Journal of Computational Mathematics, Vol.25, No.4, 2007, 485–496.

MULTISTEP FINITE VOLUME APPROXIMATIONS TO THE TRANSIENT BEHAVIOR OF A SEMICONDUCTOR DEVICE ON GENERAL 2D OR 3D MESHES [*1)]

Min Yang
(*School of Mathematics and Information Science, Yantai University, Yantai 264005, China*
Email: yangmin_math@163.com)

Abstract

In this paper, we consider a hydrodynamic model of the semiconductor device. The approximate solutions are obtained by a mixed finite volume method for the potential equation and multistep upwind finite volume methods for the concentration equations. Error estimates in some discrete norms are derived under some regularity assumptions on the exact solutions.

Mathematics subject classification: 65N30, 65M25.
Key words: Semiconductor device, Unstructured meshes, Finite volume, Multistep method, Error estimates.

1. Introduction

Let us consider a system of equations describing the mobil carrier transport in a semiconductor device in a bounded domain $\Omega \in R^d$, $d = 2, 3$:

$$-\Delta v = \nabla \cdot u = \alpha[p - e + N(x)], \tag{1.1}$$

$$\frac{\partial e}{\partial t} = \nabla \cdot [D_e(x)\nabla e - \mu_e(x)e\nabla v] - R_1(e,p), \tag{1.2}$$

$$\frac{\partial p}{\partial t} = \nabla \cdot [D_p(x)\nabla p + \mu_p(x)p\nabla v] - R_2(e,p). \tag{1.3}$$

The above system is a hydrodynamic model of the semiconductor device. Three unknowns are the electrostatic potential v, the electron mobile charge density e, and the hole mobile charge density p. $u = -\nabla v$ is the electric intensity. α is a constant related to the magnitude of electronic charge and the dielectric permittivity. $N(x) = N_D(x) - N_A(x)$, where $N_D(x)$ and $N_A(x)$ denote the donor and acceptor impurity respectively. The diffusion coefficients $D_s(x)$ $(s = e, p)$ are related to the mobilities $\mu_s(x)$ by the relation $D_s(x) = U_T\mu_s(x)$, where U_T is the thermal voltage. The recombination terms $R_i(e,p), i = 1, 2$ are Lipschitz continuous with the Lipschitz constant λ. All the coefficients appeared in (1.1)–(1.3) are positive and bounded, and $\mu_s \geq \mu_* > 0, D_s \geq D_* > 0, s = e, p$, where μ_* and D_* is positive constants.

The equations can be completed by the following initial and boundary conditions

$$e(x,0) = e_0(x),\ p(x,0) = p_0(x), \tag{1.4}$$

$$v = 0,\ e = 0,\ p = 0,\ x \in \partial\Omega,\ t \in (0,T]. \tag{1.5}$$

[*] Received November 5, 2005; final revised March 19, 2007; accepted March 28, 2007.
[1)] The research is partially supported by the National Natural Science Foundation of China (No. 10271066).

There have existed many works on the numerical solution of the above system. In [5], a finite difference method was constructed for one or two dimensional cases and the convergence analysis was given. Numerical procedures based on a mixed finite element method for the potential equation and finite element methods for the mobile charge density equations were first presented in [4, 7]. The method was then applied to a mixed initial boundary model in [12], where under some less smoothness assumptions on the exact solution, a priori error estimates were obtained. In [13], two kinds of finite element schemes, one being partly linear and another being nonlinear, were formulated and the existence of the approximate solutions was proved for both cases. The convergence analysis for the nonlinear scheme was presented in [14]. Some exponentially converging box methods, named Scharfetter-Gummel methods, were used in [10] to treat two and three dimensional semiconductor device problems. The stability of the methods and error estimates for the Slotboom variables are derived. Recently, characteristic finite element methods have been presented in [11] to avoid nonphysical oscillation and optimal error estimates were obtained there.

Finite volume method is a discretization tool used extensively in the computations for conservation laws. The method is suitable in handling general domains, which can keep local conservation properties of the numerical fluxes. We refer to [3, 6, 8, 9] and the references therein for some details. In this paper, we study a finite volume method for the semiconductor devices in multi–dimensions. We use a mixed finite volume method to treat the elliptic equation (1.1) and upwind finite volume methods to treat the convection–diffusion Eqs. (1.2)-(1.3). A multistep time discretization is considered to enhance the accuracy in temporal direction. Under the assumption that the exact solutions possess enough regularity we derive the optimal error estimates in discrete norms for the scheme.

The rest of the paper is organized as follows. In Section 2, we introduce the admissible meshes and some necessary notation. Section 3 is devoted to formulating a fully discrete finite volume scheme for Eqs. (1.1)-(1.5). In Section 4, we derive the priori error estimates for the finite volume scheme under some regularity assumptions on the exact solutions.

Throughout this paper, we use C and ϵ to denote a general positive constant and a general positive small constant, respectively, not necessarily the same in different places.

2. Meshes and Notations

Definition 2.1. (Admissible meshes) *An admissible mesh T_h of Ω is given by a family of control volumes, which are open polygonal (or polyhedral) subsets of Ω. A family $\mathcal{E}$ of subsets of $\overline{\Omega}$ contained in hyper-planes of R^d with strictly positive measure (the edges of the mesh), and a family of discrete points in Ω satisfying the following properties:*

1. *The closure of the union of all control volumes is $\overline{\Omega}$.*
2. *For any $K \in T_h$, there exists a subset $\mathcal{E}_K \subseteq \mathcal{E}$, such that $\partial K = \overline{K} \backslash K = \cup_{\sigma \in \mathcal{E}_K} \overline{\sigma}$. Furthermore, $\mathcal{E} = \cup_{K \in T_h} \mathcal{E}_K$.*
3. *For any $(K, L) \in T_h^2$ with $K \neq L$, either $\overline{K} \cap \overline{L} = 0$ or $\overline{K} \cap \overline{L} = \overline{\sigma}$. Then, we denote by $\sigma = K|L$.*
4. *The family of discrete points $\{x_K\}_{K \in T_h}$ is such that $x_K \in K$ and, if $\sigma = K|L$, it is assumed that the straight line $\overline{x_K x_L}$ is orthogonal to σ.*

Let h denote the space step of the mesh T_h. For any $K \in T_h$ and $\sigma \in \mathcal{E}$, we denote by $m(K)$ the measure of K and $m(\sigma)$ the measure of the edge σ. If $\sigma \in \mathcal{E}_K$, we denote $d_{K,\sigma}$ the Euclidean distance between the point x_K and the edge σ. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{int}$ (resp. $\mathcal{E}_{ext}$), that is, $\mathcal{E}_{int} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{ext} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). Then let $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$ if $\sigma = K|L \in \mathcal{E}_{int}$, and let $d_\sigma = d_{K,\sigma}$ if $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{ext}$.

