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EXACT AND DISCRETIZED DISSIPATIVITY OF THE PANTOGRAPH EQUATION *1)

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Abstract

The analytic and discretized dissipativity of nonlinear infinite-delay systems of the form $x'(t) = g(x(t), x(qt))(q \in (0,1), t > 0)$ is investigated. A sufficient condition is presented to ensure that the above nonlinear system is dissipative. It is proved the backward Euler method inherits the dissipativity of the underlying system. Numerical examples are given to confirm the theoretical results.

Mathematics subject classification: 65L05.
Key words: Infinite delay, Pantograph equation, Backward Euler method, Dissipativity.

1. Introduction

Let H be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the corresponding norm, X a dense continuously imbedded subspace of H. Consider the delay differential equations (DDEs)

$$\begin{cases} x'(t) = g(t, x(t), x(\alpha(t))), \quad t \geq 0, \\ x(t) = \varphi(t), t \in [\inf_{s \geq 0} \alpha(s), 0], \end{cases} \tag{1}$$

where $g : [0, +\infty) \times X \times X \to H, \varphi(t)$ and $\alpha(t)$ are given functions with $\alpha(t) \leq t$ for all $t \geq 0$.

Many dynamical systems are characterized by the property of possessing a bounded absorbing set which all trajectories enter in finite time and thereafter remain inside. In the study of dissipative systems it is often the asymptotic behaviour of the system that is of interest, and so it is highly desirable to have numerical methods that retain the dissipativity of the underlying system.

In 1994, Humphries and Stuart[5, 6] first studied the dissipativity of Runge-Kutta methods for dynamical systems without delay. Later, many results on the dissipativity of numerical methods for dynamical systems without delays were found[7, 8, 20]. For DDEs with constant delay, i.e., $\tau(t) \equiv \tau$, Huang[9, 10] gave a sufficient condition for the dissipativity of the theoretical solution, and investigated the dissipativity of (k, l)-algebraically stable[3] Runge-Kutta methods and $G(c, p, 0)$-algebraically stable[13] one-leg methods. In 2004, Tian[18] studied the dissipativity of DDEs with a bounded variable lag and the numerical dissipativity of θ-method. Moreover, Wen (Wen L.P., Numerical stability analysis for nonlinear Volterra functional differential equations in abstract spaces(in Chinese), Ph.D.Thesis, Xiangtan University, 2005.) discussed the dissipativity of Volterra functional differential equations, and further investigated the disspativity of DDEs with piecewise delays and a class of linear multistep methods.

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An interesting case of (1) is the pantograph equation, corresponding to

$$\alpha(t) = qt, q \in (0,1),$$

which can be viewed as a representative of infinite time delay. The pantograph equation arises in quite different fields of pure and applied mathematics such as number theory, dynamical systems, probability, mechanics and electrodynamics[2, 11]. In particular, it was used by Ockendon and Tayler[17] to study how the electric current is collected by the pantograph of an electric locomotive, from where it gets its name.

In early work, a constant stepsize was considered for discretization of pantograph equations. As pointed out in Liu[15, 16], however, this kind of stepsize precludes long time integration due to computer memory restrictions. In order to overcome this difficulty, Liu[15] transformed the pantograph equation into a differential equation with a constant delay by a change of variable, suggested by Jackiewicz [12]. Later, Liu[16] and Bellen, Guglielmi and Torelli [1] proposed non-constant stepsize strategies where the stepsizes are geometrically increasing and they investigated the stability of the θ-method.

Recently, many papers have dealt with exact and discretized stability of pantograph equations (see, e.g.,[1, 11, 16]). But up to now, no results of dissipativity have been known for the pantograph equation and its discrete counterpart.

In this paper, we transform the pantograph equation into a non-autonomous DDE with a constant delay by a change of variable, then investigate the dissipativity of the resulting DDE and the backward Euler method. A sufficient condition is presented to ensure that the above system is dissipative. It is shown that the backward Euler method inherits the dissipativity of the underlying system.

