Journal of Computational Mathematics, Vol.25, No.1, 2007, 67–80.

THE NONCONFORMING FINITE ELEMENT METHOD FOR SIGNORINI PROBLEM *1)

Dongying Hua
(*The First Fundamental Department, Beijing Information Technology Institute, Beijing 100101, China*
Email: huady@amss.ac.cn)

Lieheng Wang
(*LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China*
Email: wlh@lsec.cc.ac.cn)

Abstract

We present the Crouzeix-Raviart linear nonconforming finite element approximation of the variational inequality resulting from Signorini problem. We show if the displacement field is of H^2 regularity, then the convergence rate can be improved from $\mathcal{O}(h^{3/4})$ to quasi-optimal $\mathcal{O}(h|\log h|^{1/4})$ with respect to the energy norm as that of the continuous linear finite element approximation. If stronger but reasonable regularity is available, the convergence rate can be improved to the optimal $\mathcal{O}(h)$ as expected by the linear approximation.

Mathematics subject classification: 65N30.
Key words: Nonconforming finite element method, Signorini problem, Convergence rate.

1. Introduction

Signorini problem is one of the model problems considered in the theory of variational inequality [9, 13]. The continuous linear finite element approximations of this problem have been studied in many works, see, e.g., [2, 3, 10, 14]. As far as we have known that Scarpini and Vivaldi[14] first gave the $\mathcal{O}(h^{3/4})$ convergence rate under the condition that the displacement field u is of H^2 regularity. Then, Brezzi, Hager and Raviart[2] presented $\mathcal{O}(h)$ convergence rate by detailed analysis under the additional assumptions that $u|_{\partial\Omega} \in W^{1,\infty}(\partial\Omega)$ and that the number of points in the free boundary set where the constraint changes from binding to non-binding is finite. For simplicity, we call these points "the critical points". Later, Ben Belgacem [3] proved that under a weaker assumption, i.e., $u \in H^2(\Omega)$ and the number of critical points is finite, $\mathcal{O}(h|\log h|^{1/2})$ convergence order can be obtained. Recently, Ben Belgacem and Renard [5] established an improved result of $\mathcal{O}(h|\log h|^{1/4})$ convergence rate under the same assumptions as in [3]. However, the convergence rate is not optimal if stronger regularity and finite number of the critical points are not assumed. In this paper, we apply the Crouzeix-Raviart linear finite element[8] to approximate Signorini problem and achieve same results as those of the continuous linear finite element approximation. The whole process of the theoretical analysis is found more complicated and requires more technical treatments.

* Received May 16, 2005; final revised August 26, 2005; accepted September 2, 2005.
1) Supported by NSFC under grant 10671023.

The paper is organized as follows. In the next section, we describe the continuous setting of Signorini problem and its Crouzeix-Raviart linear finite element discretization. In section 3, we give some notations and lemmas for latter use. The main results and the corresponding proofs are provided in section 4. Finally, in the last section, numerical experiments are carried out to verify our theoretical results. Throughout this paper all the notations about Sobolev spaces can be found in [1]. We often use norm $\|\cdot\|_{0,p,\Omega}$ to represent $\|\cdot\|_{L^p(\Omega)}$ and use $\|\cdot\|_{\alpha,\Omega}$ to represent $\|\cdot\|_{H^\alpha(\Omega)}$. The semi-norm is used similarly. In addition, the frequently used constant C is a generic positive constant whose value may be different under different contexts.

2. Signorini Problem and its Finite Element Discretization

First, we state the continuous framework of Signorini problem. For the sake of simplicity, we only consider Signorini problem for the Poisson equation. The general continuous setting of this problem in $\mathbb{R}^2$ can be illustrated (a mathematical model) as follows.

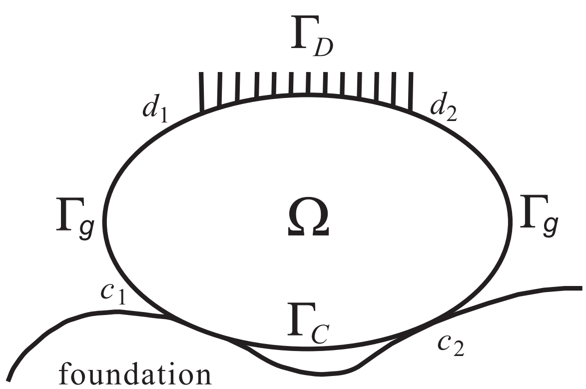

Fig. 2.1. Signorini problem.

Suppose $\Omega \subset \mathbb{R}^2$ is a Lipschitz bounded domain, which consists of three non-overlapping parts Γ_D, Γ_C and Γ_g. Here Γ_D is the fixed boundary (Dirichlet condition) with the end points d_1 and d_2, Γ_C is the contact region subjected to a rigid foundation with c_1 and c_2 as its endpoints, and Γ_g is the "glacis" with Neumann condition.

Now Signorini problem can be restated as the following mathematical model: Find

$$u \in K = \{u \in H^1_{\Gamma_D}(\Omega) : \ u \geq 0 \quad \text{a.e. on } \Gamma_C\}, \ \text{ such that}$$

$$a(u, v - u) \geq \chi(v - u), \qquad \forall\, v \in K, \tag{2.1}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad \chi(v) = \int_\Omega f\, v\, dx + \int_{\Gamma_g} g\, v\, ds.$$