The mesh T_h is quasi–uniform, i.e., there exists a constant $C_* > 0$ such that

$$d_{K,\sigma} \geq C_* h,\ \forall K \in T_h, \sigma \in \mathcal{E}_K. \tag{2.1}$$

Let $\chi(T_h)$ denote the piecewise constant space on T_h. We introduce a discrete H_0^1 norm for this space: for any $\varphi \in \chi(T_h)$,

$$\|\varphi\|_{1,h} = \left(\sum_{\sigma \in \mathcal{E}} \frac{m(\sigma)}{d_\sigma} (\Upsilon_\sigma \varphi)^2 \right)^{1/2}, \tag{2.2}$$

where $\Upsilon_\sigma \varphi = \varphi_L - \varphi_K$, if $\sigma = K|L \in \mathcal{E}_{int}$, and $\Upsilon_\sigma \varphi = -\varphi_K$, if $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{ext}$. According to [2], the discrete H_0^1 norm satisfies the following properties.

Lemma 2.1. *Let T_h be an admissible mesh defined on Ω. For any $\varphi \in \chi(T_h)$, there exists a constant $C > 0$ only depending upon Ω such that*

$$\|\varphi\|_{L^2} \leq C \|\varphi\|_{1,h}. \tag{2.3}$$

Lemma 2.2. *Let T_h be an admissible mesh defined on Ω satisfying* (2.1). *For any $\varphi \in \chi(T_h)$, there exists a constant $C > 0$ only depending upon Ω and C_*, such that*

$$\|\varphi\|_{L^\infty} \leq C(\ln|h| + 1)\|\varphi\|_{1,h}, \quad if \ \ d = 2, \tag{2.4}$$

$$\|\varphi\|_{L^\infty} \leq C h^{-1/2} \|\varphi\|_{1,h}, \quad if \ \ d = 3. \tag{2.5}$$

3. Fully Discrete Finite Volume Scheme

Let N be a positive integer. Let $\Delta t = T/N$ and $t_n = n\Delta t$. Define by

$$\varphi^n = \varphi(t_n),\ \delta\varphi^n = \varphi^n - \varphi^{n-1},\ \partial_t \varphi^n = \delta\varphi^n / \Delta t. \tag{3.1}$$

For any $K \in T_h$, we denote by η_K the unit outer normal vector to ∂K. For (1.1)-(1.3), we use the Green's formula to obtain the following integral conservation forms:

$$\int_{\partial K} u \cdot \eta_K ds = -\int_{\partial K} \nabla v \cdot \eta_K ds = \int_K \alpha[p - e + N(x)]dx, \tag{3.2}$$

$$\int_K \frac{\partial e}{\partial t} dx + \int_{\partial K} [-\mu_e(x) e u - D_e(x)\nabla e] \cdot \eta_K ds = -\int_K R_1(e,p)dx, \tag{3.3}$$

$$\int_K \frac{\partial p}{\partial t} dx + \int_{\partial K} [\mu_p(x) p u - D_p(x)\nabla p] \cdot \eta_K ds = -\int_K R_2(e,p)dx. \tag{3.4}$$

We denote by $V^n, E^n, P^n \in \chi(T_h)$ the approximations of v^n, e^n and p^n respectively. Let $U_{K,\sigma}$ be the approximation of $u \cdot \eta_K$ on the edge σ of K such that

$$U_{K,\sigma} = -\frac{V_L - V_K}{d_\sigma},\ if\ \sigma = K|L \in \mathcal{E}_{int}; \quad U_{K,\sigma} = \frac{V_K}{d_\sigma},\ if\ \sigma \in \mathcal{E}_K \cap \mathcal{E}_{ext}. \tag{3.5}$$

Let I be an extrapolator on time such that $I\varphi^{n+1}=2\varphi^n-\varphi^{n-1}$, $n\geq 1$. $I\varphi^{n+1}=\varphi^0$, $n=0$. Define the upwind values $E^{n+1}_{\sigma,+}$ of E^{n+1} and $P^{n+1}_{\sigma,+}$ of P^{n+1} on an edge σ as follows:

$$E^{n+1}_{\sigma,+}=\begin{cases} E^{n+1}_L, & IU^{n+1}_{K,\sigma}\geq 0,\\ E^{n+1}_K, & otherwise,\end{cases}\quad \sigma=K|L\in\mathcal{E}_{int},$$

$$E^{n+1}_{\sigma,+}=\begin{cases} 0, & IU^{n+1}_{K,\sigma}\geq 0,\\ E^{n+1}_K, & otherwise,\end{cases}\quad \sigma\in\mathcal{E}_K\cap\mathcal{E}_{ext}.$$

$$P^{n+1}_{\sigma,+}=\begin{cases} P^{n+1}_K, & IU^{n+1}_{K,\sigma}\geq 0,\\ P^{n+1}_L, & otherwise,\end{cases}\quad \sigma=K|L\in\mathcal{E}_{int},$$

$$P^{n+1}_{\sigma,+}=\begin{cases} P^{n+1}_K, & IU^{n+1}_{K,\sigma}\geq 0,\\ 0, & otherwise,\end{cases}\quad \sigma\in\mathcal{E}_K\cap\mathcal{E}_{ext}.$$

