Yu-Jiang Wu and Ai-Li Yang
(*School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China*
Email: myjaw@lzu.edu.cn)

Abstract

Based on the finite difference discretization of partial differential equations, we propose a kind of semi-implicit θ-schemes of incremental unknowns type for the heat equation with time-dependent coefficients. The stability of the new schemes is carefully studied. Some new types of conditions give better stability when θ is closed to 1/2 even if we have variable coefficients.

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Key words: Incremental unknowns, Semi-implicit schemes, θ-Schemes, Stability.

1. Introduction

Using finite difference discretization in the infinite dimensional dynamical systems to seek the solution of nonlinear partial differential equations and to study its long time behavior is highly stressed by many authors, see for example [4,10,12]. The Incremental Unknowns (IU) method, stemming originally from the dynamical system theory, was introduced by Temam in 1990 ([11]) for the approximation of inertial manifolds when finite differences are used to discretize a partial differential equation, see also [5,12]. It was shown that the IU method usually yields a very well conditioned matrix in the IU-type linear algebraic equations. Many articles have contributed to the analysis of the IU method and to applying the property to several kinds of differential equations.

For the heat equation of constant coefficients, Pouit [8] constructed a Y-explicit and Z-implicit IU-scheme, and Huang and Wu [7] constructed a class of weighted IU-schemes. The objective of this work is to construct a new type of semi-implicit θ-schemes for the heat equation with time-dependent coefficients which are monotonous increasing with respect to time. We will study the stability of the new schemes and give the proof of the stability theorem.

2. Semi-Implicit θ-Schemes

We consider the one-dimensional evolution equation, i.e., the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - v(t)\dfrac{\partial^2 u}{\partial x^2} = f, & 0 < x < 1, \quad 0 < t \le T, \\ u(0,t) = u(1,t) = 0, & 0 \le t \le T, \\ u(x,0) = u_0, & 0 < x < 1. \end{cases} \tag{2.1}$$

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where $v(t)$ expresses the time-dependent coefficients when t varies in $[0,T]$. Suppose that $v(t)$ is increasing and bounded on $[0,T]$ with $v(0)>0$.

At first, we discretize (2.1) by finite differences. By introducing the incremental unknowns U^m and the transfer matrix S (see, [4,7,8,11])

$$U^m=\begin{pmatrix} Y^m \\ Z^m \end{pmatrix}, \qquad S=\begin{pmatrix} I_{N-1} & 0 \\ G & I_N \end{pmatrix},$$

where the $N\times(N-1)$ matrix $G=(g_{ij})$ is given by $g_{ij}=0$ except that $g_{ii}=g_{i+1,i}=\frac{1}{2}$, we can construct the IU-type θ-scheme ([6,9,13])

$$\begin{aligned}&\left(S^TS+\theta\Delta t v^m S^TAS\right)\begin{pmatrix} Y^m \\ Z^m \end{pmatrix}\\ &=\left(S^TS-(1-\theta)\Delta t v^m S^TAS\right)\begin{pmatrix} Y^{m-1} \\ Z^{m-1} \end{pmatrix}+\Delta t S^TS\begin{pmatrix} F_Y^m \\ F_Z^m \end{pmatrix}.\end{aligned} \tag{2.2}$$

If we have the basis $(\varphi_p)_{p=1,2,\cdots,2N-1}$ in $\mathbb{R}^{2N-1}$, then the scheme (2.2) becomes

$$\sum_{p=1}^{2N-1}(1+\theta\Delta t v^m\lambda_p)\mathcal{U}_p^m\varphi_p=\sum_{p=1}^{2N-1}[(1-(1-\theta)\Delta t v^m\lambda_p)\mathcal{U}_p^{m-1}+\Delta t\mathcal{F}_p^m]\varphi_p.$$

Consequently, the solution of (2.2) can be written as

$$\mathcal{U}_p^m=\frac{1-(1-\theta)\Delta t v^m\lambda_p}{1+\theta\Delta t v^m\lambda_p}\mathcal{U}_p^{m-1}+\frac{\Delta t}{1+\theta\Delta t v^m\lambda_p}\mathcal{F}_p^m.$$

It is easy to show that

$$\frac{1-(1-\theta)\Delta t v^m\lambda_p}{1+\theta\Delta t v^m\lambda_p}<1$$

can be satisfied unconditionally. Note that since $\lambda_1<\lambda_2<\cdots<\lambda_{2N-1}$ and $\lambda_{2N-1}\sim\frac{4}{h^2}$, $(N\to\infty)$, we get the stability condition of (2.2).