The notation $H^1_{\Gamma_D} =: V$ stands for the set $\{v \in H^1(\Omega) : \ \ v = 0 \ \text{ a.e. on } \ \Gamma_D\}$. Moreover, $\partial\Omega = \Gamma_D \cup \Gamma_C \cup \Gamma_g$, and $int\,(\Gamma_D) \cap int\,(\Gamma_g) = \emptyset$, $\ int\,(\Gamma_C) \cap int\,(\Gamma_g) = \emptyset$ (see Fig 2.1). Here we only consider $u \geq 0$ a.e. on Γ_C instead of $u \geq \alpha$ a.e. on Γ_C in the closed convex set K, since our analysis can be extended to the non-zero case α easily. It is easy to check that an equivalent differential form of (2.1) is the following

$$\begin{cases} -\triangle u = f, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \Gamma_D, \\ \partial_\nu u = g, & \text{on} \quad \Gamma_g, \\ u \geq 0, \quad \partial_\nu u \geq 0, \quad \partial_\nu u \cdot u = 0, & \text{on} \quad \Gamma_C = \Gamma_C^0 \cup \Gamma_C^+, \end{cases} \tag{2.2}$$

where $\partial_\nu u = \frac{\partial u}{\partial \boldsymbol{\nu}}$ with $\boldsymbol{\nu}$ as the unit outward normal to $\partial\Omega$, $\Gamma_C^0 = \{x \in \Gamma_C : u(x) = 0\}$, and $\Gamma_C^+ = \{x \in \Gamma_C : u(x) > 0\}$. Here the differential form is in the sense of "almost everywhere" shortened for "a.e.". The existence and uniqueness of the solution to the above problem can be easily verified by the ellipticity of $a(\cdot,\cdot)$ and the continuity of χ on $H^1_{\Gamma_D}$.

Suppose $\mathcal{J}_h$ is the regular triangulation of Ω, and $T \in \mathcal{J}_h$ is the triangular element. In the following sections, we often use the subscript h to denote something related to the finite element discretization. Here for concision, suppose the domain Ω is polygonal in $\mathbb{R}^2$, so it can be exactly covered by triangular elements. It is also assumed that the triangulation $\mathcal{J}_h$ is built in such a way that the end points of Γ_D and Γ_C are always chosen as the vertices of triangular elements. Let V_h be the Crouzeix-Raviart linear finite element space corresponding to $\mathcal{J}_h$, (which is nonconforming, i.e., $V_h \nsubseteq H^1_{\Gamma_D}(\Omega) = V$), that is to say,

$$V_h = \left\{ \begin{array}{l} v_h : v_h|_T \in P_1(T),\, v_h \text{ is continuous at the midpoints of the edges of } T, \\ \forall\, T \in \mathcal{J}_h, \quad \text{and } \; v_h(a_{ij}) = 0, \text{where } a_{ij} \text{ is the midpoint of } \;\; \overline{a_i a_j} \subset \Gamma_D \end{array} \right\}. \tag{2.3}$$

Define

$$\|v_h\|_h = (\sum_T |v_h|^2_{1,T})^{1/2}, \qquad \forall\, v_h \in V_h, \tag{2.4}$$

which can be easily verified as $\|\cdot\|_h$ is a norm on V_h. Moreover, assume K_h is the following closed convex subset of V_h,

$$K_h = \{v_h \in V_h : v_h(a_{ij}) \geq 0, \text{where } a_{ij} \text{ is the midpoint of } \;\; \overline{a_i a_j} \subset \Gamma_C\}. \tag{2.5}$$

Then the finite element approximation of problem (2.1) leads to : to find $u_h \in K_h$, such that

$$a_h(u_h, v_h - u_h) \geq \chi(v_h - u_h), \qquad \forall\, v_h \in K_h, \tag{2.6}$$

where

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{J}_h} \int_T \nabla u_h \cdot \nabla v_h dx,$$

$$\chi(v_h) = \int_\Omega f v_h dx + \int_{\Gamma_g} g v_h ds.$$

As $\|\cdot\|_h$ in (2.4) is a norm in V_h, the solution of the discrete problem (2.6) uniquely exists. Moreover, it can be verified that the following abstract error estimate holds:

Lemma 2.1. *Suppose $u \in K$ is the solution of the variational Signorini problem* (2.1) *and $u_h \in K_h$ the solution of the discrete one* (2.6) *respectively. Then*

$$\|u - u_h\|_h \leq C \inf_{v_h \in K_h} \left\{ \|u - v_h\|_h^2 + a_h(u, v_h - u_h) - \chi(v_h - u_h) \right\}^{1/2}. \tag{2.7}$$

The proof is similar to that of the second Strang lemma[7], so we omit it here.

3. Notations and Lemmas

In this section, we introduce some notations and lemmas, which will be used in the next section. Let $F \subset \partial\Omega$ be the line element with respect to the triangulation $\mathcal{J}_h$, and let

$$\Gamma_{Ch} = \{F : F \subset \Gamma_C\}. \tag{3.1}$$

Then Γ_{Ch} can be divided into the following three non-overlapping sets:

$$\begin{cases} \Gamma^0_{Ch} = \{F \in \Gamma_{Ch} : F \subset \Gamma^0_C\}, \\ \Gamma^+_{Ch} = \{F \in \Gamma_{Ch} : F \subset \Gamma^+_C\}, \\ \Gamma^-_{Ch} = \big\{F \in \Gamma_{Ch} : F \cap \Gamma^0_C \neq \emptyset, F \cap \Gamma^+_C \neq \emptyset\big\}, \end{cases} \tag{3.2}$$

namely,

$$\Gamma_{Ch} = \Gamma^0_{Ch} \cup \Gamma^+_{Ch} \cup \Gamma^-_{Ch}. \tag{3.3}$$

Lemma 3.1. *For $1 < p < \infty$, the following discrete trace inequality holds:*

$$\|v\|_{0,p,\partial T} \leq C\big\{h^{-1}\|v\|^p_{0,p,T} + h^{p-1}|v|^p_{1,p,T}\big\}^{1/p}, \quad \forall\, v \in W^{1,p}(T),\ \forall\, T \in \mathcal{J}_h, \tag{3.4}$$

where C is a constant independent of v and the mesh size h.

This lemma can be established by using the same technique used by Stummel [16].

