Journal of Computational Mathematics, Vol.25, No.1, 2007, 2–12.

A BOUNDARY INTEGRAL METHOD FOR COMPUTING ELASTIC MOMENT TENSORS FOR ELLIPSES AND ELLIPSOIDS *1)

Habib Ammari
(*Centre de Mathématiques Appliquées, CNRS UMR 7641 and Ecole Polytechnique, 91128 Palaiseau Cedex, France*
Email: ammari@cmapx.polytechnique.fr)

Hyeonbae Kang Hyundae Lee
(*Department of Mathematical Sciences and RIM, Seoul National University, Seoul 151-747, Korea*
Email: hkang@math.snu.ac.kr, hdlee@math.snu.ac.kr)

Abstract

The concept of elastic moment tensor occurs in several interesting contexts, in particular in imaging small elastic inclusions and in asymptotic models of dilute elastic composites. In this paper, we compute the elastic moment tensors for ellipses and ellipsoids by using a systematic method based on layer potentials. Our computations reveal an underlying elegant relation between the elastic moment tensors and the single layer potential.

Mathematics subject classification: 45A05, 74B05.
Key words: Elastic moment tensor, Explicit formulae, Integral equations.

1. Introduction

Let B be a bounded Lipschitz domain in $\mathbb{R}^d, d = 2, 3$. Assume that the Lamé parameters of B are given by $(\widetilde{\lambda}, \widetilde{\mu})$, while those of the background $\mathbb{R}^d \setminus \overline{B}$ are given by (λ, μ). Attached to the inclusion B is a 4-tensor m_{pq}^{ij}, $i, j, p, q = 1, \ldots, d$, called the elastic moment tensor (EMT), or the elastic polarization tensor. The notion of EMT can be most simply described in the following manner. Denote by $\mathcal{L}_{\lambda,\mu}$ the Lamé operator associated with the parameters λ and μ and consider $\mathbf{H}$ to be a vector-valued function satisfying $\mathcal{L}_{\lambda,\mu}\mathbf{H} = 0$ in $\mathbb{R}^d$, $d = 2, 3$. If the field $\mathbf{H}$ is perturbed due to the presence of an elastic inclusion B with the Lamé parameters $(\widetilde{\lambda}, \widetilde{\mu})$ then the ith component of the perturbation is given by

$$\sum_{j=1}^{d} \sum_{p,q=1}^{d} \partial_p H_j(0) \partial_q \mathbf{\Gamma}(x) m_{pq}^{ij} + O(|x|^{1-d}) \quad \text{as } |x| \to \infty, \tag{1.1}$$

where H_j is the jth component of $\mathbf{H}$ and $\mathbf{\Gamma}$ is the fundamental solution to the Lamé equation with the Lamé parameters λ, μ. See [1] for a rigorous derivation of formula (1.1) which shows that through the elastic moment tensor, $M = (m_{pq}^{ij})$, we have a complete information about the leading-order term in the far-field expansion of $\mathbf{H}$. See also [6] for a representation of the perturbation by Elsheby's tensor. It is worth mentioning that the use of the EMT leads to stable and accurate algorithms for the numerical computations of the displacement field in the presence of small elastic inclusions. It is known that small size features cause difficulties in the numerical solution of the problem by the finite element or finite difference methods. This is because such features require refined meshes in their neighborhoods, with their attendant problems.

The notion of EMT also occurs naturally in several other physical contexts, in particular in asymptotic expansions of perturbations of the elastic energy [10, 11] and in models of the

* Received January 4, 2006; accepted February 16, 2006.
1) Partly supported by Korea Science and Engineering Foundation grant R02-2003-000-10012-0.

effective properties of dilute elastic composites [7, 8, 12, 13]. Recently, the EMT has been used in the inverse problem of reconstructing diametrically small elastic inclusions from boundary measurements [1, 2, 4, 8]. It turns out that we can determine the EMT of the inclusion via boundary measurements. Since EMT carries fairly good information about the size of the inclusion, the volume of the elastic inclusion can be estimated by means of the EMT.

The purpose of this paper is to present a quite general and elegant method for computing the EMT based on layer potential techniques. In particular, we apply this method to provide explicit formulae for the EMTs associated with ellipses and ellipsoids. The method reveals an interesting relation between the single layer potentials and the EMT. The EMT for ellipses has been computed in [1] using a complex analysis representation of the solutions to the two-dimensional Lamé system [14]. The method of this paper is completely different from that in [1]. Moreover, it enables us to find explicit formula for the EMT for ellipsoids. It should be mentioned that a quite similar method has been used to compute the polarization tensor associated to the conductivity problem for ellipses and ellipsoids [9].

This paper is organized as follows. In Section 2 we review the definition of the EMT in terms of layer potentials and present a general scheme to compute it. Sections 3 and 4 are devoted to the derivation of EMTs for ellipses and ellipsoids.

2. Single Layer Potential and Elastic Moment Tensor

Let $\mathbf{\Gamma} = (\Gamma_{ij})_{i,j=1}^d$ be a fundamental solution to the Lamé system, namely,

$$\Gamma_{ij}(x) := \begin{cases} -\dfrac{A}{4\pi}\dfrac{\delta_{ij}}{|x|} - \dfrac{B}{4\pi}\dfrac{x_i x_j}{|x|^3} & \text{if } d=3, \\ \dfrac{A}{2\pi}\delta_{ij}\ln|x| - \dfrac{B}{2\pi}\dfrac{x_i x_j}{|x|^2} & \text{if } d=2, \end{cases} \qquad x \neq 0,$$

where

$$A = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right) \quad \text{and} \quad B = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right).$$

Let the constants (λ, μ) denote the background Lamé coefficients, that are the elastic parameters in the absence of any inclusions. Let B be a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$. Assume that B has the pair of Lamé constants $(\widetilde{\lambda}, \widetilde{\mu})$ which is different from that of the background elastic body, (λ, μ). It is always assumed that

$$\mu > 0, \quad d\lambda + 2\mu > 0, \quad \widetilde{\mu} > 0 \quad \text{and} \quad d\widetilde{\lambda} + 2\widetilde{\mu} > 0.$$

We also assume that

$$(\lambda - \widetilde{\lambda})(\mu - \widetilde{\mu}) \geq 0, \quad ((\lambda - \widetilde{\lambda})^2 + (\mu - \widetilde{\mu})^2 \neq 0).$$

The single layer potential of the density function φ on B associated with the Lamé parameters (λ, μ) is defined by

$$\mathcal{S}_B\varphi(x) := \int_{\partial D} \mathbf{\Gamma}(x-y)\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d.$$

Analogously, we denote by $\widetilde{\mathcal{S}}_B$ the single layer potential on ∂B corresponding to the Lamé constants $(\widetilde{\lambda}, \widetilde{\mu})$.