2. Dissipativity of DDEs

Consider pantograph equation

$$\begin{cases} x'(t) = g(x(t), x(qt)), \quad t \geq 0, \\ x(0) = x_0, \end{cases} \tag{2}$$

where q is a constant with $0 < q < 1$, and g satisfies

$$\mathrm{Re}\langle u, g(u,v)\rangle \leq \gamma + \alpha\|u\|^2 + \beta\|v\|^2, \quad u, v \in X, \tag{3}$$

with γ, α and β denoting real constants.

By the change of the independent variable $y(t) = x(e^t)$(see [12, 15]), (2) can be transformed into the constant delay differential equation

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau)), \quad t \geq 0, \\ y(t) = \varphi(t), \quad t \leq 0, \end{cases} \tag{4}$$

where $\tau = -\log q$ and

$$f(t, y(t), y(t-\tau)) = e^t g(y(t), y(t-\tau)). \tag{5}$$

It follows from (3) and (5) that

$$\mathrm{Re}\langle u, f(t,u,v)\rangle \leq e^t(\gamma + \alpha\|u\|^2 + \beta\|v\|^2), \quad t \geq 0, u, v \in X. \tag{6}$$

Definition 1. *The evolutionary equation (2) is said to be dissipative in H if there is a bounded set $\mathcal{B} \subset H$ such that for all bounded sets $\Phi \subset H$ there is a time $t_0 = t_0(\Phi)$, such that for all initial values x_0 contained in Φ, the corresponding solution $x(t)$ is contained in $\mathcal{B}$ for all $t \geq t_0$. $\mathcal{B}$ is called an absorbing set in H.*

Proposition 1. *If a function g satisfies (3), then $\beta \geq 0$ and $\gamma \geq 0$.*
Lemma 1. *Suppose*

$$Y'(t) \leq 2e^t(\gamma + \alpha Y(t) + \beta Y(t-\tau)), \quad t \geq 0 \tag{7}$$

with $\alpha + \beta < 0$ and $\beta > 0$. Then

$$Y(t) \leq -\frac{\gamma}{\alpha+\beta} + Ge^{-\mu^* t}, \quad t \geq 0, \tag{8}$$

where $G = 2\sup_{t\leq 0} Y(t) > 0$, and $\mu^* > 0$ is defined as

$$\mu^* = \inf_{t\geq 0}\{\mu(t) : \mu(t) + 2e^t\alpha + 2e^t\beta e^{\mu(t)\tau} = 0\}. \tag{9}$$

Proof. For any $t \geq 0$, let

$$H(\mu) = \mu + 2e^t\alpha + 2e^t\beta e^{\mu\tau},$$

we have $H(0) = 2e^t(\alpha+\beta) < 0, H(+\infty) = +\infty$ and $H'(\mu) = 1 + 2e^t\beta\tau e^{\mu\tau} > 0$. Therefore, for this given t, there is a unique $\mu(t) > 0$ such that $\mu(t) + 2e^t\alpha + 2e^t\beta e^{\mu(t)\tau} = 0$. Hence the relation

$$\mu + 2e^t\alpha + 2e^t\beta e^{\mu\tau} = 0, \quad t \geq 0, \tag{10}$$

determines an implicit function $\mu = \mu(t)$ for $t \geq 0$. Differentiating both sides of (10) with respect to t yields

$$\frac{d\mu}{dt} + 2e^t\alpha + 2e^t\beta e^{\mu\tau} + 2e^t\beta\tau e^{\mu\tau}\frac{d\mu}{dt} = 0. \tag{11}$$

Setting $\frac{d\mu}{dt} = 0$ in (11), we have

$$\mu = \frac{1}{\tau}\log\frac{-\alpha}{\beta} > 0,$$

that is, $\mu(t) > 0$ when t satisfies $\frac{d\mu}{dt} = 0$.