Proposition 2.1. *The stability condition of the IU-type θ-scheme* (2.2) *is as follows:*

1) $0\le\theta<\frac{1}{2}$, $\quad \Delta t<\dfrac{h^2}{2(1-2\theta)v(T)}$,
2) $\theta=\frac{1}{2}$, *unconditionally stable* (it is the Crank-Nicolson scheme),
3) $\frac{1}{2}<\theta\le 1$, *unconditionally stable.*

In terms of the incremental unknowns, the semi-implicit θ-schemes of (2.1) can be written as

$$\begin{aligned}&S^TS\begin{pmatrix} Y^m \\ Z^m \end{pmatrix}+\theta v^m\Delta t S^TAS\begin{pmatrix} Y^{m-1} \\ Z^m \end{pmatrix}\\ &=(S^TS-(1-\theta)v^m\Delta t S^TAS)\begin{pmatrix} Y^{m-1} \\ Z^{m-1} \end{pmatrix}+\Delta t S^TS\begin{pmatrix} F_Y^m \\ F_Z^m \end{pmatrix}.\end{aligned} \tag{2.3}$$

Since

$$S^TS=\begin{pmatrix} B & G^T \\ G & I_N \end{pmatrix}, \qquad S^TAS=\begin{pmatrix} \frac{A^*}{2} & 0 \\ 0 & \frac{2}{h^2}I_N \end{pmatrix}, \tag{2.4}$$

$$B = \begin{pmatrix} \frac{3}{2} & \frac{1}{4} & & \\ \frac{1}{4} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{4} \\ & & \frac{1}{4} & \frac{3}{2} \end{pmatrix}, \quad A^* = A^*_{N-1} = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}, \tag{2.5}$$

the scheme (2.2) can be written as

$$\begin{cases} BY^m + G^T Z^m + \frac{1}{2}\theta \Delta t v^m A^* Y^m = BY^{m-1} + G^T Z^{m-1} \\ \qquad - \frac{1}{2}(1-\theta)\Delta t v^m A^* Y^{m-1} + \Delta t B F_Y^m + \Delta t G^T F_Z^m, \\ GY^m + Z^m + \frac{2}{h^2}\theta \Delta t v^m Z^m = GY^{m-1} + Z^{m-1} \\ \qquad - \frac{2}{h^2}(1-\theta)\Delta t v^m Z^{m-1} + \Delta t G F_Y^m + \Delta t F_Z^m. \end{cases} \tag{2.6}$$

The second equality above gives Z^m explicitly in terms of Y^m. Replacing Z^m in the first equality by the solution in the second one, we obtain the matrix to be inverted for computing Y^m:

$$M = B - \frac{h^2}{h^2 + 2\theta \Delta t v^m} G^T G + \frac{\theta \Delta t v^m}{2} A^*. \tag{2.7}$$

We now proceed in the same way for the scheme (2.3). We solve Z^m explicitly; then the matrix to be inverted for solving Y^m is

$$M_s = B - \frac{h^2}{h^2 + 2\theta \Delta t v^m} G^T G. \tag{2.8}$$

We can show easily that the matrices M and M_s are both tri-diagonal and symmetric positive definite. The condition numbers of the two matrices are, respectively,

$$cond\ (M) = \frac{h^4 + 4\theta h^2 \Delta t v^m + 4\theta^2 (v^m)^2 \Delta t^2}{h^4 + 4\theta h^2 \Delta t v^m + 4\theta^2 \pi^2 h^2 (v^m)^2 \Delta t^2}, \tag{2.9}$$

$$cond\ (M_s) = \frac{h^2 + 2\theta(2 - \pi^2 h^2)\Delta t v^m}{h^2 + 2\theta \Delta t v^m}. \tag{2.10}$$

To obtain the inequality *cond* $(M_s) <$ *cond* (M), we should let

$$2\theta \Delta t v^m [4(2\pi^2 h^2 - \pi^4 - 1)\theta^2 (v^m)^2 \Delta t^2 + 2(h^2 - \pi^2 h^4)\theta \Delta t v^m + h^4 - \pi^2 h^6] < 0. \tag{2.11}$$

Under the assumption that $\theta \neq 0$, we find that it is sufficient to set

$$v(0)\Delta t > \frac{(1+\sqrt{5})h^2}{4\theta}, \tag{2.12}$$

where v^m is replaced by $v(0)$ for $N \to \infty$. Consequently, we get the following theorem.