Lemma 3.2. *Suppose $F \subset \partial T$ is an edge of the triangular element $T \in \mathcal{J}_h$, and $v \in H^1(F)$. If, there exists some $Q^F \in F$ such that $v(Q^F) = 0$, then*

$$\|v\|_{0,F} \leq Ch\|\frac{dv}{ds}\|_{0,F} \leq Ch|v|_{1,F}, \tag{3.5}$$

$$\|v\|_{0,F} \leq Ch^{1/2}\|v\|_{1/2,F}, \tag{3.6}$$

where C is a positive constant independent of h and v, and $\frac{dv}{ds}$ denotes the derivative of v along F.

Proof. Firstly, we have

$$\begin{aligned} \|v\|^2_{0,F} \quad &= \int_F \big|v^2(s) - v^2(Q^F)\big|ds \\ &= \int_F \Big|\int_{Q^F}^s \frac{dv^2(t)}{dt}dt\Big|ds \leq 2\int_F \Big\{\int_{Q^F}^s |v(t)||\frac{dv(t)}{dt}|dt\Big\}ds \\ &\leq 2|F|\int_F |v(t)||\frac{dv(t)}{dt}|dt \leq 2h\|v\|_{0,F}\|\frac{dv}{ds}\|_{0,F} \\ &\leq 2h\|v\|_{0,F}|v|_{1,F} \end{aligned}$$

from which the estimate (3.5) is proved. Moreover

$$\begin{aligned} \|v\|^2_{0,F} \quad &\leq 2\int_F \Big|\int_{Q^F}^s v(t)\frac{dv(t)}{dt}dt\Big|ds \leq 2|F|\int_F |v(t)|\Big|\frac{dv(t)}{dt}\Big|dt \\ &\leq 2h\|v\|_{1/2,F}\|\frac{dv}{dt}\|_{-1/2,F} \leq Ch\|v\|^2_{1/2,F} \end{aligned}$$

which verifies the estimate (3.6).

Lemma 3.3. *Let u and u_h be the solutions of the problems* (2.1) *and* (2.6), *respectively. If $u \in H^2(\Omega)$, then*

$$-\sum_{F\in\Gamma_{Ch}^-}\int_F \partial_\nu u\cdot u_h ds \le Ch|u|_{2,\Omega}\|u-u_h\|_h + Ch^{3/2}\|u\|_{2,\Omega}^2, \tag{3.7}$$

where C is a positive constant independent of h.

Proof. For given $F \in \Gamma_{Ch}^-$, if $u_h \ge 0$ identically on F, then $-\int_F \partial_\nu u\cdot u_h ds \le 0$, since $\partial_\nu u \ge 0$ a.e. on Γ_C. Thus we only need to consider the case that $F \in \Gamma_{Ch}^-$ but $u_h \ge 0$ does not identically hold on F. Then, for those F, because $u_h \in K_h$ we have $u_h(m^F) \ge 0$, with m^F being the midpoint of F, and by the linearity of u_h on F, there must be some $Q^F \in F$, such that $u_h(Q^F) = 0$. Set

$$P_0^F(v) = \frac{1}{|F|}\int_F v\,ds, \qquad R_0^F(v) = v - P_0^F(v). \tag{3.8}$$

Then,

$$\begin{aligned}
-\int_F \partial_\nu u\cdot u_h ds &= -\int_F R_0^F(\partial_\nu u)u_h ds - P_0^F(\partial_\nu u)\int_F u_h ds\\
&\le -\int_F R_0^F(\partial_\nu u)u_h ds \le \|R_0^F(\partial_\nu u)\|_{0,F}\|u_h\|_{0,F}
\end{aligned} \tag{3.9}$$

since $\partial_\nu u \ge 0$ a.e. on F, $P_0^F(\partial_\nu u) \ge 0$ and $\int_F u_h ds = |F|u_h(m^F) \ge 0$. By the interpolation error estimates in [7] and (3.4) for $p = 2$, we have for $F \subset \partial T$,

$$\begin{aligned}
&\|R_0^F(\partial_\nu u)\|_{0,F}^2 \le 2\Big\{\int_F |R_0^F(\partial_1 u)|^2 ds + \int_F |R_0^F(\partial_2 u)|^2 ds\Big\}\\
&\le 2\Big\{\int_F |R_0^T(\partial_1 u)|^2 ds + \int_F |R_0^T(\partial_2 u)|^2 ds\Big\} \le Ch|u|_{2,T}^2,
\end{aligned} \tag{3.10}$$

where $R_0^T(v) = v - P_0^T(v) = v - \frac{1}{|T|}\int_T v dx$. Thus, using the fact $u_h(Q^F) = 0$, (3.5), (3.10) and Lemma 3.1 for $p = 2$, one yields

$$\begin{aligned}
&-\int_F \partial_\nu u\cdot u_h ds \le Ch^{3/2}|u|_{2,T}|u_h|_{1,F}\\
&\le Ch^{3/2}|u|_{2,T}(|u-u_h|_{1,F} + |u|_{1,F})\\
&\le Ch^{3/2}|u|_{2,T}(h^{-1}|u-u_h|_{1,T}^2 + h|u|_{2,T}^2)^{1/2} + Ch^{3/2}|u|_{2,T}|u|_{1,F}\\
&\le Ch|u|_{2,T}|u-u_h|_{1,T} + Ch^2|u|_{2,T}^2 + Ch^{3/2}|u|_{2,T}|u|_{1,F}
\end{aligned}$$

where $F \subset \partial T$. Consequently,

$$-\sum_{F\in\Gamma_{Ch}^-}\int_F \partial_\nu u\cdot u_h ds \le Ch|u|_{2,\Omega}\|u-u_h\|_h + Ch^2|u|_{2,\Omega}^2 + Ch^{3/2}|u|_{2,\Omega}\|u\|_{1,\partial\Omega},$$

from which the proof is completed by trace theorem.