For $t = 0$ in (10), we obtain $\mu(0) > 0$. (10) can be rewritten as

$$e^{-t}\mu + 2\alpha + 2\beta e^{\mu\tau} = 0.$$

It follows from the above formula that $\mu(+\infty) > 0$. Therefore,

$$\mu^* = \min\{\mu(0), \mu(+\infty), \frac{1}{\tau}\log\frac{-\alpha}{\beta}\} > 0.$$

Now we will prove by contradiction that (8) is true. Suppose there is some $\tilde{t} > 0$ such that

$$Y(\tilde{t}) > -\frac{\gamma}{\alpha+\beta} + Ge^{-\mu^*\tilde{t}}.$$

Let $Z(t) = -\frac{\gamma}{\alpha+\beta} + Ge^{-\mu^* t}, w(t) = Z(t) - Y(t)$ and

$$\varsigma = \inf\{t \geq 0 : Z(t) - Y(t) \leq 0\}. \tag{12}$$

It is obvious that $w(0) > 0, w(\tilde{t}) < 0$. Therefore, there is $\varsigma > 0$ such that $w(\varsigma) = Z(\varsigma) - Y(\varsigma) = 0$ and

$$w'(\varsigma) = Z'(\varsigma) - Y'(\varsigma) \leq 0. \tag{13}$$

On the other hand, it follows from (7) that

$$w'(\varsigma) = Z'(\varsigma) - Y'(\varsigma) \geq -G\mu^* e^{-\mu^*\varsigma} - 2e^{\varsigma}(\gamma + \alpha Y(\varsigma) + \beta Y(\varsigma - \tau)). \tag{14}$$

Let $\mu(\varsigma)$ satisfy $\mu(\varsigma) + 2e^{\varsigma}\alpha + 2e^{\varsigma}\beta e^{\mu(\varsigma)\tau} = 0$.

(i) If $\varsigma - \tau \geq 0$, it follows from (14) that

$$\begin{aligned} w'(\varsigma) &> -G\mu^* e^{-\mu^*\varsigma} - 2e^\varsigma\gamma - 2e^\varsigma\alpha(-\tfrac{\gamma}{\alpha+\beta} + Ge^{-\mu^*\varsigma}) - 2e^\varsigma\beta(-\tfrac{\gamma}{\alpha+\beta} + Ge^{-\mu^*(\varsigma-\tau)}) \\ &= Ge^{-\mu^*\varsigma}(-\mu^* - 2e^\varsigma\alpha - 2e^\varsigma\beta e^{\mu^*\tau}) \\ &= Ge^{-\mu^*\varsigma}(-\mu^* - 2e^\varsigma\alpha - 2e^\varsigma\beta e^{\mu^*\tau} + \mu(\varsigma) + 2e^\varsigma\alpha + 2e^\varsigma\beta e^{\mu(\varsigma)\tau}) \\ &= Ge^{-\mu^*\varsigma}(\mu(\varsigma) - \mu^* + 2e^\varsigma\beta(e^{\mu(\varsigma)\tau} - e^{\mu^*\tau})) \\ &\geq 0, \end{aligned}$$

which contradicts (13).

(ii) If $\varsigma - \tau < 0$, it follows from (14) that

$$\begin{aligned} w'(\varsigma) &> -G\mu^* e^{-\mu^*\varsigma} - 2e^\varsigma\gamma - 2e^\varsigma\alpha(-\tfrac{\gamma}{\alpha+\beta} + Ge^{-\mu^*\varsigma}) - 2e^\varsigma\beta G \\ &= -2e^\varsigma\gamma(1 - \tfrac{\alpha}{\alpha+\beta}) + Ge^{-\mu^*\varsigma}(-\mu^* - 2e^\varsigma\alpha - 2e^\varsigma\beta e^{\mu^*\varsigma}) \\ &> Ge^{-\mu^*\varsigma}(-\mu^* - 2e^\varsigma\alpha - 2e^\varsigma\beta e^{\mu^*\tau}) \\ &= Ge^{-\mu^*\varsigma}(-\mu^* - 2e^\varsigma\alpha - 2e^\varsigma\beta e^{\mu^*\tau} + \mu(\varsigma) + 2e^\varsigma\alpha + 2e^\varsigma\beta e^{\mu(\varsigma)\tau}) \\ &= Ge^{-\mu^*\varsigma}(\mu(\varsigma) - \mu^* + 2e^\varsigma\beta(e^{\mu(\varsigma)\tau} - e^{\mu^*\tau})) \\ &\geq 0, \end{aligned}$$

which contradicts (13). Therefore, we have

$$Y(t) \leq -\frac{\gamma}{\alpha+\beta} + Ge^{-\mu^* t} \quad t \geq 0,$$

and the proof is completed.