Theorem 2.1. *The matrix M_s in* (2.10) *is better conditioned than the matrix M in* (2.9) *if*

$$v(0)\Delta t > \frac{(1+\sqrt{5})h^2}{4\theta}.$$

3. Stability of the Semi-Implicit θ-Scheme

As in [7,8], we introduce (W_p^Y, W_p^Z, W_N^Z) to set up a basis, where

$$W_p^Y = \begin{pmatrix} (W_p)_{2j}\ , \quad j=1,2,\cdots,N-1 \\ 0 \end{pmatrix}_{p=1,2,\cdots,N-1}, \tag{3.1}$$

$$W_p^Z = \begin{pmatrix} 0 \\ (W_p)_{2j+1}\ , \quad j=0,1,\cdots,N-1 \end{pmatrix}_{p=1,2,\cdots,N}. \tag{3.2}$$

Theorem 3.1. *The necessary and sufficient condition for the stability of the scheme* (2.3) *is as follows.*

1. $\theta = 0:$ $\quad \dfrac{v(T)\Delta t}{h^2} < \dfrac{1}{2},$
2. $0 < \theta < \frac{1}{2}:$ $\quad \dfrac{v(T)\Delta t}{h^2} < \min\{\dfrac{2\theta}{1-2\theta}, \dfrac{4\theta}{1+2\theta-8\theta^2}, g(N-1), X_1, \dfrac{1}{1-\theta}\},$
3. $\theta = \frac{1}{2}:$ $\quad \dfrac{v(T)\Delta t}{h^2} < 1,$
4. $\frac{1}{2} < \theta < 1:$ $\quad \dfrac{v(T)\Delta t}{h^2} < \min\{g(N-1), X_1, \dfrac{1}{1-\theta}\},$
5. $\theta = 1:$ $\quad \dfrac{v(T)\Delta t}{h^2} < \min\{g(N-1), X_1\},$

where the positive numbers $g(N-1)$ and X_1 will be given in the proof.

Proof. To begin, we define the vector $\tilde{U}^m = S\begin{pmatrix} Y^{m-1} \\ Z^m \end{pmatrix}$. Then

$$\tilde{U}^m = S\begin{pmatrix} I_{N-1} & 0 \\ 0 & 0 \end{pmatrix} S^{-1}U^{m-1} + S\begin{pmatrix} 0 & 0 \\ 0 & I_N \end{pmatrix} S^{-1}U^m.$$

Owing to $U^m = \sum_{p=1}^{2N-1} \mathcal{U}_p^m W_p$, $\tilde{U}^m$ and the basis $(W_p)_{p=1,\cdots,2N-1}$, we can write $\tilde{U}^m$ as

$$\tilde{U}^m = \sum_{p=1}^{2N-1} \left[\mathcal{U}_p^{m-1} S\begin{pmatrix} I_{N-1} & 0 \\ 0 & 0 \end{pmatrix} S^{-1}W_p + \mathcal{U}_p^m S\begin{pmatrix} 0 & 0 \\ 0 & I_N \end{pmatrix} S^{-1}W_p \right].$$

Hence, we have the scheme

$$\sum_{p=1}^{2N-1} \mathcal{U}_p^m \left(S^T + \theta v^m \Delta t S^T A S\begin{pmatrix} 0 & 0 \\ 0 & I_N \end{pmatrix} S^{-1} \right) W_p = \sum_{p=1}^{2N-1} (\mathcal{U}_p^{m-1} T_p + \Delta t S^T \mathcal{F}_p^m W_p), \tag{3.3}$$

where

$$T_p = \left(S^T - (1-\theta)v^{m-1}\Delta t S^T A - \theta v^m \Delta t S^T A S\begin{pmatrix} I_{N-1} & 0 \\ 0 & 0 \end{pmatrix} S^{-1} \right) W_p.$$

We also set

$$R_p = \left(S^T + \theta v^m \Delta t S^T A S\begin{pmatrix} 0 & 0 \\ 0 & I_N \end{pmatrix} S^{-1} \right) W_p. \tag{3.4}$$

It is easy to obtain matrices R_p and T_p as follows (see also [7,8])

$$T_p = \begin{pmatrix} (W_p)_2 \cdot 2\left(1-\theta v^m \Delta t\lambda_p - (1-\theta)v^{m-1}\Delta t\lambda_p\right)\cos^2\frac{p\pi}{4N} \\ (W_p)_4 \cdot 2\left(1-\theta v^m \Delta t\lambda_p - (1-\theta)v^{m-1}\Delta t\lambda_p\right)\cos^2\frac{p\pi}{4N} \\ \vdots \\ (W_p)_{2N-2} \cdot 2\left(1-\theta v^m \Delta t\lambda_p - (1-\theta)v^{m-1}\Delta t\lambda_p\right)\cos^2\frac{p\pi}{4N} \\ (W_p)_1 \cdot (1-(1-\theta)v^m\Delta t\lambda_p) \\ (W_p)_3 \cdot (1-(1-\theta)v^m\Delta t\lambda_p) \\ \vdots \\ (W_p)_{2N-1} \cdot (1-(1-\theta)v^m\Delta t\lambda_p) \end{pmatrix},$$

$$R_p = \begin{pmatrix} (W_p)_2 \cdot 2\cos^2\frac{p\pi}{4N} \\ (W_p)_4 \cdot 2\cos^2\frac{p\pi}{4N} \\ \vdots \\ (W_p)_{2N-2} \cdot 2\cos^2\frac{p\pi}{4N} \\ (W_p)_1 \cdot (1+\theta v^m\Delta t\lambda_p) \\ (W_p)_3 \cdot (1+\theta v^m\Delta t\lambda_p) \\ \vdots \\ (W_p)_{2N-1} \cdot (1+\theta v^m\Delta t\lambda_p) \end{pmatrix}.$$

