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A NOTE ON THE QUADRILATERAL MESH CONDITION $RDP(N, \psi)$ *1)

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Abstract

The main aim of this paper is to show that the quadrilateral mesh condition $RDP(N, \psi)$ is only sufficient but not necessary for the optimal order error estimate of the Q_1 isoparametric element in the H^1 norm.

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Key words: Isoparametric finite element, Optimal order error estimate, Quadrilateral mesh condition.

1. Introduction

Quadrilateral meshes are widely used in the finite element method due to its simplicity and flexibility. However, numerical accuracy can not be achieved over an arbitrary mesh. Thus certain mesh conditions have been imposed in the literatures. Usually, the mesh conditions are classified into two groups [8]: One is the shape regular condition and the other is the degenerate condition. Roughly speaking, a shape regular condition requires that the element cannot be too narrow on the one hand and the interior angle of each vertex is neither too small nor too close to π on the other hand (see [2,4,5,10,13]). Meanwhile, several degenerate mesh conditions are discussed in [1,3,6,7,9,10,11,12]. However, among all mesh conditions, $RDP(N, \psi)$, the one proposed in [1] seems to be the weakest mesh condition up to now.

It has been proved in [1] that $RDP(N, \psi)$ is a sufficient condition for the following estimate:

$$\| u - Qu \|_{0,K} + h \mid u - Qu \mid_{1,K} \leq Ch^2 \mid u \mid_{2,K}, \tag{1.1}$$

where Q is the *Lagrange* interpolation operator. $\| \cdot \|$ and $| \cdot |$ denote respectively the standard norm and seminorm in *Sobolev* space.

One may ask whether or not the condition $RDP(N, \psi)$ is also necessary. Acosta and Durán[1] put it as an open problem. In this note, we will show by a counterexample that $RDP(N, \psi)$ is not necessary.

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2. The Construction of the Q_1 Isoparametric Element and Some Known Results

For the sake of convenience, we will cite the following definitions introduced in [1].

Definition 1. *We say that a quadrilateral K (resp. a triangle T) satisfies the maximum angle condition with a constant $\psi < \pi$, or shortly $MAC(\psi)$, if the angles of K(resp. T) are less than or equal to ψ.*

Definition 2. *Let K be a convex quadrilateral with diagonals d_1 and d_2. We say that K satisfies the regular decomposition property with constants $N \in \Re$ and $0 < \psi < \pi$, or shortly $RDP(N,\psi)$, if we can divide K into two triangles along one of its diagonals, that will be called always d_1, in such a way that $\frac{d_2}{d_1} \leq N$ and both triangles satisfy $MAC(\psi)$.*

For a general convex quadrilateral K and the reference element $\hat{K} = [0,1] \times [0,1]$, we will denote their vertices by M_i and $\hat{M}_i$ $(i = 1,2,3,4)$ in anticlockwise order respectively (see Fig.1 and Fig.2). We will also use the variables (ξ,η) on $\hat{K}$ and (x,y) on K.
Let $F_K : \hat{K} \to K$ be the transformation such that

$$\begin{cases} x = \sum\limits_1^4 x_i \hat{\phi}_i(\xi,\eta) \\ y = \sum\limits_1^4 y_i \hat{\phi}_i(\xi,\eta), \end{cases} \tag{2.1}$$

where $\hat{\phi}_i$ is the bilinear basis function associated with the vertex $\hat{M}_i$, i.e., $\hat{\phi}_i(\hat{M}_j) = \delta_i^j$. Now, the basis function on K, no longer bilinear in general, is defined by $\phi_i(x,y) = \hat{\phi}_i(F_K^{-1}(x,y))$ and the Q_1 isoparametric interpolation operator Q on K is defined by

$$Qu(x,y) = \hat{Q}\hat{u}(\xi,\eta),$$

where $\hat{Q}$ is the bilinear Lagrange interpolation of $\hat{u} = u \circ F_K$ on $\hat{K}$.

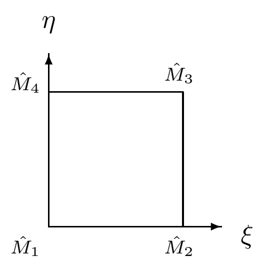

Fig. 1

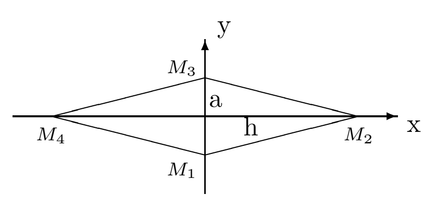

Fig.2

3. The Main Results

In this section we will show that the mesh condition $RDP(N,\psi)$ is a sufficient but not necessary condition for (1.1) to be hold.