Lemma 3.4. *Let u and u_h be the solutions of the problems* (2.1) *and* (2.6), *respectively. If $u \in H^2(\Omega)$ and the number of the critical points on Γ_C is finite, then*

$$- \sum_{F\in\Gamma_{Ch}^-} \int_F \partial_\nu u \cdot u_h ds \le Ch|u|_{2,\Omega}\|u-u_h\|_h + Ch^2|\log h|^{1/2}\|u\|_{2,\Omega}^2, \tag{3.11}$$

where C is a positive constant independent of h.

Proof. To begin with, following the same analysis used in the proof of Lemma 3.3, we only need to consider those F such that u_h has at least one zero point Q^F on F. Then, for these F, by (3.5) we have

$$\|u_h\|_{0,F} \le Ch\|\frac{du_h}{ds}\|_{0,F}. \tag{3.12}$$

It is easy to see that (3.9) and (3.10) are still valid. Therefore, from (3.9),(3.10) and (3.12) we have

$$\begin{aligned}
&-\int_F \partial_\nu u \cdot u_h ds \le Ch^{3/2}|u|_{2,T}\|\frac{du_h}{ds}\|_{0,F} \\
&\le Ch^{3/2}|u|_{2,T}(\|\frac{du_h}{ds}-\frac{du}{ds}\|_{0,F} + \|\frac{du}{ds}\|_{0,F}) \\
&\le Ch^{3/2}|u|_{2,T}(|u-u_h|_{1,F} + \|\frac{du}{ds}\|_{0,F}) \\
&\le Ch^{3/2}|u|_{2,T}(h^{-1}|u-u_h|_{1,T}^2 + h|u|_{2,T}^2)^{1/2} + Ch^{3/2}|u|_{2,T}\|\frac{du}{ds}\|_{0,F} \\
&\le Ch|u|_{2,T}|u-u_h|_{1,T} + Ch^2|u|_{2,T}^2 + Ch^{3/2}|u|_{2,T}\|\frac{du}{ds}\|_{0,F}.
\end{aligned} \tag{3.13}$$

Notice that $Du|_{\Gamma_C} \in H^{1/2}(\Gamma_C) \hookrightarrow L^p(\Gamma_C)$ with $\|v\|_{0,p,\Gamma_C} \le C\sqrt{p}\ \|v\|_{1/2,\Gamma_C}$ for any $v \in H^{1/2}(\Gamma_C)$ and $1 \le p < \infty$ (see [3, Lemma 5.1]). Therefore,

$$\begin{aligned}
\|\tfrac{du}{ds}\|_{0,F}^2 &\le \Big(\int_F 1ds\Big)^{1-2/p}\Big(\int_F |\frac{du}{ds}|^p ds\Big)^{2/p} \\
&\le Ch^{1-2/p}\|\frac{du}{ds}\|_{0,p,F}^2 \le Ch^{1-2/p}\|\frac{du}{ds}\|_{0,p,\Gamma_C}^2 \\
&\le Ch^{1-2/p}\ p\ \|Du\|_{1/2,\Gamma_C}^2 \le Ch^{1-2/p}\ p\ \|u\|_{2,\Omega}^2,
\end{aligned} \tag{3.14}$$

which implies

$$\|\frac{du}{ds}\|_{0,F} \le Ch^{1/2-1/p}\sqrt{p}\ \|u\|_{2,\Omega}. \tag{3.15}$$

From (3.13),(3.15), choosing $p = |\log h|$ and summing over all $F \in \Gamma_{Ch}^-$, we have

$$\begin{aligned}
&- \sum_{F\in\Gamma_{Ch}^-} \partial_\nu u \cdot u_h ds \\
&\le Ch|u|_{2,\Omega}\|u-u_h\|_h + Ch^2|u|_{2,\Omega}^2 + Ch^2\, h^{-1/p}\sqrt{p}\,\|u\|_{2,\Omega} \sum_{\substack{F\in\Gamma_{Ch}^- \\ F\subset\partial T}} |u|_{2,T} \\
&\le Ch|u|_{2,\Omega}\|u-u_h\|_h + Ch^2|\log h|^{1/2}\|u\|_{2,\Omega}^2.
\end{aligned}$$

since the number of the critical points is finite. The proof of this lemma is complete.

4. Main Results and Proofs

In this section, we present the main results of the error estimate for the Crouzeix-Raviart linear element approximation to Signorini problem stated in (2.1). In fact, Slimane [15] investigated the Crouzeix-Raviart approximation to the Signorini problem and obtained a convergence rate of $\mathcal{O}(h^{3/4})$. In this paper we first provide a quasi-optimal error of the nonconforming method under some reasonable assumption (Theorem 4.1). Furthermore, if additional regularity is assumed, optimality can be achieved (Theorem 4.2). Finally, if $u \in W^{2,p}(\Omega)$ with $p > 2$ we have optimal convergence rate even without the assumption that the number of the critical points is finite (Theorem 4.3).

Theorem 4.1. *Let u, u_h be the solutions of* (2.1) *and* (2.6), *respectively. If $u \in H^2(\Omega)$, and the number of the critical points on Γ_C is finite, then we have*

$$\|u - u_h\|_h \leq Ch|\log h|^{1/4}\|u\|_{2,\Omega}. \tag{4.1}$$

Theorem 4.2. *Let u, u_h be the solutions of (2.1) and* (2.6), *respectively. If $u \in H^2(\Omega)$, and the number of the critical points on Γ_C is finite and $u|_{\partial\Omega} \in W^{1,\infty}(\partial\Omega)$, then*

$$\|u - u_h\|_h \leq Ch\Big\{|u|_{2,\Omega}\big(\|u\|_{2,\Omega} + |u|_{1,\infty,\partial\Omega}\big)\Big\}^{1/2}. \tag{4.2}$$

Theorem 4.3. *Let u, u_h be the solutions of* (2.1) *and* (2.6), *respectively. If $u \in W^{2,p}(\Omega)$ with $p > 2$, then*

$$\|u - u_h\|_h \leq Ch\|u\|_{2,p,\Omega}. \tag{4.3}$$

We point out that for the continuous linear element approximation to Signorini problem the above results (2.1)-(2.2) have been obtained in [14, 5, 2]. However, for the nonconforming Crouzeix-Raviart finite element approximation we will see later that the analysis of error estimates are more difficult.