Therefore, the schemes (3.3) can be written as

$$\sum_{p=1}^{2N-1} \mathcal{U}_p^m \left(a_p W_p^Y + b_p W_p^Z\right) = \sum_{p=1}^{2N-1} \left[\mathcal{U}_p^{m-1}\left(c_p W_p^Y + d_p W_p^Z\right) + \Delta t \mathcal{F}_p^m \left(a_p W_p^Y + W_p^Z\right)\right],$$

where

$$\begin{aligned} &a_p = 2\cos^2\frac{p\pi}{4N}, \quad b_p = 1+\theta v^m \Delta t\lambda_p, \quad d_p = 1-(1-\theta)v^m\Delta t\lambda_p, \\ &c_p = 2\left(1-\theta v^m\Delta t\lambda_p - (1-\theta)v^{m-1}\Delta t\lambda_p\right)\cos^2\frac{p\pi}{4N}. \end{aligned} \tag{3.5}$$

Note that

$$W_{2N-p}^Y = -W_p^Y, \quad W_{2N-p}^Z = W_p^Z \ (p=1,\cdots,N-1), \quad W_N^Y = 0.$$

The scheme (2.3) can be rewritten as

$$\begin{aligned} &\sum_{p=1}^{N-1}\left[\left(a_p\mathcal{U}_p^m - a_{2N-p}\mathcal{U}_{2N-p}^m\right)W_p^Y + \left(b_p\mathcal{U}_p^m + b_{2N-p}\mathcal{U}_{2N-p}^m\right)W_p^Z\right] + b_N\mathcal{U}_N^m W_N^Z \\ =&\sum_{p=1}^{N-1}\left[\left(c_p\mathcal{U}_p^{m-1} - c_{2N-p}\mathcal{U}_{2N-p}^{m-1}\right)W_p^Y + \left(d_p\mathcal{U}_p^{m-1} + d_{2N-p}\mathcal{U}_{2N-p}^{m-1}\right)W_p^Z\right] + d_N\mathcal{U}_N^{m-1}W_N^Z \\ &+\sum_{p=1}^{N-1}\Delta t\left[\left(a_p\mathcal{F}_p^m - a_{2N-p}\mathcal{F}_{2N-p}^m\right)W_p^Y + \left(\mathcal{F}_p^m + \mathcal{F}_{2N-p}^m\right)W_p^Z\right] + \Delta t\mathcal{F}_N^m W_N^Z, \end{aligned} \tag{3.6}$$

or in matrix-vector form

$$\begin{bmatrix} a_1 & & & & & & -a_{2N-1} \\ & \searrow & & & & \nearrow & \\ & & a_{N-1} & & -a_{N+1} & & \\ & & & b_N & & & \\ & & b_{N-1} & & b_{N+1} & & \\ & \nearrow & & & & \searrow & \\ b_1 & & & & & & b_{2N-1} \end{bmatrix} \mathcal{U}^m$$

$$= \begin{bmatrix} c_1 & & & & & & -c_{2N-1} \\ & \searrow & & & & \nearrow & \\ & & c_{N-1} & & -c_{N+1} & & \\ & & & d_N & & & \\ & & d_{N-1} & & d_{N+1} & & \\ & \nearrow & & & & \searrow & \\ d_1 & & & & & & d_{2N-1} \end{bmatrix} \mathcal{U}^{m-1} + \Delta t \begin{bmatrix} a_1 & & & & & & -a_{2N-1} \\ & \searrow & & & & \nearrow & \\ & & a_{N-1} & & -a_{N+1} & & \\ & & & 1 & & & \\ & & 1 & & 1 & & \\ & \nearrow & & & & \searrow & \\ 1 & & & & & & 1 \end{bmatrix} \mathcal{F}^m. \quad (3.7)$$

To solve the linear system (3.7) from the row N, we have

$$\mathcal{U}_N^m = \frac{1-(1-\theta)v^m\Delta t\lambda_N}{1+\theta v^m\Delta t\lambda_N}\mathcal{U}_N^{m-1} + \frac{\Delta t}{1+\theta v^m\Delta t\lambda_N}\mathcal{F}_N^m.$$

The sufficient and necessary condition for the corresponding amplificator $\left|\frac{1-(1-\theta)v^m\Delta t\lambda_N}{1+\theta v^m\Delta t\lambda_N}\right|$ to be less than 1 is:

$$\text{(i)} \qquad 0 \le \theta < \frac{1}{2}, \quad \Delta t < \frac{h^2}{(1-2\theta)v(T)}, \quad \text{or}$$

$$\text{(ii)} \qquad \frac{1}{2} \le \theta \le 1, \quad \text{unconditionally stable.}$$

