Journal of Computational Mathematics, Vol.26, No.5, 2008, 702–715.

FINITE ELEMENT AND DISCONTINUOUS GALERKIN METHOD FOR STOCHASTIC HELMHOLTZ EQUATION IN TWO- AND THREE-DIMENSIONS*

Yanzhao Cao
Department of Mathematics, Florida A&M University, Tallahassee, FL32307, USA
Email: caoy@scs.fsu.edu
Ran Zhang and Kai Zhang
Department of Mathematics, Jilin University, Changchun 130023, China
Email: zhangran@jlu.edu.cn, kzhang@jlu.edu.cn

Abstract

In this paper, we consider the finite element method and discontinuous Galerkin method for the stochastic Helmholtz equation in $\mathbb{R}^d$ $(d = 2, 3)$. Convergence analysis and error estimates are presented for the numerical solutions. The effects of the noises on the accuracy of the approximations are illustrated. Numerical experiments are carried out to verify our theoretical results.

Mathematics subject classification: 65N30, 65N15, 65C30, 60H15.
Key words: Stochastic partial differential equation, Finite element method, Discontinuous Galerkin method, Stochastic Helmholtz equation.

1. Introduction

Many physical and engineering phenomena are modeled by partial differential equations which often contain some levels of uncertainty. The advantage of modeling using these so-called stochastic partial differential equations (SPDEs) is that SPDEs are able to more fully capture the behavior of interesting phenomena; it also means that the corresponding numerical analysis of the model will require new tools to model the systems, produce the solutions, and analyze the information stored within the solutions. In the last decade, many researchers have studied different SPDEs and various numerical methods and approximation schemes for SPDEs have also been developed, analyzed, and tested [1, 4, 7, 8, 9, 10, 12, 13, 22]. In [4, 12], the analysis based on the traditional finite element method was successfully used on partial differential equations with random coefficients, using the tensor product between the deterministic and random variable spaces. Numerical methods for SPDEs with random forcing terms have also been studied in [7, 9].

In this paper, we study the following stochastic Helmholtz equation driven by an additive white noise forcing term:

$$\begin{cases} \Delta u(x) + k^2 u(x) = -f(x) - \sigma(x)\dot{W}(x), & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded convex domain in $\mathbb{R}^d$ $(d = 2, 3)$ with smooth boundary, f is a given deterministic function and $\dot{W}$ denotes the white noise. To simplify our presentation we assume that the coefficient of the white noise is $\sigma(x) \equiv 1$. Also we assume throughout the paper that

* Received June 13, 2007 / accepted November 14, 2007 /

the wave-number k is positive and bounded away from zero, i.e., $k \geq k_0 > 0$. Following the approach of [5], the existence and uniqueness of the weak solution for (1.1) can be established by converting the problem into the Hammerstein integral equation using the Green's function. Numerical studies for Helmholtz equation have been developed for various algorithms as well and we refer to [15, 17, 21] and references therein for details about the rich literature.

The goal of this work is to construct numerical solutions of (1.1) using finite element and discontinuous Galerkin approximations and derive error estimates. As pointed out in [7], the difficulty in the error analysis of finite element methods and numerical approximations for an SPDE in general is the lack of regularity of its solution. To overcome such a difficulty, we follow the approach of [1] and [7] by first discretizing the white noise and then applying standard finite element methods and discontinuous Galerkin methods to the SPDE with discretized white noise forcing terms. To the best of our knowledge, there has been no work in the literature which studies the finite element method and discontinuous Galerkin method for the stochastic Helmholtz equation in $\mathbb{R}^d$ $(d = 2, 3)$. Here we emphasize that the discontinuous Galerkin method ([19, 20]) is particularly important for two reasons.

1. For large wave-number, the standard finite element method is inadequate for solving the Helmholtz equation, especially in the three-dimensional case, because of the pollution effect of the numerical solution.

2. To solve the stochastic Helmholtz equation using, for example, the Monte Carlo method, one needs many solves for the deterministic problem. This makes the construction of an efficient deterministic solver such as the DG method absolutely essential.