Then the multistep finite volume scheme is given by: for any $K\in T_h$,

$$\sum_{\sigma\in\mathcal{E}_K} m(\sigma)U^n_{K,\sigma}=\alpha m(K)(P^n_K-E^n_K+\bar{N}_K), \tag{3.6}$$

$$\begin{aligned}&m(K)\partial_t E^{n+1}_K-\frac{2}{3}\sum_{\sigma\in\mathcal{E}_K}\Big(m(\sigma)\bar{\mu}_{e,K}E^{n+1}_{\sigma,+}IU^{n+1}_{K,\sigma}+\bar{D}_{e,\sigma}\frac{m(\sigma)}{d_\sigma}(\Upsilon_\sigma E^{n+1})\Big)\\ =&\frac{1}{3}m(K)\partial_t E^n_K-\frac{2}{3}m(K)R_1(IE^{n+1}_K,IP^{n+1}_K),\end{aligned} \tag{3.7}$$

$$\begin{aligned}&m(K)\partial_t P^{n+1}_K+\frac{2}{3}\sum_{\sigma\in\mathcal{E}_K}\Big(m(\sigma)\bar{\mu}_{p,K}P^{n+1}_{\sigma,+}IU^{n+1}_{K,\sigma}-\bar{D}_{p,\sigma}\frac{m(\sigma)}{d_\sigma}(\Upsilon_\sigma P^{n+1})\Big)\\ =&\frac{1}{3}m(K)\partial_t P^n_K-\frac{2}{3}m(K)R_2(IE^{n+1}_K,IP^{n+1}_K),\end{aligned} \tag{3.8}$$

where

$$\bar{N}_K=\frac{1}{m(K)}\int_K N(x)dx,\quad \bar{D}_{s,\sigma}=\frac{1}{m(\sigma)}\int_\sigma D_s(x)ds,\quad \bar{\mu}_{s,K}=\mu_s(x_K),$$

for $s=e,p$, and the difference operator Υ_σ is defined in (2.2).

The initial approximate values $\{E^0_K,P^0_K\}$ can be obtained by $E^0_K=e_0(x_K)$ and $P^0_K=p_0(x_K)$, and $\{E^1_K,P^1_K\}$ can be obtained by a single step scheme similar to (3.7)-(3.8). The algorithm for the whole scheme is described as follows. Assume that the approximate solution $\{E^{n-1}_K,E^n_K,P^{n-1}_K,P^n_K\}$ are known. From (3.5)-(3.6), we can calculate $\{V^{n-1}_K,V^n_K\}$ and $\{U^{n-1}_{K,\sigma},U^n_{K,\sigma}\}$ in succession. Then we solve (3.7)-(3.8) independently and obtain $\{E^{n+1}_K,P^{n+1}_K\}$. Repeating this procedure, we can obtain all the approximations.

4. Error Estimates

First the assumptions on the regularity of the exact solutions of (1.1)-(1.5) are collected as follows

$$\begin{aligned}&v\in L^\infty(0,T;W^{2,\infty}(\Omega))\cap H^2(0,T;W^{1,\infty}(\bar{\Omega})),\\ &e,\ p\in L^\infty(0,T;L^\infty(\bar{\Omega}))\cap L^\infty(0,T;W^{1,\infty}(\Omega))\cap H^1(0,T;W^{1,\infty}(\Omega))\cap H^2(0,T;L^2(\Omega)).\end{aligned} \tag{4.1}$$

Lemma 4.1. ([1]) *For any $\varphi \in \chi(T_h)$, there exists a constant $C > 0$ depending upon various norms of φ such that*

$$\left\| \frac{2}{3}\frac{\partial \varphi^{n+1}}{\partial t} - \partial_t \varphi^{n+1} + \frac{1}{3}\partial_t \varphi^n \right\|_{L^2} \leq C\Delta t^2. \tag{4.2}$$

Lemma 4.2. *For any $\varphi \in \chi(T_h)$ and the integer $R \leq N$, we have*

$$\Delta t \sum_{n=1}^{R-1} (\partial_t \varphi_K^n)\varphi_K^{n+1} \leq \frac{3}{2}\sum_{n=1}^{R-1}(\delta\varphi_K^{n+1})^2 + (\varphi_K^R)^2 + 2(\varphi_K^1)^2 + \frac{3}{2}(\varphi_K^0)^2. \tag{4.3}$$

Proof. Note that

$$\varphi_K^n = \frac{1}{2}[(\varphi_K^n - \varphi_K^{n-1}) + (\varphi_K^n + \varphi_K^{n-1})] = \frac{1}{2}[\delta\varphi_K^n + (\varphi_K^n + \varphi_K^{n-1})].$$

We then have

$$\begin{aligned} \Delta t \sum_{n=1}^{R-1}(\partial_t\varphi_K^n)\varphi_K^{n+1} &= \sum_{n=1}^{R-1}(\delta\varphi_K^n)\varphi_K^n + \sum_{n=1}^{R-1}(\delta\varphi_K^n)(\delta\varphi_K^{n+1}) \\ = \frac{1}{2}\Big(\sum_{n=1}^{R-1}(\delta\varphi_K^n)^2 &+ \sum_{n=1}^{R-1}\big[(\varphi_K^n)^2 - (\varphi_K^{n-1})^2\big]\Big) + \sum_{n=1}^{R-1}(\delta\varphi_K^n)(\delta\varphi_K^{n+1}) \\ = \frac{1}{2}\big[\sum_{n=1}^{R-1}(\delta\varphi_K^n)^2 &+ (\varphi_K^{R-1})^2 - (\varphi_K^0)^2\big] + \sum_{n=1}^{R-1}(\delta\varphi_K^n)(\delta\varphi_K^{n+1}), \end{aligned}$$

where

$$\begin{aligned} &(\varphi_K^{R-1})^2 = (\varphi_K^R - \delta\varphi_K^R)^2 \leq 2(\delta\varphi_K^R)^2 + 2(\varphi_K^R)^2, \\ &\sum_{n=1}^{R-1}(\delta\varphi_K^n)(\delta\varphi_K^{n+1}) \leq \frac{1}{2}\sum_{n=1}^{R-1}(\delta\varphi_K^n)^2 + \frac{1}{2}\sum_{n=1}^{R-1}(\delta\varphi_K^{n+1})^2. \end{aligned}$$

It follows from above estimates that

$$\begin{aligned} \Delta t \sum_{n=1}^{R-1}(\partial_t\varphi_K^n)\varphi_K^{n+1} &\leq \sum_{n=1}^{R-1}(\delta\varphi_K^n)^2 + (\delta\varphi_K^R)^2 + (\varphi_K^R)^2 - \frac{1}{2}(\varphi_K^0)^2 + \frac{1}{2}\sum_{n=1}^{R-1}(\delta\varphi_K^{n+1})^2 \\ &= \frac{3}{2}\sum_{n=1}^{R-1}(\delta\varphi_K^{n+1})^2 + (\delta\varphi_K^1)^2 + (\varphi_K^R)^2 - \frac{1}{2}(\varphi_K^0)^2 \\ &\leq \frac{3}{2}\sum_{n=1}^{R-1}(\delta\varphi_K^{n+1})^2 + (\varphi_K^R)^2 + 2(\varphi_K^1)^2 + \frac{3}{2}(\varphi_K^0)^2. \end{aligned}$$

Thus, we get the desired result.