The special structure of the system results in solving $N-1$ systems of two equations with two unknowns

$$\begin{cases} a_p\mathcal{U}_p^m - a_{2N-p}\mathcal{U}_{2N-p}^m = c_p\mathcal{U}_p^{m-1} - c_{2N-p}\mathcal{U}_{2N-p}^{m-1} + a_p\Delta t\mathcal{F}_p^m - a_{2N-p}\Delta t\mathcal{F}_{2N-p}^m, \\ b_p\mathcal{U}_p^m + b_{2N-p}\mathcal{U}_{2N-p}^m = d_p\mathcal{U}_p^{m-1} + d_{2N-p}\mathcal{U}_{2N-p}^{m-1} + \Delta t\mathcal{F}_p^m + \Delta t\mathcal{F}_{2N-p}^m. \end{cases} \quad (3.8)$$

Therefore, we find finally for $p = 1, 2, \cdots, N-1$,

$$\mathcal{U}_{2N-p}^m = \gamma_p^m\left(\theta v^m v^{m-1}\Delta t^2\lambda_p^2\cos^2\frac{p\pi}{4N} + \theta^2 v^m(v^m - v^{m-1})\Delta t^2\lambda_p^2\cos^2\frac{p\pi}{4N}\right)\mathcal{U}_p^{m-1}$$
$$+\gamma_p^m\Gamma_{p,1}^m\mathcal{U}_{2N-p}^{m-1} - \gamma_p^m\theta\Delta t^2 v^m\lambda_p\cos^2\frac{p\pi}{4N}\mathcal{F}_p^m + \Delta t\gamma_p^m\left(1+\theta v^m\Delta t\lambda_p\sin^2\frac{p\pi}{4N}\right)\mathcal{F}_{2N-p}^m, \quad (3.9)$$

and

$$\mathcal{U}_p^m = \gamma_p^m\Gamma_{p,2}^m\mathcal{U}_p^{m-1} + \gamma_p^m\left(\theta v^m v^{m-1}\Delta t^2\lambda_{2N-p}^2\sin^2\frac{p\pi}{4N} + \theta^2 v^m(v^m - v^{m-1})\Delta t^2\sin^2\frac{p\pi}{4N}\lambda_{2N-p}^2\right)$$
$$\mathcal{U}_{2N-p}^{m-1} + \Delta t\gamma_p^m\left(1+\theta v^m\Delta t\lambda_{2N-p}\cos^2\frac{p\pi}{4N}\right)\mathcal{F}_p^m - \gamma_p^m\theta\Delta t^2 v^m\lambda_{2N-p}\sin^2\frac{p\pi}{4N}\mathcal{F}_{2N-p}^m, \quad (3.10)$$

where

$$\gamma_p^m = \left[1 + \frac{4}{h^2}\theta v^m \Delta t - 2\theta v^m \Delta t \lambda_p \cos^2\left(\frac{p\pi}{4N}\right)\right]^{-1},$$

$$\Gamma_{p,1}^m = 1 - (1-\theta)v^{m-1}\Delta t \lambda_{2N-p} - \theta v^m \Delta t \lambda_p \cos\frac{p\pi}{2N} - \theta v^m v^{m-1}\Delta t^2 \cos^2\frac{p\pi}{4N}\lambda_p^2$$
$$+\theta^2 v^m (v^{m-1} - v^m)\Delta t^2 \cos^2\frac{p\pi}{4N}\lambda_p^2,$$

$$\Gamma_{p,2}^m = 1 - (1-\theta)v^{m-1}\Delta t \lambda_p + \theta v^m \Delta t \lambda_{2N-p} \cos\frac{p\pi}{2N} - \theta v^m v^{m-1}\Delta t^2 \sin^2\frac{p\pi}{4N}\lambda_{2N-p}^2$$
$$+\theta^2 v^m (v^{m-1} - v^m)\Delta t^2 \sin^2\frac{p\pi}{4N}\lambda_{2N-p}^2.$$

Eqs. (3.9)-(3.10) can be written as

$$\begin{cases} \mathcal{U}_{2N-p}^m = a\mathcal{U}_p^{m-1} + b\mathcal{U}_{2N-p}^{m-1} + \alpha, \\ \mathcal{U}_p^m = c\mathcal{U}_p^{m-1} + d\mathcal{U}_{2N-p}^{m-1} + \beta. \end{cases} \tag{3.11}$$