The following jump relation is well-known:

$$\frac{\partial(\mathcal{S}_B\varphi)}{\partial\nu}\Big|_+ - \frac{\partial(\mathcal{S}_B\varphi)}{\partial\nu}\Big|_- = \varphi \quad \text{a.e. on } \partial B,$$

where $\partial\mathbf{u}/\partial\nu$ denotes the conormal derivative, *i.e.*,

$$\frac{\partial\mathbf{u}}{\partial\nu} := \lambda(\nabla\cdot\mathbf{u})N + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)N \quad \text{on } \partial B.$$

Here N is the outward unit normal to ∂B and the superscript T denotes the transpose of a matrix.

Let $\{\mathbf{e}_j\}_{j=1}^d$ be the standard basis for $\mathbb{R}^d$, and set the pair $(\mathbf{f}_i^j, \mathbf{g}_i^j)$ in $L^2(\partial B) \times L^2(\partial B)$ to be the unique solution to the system of integral equations [5]:

$$\begin{cases} \widetilde{\mathcal{S}}_B \mathbf{f}_i^j|_- - \mathcal{S}_B \mathbf{g}_i^j|_+ = x_i \mathbf{e}_j|_{\partial B}, \\ \dfrac{\partial}{\partial \widetilde{\nu}} \widetilde{\mathcal{S}}_B \mathbf{f}_i^j \Big|_- - \dfrac{\partial}{\partial \nu} \mathcal{S}_B \mathbf{g}_i^j \Big|_+ = \dfrac{\partial (x_i \mathbf{e}_j)}{\partial \nu}|_{\partial B}, \end{cases}$$

where $\partial/\partial\widetilde{\nu}$ is the conormal derivative corresponding to the Lamé constants $(\widetilde{\lambda}, \widetilde{\mu})$.

The (first-order) EMT is defined to be [4, 1]

$$m_{pq}^{ij} := \int_{\partial B} x_p \mathbf{e}_q \cdot \mathbf{g}_i^j \, d\sigma, \quad i, j, p, q = 1, \ldots, d,$$

or equivalently,

$$\begin{aligned} m_{pq}^{ij} &= \int_{\partial B} x_p \mathbf{e}_q \cdot \mathbf{g}_i^j \, d\sigma \\ &= \int_{\partial B} x_p \mathbf{e}_q \cdot \Big[\frac{\partial (\mathcal{S}_B \mathbf{g}_i^j)}{\partial \nu} \Big|_+ - \frac{\partial (\mathcal{S}_B \mathbf{g}_i^j)}{\partial \nu} \Big|_- \Big] \, d\sigma \\ &= - \int_{\partial B} x_p \mathbf{e}_q \cdot \frac{\partial (x_i \mathbf{e}_j)}{\partial \nu} \, d\sigma - \int_{\partial B} x_p \mathbf{e}_q \cdot \Big[\frac{\partial (\mathcal{S}_B \mathbf{g}_i^j)}{\partial \nu} \Big|_- - \frac{\partial (\widetilde{\mathcal{S}}_B \mathbf{f}_i^j)}{\partial \widetilde{\nu}} \Big|_- \Big] \, d\sigma \\ &= - \int_{\partial B} \frac{\partial (x_p \mathbf{e}_q)}{\partial \nu} \cdot x_i \mathbf{e}_j \, d\sigma - \int_{\partial B} \Big[\frac{\partial (x_p \mathbf{e}_q)}{\partial \nu} \cdot \mathcal{S}_B \mathbf{g}_i^j - \frac{\partial (x_p \mathbf{e}_q)}{\partial \widetilde{\nu}} \cdot \widetilde{\mathcal{S}}_B \mathbf{f}_i^j \Big] \, d\sigma \\ &= \int_{\partial B} \Big[\frac{\partial (x_p \mathbf{e}_q)}{\partial \widetilde{\nu}} - \frac{\partial (x_p \mathbf{e}_q)}{\partial \nu} \Big] \cdot \mathcal{S}_B \mathbf{g}_i^j \, d\sigma + \int_{\partial B} \frac{\partial (x_p \mathbf{e}_q)}{\partial \widetilde{\nu}} \cdot x_i \mathbf{e}_j - x_p \mathbf{e}_q \cdot \frac{\partial (x_i \mathbf{e}_j)}{\partial \nu} \\ &= \int_{\partial B} \Big[\mathcal{S}_B \left(\frac{\partial (x_p \mathbf{e}_q)}{\partial \widetilde{\nu}} \right) - \mathcal{S}_B \left(\frac{\partial (x_p \mathbf{e}_q)}{\partial \nu} \right) \Big] \cdot \mathbf{g}_i^j \, d\sigma + \int_{\partial B} \frac{\partial (x_p \mathbf{e}_q)}{\partial \widetilde{\nu}} \cdot x_i \mathbf{e}_j - x_p \mathbf{e}_q \cdot \frac{\partial (x_i \mathbf{e}_j)}{\partial \nu}. \end{aligned}$$

Since

$$\begin{aligned} X_{pq}^{ij} :&= \int_{\partial B} \frac{\partial (x_p \mathbf{e}_q)}{\partial \widetilde{\nu}} \cdot x_i \mathbf{e}_j - x_p \mathbf{e}_q \cdot \frac{\partial (x_i \mathbf{e}_j)}{\partial \nu} \\ &= |B| \big[(\widetilde{\lambda} - \lambda) \delta_{pq} \delta_{ij} + (\widetilde{\mu} - \mu)(\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}) \big], \end{aligned}$$

where $|B|$ denotes the volume of B, we arrive at the following equation which has to be solved:

$$m_{pq}^{ij} - \int_{\partial B} \Big[\mathcal{S}_B \left(\frac{\partial (x_p \mathbf{e}_q)}{\partial \widetilde{\nu}} \right) - \mathcal{S}_B \left(\frac{\partial (x_p \mathbf{e}_q)}{\partial \nu} \right) \Big] \cdot \mathbf{g}_i^j \, d\sigma = X_{pq}^{ij}. \tag{2.1}$$