The key to the error estimates is the Lipschitz type regularity properties of the Green functions in the L^2 norm. In the two-dimensional case, such an regularity result was obtained in [7]. In the three-dimensional case, a similar result was obtained in [5]. In this paper we derive a new estimate which is sharper than the one in [5] for the regularity of the Green function. As a result we obtain error estimates for the finite element and discontinuous Galerkin approximations in the 3-D case which are comparable to finite difference error estimates (see [11]).

The paper is organized as follows. In Section 2, we study the approximation of (1.1) using discretized white noises. We shall establish the estimate of the approximate solutions in H^2-norm and their error estimates in the L^2-norm. In Section 3, we study finite element approximations and discontinuous Galerkin approximations of the stochastic Helmholtz equation with discretized white noise forcing terms, and obtain the L^2 error estimates between the finite element solutions and the exact solution of (1.1). In Section 4, we present numerical simulations using the finite element method and discontinuous Galerkin method constructed in Section 3. Finally in Section 5, we conclude the paper with a few concluding remarks.

2. The Approximation Problem

In this section, we first introduce an approximate problem of (1.1) by replacing the white noise $\dot{W}$ with its piecewise constant approximation $\dot{W}^s$. Then we establish the regularity of the solution of the approximate problem and its error estimates. For the simplicity of presentation, we assume that Ω is a convex polygonal domain.

Let $\{\mathcal{T}_h\}$ be a family of triangulations on Ω consisting of simplices. For $K \in \{\mathcal{T}_h\}$, let $h_K = \text{diam}K$ and $\rho_K =$ the radius of the largest ball inscribed in K. We say an element

$K \in \mathcal{T}_h$ is σ_0-*shape regular* if

$$h_K/\rho_K \le \sigma_0$$

and $\mathcal{T}_h$ is σ_0-shape regular if all its elements are σ_0-shape regular. Here σ_0 is a positive constant. Denote $\overline{h} = \max_{K\in\mathcal{T}_h} h_K$ and $\underline{h} = \min_{K\in\mathcal{T}_h} h_K$. We say $\mathcal{T}_h$ is quasi-uniform if it is shape regular and there exists a constant $\sigma_1 > 0$ such that

$$\overline{h} \le \sigma_1 \underline{h}. \tag{2.1}$$

Write

$$\xi_K = \frac{1}{\sqrt{|K|}} \int_K 1 dW(x)$$

for each triangle $K \in \mathcal{T}_h$, where $|K|$ denotes the area of K. It is well-known that $\{\xi_K\}_{K\in\mathcal{T}_h}$ is a family of independent identically distributed normal random variables with mean 0 and variance 1 (see [18]). Then the piecewise constant approximation to $\dot{W}(x)$ is given by

$$\dot{W}^s(x) = \sum_{K\in\mathcal{T}_h} |K|^{-\frac{1}{2}} \xi_K \chi_K(x), \tag{2.2}$$

where χ_K is the characteristic function of K. It is easy to see that $\dot{W}^s \in L^2(\Omega)$. However, we have the following estimate which shows that the L^2 norm $\|\dot{W}^s\|$ of $\dot{W}^s$ is unbounded as $h \to 0$ (cf. [7]).

Lemma 2.1. *There exist positive constants C_1 and C_2 independent of h such that*

$$C_1 h^{-2} \le E(\|\dot{W}^s\|^2) \le C_2 h^{-2}. \tag{2.3}$$

Replacing $\dot{W}$ by $\dot{W}^s$ in (1.1), we have the following stochastic Helmholtz equation with discretized white noise forcing term:

$$\begin{cases} \Delta u^s(x) + k^2 u^s(x) = -f(x) - \dot{W}^s(x), & x \in \Omega, \\ u^s(x) = g(x), & x \in \partial\Omega. \end{cases} \tag{2.4}$$

Its variational form is: Find $u^s \in H_g^1(\Omega) := \{u \in H^1(\Omega) : u = g \ on \ \partial\Omega\}$ such that

$$a(u^s, v) = (F^s, v), \quad v \in H_0^1(\Omega), \tag{2.5}$$

where $F^s = f + \dot{W}^s$, $(\cdot,\cdot)$ denotes the inner product on $L^2(\Omega)$, and

$$a(u, v) = (\nabla u, \nabla v) - k^2(u, v). \tag{2.6}$$

We equip the space $H^1(\Omega)$ with the norm

$$|||u||| := (|u|_{1,\Omega}^2 + k^2\|u\|_{0,\Omega}^2)^{1/2},$$

which is obviously equivalent to the H^1-norm. In the rest of this section we shall show that (2.4) has a unique solution u^s and then establish an estimate for the error $u - u^s$.