Theorem 1. *Suppose $y(t)$ is a solution of (4) where f satisfies (6) and $\alpha+\beta<0$. Then for any given $\epsilon>0$ there exists $t = \check{t}(\bar{\varphi}, \epsilon), \bar{\varphi} = \sup_{t\leq 0} \|\varphi(t)\|^2$, such that for all $t > \check{t}$,*

$$\|y(t)\|^2 < -\frac{\gamma}{\alpha+\beta} + \epsilon. \tag{15}$$

Hence the system is dissipative, and the open ball $B = B(0, \sqrt{-\frac{\gamma}{\alpha+\beta}+\epsilon})$ is an absorbing set for any $\epsilon > 0$.

Proof. Define

$$Y(t) := \|y(t)\|^2 = \langle y(t), y(t)\rangle. \tag{16}$$

Then

$$\begin{aligned} Y'(t) &= 2\mathrm{Re}\langle y(t), y'(t)\rangle = 2\mathrm{Re}\langle y(t), f(t, y(t), y(t-\tau))\rangle \\ &\leq 2e^t(\gamma + \alpha Y(t) + \beta Y(t-\tau)), \quad t \geq 0. \end{aligned} \tag{17}$$

If $\beta > 0$, the conclusion follows directly from (17) and Lemma 1.

If $\beta = 0$, (17) yields

$$e^{-2\alpha\int_0^t e^s ds}(Y'(t) - 2\alpha e^t Y(t)) \leq 2e^{-2\alpha\int_0^t e^s ds} e^t\gamma. \tag{18}$$

Integrating (18) from 0 to t gives

$$Y(t) \leq e^{2\alpha\int_0^t e^s ds} Y(0) + (1 - e^{2\alpha\int_0^t e^s ds})\frac{\gamma}{-\alpha}, \quad t > 0,$$

which shows that (15) holds for any $t > \check{t}$. The proof is completed.

Corollary 1. *Assume that system (2) satisfies (3) and $\alpha+\beta<0$. Then the system is dissipative.*

Remark 1. Theorem 1 is different from the dissipativity results obtained by Tian[18] and Wen (see footnote 1) on page 2). Their studies were restricted to the case: either $\gamma(t)$ is bounded, or $\gamma(t), \alpha(t)$ and $\beta(t)$ are all bounded. In this paper, $\gamma(t) = e^t\gamma, \alpha(t) = e^t\alpha$ and $\beta(t) = e^t\beta$ (see formula (6)) are not restricted by the conditions. Moreover, the pantograph equation does not belong to the system investigated by Tian[18] and Wen (Wen L.P., Numerical stability analysis for nonlinear Volterra functional differential equations in abstract spaces(in Chinese), Ph.D.Thesis, Xiangtan University, 2005).

Remark 2. There exist some important differences between the condition (3) and the monotonicity conditions[2, 19]

$$\mathrm{Re}\langle u_1 - u_2, g(u_1, v) - g(u_2, v)\rangle \le a\|u_1 - u_2\|^2, \quad t \ge 0, u_1, u_2, v \in X, \tag{19}$$

$$\|g(u, v_1) - g(u, v_2)\| \le b\|v_1 - v_2\|, \quad t \ge 0, u, v_1, v_2 \in X. \tag{20}$$

In fact, as an example without delays, Humphries and Stuart[6] proved that after translation of the origin, the Lorenz equations are dissipative, but do not satisfy the condition (19).

3. Dissipativity of the Backward Euler Method

For solving problem (4), we consider the backward Euler method

$$y_n = y_{n-1} + hf(t_n, y_n, \bar{y}_n), \quad n = 1, 2, \cdots, \tag{21}$$

where $h > 0$ is the stepsize, y_n is an approximation to the exact solution $y(t_n)$ with $t_n = nh$, and $\bar{y}_n$ is an approximation to $y(t_n - \tau)$ that is obtained by a specific interpolation at the point $t_n - \tau$.