Now we define $\pi, \xi, \vartheta \in \chi(T_h)$ such that for any $x \in K$, $K \in T_h$,

$$\pi = V_K - v(x_K), \quad \xi = E_K - e(x_K), \quad \vartheta = P_K - p(x_K). \tag{4.4}$$

Lemma 4.3. *Suppose that $v \in L^\infty(0,T;W^{2,\infty}(\Omega))$ and $e, p \in L^\infty(0,T;W^{1,\infty}(\Omega))$. Then there exists a constant $C > 0$ such that*

$$\|\pi^n\|_{1,h}^2 \leq C(\|\vartheta^n\|_{L^2}^2 + \|\xi^n\|_{L^2}^2 + h^2). \tag{4.5}$$

Proof. Similar to (3.5) we define that

$$u_{K,\sigma} = -\frac{v(x_L) - v(x_K)}{d_\sigma},\ if\ \sigma = K|L \in \mathcal{E}_{int};\quad u_{K,\sigma} = \frac{v(x_K)}{d_\sigma},\ if\ \sigma \in \mathcal{E}_K \cap \mathcal{E}_{ext}. \tag{4.6}$$

Then from (3.2) and (3.6), we can obtain the following error equation.

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}_K} m(\sigma)(U^n_{K,\sigma} - u^n_{K,\sigma}) = &\sum_{\sigma\in\mathcal{E}_K}\int_\sigma (u^n\cdot\eta_K - u^n_{K,\sigma})ds + \alpha m(K)(\vartheta^n_K - \xi^n_K)\\ &+\int_K \alpha\big([p^n(x_K) - p^n] - [e^n(x_K) - e^n]\big)dx. \end{aligned}\tag{4.7}$$

Multiply (4.7) by π^n_K and sum the result on T_h. First, we see that

$$\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K} m(\sigma)(U^n_{K,\sigma} - u^n_{K,\sigma})\pi^n_K = \|\pi^n\|^2_{1,h}. \tag{4.8}$$

The terms on the right hand side of (4.7) will be estimated in sequence. Reordering by interior edges and using Young inequality, we have

$$\begin{aligned}&\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}\int_\sigma (u^n\cdot\eta_K - u^n_{K,\sigma})ds\pi^n_K\\ =&\sum_{K\in T_h}\bigg(\frac{1}{2}\sum_{\sigma=K|L\in\mathcal{E}_K}\int_\sigma (u^n\cdot\eta_K - u^n_{K,\sigma})ds(\pi^n_K - \pi^n_L) + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{ext}}\int_\sigma (u^n\cdot\eta_K - u^n_{K,\sigma})ds(\pi^n_K)\bigg)\\ \le& C\{\|v\|_{L^\infty(W^{2,\infty})}\}\sum_{\sigma\in\mathcal{E}} m(\sigma)d_\sigma h^2 + \epsilon\|\pi^n\|^2_{1,h} = C\{d, m(\Omega)\}h^2 + \epsilon\|\pi^n\|^2_{1,h}.\end{aligned}\tag{4.9}$$

By Lemma 2.1, we have

$$\begin{aligned}&\sum_{K\in T_h}\bigg(\alpha m(K)(\vartheta^n_K - \xi^n_K) + \int_K \big([p^n(x_K) - p^n] - [e^n(x_K) - e^n]\big)dx\bigg)\pi^n_K\\ \le& C\{\|e\|_{L^\infty(W^{1,\infty})}, \|p\|_{L^\infty(W^{1,\infty})}\}(\|\vartheta^n\|_{L^2} + \|\xi^n\|_{L^2} + h)\|\pi^n\|_{L^2}\\ \le& C(\|\vartheta^n\|^2_{L^2} + \|\xi^n\|^2_{L^2} + h^2) + \epsilon\|\pi^n\|^2_{1,h}.\end{aligned}\tag{4.10}$$

So combining (4.8)-(4.10) and choosing ϵ small enough, we get the desired result.

In the error analysis below, we will use the estimates

$$\sup_{0\le n\le N}\|\xi^n\|_{L^\infty}\to 0,\quad \sup_{0\le n\le N}\|\vartheta^n\|_{L^\infty}\to 0,\quad (h,\Delta t)\to 0. \tag{4.11}$$

From (4.4), we have $\xi^0 = \vartheta^0 = 0$ and (4.11) is trivial for $N = 0$. Next we make the induction hypothesis that (4.11) holds for $0 \le N \le R-1$. We will prove that (4.11) holds for $N = R$ by an induction argument later. Let us consider the error estimates between (3.3) and (3.7).

Lemma 4.4. *If v, e and p satisfy the regularity assumption* (4.1), *then*

$$\begin{aligned}&\|\xi^R\|^2_{L^2} + \Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|^2_{1,h} + \Delta t\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K} m(\sigma)|IU^{n+1}_{K,\sigma}|(\Upsilon_\sigma\xi^{n+1})^2\\ \le& C\bigg(\Delta t^4 + h^2 + \Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|^2_{L^2} + \Delta t\sum_{n=1}^{R-1}\|\vartheta^{n+1}\|^2_{L^2} + \|\xi^1\|^2_{L^2} + \|\vartheta^1\|^2_{L^2}\bigg).\end{aligned}\tag{4.12}$$

Proof. For any $K \in T_h$, $\sigma \in \mathcal{E}_K$, we define the upwind value $e_{\sigma,+}^{n+1}$ as follows:

$$e_{\sigma,+}^{n+1} = \begin{cases} e^{n+1}(x_L), & IU_{K,\sigma}^{n+1} \geq 0, \\ e^{n+1}(x_K), & otherwise, \end{cases} \quad \sigma = K|L \in \mathcal{E}_{int},$$

$$e_{\sigma,+}^{n+1} = \begin{cases} 0, & IU_{K,\sigma}^{n+1} \geq 0, \\ e^{n+1}(x_K), & otherwise, \end{cases} \quad \sigma \in \mathcal{E}_K \cap \mathcal{E}_{ext}.$$