Lemma 2.2. *Let Ω be a bounded convex domain with smooth boundary. If k^2 is not an interior eigenvalue, then there is a unique solution $u^s \in H^2(\Omega)$ of (2.4) which satisfies*

$$E(\|u^s\|_2^2) \le C_2 h^{-2}, \tag{2.7}$$

where C_2 is a positive constant independent of h.

Proof. By the standard technique, we know that there is a unique solution u^s of (2.4) which satisfies

$$|||u^s||| \le C_1 C_{fg}, \quad |u^s|_2 \le C_1\big(C_{fg}k + \|g\|_{H^{1/2}(\partial\Omega)}\big),$$

with

$$C_{fg} := \|F^s\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)},$$

for any $F^s \in L^2(\Omega)$, $g \in H^{\frac{1}{2}}(\Omega)$, where $C_1 > 0$ depends only on Ω. Notice that u^s is the weak solution of the boundary value problem

$$\begin{cases} -\Delta u^s(x) = R_h, & x \in \Omega, \\ u^s(x) = g(x), & x \in \partial\Omega, \end{cases} \tag{2.8}$$

where $R_h = f + \dot{W}^s + k^2 u^s$. Therefore, by the results of the solution regularity of (2.8), we have that $u^s \in H^2(\Omega)$ and

$$\|u^s\|_2^2 \le C\|R_h\|^2.$$

The estimate (2.7) then follows from the above inequality and (2.3).

Next we estimate the error between the weak solution u of (1.1) and its approximation u^s. Recall that u and u^s are the unique solutions of the following Hammerstein integral equations, respectively (cf. [3]):

$$u = Kf + K\dot{W} + \int_\Omega \frac{\partial G(x,y)}{\partial\nu} g(y)ds(y), \tag{2.9}$$

$$u^s = Kf + K\dot{W}^s + \int_\Omega \frac{\partial G(x,y)}{\partial\nu} g(y)ds(y), \tag{2.10}$$

where

$$K\varphi(x) = \int_\Omega G(x,y)\varphi(y)dy$$

is the convolution operator and G is the Green function of the Helmholtz equation. It is well known that

$$G(x,y) = \begin{cases} G_2(x,y) = \dfrac{1}{2\pi}\log\dfrac{1}{|x-y|} + V_2(x,y), & d=2, \\ G_3(x,y) = \dfrac{e^{ik|x-y|}}{4\pi|x-y|} + V_3(x,y), & d=3, \end{cases} \tag{2.11}$$

where $V_i = V(x,y)$, $i = 2,3$ are Lipschitz continuous functions of x and y (cf. [6]).

The following lemma regarding the regularity of the Green function G defined in (2.11) will play an important role in our error estimate between u and u^s.

Lemma 2.3. *(a) For $d = 2$, there exists a positive number C independent of $\alpha \in (0,1)$ such that*

$$\int_\Omega |G_2(x,y) - G_2(x,z)|^2 dx \le C\alpha^{-1}|y-z|^{2-\alpha}, \qquad \forall\, y,z \in \Omega. \tag{2.12}$$

(b) For $d = 3$, there exists a positive number C independent of $\gamma = \min\{3-\beta, \beta\}$ such that

$$\int_\Omega |G_3(x,y) - G_3(x,z)|^\beta dx \le C|y-z|^\gamma, \qquad \forall\, y,z \in \Omega,\ \beta \in (1,3). \tag{2.13}$$