Before verifying the main results, we present the following lemmas.

Lemma 4.1. *Let u, u_h be the solutions of* (2.1) *and* (2.6), *respectively. If $u \in H^2(\Omega)$, then we have*

$$\|u - u_h\|_h^2 \leq C\big\{h^2|u|_{2,\Omega}^2 + \sum_{F\in\Gamma_{Ch}} \int_F \partial_\nu u(\Pi_h u - u_h)ds\big\}, \tag{4.4}$$

where Π_h is the linear interpolator operator of Crouzeix-Raviart finite element.

Proof. By the abstract error estimate (2.7) as well as (2.2), we get

$$\begin{aligned}
&E_h(u,v_h-u_h):=a_h(u,v_h-u_h)-\chi(v_h-u_h)\\
&=\sum_T\int_T\nabla u\cdot\nabla(v_h-u_h)dx-\int_\Omega f(v_h-u_h)dx-\int_{\Gamma_g}g(v_h-u_h)ds\\
&=-\int_\Omega(\triangle u+f)(v_h-u_h)dx+\sum_T\int_{\partial T}\partial_\nu u(v_h-u_h)ds-\int_{\Gamma_g}g(v_h-u_h)ds\\
&=\sum_T\sum_{\substack{F\subset\partial T\\ F\not\subseteq\partial\Omega}}\int_F\partial_\nu u(v_h-u_h)ds+\sum_{F\in\Gamma_{Dh}}\int_F\partial_\nu u(v_h-u_h)ds+\sum_{F\in\Gamma_{Ch}}\int_F\partial_\nu u(v_h-u_h)ds\\
&=:I_1+I_2+I_3.
\end{aligned}\tag{4.5}$$

Let $w_h=v_h-u_h$. By the standard interpolation error estimates of Crouzeix-Raviart finite element[8], we have

$$I_1=\sum_T\sum_{\substack{F\subset\partial T\\ F\not\subseteq\partial\Omega}}\int_F\partial_\nu u\cdot w_hds\le Ch|u|_{2,\Omega}\|w_h\|_h,\tag{4.6}$$

$$I_2=\sum_{F\in\Gamma_{Dh}}\int_F\partial_\nu u\cdot w_hds\le Ch|u|_{2,\Omega}\|w_h\|_h.\tag{4.7}$$

Combining (4.5)-(4.7) gives

$$E_h(u,v_h-u_h)\le Ch|u|_{2,\Omega}\|v_h-u_h\|_h+I_3.\tag{4.8}$$

Thus, by (4.7), the definition of $E_h(u,v_h-u_h)$, (4.8) together with the triangular inequality, it is easy to see

$$\|u-u_h\|_h^2\le C\inf_{v_h\in K_h}\Big\{\|u-v_h\|_h^2+Ch|u|_{2,\Omega}\big(\|u-v_h\|_h+\|u-u_h\|_h\big)+I_3\Big\}.$$

Using Young's inequality

$$ab\le\frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2,\qquad\forall\,\varepsilon>0,$$

we obtain

$$\|u-u_h\|_h^2\le C\inf_{v_h\in K_h}\Big\{\|u-v_h\|_h^2+I_3\Big\}+Ch^2|u|_{2,\Omega}^2.$$

Notice also that $\Pi_hv\in K_h$ for all $v\in K$. Choosing $v_h=\Pi_hu$ in the above inequality and using the standard interpolation error estimates of Crouzeix-Raviart linear finite element[8], we derive

$$\|u-u_h\|_h^2\le C\big(h^2|u|_{2,\Omega}^2+I_3\big)=C\big\{h^2|u|_{2,\Omega}^2+\sum_{F\in\Gamma_{Ch}}\int_F\partial_\nu u(\Pi_hu-u_h)ds\big\},\tag{4.9}$$

which completes our proof.

With Lemma 4.1 at hand, in order to prove the above theorems, we only need to handle the last term of the right-hand side of (4.9), i.e., $I_3=\sum_{F\in\Gamma_{Ch}}\int_F\partial_\nu u(\Pi_hu-u_h)\,ds$.

4.1. Proof of Theorem 4.1

Note from (2.2) that $\partial_\nu u = 0$ on Γ_{Ch}^+ and $\partial_\nu u \cdot u = 0$ on Γ_C,

$$\begin{aligned} I_3 &= \sum_{F\in\Gamma_{Ch}} \int_F \partial_\nu u(\Pi_h u - u_h)ds \\ &= \sum_{F\in\Gamma_{Ch}^0} \int_F \partial_\nu u(\Pi_h u - u_h)ds + \sum_{F\in\Gamma_{Ch}^-} \int_F \partial_\nu u(\Pi_h u - u_h)ds \\ &= \sum_{F\in\Gamma_{Ch}^0} \int_F \partial_\nu u(\Pi_h u - u_h)ds + \sum_{F\in\Gamma_{Ch}^-} \int_F \partial_\nu u(\Pi_h u - u)ds - \sum_{F\in\Gamma_{Ch}^-} \int_F \partial_\nu u \cdot u_h ds \\ &=: D_1 + D_2 + A. \end{aligned} \tag{4.10}$$