To localize the eigenvalues of the amplification matrix, we need to solve

$$P(\sigma_p) \equiv \det\begin{pmatrix} a-\sigma_p & b \\ c & d-\sigma_p \end{pmatrix} = \sigma_p^2 - (a+d)\sigma_p + ad - bc = 0,$$

namely, we need to find the roots of the following equation:

$$\sigma_p^2 - \gamma_p^m\left(\Gamma_{p,2}^m + \Gamma_{p,1}^m\right)\sigma_p + \left(\gamma_p^m\right)^2\Big\{\Gamma_{p,2}^m \cdot \Gamma_{p,1}^m$$
$$-\left(\theta v^m v^{m-1}\Delta t^2 \lambda_p^2 \cos^2\frac{p\pi}{4N} + \theta^2 v^m (v^m - v^{m-1})\Delta t^2 \lambda_p^2 \cos^2\frac{p\pi}{4N}\right)$$
$$\cdot\left(\theta v^m v^{m-1}\Delta t^2 \lambda_{2N-p}^2 \sin^2\frac{p\pi}{4N} + \theta^2 v^m (v^m - v^{m-1})\Delta t^2 \sin^2\frac{p\pi}{4N}\lambda_{2N-p}^2\right)\Big\} = 0. \tag{3.12}$$

Because $\sin^4\frac{p\pi}{2N}\cos^2\frac{p\pi}{2N}\left(\cos^2\frac{p\pi}{2N} - 1\right) < 0$ holds true unconditionally, by direct computation, the necessary and sufficient condition of stability for our scheme can be written as: the minimum point of $P(\sigma_p)$ is between -1 and 1, $P(1) > 0$ and $P(-1) > 0$.

(i) The Minimum Point of $P(\sigma_p)$ between -1 and 1. The minimum of $P(\sigma_p)$ is obtained at

$$\sigma_{min} = \frac{1}{2}\gamma_p^m\left(2 - \frac{4(1-\theta)v^{m-1}\Delta t}{h^2} + \frac{4\theta v^m \Delta t}{h^2}\cos^2\frac{p\pi}{2N}\right.$$
$$\left.-\theta v^m v^{m-1}\Delta t^2 \lambda_p \lambda_{2N-p} + \theta^2 v^m (v^{m-1} - v^m)\Delta t^2 \frac{4}{h^4}\sin^2\frac{p\pi}{2N}\right).$$

It is evident that $\sigma_{min} < 1$.

As for the condition $\sigma_{min} > -1$, it is equivalent to

$$2 - \frac{4(1-\theta)v^{m-1}\Delta t}{h^2} + \frac{4\theta v^m \Delta t}{h^2}\cos^2\frac{p\pi}{2N} - \theta v^m v^{m-1}\Delta t^2 \lambda_p \lambda_{2N-p}$$
$$+\theta^2 v^m (v^{m-1} - v^m)\Delta t^2 \frac{4}{h^4}\sin^2\frac{p\pi}{2N} + 2\left(\gamma_p^m\right)^{-1} > 0. \tag{3.13}$$

Let us denote the left-hand side of the inequality by $Q(v^{m-1})$. Then

$$Q'(v^{m-1}) = -\frac{4(1-\theta)\Delta t}{h^2} - \theta(1-\theta)v^m \Delta t^2 \frac{4}{h^4}\sin^2\frac{p\pi}{2N} < 0.$$

So, $Q(v^{m-1})$ is decreasing about v^{m-1}. Replacing v^{m-1} by v^m in (3.13), we obtain that

$$2-\frac{4(1-\theta)v^m\Delta t}{h^2}+\frac{4\theta v^m\Delta t}{h^2}\cos^2\frac{p\pi}{2N}-\theta(v^m)^2\Delta t^2\lambda_p\lambda_{2N-p}+2\left(\gamma_p^m\right)^{-1}>0. \tag{3.14}$$

Denoting $X=v^m\Delta t/h^2>0$, we obtain the above condition with the help of a polynomial $Q_p(X)$:

$$Q_p(X)=\theta\sin^2\frac{p\pi}{2N}X^2-\left(1+2\theta\cos^2\frac{p\pi}{2N}\right)X-1<0.$$

Its discriminant is $\delta=1+4\theta+4\theta^2\cos^4\frac{p\pi}{2N}>0$, so the condition $Q_p(X)<0$ is realized if and only if

$$X<\frac{1+2\theta\cos^2\frac{p\pi}{2N}+\sqrt{1+4\theta+4\theta^2\cos^4\frac{p\pi}{2N}}}{2\theta\sin^2\frac{p\pi}{2N}}=:g(p).$$

It is easy to see that $g(p)\geq g(N-1)$, $\forall p$. At last, the necessary and sufficient condition for $\sigma_{min}>-1$ can be written as

$$\frac{v(T)\Delta t}{h^2}<\frac{1+2\theta\sin^2\frac{\pi}{2N}+\sqrt{1+4\theta+4\theta^2\sin^4\frac{\pi}{2N}}}{2\theta\cos^2\frac{\pi}{2N}}=g(N-1), \tag{3.15}$$

with $\lim_{N\to\infty}g(N-1)=(1+\sqrt{1+4\theta})/(2\theta)$.