Proof. We only prove that (2.13) holds. The proof of (2.12) is similar to that of Lemma 2 in [7]. To prove (2.13) it suffices to prove that it holds for the singular part of G_3. Let $\xi = (y+z)/2$, $r = |y-z|$, $B_\rho(x)$ denote the ball with center at x and radius as ρ. Obviously, we have

$$\int_\Omega \left|\frac{1}{|x-y|} - \frac{1}{|x-z|}\right|^\beta dx = I + II + III + IV$$
$$:= \int_{B_{\frac{r}{4}}(y)} + \int_{B_{\frac{r}{4}}(z)} + \int_{B_{5r}(\xi)\setminus B_{\frac{r}{4}}(y)\cup B_{\frac{r}{4}}(z)} + \int_{\Omega\setminus B_{5r}(\xi)} \left|\frac{1}{|x-y|} - \frac{1}{|x-z|}\right|^\beta dx.$$

Notice that for $x \in B_{\frac{r}{4}}$, $|x-z| > \frac{r}{2}$. Thus,

$$\begin{aligned} I &:= \int_{B_{\frac{r}{4}}(y)} \left|\frac{1}{|x-y|} - \frac{1}{|x-z|}\right|^\beta dx = \int_{B_{\frac{r}{4}}(y)} \frac{||x-z|-|x-y||^\beta}{|x-y|^\beta |x-z|^\beta} dx \\ &\le 2^\beta \int_{B_{\frac{r}{4}}(y)} \frac{|y-z|^\beta}{|x-y|^\beta r^\beta} dx = 2^\beta \int_{B_{\frac{r}{4}}(y)} \frac{dx}{|x-y|^\beta} \\ &\le C \int_0^{\frac{r}{4}} \frac{s^2}{s^\beta} ds \le C r^{3-\beta}. \end{aligned}$$

Similarly, we have $II \le C r^{3-\beta}$. For III, a simple calculation gives

$$\begin{aligned} III &:= \int_{B_{5r}(\xi)\setminus B_{\frac{r}{4}}(y)\cup B_{\frac{r}{4}}(z)} \left|\frac{1}{|x-y|} - \frac{1}{|x-z|}\right|^\beta dx \\ &\le \frac{C}{r^\beta} \int_{B_{5r}(\xi)\setminus B_{\frac{r}{4}}(y)\cup B_{\frac{r}{4}}(z)} 1 dx \le C r^{3-\beta}. \end{aligned}$$

To estimate IV we notice that

$$\left|\frac{|x-\xi|}{|x-y|} - 1\right| \le \frac{|\xi - y|}{|x-y|} \le \frac{r/2}{5r-(r/2)} = \frac{1}{9}.$$

Consequently,

$$\begin{aligned} \frac{8}{9}|x-y| \le |x-\xi| \le \frac{10}{9}|x-y|, \\ \frac{8}{9}|x-z| \le |x-\xi| \le \frac{10}{9}|x-z|. \end{aligned}$$

Therefore,

$$\begin{aligned} IV &:= \int_{\Omega\setminus B_{5r}(\xi)} \left|\frac{1}{|x-y|} - \frac{1}{|x-z|}\right|^\beta dx \le \int_{\Omega\setminus B_{5r}(\xi)} \frac{r^\beta}{|x-y|^\beta |x-z|^\beta} dx \\ &\le C r^\beta \int_{\Omega\setminus B_{5r}(\xi)} \frac{1}{|x-\xi|^{2\beta}} dx \le C r^\beta \int_{5r}^R \frac{s^2}{s^{2\beta}} ds \le C r^\beta (r^{3-2\beta} + R^{3-2\beta}) \\ &\le C(r^\beta + r^{3-\beta}), \end{aligned}$$

where R is a constant such that $\Omega \subset B_R(0)$. Combining all the above inequalities, we obtain the desired estimate (2.13) by setting $\gamma = \min\{3-\beta, \beta\}$. □

Remark 2.1. Setting $\alpha = \big|\log|y-z|\big|^{-1}$ and $\beta = 2$ in (2.12) and (2.13) respectively, we obtain

$$\int_\Omega |G_2(x,y)-G_2(x,z)|^2 dx \le C|y-z|^2|\log|y-z||, \qquad \forall\, y,z\in\Omega, \tag{2.14}$$

$$\int_\Omega |G_3(x,y)-G_3(x,z)|^2 dx \le C|y-z|, \qquad \forall\, y,z\in\Omega. \tag{2.15}$$

Now we are in a position to establish an error estimate between u and u^s.