Moreover, for any $F \in \Gamma_{Ch}^0$,

$$\int_F (\Pi_h u - u_h)ds = |F|(\Pi_h u - u_h)(m^F) = -u_h(m^F)|F| \le 0,$$

where m^F is the midpoint of F. Then by Lemma 3.1 for $p = 2$, it follows easily that

$$\begin{aligned} &\int_F \partial_\nu u(\Pi_h u - u_h)ds \le \int_F R_0^F(\partial_\nu u)(\Pi_h u - u_h)ds \\ &= \int_F R_0^F(\partial_\nu u)(\Pi_h u - u)ds + \int_F R_0^F(\partial_\nu u) R_0^F(u - u_h)ds \\ &\le \|R_0^F(\partial_\nu u)\|_{0,F}(\|\Pi_h u - u\|_{0,F} + \|R_0^F(u - u_h)\|_{0,F}) \\ &\le Ch^{1/2}|u|_{2,T}(h^{3/2}|u|_{2,T} + h^{1/2}|u - u_h|_{1,T}) \le Ch^2|u|_{2,T}^2 + Ch|u|_{2,T}|u - u_h|_{1,T}. \end{aligned}$$

Hence,

$$D_1 \le Ch^2|u|_{2,\Omega}^2 + Ch|u|_{2,\Omega}\|u - u_h\|_h. \tag{4.11}$$

Now we turn to estimate D_2. Notice that for the interpolation error estimates of the Crouzeix-Raviart element, we have

$$\begin{aligned} &\|\Pi_h u - u\|_{0,T} \le Ch^2|u|_{2,T}, \qquad \|\Pi_h u - u\|_{1,T} \le Ch|u|_{2,T}, \\ &\|\Pi_h u - u\|_{2,T} \le C|u|_{2,T}. \end{aligned}$$

Then using the theory of interpolation spaces [6] yields

$$\|\Pi_h u - u\|_{0,p,T} \le Ch^{1+2/p}|u|_{2,T}, \qquad |\Pi_h u - u|_{1,p,T} \le Ch^{2/p}|u|_{2,T}. \tag{4.12}$$

Consequently, by Lemma 3.1, for any $F \in \Gamma_{Ch}^-$,

$$\begin{aligned} &\int_F \partial_\nu u(\Pi_h u - u)ds \le \|\partial_\nu u\|_{0,p',F}\|\Pi_h u - u\|_{0,p,F} \\ &\le C\|\partial_\nu u\|_{0,p',F}(h^{-1}\|\Pi_h u - u\|_{0,p,T}^p + h^{p-1}|\Pi_h u - u|_{1,p,T}^p)^{1/p} \\ &\le C\|\partial_\nu u\|_{0,p',F}(h^{p+1}|u|_{2,T}^p)^{1/p} \\ &\le Ch^{1+1/p}|u|_{2,T}\|\partial_\nu u\|_{0,p',F} \le Ch^{1+1/p}|u|_{2,T}\|\partial_\nu u\|_{0,p',\Gamma_C}. \end{aligned}$$

Note that $H^{1/2}(\Gamma_C) \hookrightarrow L^{p'}(\Gamma_C)$, $(1 \le p' < +\infty)$ and $\|\partial_\nu u\|_{0,p',\Gamma_C} \le C\sqrt{p'}\,\|\partial_\nu u\|_{1/2,\Gamma_C}$(see [3]). Then under the assumption of finite number of critical points on Γ_C, we obtain

$$\begin{aligned} D_2 &= \sum_{F\in\Gamma_{Ch}^-} \int_F \partial_\nu u(\Pi_h u - u)ds \le Ch^{1+1/p}\|\partial_\nu u\|_{0,p',\Gamma_C} \sum_{F\in\Gamma_{Ch}^-} |u|_{2,T} \\ &\le Ch^{1+1/p}\sqrt{p'}\|u\|_{2,\Omega}^2 \le C\sqrt{p'}h^{-1/p'}h^2\|u\|_{2,\Omega}^2, \end{aligned}$$

where $1/p + 1/p' = 1$. Choosing $p' = |\log h|$, we obtain

$$D_2 \le Ch^2|\log h|^{1/2}\|u\|_{2,\Omega}^2. \tag{4.13}$$

Finally, using Lemma 3.4, we can estimate the last term in (4.10)

$$A \le Ch|u|_{2,\Omega}\|u - u_h\|_h + Ch^2|\log h|^{1/2}\|u\|_{2,\Omega}^2. \tag{4.14}$$

We can finish the proof of Theorem 4.1 by using (4.10), (4.11), (4.13), (4.14) together with (4.9) and Young's inequality.

4.2. Proof of Theorem 4.2

Following the proof of Theorem 4.1, to improve the convergence rate from $\mathcal{O}(h|\log h|^{1/4})$ to $\mathcal{O}(h)$, it is sufficient to re-estimate the term D_2 and A in I_3 since the estimate of D_1 in (4.11) is optimal and still valid. For all $F \in \Gamma_{Ch}^-$, by Lemma 3.1 with $p = 2$ and the interpolation error,

$$\begin{aligned} \int_F \partial_\nu u(\Pi_h u - u)ds &\le \|\partial_\nu u\|_{0,\infty,F}\int_F |\Pi_h u - u|ds \\ &\le Ch^{1/2}\|\partial_\nu u\|_{0,\infty,F}\|u - \Pi_h u\|_{0,F} \\ &\le Ch^2|u|_{1,\infty,F}|u|_{2,T}, \end{aligned}$$

from which we deduce that

$$\begin{aligned} D_2 &= \sum_{F\in\Gamma_{Ch}^-} \int_F \partial_\nu u(\Pi_h u - u)ds \\ &\le Ch^2|u|_{1,\infty,\Gamma_C} \sum_{F\in\Gamma_{Ch}^-} |u|_{2,T} \le Ch^2|u|_{1,\infty,\Gamma_C}|u|_{2,\Omega}. \end{aligned} \tag{4.15}$$

Here we have used the assumption that the number of the critical points is finite. Furthermore, following the same line of the proof of Lemma 3.4, (3.13) is still valid in this case. Namely,

$$-\int_F \partial_\nu u \cdot u_h ds \le Ch|u|_{2,T}|u - u_h|_{1,T} + Ch^2|u|_{2,T}^2 + Ch^{3/2}|u|_{2,T}\|\frac{du}{ds}\|_{0,F}. \tag{4.16}$$

Now

$$\|\frac{du}{ds}\|_{0,F}^2 \le Ch\|\frac{du}{ds}\|_{0,\infty,F}^2 \le Ch|u|_{1,\infty,\Gamma_C}^2. \tag{4.17}$$

Thus, by (4.16)(4.17) and summing over all $F \in \Gamma_{Ch}^-$ yields

$$A \le Ch|u|_{2,\Omega}\|u - u_h\|_h + Ch^2|u|_{2,\Omega}^2 + Ch^2|u|_{2,\Omega}|u|_{1,\infty,\Gamma_C}. \tag{4.18}$$

Therefore, the estimate (4.2) follows from (4.9)-(4.11), (4.15), (4.18) and Young's inequality. Hence, the proof is complete.