Let $\Gamma_0(x)$ be the fundamental solution for the Laplacian, that is,

$$\Gamma_0(x) := \begin{cases} -\dfrac{1}{4\pi} \dfrac{1}{|x|} & \text{if } d = 3, \\ \dfrac{1}{2\pi} \ln |x| & \text{if } d = 2, \end{cases} \qquad x \neq 0.$$

Since

$$\begin{aligned} \frac{\partial (x_p \mathbf{e}_q)}{\partial \nu} &= \lambda \nabla \cdot (x_p \mathbf{e}_q) N + \mu (\nabla (x_p \mathbf{e}_q) + \nabla (x_p \mathbf{e}_q)^T) N \\ &= \lambda \delta_{pq} N + \mu (n_p \mathbf{e}_q + n_q \mathbf{e}_p), \end{aligned}$$

where the unit outward normal $N = (n_1, \ldots, n_d)$, we get, for $k = 1, \ldots, d$,

$$\begin{aligned} &\mathcal{S}_B \left(\frac{\partial (x_p \mathbf{e}_q)}{\partial \nu} \right) (x)_k \\ &= \lambda \delta_{pq} \left(\int_{\partial B} \Gamma(x - y) N(y) \right)_k + \mu \left(\int_{\partial B} \Gamma(x - y)(n_p \mathbf{e}_q + n_q \mathbf{e}_p) \right)_k \\ &= \lambda \delta_{pq} \sum_{l=1}^d \int_{\partial B} \Gamma_{kl}(x - y) n_l(y) + \mu \int_{\partial B} \Gamma_{kq}(x - y) n_p(y) + \Gamma_{kp}(x - y) n_q(y). \end{aligned}$$

By the divergence theorem, we obtain

$$\begin{aligned}\sum_{l=1}^{d} \int_{\partial B} \Gamma_{kl}(x-y) n_l(y) &= A \int_{\partial B} \Gamma_0(x-y) n_k(y) - \frac{B}{\omega_d} \int_{\partial B} (x_k - y_k) \frac{\langle x-y, n(y) \rangle}{|x-y|^d} \\ &= (A-B) \int_{\partial B} \Gamma_0(x-y) n_k(y).\end{aligned}$$

We also have

$$\begin{aligned}&\int_{\partial B} \Gamma_{kq}(x-y) n_p(y) + \Gamma_{kp}(x-y) n_q(y) \\ &= A\delta_{kp} \int_{\partial B} \Gamma_0(x-y) n_q(y) + A\delta_{kq} \int_{\partial B} \Gamma_0(x-y) n_p(y) - B\Big(I_{kpq}(x) + I_{kqp}(x) \Big),\end{aligned}$$

where

$$I_{kpq}(x) := \frac{1}{\omega_d} \int_{\partial B} \frac{(x_k - y_k)(x_p - y_p)}{|x-y|^d} n_q(y) d\sigma(y).$$

Now, we make two basic assumptions for this paper.
Basic assumptions. Assume that for $x \in \partial B$

$$\int_{\partial B} \Gamma_0(x-y) n_k(y) = \sum_{l=1}^{d} c_{kl} x_l, \quad k = 1, \ldots, d, \tag{2.2}$$

and

$$I_{kpq}(x) + I_{kqp}(x) = \sum_{l=1}^{d} d_{pq}^{kl} x_l, \quad k, p, q = 1, \ldots, d, \tag{2.3}$$

for some constants c_{kl} and d_{pq}^{kl}.

The assumptions (2.2) and (2.3) are of geometric nature. It will be interesting and important to find the class of domains B for which these assumptions hold. We will see in following sections that these assumptions hold if B is an ellipse or an ellipsoid.

Under these assumptions, we write

$$\sum_{l=1}^{d} \int_{\partial B} \Gamma_{kl}(x-y) n_l(y) = (A-B) \sum_{l=1}^{d} c_{kl} x_l,$$

and

$$\int_{\partial B} \Gamma_{kq}(x-y) n_p(y) + \Gamma_{kp}(x-y) n_q(y) = A(\delta_{kp} \sum_{l=1}^{d} c_{ql} x_l + \delta_{kq} \sum_{l=1}^{d} c_{pl} x_l) - B \sum_{l=1}^{d} d_{pq}^{kl} x_l,$$

and obtain the following lemma.

Lemma 2.1. *Under the assumptions* (2.2) *and* (2.3), *we have*

$$\begin{aligned}&\mathcal{S}_B \left(\frac{\partial (x_p \mathbf{e}_q)}{\partial \nu} \right)(x) \\ &= \sum_{k=1}^{d} \Big[\lambda(A-B) \delta_{pq} \sum_{l=1}^{d} c_{kl} x_l + \mu A(\delta_{kp} \sum_{l=1}^{d} c_{ql} x_l + \delta_{kq} \sum_{l=1}^{d} c_{pl} x_l) - \mu B \sum_{l=1}^{d} d_{pq}^{kl} x_l \Big] \mathbf{e}_k.\end{aligned}$$

It then follows that

$$\int_{\partial B}\Big[\mathcal{S}_B\left(\frac{\partial(x_p\mathbf{e}_q)}{\partial\widetilde{\nu}}\right)-\mathcal{S}_B\left(\frac{\partial(x_p\mathbf{e}_q)}{\partial\nu}\right)\Big]\cdot\mathbf{g}_i^j\,d\sigma$$
$$=\delta_{pq}(\widetilde{\lambda}-\lambda)(A-B)\sum_{k,l}c_{kl}\int_{\partial B}x_l\mathbf{e}_k\cdot g_i^j$$
$$+(\widetilde{\mu}-\mu)A\sum_{k,l}(\delta_{kp}c_{ql}+\delta_{kq}c_{pl})\int_{\partial B}x_l\mathbf{e}_k\cdot g_i^j-(\widetilde{\mu}-\mu)B\sum_{k,l}d_{pq}^{kl}\int_{\partial B}x_l\mathbf{e}_k\cdot g_i^j$$
$$=\sum_{k,l}\big[\delta_{pq}(\widetilde{\lambda}-\lambda)(A-B)c_{kl}+(\widetilde{\mu}-\mu)A(\delta_{kp}c_{ql}+\delta_{kq}c_{pl})-(\widetilde{\mu}-\mu)Bd_{pq}^{kl}\big]m_{lk}^{ij}.$$