Theorem 2.1. *Let u and u^s be the solution of* (1.1) *and* (2.4) *respectively. We have*

$$E(\|u-u^s\|^2) = \begin{cases} Ch^2|\log h|, & d=2, \\ Ch, & d=3, \end{cases} \tag{2.16}$$

where C is a positive constant independent of u and h.

Proof. Subtracting (2.9) from (2.10), we obtain

$$u - u^s = K\dot{W} - K\dot{W}^s. \tag{2.17}$$

Using Ito's isometry gives

$$\begin{aligned}
E(\|K\dot{W} - K\dot{W}^s\|^2) &= E\Big(\int_\Omega \Big[\int_\Omega G(x,y)dW(y) - \int_\Omega G(x,y)dW^s(y)\Big]^2 dx\Big) \\
&= E\Big(\int_\Omega \Big[\sum_{K\in\mathcal{T}_h}\int_K G(x,y)dW(y) - |K|^{-1}\sum_{K\in\mathcal{T}_h}\int_K G(x,z)dz\int_K 1dW(y)\Big]^2 dx\Big) \\
&= E\Big(\int_\Omega \Big[\sum_{K\in\mathcal{T}_h}\int_K\int_K |K|^{-1}(G(x,y)-G(x,z))dzdW(y)\Big]^2 dx\Big) \\
&= \int_\Omega \Big(\sum_{K\in\mathcal{T}_h}\int_K \Big[|K|^{-1}\int_K (G(x,y)-G(x,z))dz\Big]^2 dy\Big)dx.
\end{aligned}$$

It follows from the Hölder inequality that

$$\begin{aligned}
E(\|K\dot{W} - K\dot{W}^s\|^2) &\le \int_\Omega \Big(\sum_{K\in\mathcal{T}_h}|K|^{-1}\int_K\int_K (G(x,y)-G(x,z))^2 dzdy\Big)dx \\
&= \sum_{K\in\mathcal{T}_h}|K|^{-1}\int_K\int_K\int_\Omega (G(x,y)-G(x,z))^2 dxdzdy.
\end{aligned} \tag{2.18}$$

Then the desired result (2.16) follows from (2.17), (2.18) and Remark 2.1. □

3. Finite Element and Discontinuous Galerkin Method

In this section, we consider the finite element and discontinuous Galerkin approximations of variational problem (2.5) for low wave-numbers as well as high wave-numbers and establish their error estimates.

3.1. Finite element methods

Let V_h be a family of linear finite element subspaces of $H^1_g(\Omega)$ with respect to the triangulation $\{\mathcal{T}_h\}$ specified in Section 2. Then the finite element approximation to (2.4) is: Find $u^s_h \in V_h$ such that

$$(\nabla u^s_h, \nabla v) - k^2(u^s_h, v) = (f + \dot{W}^s, v), \qquad \forall\ v \in H^1_0(\Omega). \tag{3.1}$$

We assume the approximation property for piecewise linear finite elements ([14]): There exists a constant C depending only on Ω and the minimal angles in the triangulation such that, for all $u^s \in H^2(\Omega)$, there holds

$$\inf_{\chi \in V_h} (\|u^s - \chi\| + h|u^s - \chi|_1) \le C(\Omega)h^2\big(|u^s|_2 + (1+k)|||u^s|||\big). \tag{3.2}$$

The approximate variational problem (3.1) has a unique solution (by the Gårding inequality) and we have the following lemma on the error estimate for $u - u^s_h$.

Theorem 3.1. *Let Ω be a bounded convex domain with smooth boundary, u and u^s_h be the solution of* (1.1) *and* (3.1) *respectively. If the mesh satisfies $hk^2 \lesssim 1$ and k^2 is not an eigenvalue of $-\Delta$, then we have*

$$E(\|u - u^s_h\|^2) = \begin{cases} Ch^2|\log h| + Ch^2k^2, & d = 2, \\ Ch + Ch^2k^2, & d = 3, \end{cases} \tag{3.3}$$

where C is a positive constant independent of u and h.