Let $\tau = (m - \delta)h$ with integer $m \ge 1$ and $\delta \in [0, 1)$. We define

$$\bar{y}_n = \delta y_{n-m+1} + (1 - \delta)y_{n-m}, \tag{22}$$

where $y_l = \varphi(lh)$ for $l < 0$. It is well known that backward Euler method is of order 1. So the linear interpolation procedure (22) will not lead to order reduction for every stepsize $h > 0$.

Definition 2. *A numerical method is said to be dissipative if, when the method is applied to (4), (6) with $\alpha + \beta < 0$, there exists a constant r such that, for any initial function $\varphi(t)$, there exists an n_0, depending only on $\varphi(t)$ and h, such that*

$$\|y_n\| \le r, \quad n \ge n_0, \tag{23}$$

holds.

Theorem 2. *The backward Euler method is dissipative.*

Proof. Making the inner products of (21) with y_n and using (6), we have

$$\begin{aligned} \|y_n\|^2 &= \mathrm{Re}\langle y_n, y_{n-1}\rangle + h\mathrm{Re}\langle y_n, f(t_n, y_n, \bar{y}_n)\rangle \\ &\le \|y_n\|\|y_{n-1}\| + he^{t_n}(\gamma + \alpha\|y_n\|^2 + \beta\|\bar{y}_n\|^2) \\ &\le \tfrac{1}{2}\|y_{n-1}\|^2 + \tfrac{1}{2}\|y_n\|^2 + he^{t_n}(\gamma + \alpha\|y_n\|^2 + \beta\|\bar{y}_n\|^2). \end{aligned} \tag{24}$$

In view of (22) we obtain

$$\begin{aligned} \|\bar{y}_j\|^2 &= \delta^2\|y_{j-m+1}\|^2 + (1-\delta)^2\|y_{j-m}\|^2 + 2\delta(1-\delta)\mathrm{Re}\langle y_{j-m+1}, y_{j-m}\rangle \\ &\le \delta^2\|y_{j-m+1}\|^2 + (1-\delta)^2\|y_{j-m}\|^2 + \delta(1-\delta)(\|y_{j-m+1}\|^2 + \|y_{j-m}\|^2) \\ &= \delta\|y_{j-m+1}\|^2 + (1-\delta)\|y_{j-m}\|^2. \end{aligned} \tag{25}$$

A combination of (24) and (25) leads to

$$\|y_n\|^2 \le \tfrac{1}{2}\|y_{n-1}\|^2 + \tfrac{1}{2}\|y_n\|^2 + he^{t_n}(\gamma + \alpha\|y_n\|^2 + \beta\delta\|y_{n-m+1}\|^2 + \beta(1-\delta)\|y_{n-m}\|^2). \tag{26}$$

(i) If $m = 1$, it follows from (26) and $\alpha + \beta\delta < 0$ that

$$\|y_n\|^2 \le \frac{he^{t_n}\gamma}{\frac{1}{2} - he^{t_n}(\alpha + \beta\delta)} + \frac{\frac{1}{2} + he^{t_n}\beta(1-\delta)}{\frac{1}{2} - he^{t_n}(\alpha + \beta\delta)}\|y_{n-1}\|^2. \tag{27}$$

It is easily seen that

$$\sup_{n\ge 0} \frac{he^{t_n}\gamma}{\frac{1}{2} - he^{t_n}(\alpha + \beta\delta)} = \frac{\gamma}{-(\alpha + \beta\delta)}, \quad \sup_{n\ge 0} \frac{\frac{1}{2} + he^{t_n}\beta(1-\delta)}{\frac{1}{2} - he^{t_n}(\alpha + \beta\delta)} = \frac{\frac{1}{2} + h\beta(1-\delta)}{\frac{1}{2} - h(\alpha + \beta\delta)}. \tag{28}$$