Therefore, (2.1) becomes

$$m_{pq}^{ij}-\sum_{k,l}\big[\delta_{pq}(\widetilde{\lambda}-\lambda)(A-B)c_{kl}+(\widetilde{\mu}-\mu)A(\delta_{kp}c_{ql}+\delta_{kq}c_{pl})-(\widetilde{\mu}-\mu)Bd_{pq}^{kl}\big]m_{lk}^{ij}=X_{pq}^{ij}\,.$$

Note that (m_{pq}^{ij}) has symmetry, namely,

$$m_{pq}^{ij}=m_{qp}^{ij},\ m_{pq}^{ij}=m_{pq}^{ji},\ m_{pq}^{ij}=m_{ij}^{pq},$$

and so does (X_{pq}^{ij}). Using these properties, we can symmetrize the equation and state the following theorem.

Theorem 2.2. *Assume that* (2.2) *and* (2.3) *hold on B. Let $C_B=(C_{pq}^{kl})$ be defined by*

$$C_{pq}^{kl}=\frac{1}{2}(\widetilde{\lambda}-\lambda)(A-B)\delta_{pq}(c_{kl}+c_{lk})$$
$$+\frac{1}{2}(\widetilde{\mu}-\mu)A(\delta_{kp}c_{ql}+\delta_{kq}c_{pl}+\delta_{lp}c_{qk}+\delta_{lq}c_{pk})-\frac{1}{2}(\widetilde{\mu}-\mu)B(d_{pq}^{kl}+d_{pq}^{lk}),\tag{2.4}$$

and let $X=(X_{pq}^{ij})$. Then the EMT $M=(m_{pq}^{ij})$ satisfies

$$(I-C_B)M=X.\tag{2.5}$$

3. Elastic Moment Tensor of Ellipses

We now compute the EMT of ellipses. To this end we prove that (2.2) and (2.3) hold when B is an ellipse and compute the tensor C_B defined by (2.4).

We begin with briefly reviewing elliptic coordinates. Assume that B is an ellipse of the form

$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1,\quad a\geq b>0.\tag{3.1}$$

Then the focal line $[-R,R]$, where $R=\sqrt{a^2-b^2}$, lies on the x_1-axis. The elliptic coordinates (r,ω) are defined by

$$x_1=R\cos\omega\cosh r,\quad x_2=R\sin\omega\sinh r,\quad r\geq 0,\ 0\leq\omega\leq 2\pi,$$

in which the ellipse B is given by $B=\{(r,\omega):r<\rho\}$, where ρ is defined to be $a=R\cosh\rho$ and $b=R\sinh\rho$. Separation of variables shows that harmonic functions in $\mathbb{R}^2\setminus[-R,R]$ take the form

$$\psi(r,\omega)=A_0+\sum_{j=1}^{+\infty}\Big(A_j\cos j\omega\,e^{-jr}+B_j\sin j\omega\,e^{-jr}+C_j\cos j\omega\,e^{jr}+D_j\sin j\omega\,e^{jr}\Big).$$

For the solution in $B\setminus[-R,R]$ to be extended to a harmonic function in B we must furthermore require that

$$\psi(0,\omega)=\psi(0,-\omega)\quad\text{and}\quad\frac{\partial\psi}{\partial r}(0,\omega)=-\frac{\partial\psi}{\partial r}(0,-\omega).$$

Let $\mathcal{S}_B^0\varphi$ be the single layer potential for the Laplacian, *i.e.*,

$$\mathcal{S}_B^0\varphi(x)=\int_{\partial B}\Gamma_0(x-y)\varphi(y)d\sigma(y),\quad x\in\mathbb{R}^d,$$

where $\Gamma_0(x)$ is the fundamental solution for the Laplacian.

Using elliptic coordinates we can prove the following lemma.

Lemma 3.1. *If h is a harmonic polynomial of homogeneous degree j, then $\mathcal{S}_B^0(\nabla h \cdot n)(x)$ is also a harmonic polynomial of the same degree j for $x \in B$. Moreover, if $h(x) = \cos j\omega(e^{jr} + e^{-jr})$ in elliptic coordinates, then*

$$\mathcal{S}_B^0(\nabla h \cdot n)(x) = -\frac{1}{2}(1 - m^j)h(x), \quad x \in B, \tag{3.2}$$

and if $h(x) = \sin j\omega(e^{jr} - e^{-jr})$, then

$$\mathcal{S}_B^0(\nabla h \cdot n)(x) = -\frac{1}{2}(1 + m^j)h(x), \quad x \in B, \tag{3.3}$$

where

$$m = \frac{a-b}{a+b}. \tag{3.4}$$

Proof. If h is a harmonic polynomial of homogeneous degree j, then it is a linear combination of $\cos i\omega(e^{ir} + e^{-ir})$ and $\cos i\omega(e^{ir} - e^{-ir})$ for $i = 0, \cdots, j$. Thus it suffices to prove (3.2) and (3.3).

Assume that $h(x) = \cos j\omega(e^{jr} + e^{-jr})$. Let

$$u(x) := \begin{cases} h(x) + \mathcal{S}_B^0(\nabla h \cdot n)(x), & x \in B, \\ \mathcal{S}_B^0(\nabla h \cdot n)(x), & x \in \mathbb{R}^2 \setminus \overline{B}. \end{cases}$$

Then u is the unique solution to the following problem:

$$\begin{cases} \Delta u = 0 & \text{in } B \cup (\mathbb{R}^3 \setminus \overline{B}), \\ u|_+ - u|_- = -h(x) & \text{on } \partial B, \\ \dfrac{\partial u}{\partial n}\Big|_+ - \dfrac{\partial u}{\partial n}\Big|_- = 0 & \text{on } \partial B, \\ u(x) = O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases} \tag{3.5}$$

The transmission conditions on ∂B in (3.5) come from the continuity of $\mathcal{S}_B^0(\nabla h \cdot n)$ across ∂B and the jump relation for the single layer potential.