Proof. By a standard argument, under the assumption $(1 + k^2)h < C$, the inf-sup condition

$$\inf_{u \in V_h \setminus \{0\}} \sup_{v \in V_h \setminus \{0\}} \frac{\Re e\ a(u,v)}{|||u|||\ |||v|||} \ge \frac{C}{1+k}$$

holds. Also, the finite element solution u^s_h satisfies

$$|||u^s - u^s_h||| \le C \inf_{\chi \in V_h} |||u^s - \chi||| \le Chk(\|F^s\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}),$$

where C only depends on k_0. Using the Aubin-Nitsche technique, we can get the following L^2-estimate:

$$\|u^s - u^s_h\|_{L^2(\Omega)} \le Chk|||u^s - u^s_h||| \le Ch^2k^2,$$

This, together with Theorem 2.1, implies the conclusion of the theorem. □

3.2. Discontinuous Galerkin method

The standard finite element method provides a quasi-optimal numerical approximation for elliptic boundary value problems in the sense that the accuracy of the numerical solution differs only by a constant multiple from the best approximation of the finite element space. While this property guarantees good performances of computations at any mesh resolution for the Laplace operator, it can not be preserved for the Helmholtz equation. The reason is that the second term in (3.3) increases with the wave-number k. This phenomenon is well-known as the pollution effect. It is due to numerical dispersion errors. FEM is able to cope with large wave-numbers only if the mesh resolution is also increased suitably (under the constraint $hk^2 \lesssim 1$). In order to avoid the pollution effect, numerous discretization techniques have been developed. They include the weak element method for the Helmholtz equation, the Galerkin/least-squares method, the quasi-stabilized finite element method, the partition of unity method, the residual-free bubbles for the Helmholtz equation, the ultra-weak variational method, the least squares method. Recently a discontinuous Galerkin method has been introduced in numerical simulations by Alvarez et al. (cf. [2]).

Here we shall analyze discontinuous Galerkin (DG) discretizations of the stochastic Helmholtz equation and give an error estimate. Let V_h and $\mathcal{T}_h$ be specified in Section 3.1. We denote by $\mathcal{E}_I$ the union of all interior faces of $\mathcal{T}_h$, by $\mathcal{E}_B$ the union of all boundary faces, and set $\mathcal{E} = \mathcal{E}_I \cup \mathcal{E}_B$. Consider an interior face e shared by two elements K^+ and K^-. Denoting by $v^\pm$ and $\mathbf{r}^\pm$ the traces on $K^\pm$ of functions v and $\mathbf{r}$ that are smooth in $K^\pm$, we define the averages and jumps of v and $\mathbf{r}$ across e by

$$\overline{v} = (v^+ + v^-)/2, \quad [v] = v^+\mathbf{n}_{K^+} + v^-\mathbf{n}_{K^-}, \quad \overline{\mathbf{r}} = (\mathbf{r}^+ + \mathbf{r}^-)/2, \quad [\mathbf{r}] = \mathbf{r}^+\mathbf{n}_{K^+} + \mathbf{r}^-\mathbf{n}_{K^-}.$$

For v belonging to $V(h) := V_h + H^1(\Omega)$, we define $\mathcal{L}(v) \in (V_h)^3$ by

$$\int_\Omega \mathcal{L}(v)\cdot \mathbf{r}dx = \int_{\mathcal{E}_I} (\overline{\mathbf{r}} - b[\mathbf{r}])\cdot [v]ds + \int_{\mathcal{E}_B} v\mathbf{r}\cdot \mathbf{n}ds, \quad \forall\, \mathbf{r} \in (V_h)^3$$

with parameters b to be properly chosen. Then the DG approximation to (2.4) is: Find $u_h^s \in V_h$ such that

$$\mathcal{B}_h(u_h^s, \chi) - k^2(u_h^s, \chi) = (f + \dot{W}^s, \chi), \qquad \forall\, \chi \in H_0^1(\Omega), \tag{3.4}$$

where

$$\mathcal{B}_h(u,\chi) = \int_\Omega (\nabla u - \mathcal{L}(u))\cdot(\nabla\chi - \mathcal{L}(\chi))dx.$$