Let $k_1 = \frac{\frac{1}{2}+h\beta(1-\delta)}{\frac{1}{2}-h(\alpha+\beta\delta)}, k_2 = \frac{\gamma}{-(\alpha+\beta\delta)}$. Considering (27) and (28), we have

$$\begin{aligned}\|y_n\|^2 &\leq k_1^n\|y_0\|^2 + k_2\sum_{i=0}^{n-1} k_1^i \leq k_1^n\|y_0\|^2 + \tfrac{k_2}{1-k_1} \\ &= k_1^n\|y_0\|^2 + \tfrac{\gamma}{-(\alpha+\beta)}\cdot\tfrac{1-2h(\alpha+\beta\delta)}{-2h(\alpha+\beta\delta)}.\end{aligned} \tag{29}$$

(ii) If $m > 1$, it follows from (26) that

$$\begin{aligned}\|y_n\|^2 &\leq \tfrac{2he^{t_n}\gamma}{1-2he^{t_n}\alpha} + \tfrac{1}{1-2he^{t_n}\alpha}\|y_{n-1}\|^2 + \tfrac{2he^{t_n}\beta}{1-2he^{t_n}\alpha}(\delta\|y_{n-m+1}\|^2 + (1-\delta)\|y_{n-m}\|^2) \\ &\leq \tfrac{2he^{t_n}\gamma}{1-2he^{t_n}\alpha} + \tfrac{1+2he^{t_n}\beta}{1-2he^{t_n}\alpha}\max_{1\leq j\leq m}\|y_{n-j}\|^2.\end{aligned} \tag{30}$$

A straightforward calculation shows that

$$\sup_{n\geq 0}\frac{1+2he^{t_n}\beta}{1-2he^{t_n}\alpha} = \frac{1+2h\beta}{1-2h\alpha}, \quad \sup_{n\geq 0}\frac{2he^{t_n}\gamma}{1-2he^{t_n}\alpha} = \frac{\gamma}{-\alpha}. \tag{31}$$

Let $l_1 = \frac{1+2h\beta}{1-2h\alpha}, l_2 = \frac{\gamma}{-\alpha}$. In view of (30) and (31), we obtain

$$\|y_n\|^2 \leq l_2 + l_1\max_{1\leq j\leq m}\|y_{n-j}\|^2, \tag{32}$$

which yields

$$\begin{aligned}\|y_{pm+j}\|^2 &\leq l_2 + l_1\max_{1\leq i\leq m}\|y_{pm+j-i}\|^2 \\ &\leq \cdots \leq l_2(1+l_1+\cdots+l_1^{pm+j-1}) + l_1^p\max_{-m+1\leq i\leq 0}\|y_i\|^2 \\ &\leq \tfrac{l_2}{1-l_1} + l_1^p\max_{t\leq 0}\|\varphi(t)\|^2 \\ &\leq \tfrac{\gamma}{-(\alpha+\beta)}\cdot\tfrac{1-2h\alpha}{-2h\alpha} + l_1^p\max_{t\leq 0}\|\varphi(t)\|^2, \quad j=0,1,\cdots,m-1, p=1,2,\cdots\end{aligned} \tag{33}$$

where we have used the fact that $l_1 < 1$. Noting that the fact $k_1 < 1, l_1 < 1$ and $\frac{1-2h(\alpha+\beta\delta)}{-2h(\alpha+\beta\delta)} \geq \frac{1-2h\alpha}{-2h\alpha}$, for any $\epsilon > 0$, letting

$$r = \sqrt{\frac{\gamma}{-(\alpha+\beta)}\frac{1-2h(\alpha+\beta\delta)}{-2h(\alpha+\beta\delta)}+\epsilon},$$

and using (29) and (33), we have that there exists an n_0, which depends on $\varphi(t)$ and h, such that

$$\|y_n\| \leq r, \quad n \geq n_0.$$

This completes the proof.