Let $\xi_{\sigma,+}^{n+1} = E_{\sigma,+}^{n+1} - e_{\sigma,+}^{n+1}$. From (3.3) and (3.7) we have the following error equation:

$$m(K)\partial_t \xi_K^{n+1} - \frac{2}{3}\sum_{\sigma\in\mathcal{E}_K}\big(m(\sigma)\bar{\mu}_{e,K}\xi_{\sigma,+}^{n+1} IU_{K,\sigma}^{n+1} + \bar{D}_{e,\sigma}\frac{m(\sigma)}{d_\sigma}\Upsilon_\sigma \xi^{n+1}\big) = \sum_{i=1}^{8} S_i. \tag{4.13}$$

Multiply (4.13) by $\Delta t \xi_K^{n+1}$, and sum the resulting equation over T_h for $1 \leq n \leq R-1$. Note that $\xi^0 = 0$ and an application of Lemma 4.2 gives

$$\Delta t \sum_{K\in T_h}\sum_{n=1}^{R-1} S_1 \xi_K^{n+1} = \frac{1}{3}\Delta t \sum_{K\in T_h}\sum_{n=1}^{R-1} m(K)(\partial_t \xi_K^n)\xi_K^{n+1}$$
$$\leq \frac{1}{2}\sum_{n=1}^{R-1}\|\delta\xi^{n+1}\|_{L^2}^2 + \frac{1}{3}\|\xi^R\|_{L^2}^2 + \frac{2}{3}\|\xi^1\|_{L^2}^2. \tag{4.14}$$

Using Taylor expansion, we have

$$\Delta t \sum_{K\in T_h}\sum_{n=1}^{R-1} S_2 \xi_K^{n+1}$$
$$= \Delta t \sum_{n=1}^{R-1}\sum_{K\in T_h}\bigg(\int_K \partial_t\big(e^{n+1} - e^{n+1}(x_K)\big)dx - \frac{1}{3}\int_K \partial_t\big(e^n - e^n(x_K)\big)dx\bigg)\xi_K^{n+1}$$
$$\leq C\{\|e\|_{H^1(W^{1,\infty})}\}(h^2 + \Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{L^2}^2). \tag{4.15}$$

It follows from Lemma 4.1 that

$$\Delta t \sum_{K\in T_h}\sum_{n=1}^{R-1} S_3 \xi_K^{n+1} = \Delta t \sum_{n=1}^{R-1}\sum_{K\in T_h}\int_K\bigg(\frac{2}{3}\frac{\partial e^{n+1}}{\partial t} - \partial_t e^{n+1} + \frac{1}{3}\partial_t e^n\bigg)\xi_K^{n+1}dx$$
$$\leq C(\Delta t^4 + \Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{L^2}^2). \tag{4.16}$$

Next, we have

$$\Delta t \sum_{K\in T_h}\sum_{n=1}^{R-1} S_4 \xi_K^{n+1} = \frac{2\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K} IU_{K,\sigma}^{n+1}\bar{\mu}_{e,K}\int_\sigma\big(e_{\sigma,+}^{n+1} - e^{n+1}\big)ds\xi_K^{n+1}$$
$$= \frac{2\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K} Iu_{K,\sigma}^{n+1}\bar{\mu}_{e,K}\int_\sigma\big(e_{\sigma,+}^{n+1} - e^{n+1}\big)ds\xi_K^{n+1}$$
$$+\frac{2\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}(IU_{K,\sigma}^{n+1} - Iu_{K,\sigma}^{n+1})\bar{\mu}_{e,K}\int_\sigma\big(e_{\sigma,+}^{n+1} - e^{n+1}\big)ds\xi_K^{n+1}$$
$$= \frac{2\Delta t}{3}\sum_{n=1}^{R-1}(S_{41} + S_{42}).$$

By (4.6) we have $|Iu^{n+1}_{K,\sigma}\bar{\mu}_{e,K}| \le 3\|v\|_{L^\infty(W^{1,\infty})}\|\mu_e\|_{L^\infty}$. Reordering by interior edges gives

$$\begin{aligned}
S_{41} &= \sum_{K\in T_h}\Big(\frac{1}{2}\sum_{\sigma=K|L\in\mathcal{E}_K} Iu^{n+1}_{K,\sigma}\bar{\mu}_{e,K}\int_\sigma \big(e^{n+1}_{\sigma,+}-e^{n+1}\big)ds(\xi^{n+1}_K-\xi^{n+1}_L)\\
&\quad+\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{ext}} Iu^{n+1}_{K,\sigma}\bar{\mu}_{e,K}\int_\sigma \big(e^{n+1}_{\sigma,+}-e^{n+1}\big)ds\xi^{n+1}_K\Big)\\
&\le C\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}\frac{d_\sigma}{m(\sigma)}\Big[\int_\sigma(e^{n+1}_{\sigma,+}-e^{n+1})ds\Big]^2+\epsilon\sum_{\sigma\in\mathcal{E}}\frac{m(\sigma)}{d_\sigma}(\Upsilon_\sigma\xi^{n+1})^2\\
&\le C\{\|e\|_{L^\infty(W^{1,\infty})}\}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K} m(\sigma)d_\sigma h^2+\epsilon\|\xi^{n+1}\|^2_{1,h}\\
&\le C\{d,m(\Omega)\}h^2+\epsilon\|\xi^{n+1}\|^2_{1,h}.
\end{aligned}$$

Note that

$$\sum_{\sigma\in\mathcal{E}_K}\frac{m(\sigma)}{d_\sigma}h^2 \le C\{C_*\}\sum_{\sigma\in\mathcal{E}_K} m(\sigma)d_{K,\sigma}\le C\{C_*,d\}m(K).$$

Therefore, it follows from (3.5) and (4.6) that

$$\begin{aligned}
S_{42} &= \sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}(IU^{n+1}_{K,\sigma}-Iu^{n+1}_{K,\sigma})\bar{\mu}_{e,K}\int_\sigma\big(e^{n+1}_{\sigma,+}-e^{n+1}\big)ds\xi^{n+1}_K\\
&\le C\{\|e\|_{L^\infty(W^{1,\infty})},\|\mu_e\|_{L^\infty}\}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}\frac{m(\sigma)h}{d_\sigma}(2|\Upsilon_\sigma\pi^n|+|\Upsilon_\sigma\pi^{n-1}|)|\xi^{n+1}_K|\\
&\le C(\|\pi^n\|^2_{1,h}+\|\pi^{n-1}\|^2_{1,h})+C\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}\frac{m(\sigma)}{d_\sigma}h^2(\xi^{n+1}_K)^2\\
&\le C(\|\pi^n\|^2_{1,h}+\|\pi^{n-1}\|^2_{1,h})+C\|\xi^{n+1}\|^2_{L^2}.
\end{aligned}$$