We point out that if $\theta=0$, then we have $\sigma_{min}>-1$ absolutely.

(ii) The Condition $P(1)>0$. Let be $\sigma_p=1$ in (3.12). We get

$$\begin{aligned}\frac{\partial P(1)}{\partial(v^{m-1})}=\left(\gamma_p^m\right)^2&\left[\frac{8}{h^4}(1-\theta)^2\Delta t^2v^{m-1}\sin^2\frac{p\pi}{2N}\right.\\&+\frac{8}{h^4}\theta(1-\theta)\Delta t^2v^m\sin^2\frac{p\pi}{2N}+\frac{16}{h^6}\theta^2(1-\theta)\Delta t^3(v^m)^2\sin^2\frac{p\pi}{2N}\left(1+\cos^2\frac{p\pi}{2N}\right)\\&\left.+\frac{16}{h^6}\theta(1-\theta)^2\Delta t^3v^mv^{m-1}\sin^2\frac{p\pi}{2N}\left(1+\cos^2\frac{p\pi}{2N}\right)\right]>0.\end{aligned} \tag{3.16}$$

Note that $P(1)$ is increasing about v^{m-1}. Since both inequalities $v^{m-1}>0$ and

$$P(1)|_{v^{m-1}=0}=h^{-4}\theta^2(v^m)^2\Delta t^2\left(\gamma_p^m\right)^2\left[4\sin^2\frac{p\pi}{2N}+8\theta\ v^m\Delta t\left(2\sin^2\frac{p\pi}{2N}-\sin^4\frac{p\pi}{2N}\right)\right]>0,$$

are unconditionally satisfied, so we have $P(1)>0$ unconditionally.

(iii) The Condition $P(-1)>0$. Taking $\sigma_p=-1$ in (3.12) we get, based on $0<v^{m-1}\leq v^m$, that

$$\begin{aligned}v^m\frac{\partial P(-1)}{\partial(v^{m-1})}\leq\left(\gamma_p^m\right)^2&\left[-(1-\theta)\Delta t\frac{8v^m}{h^2}-\frac{8}{h^4}\theta(1-\theta)\Delta t^2(v^m)^2\left(2+2\cos^2\frac{p\pi}{2N}\right)\right.\\&\left.+\frac{8}{h^4}(1-\theta)^2\Delta t^2(v^m)^2\sin^2\frac{p\pi}{2N}+\frac{16}{h^6}\theta(1-\theta)^2\Delta t^3(v^m)^3\sin^2\frac{p\pi}{2N}\left(1+\cos^2\frac{p\pi}{2N}\right)\right].\end{aligned} \tag{3.17}$$

Setting $X=v^m\Delta t/h^2$, then the proof of the inequality $\partial P(-1)/\partial(v^{m-1})<0$ is equivalent to that of the following inequality:

$$2\theta(1-\theta)\sin^2\frac{p\pi}{2N}\left(1+\cos^2\frac{p\pi}{2N}\right)X^2+\left(\sin^2\frac{p\pi}{2N}-3\theta-\theta\cos^2\frac{p\pi}{2N}\right)X-1<0.$$

The left-hand side of the above inequality attains its maximum at $P=N$. Letting $P=N$ and $N\to\infty$, we get $2\theta(1-\theta)X^2+(1-3\theta)X-1<0$. Therefore, $P(-1)$ is decreasing about v^{m-1}, while $0<X<1/(1-\theta)$.

Now, we replace v^{m-1} by v^m in $P(-1)$, and denote this new equality by $P_0(-1)$:

$$\begin{aligned}P_0(-1) = 1 + \gamma_p^m \Big(&1 - (1-\theta)v^m\Delta t\lambda_p + \theta v^m \Delta t \lambda_{2N-p}\cos\frac{p\pi}{2N} - \theta(v^m)^2\Delta t^2 \sin^2\frac{p\pi}{4N}\lambda_{2N-p}^2 \\ &+1 - (1-\theta)v^m\Delta t\lambda_{2N-p} - \theta v^m\Delta t\lambda_p\cos\frac{p\pi}{2N} - \theta(v^m)^2\Delta t^2\cos^2\frac{p\pi}{4N}\lambda_p^2\Big) \\ &+ \left(\gamma_p^m\right)^2 \Big\{\Big(1 - (1-\theta)v^m\Delta t\lambda_p + \theta v^m\Delta t\lambda_{2N-p}\cos\frac{p\pi}{2N} \\ &-\theta(v^m)^2\Delta t^2\sin^2\frac{p\pi}{4N}\lambda_{2N-p}^2\Big)\cdot\Big(1-(1-\theta)v^m\Delta t\lambda_{2N-p} - \theta v^m\Delta t\lambda_p\cos\frac{p\pi}{2N} \\ &-\theta(v^m)^2\Delta t^2\cos^2\frac{p\pi}{4N}\lambda_p^2\Big) - \theta^2(v^m)^4\Delta t^4\lambda_p^2\lambda_{2N-p}^2\sin^2\frac{p\pi}{4N}\cos^2\frac{p\pi}{4N}\Big\}. \qquad (3.18)\end{aligned}$$

Naturally we have $P_0(-1) < P(-1)$, so $P_0(-1) > 0$ implies $P(-1) > 0$.