On the other hand, elementary but tedious computations show that the function v defined by

$$v(r, \omega) = \begin{cases} \alpha \cos j\omega(e^{jr} + e^{-jr}), & r \le \rho, \\ \beta \cos j\omega e^{-jr}, & r > \rho, \end{cases}$$

where

$$\alpha = \frac{e^{j\rho} + e^{-j\rho}}{2e^{j\rho}}, \quad \beta = -(e^{2j\rho} - e^{-2j\rho}),$$

is a solution of (3.5). By the uniqueness of a solution to (3.5), we have

$$\mathcal{S}_B^0(\nabla h \cdot n)(x) = (\alpha - 1)h(x), \quad x \in B.$$

Since $e^{-2\rho} = m$, we get (3.2).

Identity (3.3) can be proved in the same way. This completes the proof.

As immediate consequences of (3.2) and (3.3), we get the following identities: for $x \in B$,

$$\begin{cases} \mathcal{S}_B^0(n_1)(x) = s_1 x_1, \quad s_1 = -\dfrac{1}{2}(1 - m), \\ \mathcal{S}_B^0(n_2)(x) = s_2 x_2, \quad s_2 = -\dfrac{1}{2}(1 + m), \\ \mathcal{S}_B^0(\nabla(x_1 x_2) \cdot n)(x) = s_3 x_1 x_2, \quad s_3 = -\dfrac{1}{2}(1 + m^2). \end{cases} \tag{3.6}$$

It then follows from (3.6) that

$$c_{kl} = \delta_{kl} s_k, \quad k, l = 1, 2.$$

We now compute $I_{kpq}(x)$. Observe that, for $x \in B$,

$$
\begin{aligned}
I_{kpq}(x) &= \frac{1}{2\pi} \int_{\partial B} \frac{(x_k - y_k)(x_p - y_p)}{|x-y|^2} n_q(y) d\sigma \\
&= \frac{1}{2\pi} \int_B \frac{\partial}{\partial y_q} \frac{(x_k - y_k)(x_p - y_p)}{|x-y|^2} dy \\
&= -\frac{1}{2\pi} \int_B \frac{\delta_{kq}(x_p - y_p) + \delta_{pq}(x_k - y_k)}{|x-y|^2} dy + \frac{1}{\pi} \int_B \frac{(x_k - y_k)(x_p - y_p)(x_q - y_q)}{|x-y|^4} dy \\
&= \delta_{kq} \mathcal{S}_B^0(n_p)(x) + \delta_{pq} \mathcal{S}_B^0(n_k)(x) + \frac{1}{\pi} \int_B \frac{(x_k - y_k)(x_p - y_p)(x_q - y_q)}{|x-y|^4} dy.
\end{aligned}
$$

Thus we have

$$
I_{kpq}(x) - \delta_{kq} s_p x_p = I_{kqp}(x) - \delta_{kp} s_q x_q. \tag{3.7}
$$

On the other hand, for $x \in B$,

$$
\begin{aligned}
I_{kpq}(x) &= x_p \frac{1}{2\pi} \int_{\partial B} \frac{x_k - y_k}{|x-y|^2} n_q(y) d\sigma(y) - \frac{1}{2\pi} \int_{\partial B} \frac{x_k - y_k}{|x-y|^2} y_p n_q(y) d\sigma(y) \\
&= x_p \frac{\partial}{\partial x_k} \mathcal{S}_B^0(n_q)(x) - \frac{\partial}{\partial x_k} \mathcal{S}_B^0(y_p n_q)(x) \\
&= \delta_{kq} s_q x_p - \frac{\partial}{\partial x_k} \mathcal{S}_B^0(y_p n_q)(x).
\end{aligned}
$$

Therefore, if $p \neq q$, then (3.6) yields

$$
\begin{aligned}
I_{kpq} + I_{kqp} &= \delta_{kq} s_q x_p + \delta_{kp} s_p x_q - \frac{\partial}{\partial x_k} \mathcal{S}_B^0(\nabla(y_1 y_2) \cdot n)(x) \\
&= \delta_{kq} s_q x_p + \delta_{kp} s_p x_q - s_3 \frac{\partial (x_1 x_2)}{\partial x_k}.
\end{aligned} \tag{3.8}
$$

Combining (3.7) and (3.8), we obtain

$$
2 I_{kpq} = \delta_{kq}(s_p + s_q) x_p + \delta_{kp}(s_p - s_q) x_q - s_3 \frac{\partial (x_1 x_2)}{\partial x_k}. \tag{3.9}
$$

If $k \neq q$, then using the simple fact that $I_{kpq} = I_{pkq}$ we can compute I_{kpq}. It now remains to compute I_{111} and I_{222}. Since

$$
\begin{aligned}
I_{111}(x) + I_{221}(x) &= \frac{1}{2\pi} \int_{\partial B} n_1(y) = 0, \\
I_{222}(x) + I_{112}(x) &= \frac{1}{2\pi} \int_{\partial B} n_2(y) = 0,
\end{aligned}
$$

we can compute I_{111} and I_{222}. In short, we get

$$
\begin{aligned}
2 I_{111} &= (-m + s_3) x_1, \quad 2 I_{222} = (m + s_3) x_2, \\
2 I_{211} &= -(1 + s_3) x_2, \quad 2 I_{122} = -(1 + s_3) x_1 \\
I_{112} + I_{121} &= (s_1 - s_3) x_2, \quad I_{212} + I_{221} = (s_2 - s_3) x_1,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&d_{11}^{11} = -m + s_3, \ d_{11}^{12} = 0, \ d_{11}^{21} = 0, \ d_{11}^{22} = -(1 + s_3), \ d_{12}^{11} = 0, \ d_{12}^{12} = s_1 - s_3, \\
&d_{12}^{21} = s_2 - s_3, \ d_{12}^{22} = 0, \ d_{22}^{11} = -(1 + s_3), \ d_{22}^{12} = 0, \ d_{22}^{21} = 0, \ d_{22}^{22} = m + s_3.
\end{aligned}
$$