Theorem 3.1[1]**.** *Let K be a convex quadrilateral with diameter h. For any $u \in H^2(\Omega)$, there exists a constant C_0 independent of K such that*

$$\| u - Qu \|_{0,K} \leq C_0 h^2 \mid u \mid_{2,K} \tag{3.1}$$

and, if K satisfies $RDP(N,\psi)$, then there exists a constant $C = C(N,\psi)$ such that

$$\mid u - Qu \mid_{1,K} \leq Ch \mid u \mid_{2,K} . \tag{3.2}$$

Consider the element K with vertices $M_1(0,-a), M_2(h,0), M_3(0,a)$ and $M_4(-h,0)$ like in Fig.2. We assume that $a \ll h$. Then this element does not satisfy $RDP(N,\psi)$([8]). In fact, if we decompose K by the diagonal M_1M_3, then the triangle $\triangle M_1M_2M_3$ and $\triangle M_1M_3M_4$ indeed satisfy the maximal angle condition since all interior angles in these two triangles are bounded from above by $\frac{\pi}{2}$. However, $\frac{|M_2M_4|}{|M_1M_3|} = \frac{h}{a}$ which cannot be bounded by any constant as a tends to zero. If we decompose K by the diagonal M_2M_4, a simple computation leads to $\sin \angle M_4M_1M_2 = \frac{2ah}{a^2+h^2}$, so the angle $\angle M_4M_1M_2$ approaches π as a tends to zero, thus it violates $RDP(N,\psi)$.

For the element K like in Fig.2, it can be checked that the transformation $F_K : \hat{K} \to K$ can be expressed as

$$\begin{cases} x = h(\xi - \eta) \\ y = a(\xi + \eta - 1). \end{cases} \tag{3.3}$$

Taking $u = u(x,y) = x^3$, we have

$$\hat{Q}\hat{u}(\xi,\eta) = h^3(\xi - \eta). \tag{3.4}$$

An elementary manipulation yields

$$\begin{aligned} \mid u - Qu \mid_{1,K}^2 &= \int_K [(\frac{\partial(u-Qu)}{\partial x})^2 + (\frac{\partial(u-Qu)}{\partial y})^2]dxdy \\ &= \int_{\hat{K}} [(\frac{\partial(\hat{u}-\hat{Q}\hat{u})}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial(\hat{u}-\hat{Q}\hat{u})}{\partial \eta}\frac{\partial \eta}{\partial x})^2 \\ &\quad + (\frac{\partial(\hat{u}-\hat{Q}\hat{u})}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial(\hat{u}-\hat{Q}\hat{u})}{\partial \eta}\frac{\partial \eta}{\partial y})^2]2ahd\xi d\eta \\ &= 2ah^5\int_{\hat{K}}[3(\xi-\eta)^2 - 1]^2 d\xi d\eta \\ &= \frac{6}{5}ah^5, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \mid u \mid_{2,K}^2 &= \int_K [(\frac{\partial u^2}{\partial x^2})^2 + (\frac{\partial u^2}{\partial y \partial x})^2 + (\frac{\partial u^2}{\partial x \partial y})^2 + (\frac{\partial u^2}{\partial y^2})^2]dxdy \\ &= \int_K 36x^2 dxdy = 36\int_{\hat{K}} h^2(\xi-\eta)^2 2ahd\xi d\eta \\ &= 12ah^3. \end{aligned} \tag{3.6}$$

Therefore

$$\frac{\mid u - Qu \mid_{1,K}^2}{\mid u \mid_{2,K}^2} = \frac{1}{10}h^2 < h^2, \tag{3.7}$$

which yields

$$\mid u - Qu \mid_{1,K} < h \mid u \mid_{2,K} . \tag{3.8}$$

That is to say, although the element K does not satisfy $RDP(N,\psi)$, we can derive the optimal error estimate for the function $u = x^3$. This reveals that the mesh condition $RDP(N,\psi)$ is just sufficient but not necessary for (1.1) to hold for the Q_1 isoparameteric element.

We will end this note with an open problem: what is really the weakest necessary condition for (1.1) to hold for the Q_1 isoparameteric element?

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