Thus combining the two estimates for S_{41} and S_{42} and using Lemma 4.3 yield

$$\begin{aligned}
&\Delta t\sum_{K\in T_h}\sum_{n=1}^{R-1}S_4\xi^{n+1}_K\\
&\le C\Delta t\sum_{n=0}^{R}(\|\vartheta^n\|^2_{L^2}+\|\xi^n\|^2_{L^2})+Ch^2+\frac{2\epsilon\Delta t}{3}\sum_{n=1}^{R-1}\|\xi^{n+1}\|^2_{1,h}.
\end{aligned}\tag{4.17}$$

Taking a similar argument yields

$$\begin{aligned}
\Delta t\sum_{K\in T_h}\sum_{n=1}^{R-1}S_5\xi^{n+1}_K &= \frac{2\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}IU^{n+1}_{K,\sigma}\int_\sigma\big(\bar{\mu}_{e,K}-\mu_e\big)e^{n+1}ds\xi^{n+1}_K\\
&\le C\Delta t\sum_{n=0}^{R}(\|\vartheta^n\|^2_{L^2}+\|\xi^n\|^2_{L^2})+Ch^2+\frac{2\epsilon\Delta t}{3}\sum_{n=1}^{R-1}\|\xi^{n+1}\|^2_{1,h}.
\end{aligned}\tag{4.18}$$

where C depends on $\|\mu_e\|_{L^\infty(W^{1,\infty})}$, $\|e\|_{L^\infty(L^\infty)}$, $\|v\|_{L^\infty(W^{1,\infty})}$, C_*, d and $m(\Omega)$.
For S_6, we have

$$\Delta t\sum_{K\in T_h}\sum_{n=1}^{R-1}S_6\xi_K^{n+1}=-\frac{2\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}\int_\sigma\mu_e\big(u^{n+1}\cdot\eta_K-IU_{K,\sigma}^{n+1}\big)e^{n+1}ds\xi_K^{n+1}$$
$$=-\frac{2\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}\int_\sigma\mu_e(u^{n+1}-Iu^{n+1})\cdot\eta_K e^{n+1}ds\xi^{n+1}$$
$$-\frac{2\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}\int_\sigma\mu_e(Iu^{n+1}\cdot\eta_K-Iu_{K,\sigma}^{n+1})e^{n+1}ds\xi^{n+1}$$
$$-\frac{2\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}\int_\sigma\mu_e(Iu_{K,\sigma}^{n+1}-IU_{K,\sigma}^{n+1})e^{n+1}ds\xi^{n+1}.$$

Then taking a similar argument as for (4.9) and (4.17) gives

$$\Delta t\sum_{K\in T_h}\sum_{n=1}^{R-1}S_6\xi_K^{n+1}\le C(\Delta t^4+h^2+\Delta t\sum_{n=0}^{R-1}(\|\vartheta^n\|_{L^2}^2+\|\xi^n\|_{L^2}^2))+\epsilon\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{1,h}^2, \tag{4.19}$$

where C depends on $\|e\|_{L^\infty(L^\infty(\bar\Omega))}$, $\|v\|_{W^{2,\infty}(W^{1,\infty}(\bar\Omega))}$, $\|v\|_{L^\infty(W^{2,\infty})}$, C_*, d and $m(\Omega)$.
For the term S_7, we reorder by interior edges to have

$$\Delta t\sum_{K\in T_h}\sum_{n=1}^{R-1}S_7\xi_K^{n+1}=\frac{2\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}\Bigg(\sum_{\sigma=K|L\in\mathcal{E}_K}\int_\sigma D_e(x)\big(\frac{e^{n+1}(x_L)-e^{n+1}(x_K)}{d_\sigma}$$
$$-\nabla e^{n+1}\cdot\eta_K\big)ds\xi_K^{n+1}+\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{ext}}\int_\sigma D_e(x)\big(\frac{-e^{n+1}(x_K)}{d_\sigma}-\nabla e^{n+1}\cdot\eta_K\big)ds\xi_K^{n+1}\Bigg)$$
$$\le C\{\|e\|_{L^\infty(W^{2,\infty})}\}\Delta t\sum_{n=1}^{R-1}\sum_{\sigma\in\mathcal{E}}m(\sigma)d_\sigma h^2+\epsilon\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{1,h}^2$$
$$\le C\{d,m(\Omega)\}h^2+\epsilon\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{1,h}^2. \tag{4.20}$$

At last, using the triangle inequality and the Lipschitz continuity of R_1, we have

$$\Delta t\sum_{K\in T_h}\sum_{n=1}^{R-1}S_8\xi_K^{n+1}=-\frac{2\Delta t}{3}\sum_{n=1}^{R-1}\int_K\big(R_1(IE^{n+1},IP^{n+1})-R_1(e^{n+1},p^{n+1})\big)dx\xi_K^{n+1}$$
$$\le C\{\lambda,\|e\|_{H^2(L^2)},\|e\|_{L^\infty(W^{1,\infty})},\|p\|_{H^2(L^2)},\|p\|_{L^\infty(W^{1,\infty})}\}$$
$$\left(\Delta t^4+h^2+\Delta t\sum_{n=0}^{R}\|\xi^n\|_{L^2}^2+\Delta t\sum_{n=0}^{R-1}\|\vartheta^n\|_{L^2}^2\right). \tag{4.21}$$

Now we estimate the terms on the left-hand side of (4.13).

$$\Delta t\sum_{n=1}^{R-1}\sum_{K\in T_h}m(K)(\partial_t\xi_K^{n+1})\xi_K^{n+1}=\frac{1}{2}\|\xi^R\|_{L^2}^2-\frac{1}{2}\sum_{K\in T_h}\|\xi^1\|_{L^2}^2+\frac{1}{2}\sum_{n=1}^{R-1}\|\delta\xi^{n+1}\|_{L^2}^2. \tag{4.22}$$

Reordering by interior edges, we have

$$-\frac{2\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}(\bar{D}_{e,\sigma}\frac{m(\sigma)}{d_\sigma}\Upsilon_\sigma\xi^{n+1})\xi_K^{n+1}\geq\frac{2D_*}{3}\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{1,h}^2. \tag{4.23}$$