We use the conventional identification

$$
(11) \to 1, \ (22) \to 2, \ (12) \to 3.
$$

Since $s_1 + s_2 = -1$, we can identify

$$
\frac{1}{2}\Big(\delta_{pq}(c_{kl} + c_{lk})\Big) \to \begin{pmatrix} s_1 & s_2 & 0 \\ s_1 & s_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} := C_1, \tag{3.10}
$$

$$\frac{1}{2}\Big(\delta_{kp}c_{ql}+\delta_{kq}c_{pl}+\delta_{lp}c_{qk}+\delta_{lq}c_{pk}\Big)\rightarrow\begin{pmatrix}2s_1 & 0 & 0\\ 0 & 2s_2 & 0\\ 0 & 0 & -1\end{pmatrix}:=C_2, \tag{3.11}$$

and

$$\frac{1}{2}\Big(d^{kl}_{pq}+d^{lk}_{pq}\Big)\rightarrow\begin{pmatrix}-m+s_3 & -1-s_3 & 0\\ -1-s_3 & m+s_3 & 0\\ 0 & 0 & m^2\end{pmatrix}:=C_3. \tag{3.12}$$

Note that in the matrices on the right-hand sides above the $3-3$ entries are $c^{12}_{12}+c^{21}_{21}$.

Let

$$\Lambda=(\widetilde{\lambda}-\lambda)(A-B)=\frac{(\widetilde{\lambda}-\lambda)}{2\mu+\lambda},\ \Theta=(\widetilde{\mu}-\mu)A,\ \Xi=(\widetilde{\mu}-\mu)B.$$

Since

$$C_B=\Lambda C_1+\Theta C_2-\Xi C_3,$$

we get from (2.4), (3.10), (3.11), and (3.12) that

$$C_B=\begin{pmatrix}(\Lambda+2\Theta)s_1-\Xi(-m+s_3) & s_2\Lambda+\Xi(1+s_3) & 0\\ s_1\Lambda+\Xi(1+s_3) & (\Lambda+2\Theta)s_2-\Xi(m+s_3) & 0\\ 0 & 0 & -\Theta-\Xi m^2\end{pmatrix}.$$

Note that

$$\begin{cases}X^{11}_{11}=X^{22}_{22}=|B|[(\widetilde{\lambda}-\lambda)+2(\widetilde{\mu}-\mu)],\\ X^{11}_{22}=X^{22}_{11}=|B|(\widetilde{\lambda}-\lambda),\\ X^{12}_{12}=|B|(\widetilde{\mu}-\mu),\\ X^{12}_{11}=X^{12}_{22}=0.\end{cases}$$

Set

$$X:=|B|\begin{pmatrix}(\widetilde{\lambda}-\lambda)+2(\widetilde{\mu}-\mu) & \widetilde{\lambda}-\lambda & 0\\ \widetilde{\lambda}-\lambda & (\widetilde{\lambda}-\lambda)+2(\widetilde{\mu}-\mu) & 0\\ 0 & 0 & \widetilde{\mu}-\mu\end{pmatrix}.$$

We obtain the following theorem.

Theorem 3.2. *Let (m^{ij}_{pq}) be the EMT for B and $\widetilde{\lambda},\widetilde{\mu},\lambda,\mu$ be respectively the Lamé constants for the ellipse B given by* (3.1) *and the background. Then we have*

$$m^{11}_{11}=|B|(\lambda+2\mu)\frac{(\tilde{\mu}-\mu)k_1[m^2-2(\kappa-1)m]-k_1(\mu+\kappa\tilde{\mu})+k_2(\kappa-1)(\tilde{\mu}-\mu)}{(\tilde{\mu}-\mu)[\mu-\kappa(\tilde{\lambda}+\tilde{\mu})]m^2+k_2(\mu+\kappa\tilde{\mu})},$$

$$m^{22}_{22}=|B|(\lambda+2\mu)\frac{(\tilde{\mu}-\mu)k_1[m^2+2(\kappa-1)m]-k_1(\mu+\kappa\tilde{\mu})+k_2(\kappa-1)(\tilde{\mu}-\mu)}{(\tilde{\mu}-\mu)[\mu-\kappa(\tilde{\lambda}+\tilde{\mu})]m^2+k_2(\mu+\kappa\tilde{\mu})},$$

$$m^{11}_{22}=|B|\frac{(\lambda+2\mu)[(\tilde{\mu}-\mu)k_1m^2+(\tilde{\lambda}-\lambda)(\tilde{\mu}+\kappa\mu)+(\tilde{\mu}-\mu)^2]}{(\tilde{\mu}-\mu)[\mu-\kappa(\tilde{\lambda}+\tilde{\mu})]m^2+k_2(\mu+\kappa\tilde{\mu})},$$

$$m^{12}_{12}=|B|\frac{\mu(\tilde{\mu}-\mu)(\kappa+1)}{(\tilde{\mu}-\mu)m^2+\mu+\kappa\tilde{\mu}},$$

where m is defined by (3.4), $|B|$ denotes the volume of B, and

$$k_1=\lambda-\tilde{\lambda}+\mu-\tilde{\mu},\ k_2=\mu+\tilde{\lambda}+\tilde{\mu},\ \kappa=\frac{\lambda+3\mu}{\lambda+\mu}.$$

In particular, if $m = 0$, *i.e.*, B is a disk, then

$$m_{22}^{11} = |B| \frac{(\lambda + 2\mu)[(\tilde{\lambda} - \lambda)(\tilde{\mu} + \kappa\mu) + (\tilde{\mu} - \mu)^2]}{(\mu + \tilde{\lambda} + \tilde{\mu})(\mu + \kappa\tilde{\mu})},$$

$$m_{12}^{12} = |B| \frac{\mu(\widetilde{\mu} - \mu)(\kappa + 1)}{\mu + \kappa\widetilde{\mu}}.$$

This is exactly the formula provided in [1].