4. Numerical Examples

Consider the nonlinear problem

$$\begin{cases} x'(t) = -ax(t) + \frac{bx(qt)}{1+x(qt)^N}, & t \geq 0, \\ x(0) = x_0, \end{cases} \tag{34}$$

where q is a constant with $0 < q < 1$, $a > 0$ and b are real parameters, and N is an even positive integer. This equation is a modification of the model[4] for respiratory diseases. The constant delay $\tau(t) = \tau$ is replaced by the infinite delay $\tau(t) = (1-q)t$. For $N = 2, 4$, an application of Theorem 3.2 in Li[14] gives that the system (34) is asymptotically stable when $|\, b \,| < a$. On the other hand, for certain values of the parameters and of the delay, the solution is oscillatory, and sometimes it oscillates even chaotically. In these cases the system is not asymptotically stable. However, for any $c \in (0, a)$ we can choose $\alpha = -a + c, \beta = 0, \gamma = b^2/(4c)$ such that (3) holds. Therefore, the system is dissipative.

By the transformation $y(t) = x(e^t)$, the pantograph equation (34) can be transformed into a DDE with constant delay,

$$\begin{cases} y'(t) = e^t(-ay(t) + b\frac{y(t-\tau)}{1+y(t-\tau)^N}), \quad t \geq 0, \\ y(t) = \varphi(t), \quad t \leq 0, \end{cases} \tag{35}$$

where $\tau = -\log q$.

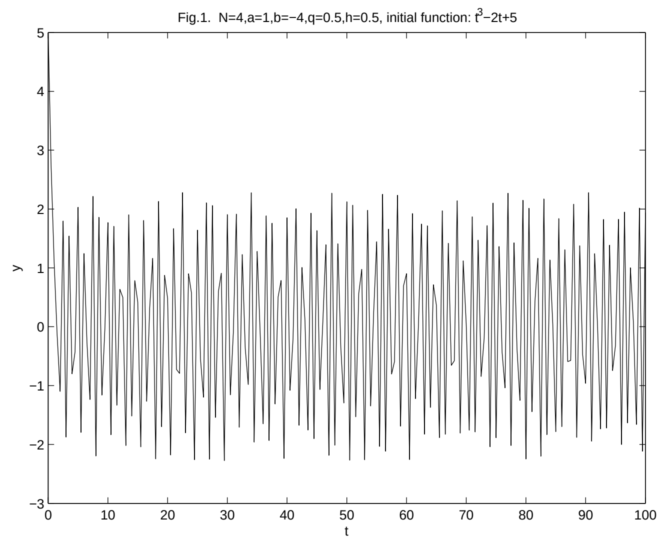

Fig. 4.1. Numerical results for $b = -4$

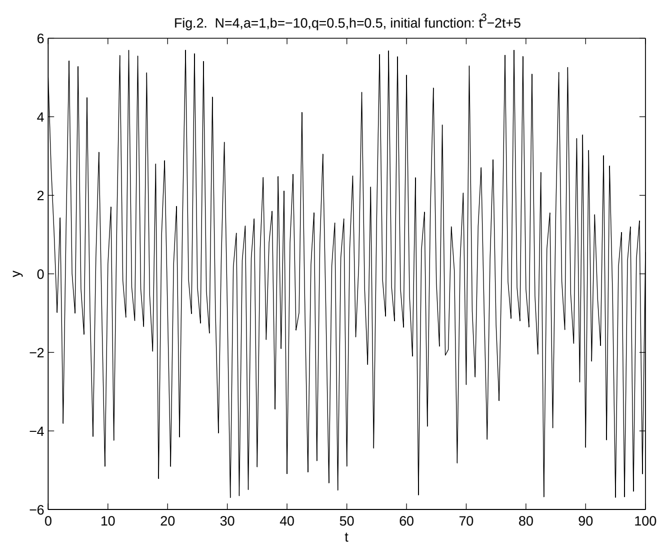

Fig. 4.2. Numerical results for $b = -10$

We apply the backward Euler method to the problem (35). For $N = 4$, the numerical results are shown in Fig.1-2, where $a = 1, q = 0.5, h = 0.5, \varphi(t) = t^3 - 2t + 5$. Fig.1-2 show that the problem (35) with the given parameters is dissipative. Therefore, these numerical examples confirm our theoretical results.

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