4.3. Proof of Theorem 4.3

We still denote

$$I_3 = D_1 + D_2 + A. \tag{4.19}$$

The bound of D_1 given by (4.11) is still valid. Observe that now $u \in W^{2,p}(\Omega)$, $p > 2$, so $u|_{\partial\Omega} \in W^{2-1/p,p}(\partial\Omega)$. By Sobolev imbedding theorem[1], we know $\partial_\nu u|_{\partial\Omega} \in W^{1-1/p,p}(\partial\Omega) \hookrightarrow C^0(\partial\Omega)$. For any $F \in \Gamma_{Ch}^-$, by (2.2) we deduce that there exists some $x^F \in F$ such that $\partial_\nu u(x^F) = 0$. Moreover, D_2 can be re-estimated as

$$\begin{aligned} D_2 &= \sum_{F\in\Gamma_{Ch}^-} \int_F \partial_\nu u\, (\Pi_h u - u) ds = \sum_{F\in\Gamma_{Ch}^-} \int_F (\partial_\nu u - \partial_\nu u(x^F))\, (\Pi_h u - u) ds \\ &\le \sum_{F\in\Gamma_{Ch}^-} \|\partial_\nu u - \partial_\nu u(x^F)\|_{0,F} \|\Pi_h u - u\|_{0,F} \\ &\le \sum_{F\in\Gamma_{Ch}^-} Ch^{1-1/p} |\partial_\nu u|_{1-1/p,F}\, Ch^{3/2} |u|_{2,T} \\ &\le Ch^{5/2-1/p} |\partial_\nu u|_{1-1/p,\Gamma_C} \|u\|_2 \le Ch^2 \|u\|_{2,p}^2, \end{aligned} \tag{4.20}$$

where we have used Lemma 8.1 in [4], the discrete trace inequality (3.4) and the standard interpolation error estimates. Next, we turn to bound A. As discussed above, we only need to consider those F such that u_h has at least one zero point on it. Thus, by (3.5) one yields

$$\|u_h\|_{0,F} \le Ch \Big\| \frac{du_h}{ds} \Big\|_{0,F}. \tag{4.21}$$

For $F \in \Gamma_{Ch}^-$, u also has at least one zero point which we may denote by Q^F, i.e., $u(Q^F) = 0$. If there exists some neighborhood $W \subset F$ of Q^F such that $u(x)|_W = 0$, then there must be $\frac{du}{ds}(Q^F) = 0$. Otherwise, there exists some neighborhood $W \subset F$ of Q^F such that $u(x) > 0$ in W except Q^F itself. Under the latter case, it is easy to check Q^F is the minimum point in W which implies $\frac{du}{ds}(Q^F) = 0$ since $\frac{du}{ds} \in C^0(\partial\Omega)$. In short, there exists some $Q^F \in F$ such that $\frac{du}{ds}(Q^F) = 0$. In addition, following the proof of Lemma 3.4 we know (3.13) still holds. Thus, it follows from (3.6) that

$$\Big\| \frac{du}{ds} \Big\|_{0,F} \le Ch^{1/2} \Big\| \frac{du}{ds} \Big\|_{1/2,F} \le Ch^{1/2} |u|_{3/2,F}. \tag{4.22}$$

Combining (3.13) and (4.22) and summing over all $F \in \Gamma_{Ch}^-$, it follows that

$$A = -\sum_{F\in\Gamma_{Ch}^-} \int_F \partial_\nu u \cdot u_h ds \le Ch|u|_{2,\Omega} \|u - u_h\|_h + Ch^2 \|u\|_{2,\Omega}^2. \tag{4.23}$$

Finally, using (4.9),(4.11),(4.19),(4.20),(4.23) and Young's inequality we complete the proof of Theorem 4.3.

5. Numerical Experiments

In this section, we will show by numerical experiments the convergence of the nonconforming Crouzeix-Raviart finite element approximation of Signorini problem. We make a comparison between our Crouzeix-Raviart approximation and the continuous linear approximation.

We consider Signorini problem in $\Omega = [0,1]\times[0,1]$ where the Dirichlet boundary Γ_D is $[0,1]\times\{1\}$, the contact boundary Γ_C is $[0,1]\times\{0\}$, while the rest of $\partial\Omega$, that is Γ_g, is subjected to Neumman boundary condition. To be more specific,

$$\begin{cases} -\triangle u = 2\pi\sin(2\pi x), & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D, \\ \partial_\nu u = 0, & \text{in } \Gamma_g, \\ u \geq 0, \quad \partial_\nu u \geq 0, \quad \partial_\nu u \cdot u = 0, & \text{on } \Gamma_C. \end{cases} \tag{5.1}$$

Since an explicit solution of Signorini problem is not available, we use the discrete solution by the continuous linear elements ($h = 1/512$) as the reference solution. Then on the coarser mesh size of $h = 1/2, 1/4, \cdots, 1/64$, we compute the approximation solution u_h using the Crouzeix-Raviart and continuous linear elements, respectively. Fig 5.1 is the continuous linear element reference solution on uniform mesh size $h = 1/512$. It is plotted by the interpolation of the reference solution on mesh size $h = 1/32$.