4. Elastic Moment Tensor of Ellipsoids

We now compute the EMT of ellipsoids. To this end we compute the matrix C_B when B is the ellipsoid given by

$$\frac{x_1^2}{c_1^2} + \frac{x_2^2}{c_2^2} + \frac{x_3^2}{c_3^2} = 1. \tag{4.1}$$

Lemma 4.1. *For $i, j = 1, 2, 3,\ i \neq j$, we have*

$$\mathcal{S}_B(n_i) = s_i x_i, \tag{4.2}$$

$$\mathcal{S}_B(\nabla(x_i x_j) \cdot n) = s_{ij} x_i x_j, \tag{4.3}$$

where

$$s_i = -\frac{c_1 c_2 c_3}{2} \int_0^\infty \frac{ds}{(c_i^2 + s)\sqrt{g(s)}},$$

$$s_{ij} = -\frac{c_1 c_2 c_3 (c_i^2 + c_j^2)}{2} \int_0^\infty \frac{ds}{(c_i^2 + s)(c_j^2 + s)\sqrt{g(s)}},$$

$$g(s) = (c_1^2 + s)(c_2^2 + s)(c_3^2 + s).$$

Proof. We note that (4.2) was proved in [9]. To prove (4.3), we set, for a harmonic function h,

$$u(x) := \begin{cases} h(x) + \mathcal{S}_B(\nabla u \cdot n), & x \in B, \\ \mathcal{S}_B(\nabla h \cdot n), & x \in \mathbb{R}^3 \setminus \overline{B}. \end{cases}$$

Then u is the unique solution to the following problem:

$$\begin{cases} \Delta u = 0 & \text{in } B \bigcup (\mathbb{R}^3 \setminus \overline{B}), \\ u|_+ - u|_- = -h & \text{on } \partial B, \\ \nabla u \cdot n|_+ - \nabla u \cdot n|_- = 0 & \text{on } \partial B, \\ u(x) = O(|x|^{-2}) & \text{as } |x| \to \infty. \end{cases} \tag{4.4}$$

We now construct explicitly the solution of (4.4) using ellipsoidal coordinates. The confocal ellipsoidal coordinates $\rho,\ \mu,\ \xi$ satisfy

$$\frac{x_1^2}{c_1^2 + \rho} + \frac{x_2^2}{c_2^2 + \rho} + \frac{x_3^2}{c_3^2 + \rho} = 1,$$

$$\frac{x_1^2}{c_1^2 + \mu} + \frac{x_2^2}{c_2^2 + \mu} + \frac{x_3^2}{c_3^2 + \mu} = 1,$$

$$\frac{x_1^2}{c_1^2 + \xi} + \frac{x_2^2}{c_2^2 + \xi} + \frac{x_3^2}{c_3^2 + \xi} = 1,$$

and are subject to the conditions $-c_3^2 < \xi < -c_2^2 < \mu < -c_1^2 < \rho$. The boundary ∂B of B is the surface $\rho = 0$. Written in ellipsoidal coordinates, the Laplace operator becomes

$$\Delta u = \frac{4\sqrt{g(\rho)}}{(\rho - \mu)(\rho - \xi)} \frac{\partial}{\partial \rho}\left[\sqrt{g(\rho)} \frac{\partial u}{\partial \rho}\right] + \frac{4\sqrt{g(\rho)}}{(\mu - \rho)(\mu - \xi)} \frac{\partial}{\partial \mu}\left[\sqrt{g(\mu)} \frac{\partial u}{\partial \mu}\right] + \frac{4\sqrt{g(\xi)}}{(\xi - \mu)(\xi - \rho)} \frac{\partial}{\partial \xi}\left[\sqrt{g(\xi)} \frac{\partial u}{\partial \xi}\right].$$

Since h is harmonic, the Laplace equation of functions of the form $\phi(\rho)h$ becomes

$$\begin{aligned}
0 &= \Delta(\phi(\rho)h) = \Delta(\phi(\rho)h) - \phi\Delta h \\
&= \frac{4\sqrt{g(\rho)}}{(\rho-\mu)(\rho-\xi)}\frac{\partial}{\partial\rho}\left[\sqrt{g(\rho)}\left(h\frac{\partial\phi}{\partial\rho} + \phi\frac{\partial h}{\partial\rho}\right)\right] - \frac{4\phi\sqrt{g(\rho)}}{(\rho-\mu)(\rho-\xi)}\frac{\partial}{\partial\rho}\left(\sqrt{g(\rho)}\frac{\partial h}{\partial\rho}\right) \\
&= \frac{4\sqrt{g(\rho)}}{(\rho-\mu)(\rho-\xi)}\frac{\partial}{\partial\rho}\left(\sqrt{g(\rho)}h\frac{\partial\phi}{\partial\rho}\right) + \frac{4\sqrt{g(\rho)}}{(\rho-\mu)(\rho-\xi)}\sqrt{g(\rho)}\frac{\partial h}{\partial\rho}\frac{\partial\phi}{\partial\rho} \\
&= \frac{4\sqrt{g(\rho)}h}{(\rho-\mu)(\rho-\xi)}\left[\frac{d}{d\rho}\left(\sqrt{g(\rho)}\frac{d\phi}{d\rho}\right) + 2\frac{\sqrt{g(\rho)}}{h}\frac{\partial h}{\partial\rho}\frac{d\phi}{d\rho}\right].
\end{aligned}$$

Let $h(x) = x_i x_j$, $i \neq j$ and v be defined by

$$v(x) := \begin{cases} (\psi(0)+1)x_i x_j, & x \in B \ (\rho < 0), \\ \psi(\rho)x_i x_j, & x \in \mathbb{R}^3 \setminus \overline{B} \ (\rho \geq 0). \end{cases}$$

Then, using the identity $\dfrac{\partial x_i}{\partial\rho} = \dfrac{x_i}{2(c_i^2+\rho)}$, we can easily see that v is a solution of (4.4) for $h = x_i x_j$, $i \neq j$, if we set

$$\psi(\rho) = -\frac{c_1c_2c_3(c_i^2+c_j^2)}{2}\int_\rho^\infty \frac{1}{(c_i^2+s)(c_j^2+s)\sqrt{g(s)}}ds.$$

Therefore, we have

$$\mathcal{S}_B(\nabla(x_i x_j)\cdot n) = \psi(0)x_i x_j,$$

which completes the proof.