Let $\xi_{\sigma,-}^{n+1}$ denote the downwind value of ξ^{n+1} on an edge σ of K such that

$$\xi_{\sigma,-}^{n+1}=\begin{cases}\xi_K^{n+1}, & IU_{K,\sigma}^{n+1}\geq 0,\\ \xi_L^{n+1}, & otherwise,\end{cases}\quad \sigma=K|L\in\mathcal{E}_{int},$$

$$\xi_{\sigma,-}^{n+1}=\begin{cases}\xi_K^{n+1}, & IU_{K,\sigma}^{n+1}\geq 0,\\ 0, & otherwise,\end{cases}\quad \sigma\in\mathcal{E}_K\cap\mathcal{E}_{ext}.$$

Then,

$$\begin{aligned}
&-\frac{2\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K}m(\sigma)\bar{\mu}_{e,K}\xi_{\sigma,+}^{n+1}IU_{K,\sigma}^{n+1}\xi_K^{n+1}\\
=&\frac{\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}(\sum_{\sigma=K|L\in\mathcal{E}_K}+2\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{ext}})m(\sigma)\bar{\mu}_{e,K}|IU_{K,\sigma}^{n+1}|\xi_{\sigma,+}^{n+1}(\xi_{\sigma,+}^{n+1}-\xi_{\sigma,-}^{n+1})\\
=&\frac{\Delta t}{6}\sum_{n=1}^{R-1}\sum_{K\in T_h}(\sum_{\sigma=K|L\in\mathcal{E}_K}+2\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{ext}})m(\sigma)\bar{\mu}_{e,K}|IU_{K,\sigma}^{n+1}|\\
&\Big((\xi_{\sigma,+}^{n+1}-\xi_{\sigma,-}^{n+1})^2+\big[(\xi_{\sigma,+}^{n+1})^2-(\xi_{\sigma,-}^{n+1})^2\big]\Big)\\
=&\frac{\Delta t}{6}\sum_{n=1}^{R-1}\sum_{K\in T_h}(\sum_{\sigma=K|L\in\mathcal{E}_K}+2\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{ext}})m(\sigma)\bar{\mu}_{e,K}|IU_{K,\sigma}^{n+1}|(\xi_{\sigma,+}^{n+1}-\xi_{\sigma,-}^{n+1})^2\\
&-\frac{\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}(\xi_K^{n+1})^2\bar{\mu}_{e,K}\sum_{\sigma\in\mathcal{E}_K}m(\sigma)IU_{K,\sigma}^{n+1}\\
=&\frac{\Delta t}{6}\sum_{n=1}^{R-1}\sum_{K\in T_h}\Bigg(\sum_{\sigma=K|L\in\mathcal{E}_K}m(\sigma)\bar{\mu}_{e,K}|IU_{K,\sigma}^{n+1}|(\xi_L^{n+1}-\xi_K^{n+1})^2\\
&+2\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{ext}}m(\sigma)\bar{\mu}_{e,K}|IU_{K,\sigma}^{n+1}|(\xi_K^{n+1})^2\Bigg)\\
&-\frac{\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}(\xi_K^{n+1})^2\bar{\mu}_{e,K}\sum_{\sigma\in\mathcal{E}_K}m(\sigma)IU_{K,\sigma}^{n+1},
\end{aligned}$$

where from (3.6) and the triangle inequality, we have

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{E}_K}m(\sigma)IU_{K,\sigma}^{n+1}\\
&\leq|\alpha|\Big(2\|E^n\|_{L^\infty}+|E^{n-1}\|_{L^\infty}+2\|P^n\|_{L^\infty}+\|P^{n-1}\|_{L^\infty}+\|N\|_{L^\infty}\Big)m(K).
\end{aligned}$$

Note that $\|E^n\|_{L^\infty}\leq\|e\|_{L^\infty(L^\infty)}+\|\xi^n\|_{L^\infty}$ and $\|P^n\|_{L^\infty}\leq\|p\|_{L^\infty(L^\infty)}+\|\vartheta^n\|_{L^\infty}$. Using the induction hypothesis, we know when h and Δt are small enough,

$$2\|E^n\|_{L^\infty}+\|E^{n-1}\|_{L^\infty}+2\|P^n\|_{L^\infty}+\|P^{n-1}\|_{L^\infty}\leq C_0,\quad n\leq R-1,$$

where C_0 is a fixed positive constant. Hence,

$$
\begin{aligned}
&-\frac{2\Delta t}{3}\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K} m(\sigma)\bar{\mu}_{e,K}\xi_{\sigma,+}^{n+1}IU_{K,\sigma}^{n+1}\xi_K^{n+1}\\
\geq\ &\frac{\Delta t}{6}\mu_*\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K} m(\sigma)|IU_{K,\sigma}^{n+1}|(\Upsilon_\sigma\xi^{n+1})^2 - C\{C_0,\mu_*\}\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{L^2}^2. \quad (4.24)
\end{aligned}
$$

Since $\xi^0=\vartheta^0=0$, combining (4.13)-(4.24) and choosing ϵ small enough yield the desired result. This completes the proof of Lemma 4.4.

For Eqs. (3.4) and (3.8), we have the similar estimates as follows.

Lemma 4.5. *Suppose that* $v\in L^\infty(0,T;W^{2,\infty}(\Omega))\cap W^{2,\infty}(0,T;W^{1,\infty}(\bar{\Omega}))$, $e\in L^\infty(0,T;W^{1,\infty}(\Omega))\cap H^2(0,T;L^2(\Omega))$, *and* $p\in L^\infty(0,T;L^\infty(\bar{\Omega}))\cap L^\infty(0,T;W^{1,\infty})\cap H^2(0,T;L^2(\Omega))\cap H^1(0,T;W^{1,\infty}(\Omega))$. *Then,*

$$
\begin{aligned}
&\|\vartheta^R\|_{L^2}^2+\Delta t\sum_{n=1}^{R-1}\|\vartheta^{n+1}\|_{1,h}^2+\Delta t\sum_{n=1}^{R-1}\sum_{K\in T_h}\sum_{\sigma\in\mathcal{E}_K} m(\sigma)|IU_{K,\sigma}^{n+1}|(\Upsilon_\sigma\vartheta^{n+1})^2\\
\leq\ &C\bigg(\Delta t^4+h^2+\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{L^2}^2+\Delta t\sum_{n=1}^{R-1}\|\vartheta^{n+1}\|_{L^2}^2+\|\xi^1\|_{L^2}^2+\|\vartheta^1\|_{L^2}^2\bigg). \quad (4.25)
\end{aligned}
$$