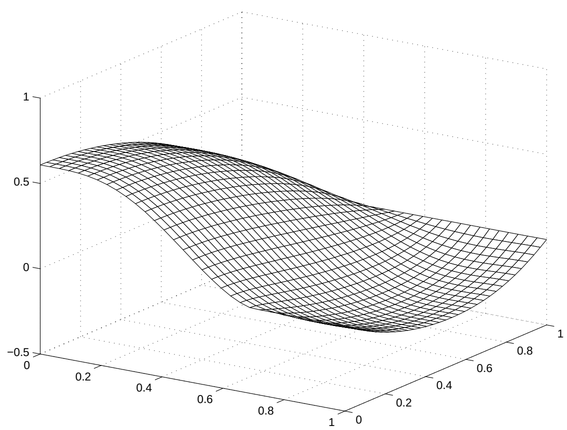

Fig. 5.1. The reference solution for problem (5.1).

From Fig 5.1 we can see that there is only one point that changes from binding to non-binding, i.e., the number of the critical points is one. Next, we compare the errors comparison between Crouzeix-Raviart approximation and the continuous linear approximation. The notation CF stands for conforming solution while NCF the nonconforming one.

To consider the convergence rate of Crouzeix-Raviart approximation, we plot the errors using the logarithm scales (see Fig 5.2 and 5.3). It can be checked that the slope of the curve represents the convergence rate.

Finally, we can see that the average slope of the continuous linear element approximation is about 1.89 in Fig 5.2 and 0.98 in Fig 5.3, while the corresponding average slopes of

	L^2 error		H^1 error	
N	CF	NCF	CF	NCF
2	0.10075495	0.14182150	0.69048776	0.72540780
4	0.02822581	0.02318398	0.34568311	0.31868764
8	0.00858278	0.00637450	0.18152995	0.16304303
16	0.00199802	0.00163432	0.09264809	0.08286176
32	0.00053274	0.00042440	0.04649183	0.04174806
64	0.00014134	0.00009866	0.02317579	0.02100774

Table 5.1: Numerical errors in L^2 and H^1 norm error. CF, stands for conforming solution, and NCF is the nonconforming solution.

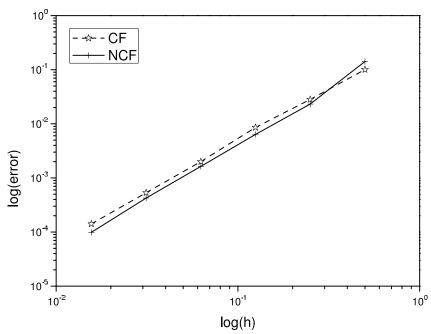

Fig. 5.2. L^2-norm error.

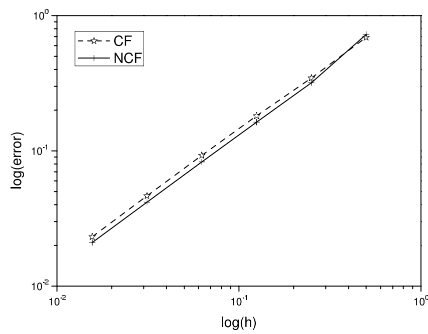

Fig. 5.3. H^1-norm error.

Crouzeix-Raviart linear element approximation are about 2.10 and 1.02, respectively. Therefore, numerical experiments show that the convergence rates of the two methods are of $\mathcal{O}(h^2)$ in L^2 norm and $\mathcal{O}(h)$ in H^1 norm, respectively.

References

[1] D.A. Adams, Sobolev Spaces, Academic press, New York, 1975.

[2] F. Brezzi, W.W. Hager, P.A. Raviart, Error estimates for the finite element solution of varitational inequalities, Part I, Primal theory, *Numer. Math.,* **28** (1977), 431-443.

[3] F. Ben Belgacem, Numerical simulation of some variational inequalityies arisen from unilateral contact problems by the finite element method, *SIAM J. Numer. Anal.,* **37** (2000), 1198-1216.

[4] Z. Belhachmi, F. Ben Belgacem, Quadratic finite element approximation of the Signorini problem, *Math. Comput.,* **72**:241 (2001), 83-104.

[5] F. Ben Belgacem, Y. Renard, Hybrid finite element methods for the Signorini problem, *Math. Comput.,* **72**:243 (2003), 1117-1145.

[6] J. Bergh, J. Löfström, Interpolation Spaces, Springer-Verlag, 1976.

[7] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North Holland, 1978.

[8] M. Crouzeix, P.A. Raviart, Conforming and non-conforming finite element methods for solving the stationary Stokes equations, *R.A.I.R.O. Anal. Num.,* **7** (1973), 33-76.

[9] G. Duvaut, J.L. Lions, Les Inéquations en Mécanique et en Physique, Dunod, Paris, 1972.

[10] R.S. Falk, Error estimates for the approximation of a class of variational inequalities, *Math. Comput.,* **28** (1974), 963-971.

[11] R.S. Falk, B. Mercier, Error estimates for elastoplastic problems, *R.A.I.R.O. Anal. Num.,* **11** (1997), 117-134.

[12] J.L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Vol. I, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[13] J.L. Lions, G. Stampacchia, Variational inequalityies, *Comm. Pure Appl. Math.,* **20** (1967), 493-519.

[14] F. Scarpini, M.A. Vivaldi, Error estimates for the approximation of some unilateral problems, *R.A.I.R.O. Anal. Num.,* **11** (1977), 197-208.

[15] L. Slimane, Méthodes mixtes et traitement du vérouillage numérique pour la résolution des inéquations variationnelles, Thèse de l'Institut National des Sciences Appliquées de Toulouse, 2001.

[16] F. Stummel, The generalized patch test, *SIAM J. Numer. Anal.,* **16** (1979), 449-471.