By the same argument as in (3.9), we have

$$\begin{aligned}
&I_{kpq} = I_{pkq}, \\
&2I_{kpq} = \delta_{kq}(s_p+s_q)x_p + \delta_{kp}(s_p-s_q)x_q - s_{pq}\frac{\partial(x_p x_q)}{\partial x_k}, \\
&\sum_{k=1}^{3} I_{kkp} = -s_p x_p.
\end{aligned}$$

Using these relations, we know that the nonzero terms in d_{pq}^{ij}, for $p \neq q$, are given by

$$\begin{aligned}
&d_{pq}^{pq} = s_p - s_{pq}, \quad d_{pq}^{qp} = s_q - s_{pq}, \\
&d_{qq}^{pp} = s_p + s_q - s_{pq}, \quad d_{pp}^{pp} = -\sum_{k\neq p} s_k + \sum_{k\neq p} s_{pk}.
\end{aligned}$$

To state our main result in this section, we consider the following 3×3 submatrices of M, C_B, X:

$$M_s := (m_{jj}^{ii}),$$

$$X_s := |B|\begin{pmatrix} (\tilde{\lambda}-\lambda)+2(\tilde{\mu}-\mu) & \tilde{\lambda}-\lambda & \tilde{\lambda}-\lambda \\ \tilde{\lambda}-\lambda & (\tilde{\lambda}-\lambda)+2(\tilde{\mu}-\mu) & \tilde{\lambda}-\lambda \\ \tilde{\lambda}-\lambda & \tilde{\lambda}-\lambda & (\tilde{\lambda}-\lambda)+2(\tilde{\mu}-\mu) \end{pmatrix}$$

and

$$C_s := \begin{pmatrix} (\Lambda+2\Theta)s_1 - \Xi t_1 & \Lambda s_2 - \Xi(s_1+s_2-s_{12}) & \Lambda s_3 - \Xi(s_1+s_3-s_{13}) \\ \Lambda s_1 - \Xi(s_1+s_2-s_{12}) & (\Lambda+2\Theta)s_2 - \Xi t_2 & \Lambda s_3 - \Xi(s_2+s_3-s_{23}) \\ \Lambda s_1 - \Xi(s_1+s_3-s_{13}) & \Lambda s_2 - \Xi(s_2+s_3-s_{23}) & (\Lambda+2\Theta)s_3 - \Xi t_3 \end{pmatrix}$$

for $t_p = -\sum_{k\neq p} s_k + \sum_{k\neq p} s_{pk}$, $p = 1, 2, 3$. Formula (2.5) yields now the following theorem.

Theorem 4.2. *Let* (m_{pq}^{ij}) *be the EMT for the ellipsoid* B *given by* (4.1) *and* $\widetilde{\lambda}, \widetilde{\mu}, \lambda, \mu$ *be the Lamé constants. We have*

$$M_s = (I - C_s)^{-1} X_s,$$

and

$$[1 - \Theta(s_1 + s_2) + \Xi(s_1 + s_2 - 2s_{12})] m_{12}^{12} = |B|(\tilde{\mu} - \mu),$$

$$[1 - \Theta(s_1 + s_3) + \Xi(s_1 + s_3 - 2s_{13})] m_{13}^{13} = |B|(\tilde{\mu} - \mu),$$

$$[1 - \Theta(s_2 + s_3) + \Xi(s_2 + s_3 - 2s_{23})] m_{23}^{23} = |B|(\tilde{\mu} - \mu),$$

while the remaining terms are all zeros.

5. Conclusion

In this paper we have explicitly computed the elastic moment tensor for ellipses in the plane and ellipsoids in three-dimensional space. Our formulae can be extended to hard elastic inclusions and holes. Our derivations are based on an elegant layer potential technique and reveal new interesting identities. The expressions of the elastic moment tensor provided in this paper can be used for developing efficient algorithms to reconstruct elastic inclusions of small volume.

References

[1] H. Ammari and H. Kang, Reconstruction of Small Inhomogeneities from Boundary Measurements, Lecture Notes in Mathematics, Volume 1846, Springer-Verlag, Berlin, 2004.

[2] H. Ammari and H. Kang, Reconstruction of elastic inclusions of small volume via dynamic measurements, *Appl. Math. Opt.*, **54** (2006), 223–235.

[3] H. Ammari, H. Kang, and M. Lim, Effective parameters of elastic composites, *Indiana Univ. J. Math.*, **55** (2006), 903–922.

[4] H. Ammari, H. Kang, G. Nakamura, and K. Tanuma, Complete asymptotic expansions of solutions of the system of elastostatics in the presence of an inclusion of small diameter and detection of an inclusion, *J. Elasticity*, **67** (2002), 97–129.

[5] L. Escauriaza and J.K. Seo, Regularity properties of solutions to transmission problems, *Trans. Amer. Math. Soc.*, **338**:1 (1993), 405–430.

[6] J.D. Eshelby, The determination of the elastic field of an ellipsoidal inclusion, and related problems, *Proc. Royal Soc. London*, Series A, **246**:1226 (1957), 376–396.

[7] V.V. Jikov, S.M. Kozlov, and O.A. Oleinik, Homogenization of Differential Operators and Integral Functionals, Springer-Verlag, Berlin, 1994.

[8] H. Kang, E. Kim, and J. Lee, Identification of Elastic Inclusions and Elastic Moment Tensors by Boundary Measurements, *Inverse Problems*, **19** (2003), 703–724.

[9] H. Kang and K. Kim, Anisotropic polarization tensors for ellipses and ellipsoids, preprint, 2005.

[10] T. Lewiński and Sokolowski, Energy change due to the appearance of cavities in elastic solids, *International J. Solids Structures*, **40** (2003), 1765–1803.

[11] V.G. Maz'ya and S.A. Nazarov, The asymptotic behavior of energy integrals under small perturbations of the boundary near corner points and conical points (in Russian), Trudy Moskovsk. Matem. Obshch. Vol. 50, English Translation: Trans. Moscow Math. Soc. (1988), 77–127.

[12] G.W. Milton, The Theory of Composites, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 2001.

[13] A.B. Movchan and S.K. Serkov, The Pólya-Szegö matrices in asymptotic models of dilute composite, *European J. Appl. Math.*, **8** (1997), 595–621.

[14] N.I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity, English translation, Noordhoff International Publishing, Leyden, 1977.