Theorem 4.1. *Suppose that the exact solutions of* (1.1)-(1.5) *satisfy the smooth condition* (4.1) *and* $\Delta t=\mathcal{O}(h^{1/2})$. *For the fully discrete finite volume scheme* (3.5)-(3.8), *we have*

$$
\sup_{0\leq n\leq N}(\|\xi^n\|_{L^2}+\|\vartheta^n\|_{L^2}+\|\pi^n\|_{1,h})+\Delta t\sum_{n=0}^{N}(\|\xi^n\|_{1,h}^2+\|\vartheta^n\|_{1,h}^2)^{1/2}\leq C(\Delta t^2+h). \quad (4.26)
$$

Proof. Using Lemmas 4.4 and 4.5 yields

$$
\begin{aligned}
&\|\xi^R\|_{L^2}^2+\|\vartheta^R\|_{L^2}^2+\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{1,h}^2+\Delta t\sum_{n=1}^{R-1}\|\vartheta^{n+1}\|_{1,h}^2\\
\leq\ &C\bigg(\Delta t^4+h^2+\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{L^2}^2+\Delta t\sum_{n=1}^{R-1}\|\vartheta^{n+1}\|_{L^2}^2+\|\xi^1\|_{L^2}^2+\|\vartheta^1\|_{L^2}^2\bigg). \quad (4.27)
\end{aligned}
$$

Applying Gronwall's inequality gives

$$
\begin{aligned}
&\|\xi^R\|_{L^2}^2+\|\vartheta^R\|_{L^2}^2+\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{1,h}^2+\Delta t\sum_{n=1}^{R-1}\|\vartheta^{n+1}\|_{1,h}^2\\
\leq\ &C\bigg(\Delta t^4+h^2+\|\xi^1\|_{L^2}^2+\|\vartheta^1\|_{L^2}^2\bigg). \quad (4.28)
\end{aligned}
$$

For the single step scheme used to determine E^1 and P^1, a similar argument as that used in the proof of Lemma 4.4 will give $\|\xi^1\|_{L^2}^2+\|\vartheta^1\|_{L^2}^2\leq C(\Delta t^4+h^2)$. Thus, we have

$$
\|\xi^R\|_{L^2}^2+\|\vartheta^R\|_{L^2}^2+\Delta t\sum_{n=1}^{R-1}\|\xi^{n+1}\|_{1,h}^2+\Delta t\sum_{n=1}^{R-1}\|\vartheta^{n+1}\|_{1,h}^2\ \leq C(\Delta t^4+h^2). \quad (4.29)
$$

If $\Delta t = \mathcal{O}(h^{1/2})$, from (4.29) we see that $\|\xi^R\|_{1,h} \le Ch^{3/4}$. Then by Lemma 2.2,

$$\|\xi^R\|_{L^\infty} \le C(\ln|h|+1)\|\xi^R\|_{1,h} \le C(\ln|h|+1)h^{3/4} \to 0,\ d=2,$$
$$\|\xi^R\|_{L^\infty} \le Ch^{-1/2}\|\xi^R\|_{1,h} \le Ch^{1/4} \to 0,\ d=3.$$

Similarly, we can verify that $\|\vartheta^R\|_{L^\infty} \to 0$. So (4.11) holds for $N = R$. Now the induction argument is completed and we obtain the estimate (4.11). Therefore (4.29) holds for any integer R. At last combining with (4.5) gives the desired result.

Acknowledgment. The author is very grateful to the referee for the constructive comments.

References

[1] J.H. Bramble, R.E. Ewing and G. Li, Alternating direction multistep methods for parabolic problems-iterative stabilization, *SIAM J. Numer. Anal.*, **26** (1989), 904-919.

[2] Y. Coudière, T. Gallouët and R. Herbin, Discrete Sobolev inequalities and L^p error estiamtes for approximate finite volume solutions of convection diffusion equation, *M2AN Math. Model. Numer. Anal.*, **35** (2001), 767-778.

[3] X. Cui, Finite volume numerical analysis for parabolic equation with Robin boundary condition, *J. Comput. Math.*, **23** (2005), 165-176.

[4] J. Douglas Jr., I. Gamba and M. Squeff, Simulation of the transient behavior of a one dimensional semiconductor device, *Math. Appl. Comput.*, **5** (1986), 103-122.

[5] J. Douglas Jr. and Y. Yuan, Finite difference methods for the transient behavior of a semiconductor device, *Math. Appl. Comput.*, **6** (1987), 25-38.

[6] R. Eymard, T. Gallouet and R. Herbin, *Finite Volume Methods, Handbook for Numerical Analysis*, North Holland, 2000.

[7] I. Gamba and M. Squeff, Simulation of the transient behavior of a one dimensional semiconductor device II, *SIAM J. Numer. Anal.*, **26** (1989), 539-552.

[8] R. Li, Z. Chen and W. Wu, *Generalized Difference Methods for Differential Equations, Numerical Analysis of Finite Volume Methods*, Marcel Dekker, New York, 2000.

[9] H. Man and Z. Shi, P1-nonconforming quadrilateral finite volume element method and its cascadic multigrid algorithm, *J. Comput. Math.*, **24** (2006), 59-80.

[10] J. Miller and S. Wang, A tetrahedral mixed finite element method for the stationary semiconductor continuity equations, *SIAM J. Numer. Anal.*, **31** (1994), 196-216.

[11] Y. Yuan, Characteristic finite element method and analysis for the numerical simulation of a semiconductor device, *Acta Math. Sci.* **19** (1999), 241–251.

[12] J. Zhu, H. Wu and Y. Wang, A mixed method for the mixed initial boundary value problems of equations of semiconductor devices, *SIAM J. Numer. Anal.*, **31** (1994), 731-744.

[13] M. Zlámal, Finite element solution of the fundamental equations of semiconductor devices I, *Math. Comp.*, **46** (1986), 27-43.

[14] M. Zlámal, Finite element solution of the fundamental equations of semiconductor devices II, *Appl. Math.*, **46** (2001), 251-294.