Let us replace $P_0(-1)$, $v^m\Delta t/h^2$ and $\cos^2(p\pi/2N)$ by $\Phi_X(y)$, X and y, respectively, and set $\sigma_p = -1$ in (3.12). We obtain that

$$\begin{aligned}\Phi_X(y) = &(4\theta^2X^2 - 2\theta X^3 + 4\theta^2X^3)y^2 + \left[(8\theta^2 - 2\theta - 1)X^2 + 4\theta X\right]y \\ &+2\theta(1-2\theta)X^3 + (4\theta^2 - 6\theta + 1)X^2 + (4\theta - 2)X + 1. \qquad (3.19)\end{aligned}$$

If $\theta = 0$, the condition of $P_0(-1) > 0$ is guaranteed by $X < \frac{1}{2}$. If $\theta \neq 0$, we have generally

$$\Phi_X'(y) = (8\theta^2X^2 - 4\theta X^3 + 8\theta^2X^3)y + (8\theta^2 - 2\theta - 1)X^2 + 4\theta X$$

and

$$\Phi_X''(y) = (8\theta^2 - 4\theta)X^3 + 8\theta^2X^2.$$

It follows that

(1) $1 \geq \theta \geq \frac{1}{2}$: $\Phi_X''(y) \geq 0$ hold true unconditionally,

(2) $\frac{1}{2} > \theta > 0$: if $X \leq \dfrac{2\theta}{1-2\theta}$, i.e., $\dfrac{v(T)\Delta t}{h^2} \leq \dfrac{2\theta}{1-2\theta}$, then $\Phi_X''(y) \geq 0$.

Hence, we know that $\Phi_X'(y)$ is increasing. Therefore, the inequality

$$\Phi_X'(0) = (8\theta^2 - 2\theta - 1)X^2 + 4\theta X > 0$$

implies $\Phi_X'(y) > 0$. It is now easy to find that

(1) $1 \geq \theta \geq \frac{1}{2}$: $\Phi_X'(0) > 0$ are satisfied unconditionally,

(2) $\frac{1}{2} > \theta > 0$: if $X \leq \dfrac{4\theta}{1+2\theta-8\theta^2}$, i.e., $\dfrac{v(T)\Delta t}{h^2} \leq \dfrac{4\theta}{1+2\theta-8\theta^2}$, then $\Phi_X'(0) > 0$.

That yields the conclusion that $\Phi_X(y)$ is increasing. We can get $\Phi_X(y) > 0$, provided that

$$\Phi_X(0) = dX^3 + eX^2 + fX + 1 > 0,$$

where $d = 2\theta(1-2\theta)$, $e = 4\theta^2 - 6\theta + 1$, $f = 4\theta - 2$. In particular, if $\theta = \frac{1}{2}$, then $\Phi_X(0) > 0$ holds true with $X < 1$.

Now we suppose that $\theta > \frac{1}{2}$. Note that $\Phi_X(0)$ has a positive root

$$X_1 = K^{\frac{1}{3}} - \frac{3fd - e^2}{9d^2K^{\frac{1}{3}}} - \frac{e}{3d},$$

with

$$K = \frac{9fed - 27d^2 - 2e^3}{54d^3} + \frac{\sqrt{12f^3d - 3f^2e^2 - 54fed + 81d^2 + 12e^3}}{18d^2}.$$

Therefore, we know that if $X < X_1$ and $\Phi_X(0) > 0$ hold true. Then the condition for $P(-1) > 0$ reads,

(i) $\theta = 0$: $\quad \frac{v(T)\Delta t}{h^2} < \frac{1}{2}$,

(ii) $0 < \theta < \frac{1}{2}$: $\quad \frac{v(T)\Delta t}{h^2} < \min\left\{\frac{4\theta}{1+2\theta-8\theta^2},\ \frac{2\theta}{1-2\theta},\ X_1,\ \frac{1}{1-\theta}\right\}$,

(iii) $\theta = \frac{1}{2}$: $\quad \frac{v(T)\Delta t}{h^2} < 1$,

(iv) $\frac{1}{2} < \theta < 1$: $\quad \frac{v(T)\Delta t}{h^2} < \min\left\{X_1,\ \frac{1}{1-\theta}\right\}$,

(v) $\theta = 1$: $\quad \frac{v(T)\Delta t}{h^2} < X_1$.

This completes the proof of Theorem 3.1.

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