We follow [16] to define the DG norm and the weighted DG norm as

$$\|v\|_{DG}^2 = \|\nabla_h v\|_{0,\Omega}^2 + \|h^{-1/2}[[v]]_N\|_{0,\mathcal{E}_h}^2, \quad |||v|||_{DG}^2 = \|v\|_{DG}^2 + k^2\|v\|_{0,\Omega}^2.$$

If u^s is the solution to (2.4), the residual of the DG formulation is defined by

$$\mathcal{R}_h(u,\chi) = (f,\chi) - \mathcal{B}_h(u,\chi) + k^2(u,\chi), \quad \forall \chi \in V_h.$$

We state our main result of this section as follows:

Theorem 3.2. *Let u and u_h^s be the solution of* (1.1) *and* (3.4)*, respectively. If the mesh satisfies $hk^2(1+hk) \gtrsim 1$ and h sufficiently small, then*

$$E(\|u - u_h^s\|^2) = \begin{cases} Ch^2|\log h| + Ch^2k^2, & d=2, \\ Ch + Ch^2k^2, & d=3, \end{cases} \tag{3.5}$$

where $C>0$ is a constant independent of h and k.

Proof. Let u^s and u_h^s be the solution to (2.4) and (3.4) respectively. We have

$$\begin{aligned} &|||u^s - u_h^s|||_{DG} \\ \le& C\left(\inf_{\chi\in V_h} |||u^s - \chi|||_{DG} + k^2 \sup_{0\ne\chi\in V_h} \frac{(u^s - u_h^s, \chi)}{\|\chi\|_{0,\Omega}} + \sup_{0\ne\chi\in V_h} \frac{\mathcal{R}_h(u^s,\chi)}{\|\chi\|_{DG}}\right), \end{aligned} \tag{3.6}$$

with $C > 0$ independent of h and k. Next, we consider the error estimates of the three terms on the right-hand side of (3.6) separately. The first term is just the best approximation error. By the standard Aubin-Nitsche technique, we can get following estimate for the second term,

$$(u^s - u_h^s, \chi) \le Ch \left((1+kh)|||u^s - u_h^s|||_{DG} + \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|u^s\|_{2,K}^2 \right)^{1/2} \right) \|\chi\|_{0,\Omega}. \tag{3.7}$$

For the third term, by the DG method for the residual term [16], we have

$$\mathcal{R}_h(u^s, \chi) \le C \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|u^s\|_{2,K}^2 \right)^{1/2} \|h^{-1/2}[[\chi]]_N\|_{0,\mathcal{E}_h}^2 \quad \forall\, \chi \in V_h. \tag{3.8}$$

Insert the estimate (3.7) into (3.6) and subtract $Ch\|u^s - u_h^s\|_{DG}$ from both sides of (3.6). Using the best approximation error and (3.8), we obtain

$$|||u^s - u_h^s|||_{DG} \le C \left(\sum_{K \in \mathcal{T}_h} h_K^2 \|u^s\|_{2,K}^2 \right)^{1/2},$$

provided that $hk^2(1+hk) \gtrsim 1$. The above inequality and Theorem 2.1 lead to (3.5). □

4. Numerical Results for Some Model Equations

In this section, we present numerical examples to demonstrate our theoretical results in the previous section. We will consider both the finite element method and the discontinuous Galerkin method.

The Gaussian random process $\dot{W}^s$ shall be simulated using the random number generator. Theoretically, the number of samples M should be chosen so that the error generated by the Monte Carlo method is in the same magnitude of the errors generated by the finite element approximation and the discontinuous Galerkin method. Although for the linear problem, $E(u_h^s)$ is the finite element or discontinuous Galerkin approximation of the deterministic solution, we shall evaluate $E(u_h^s)$ by using the Monte Carlo method to examine

$$e_1(h) = \|E(u) - E(u_h^s)\|,$$

to ensure that we have used enough samples. We also employ the following type of errors

$$e_2(h) = |E(\|u\|^2) - E(\|u_h^s\|^2)|$$

to check the error estimates for the finite element method and the discontinuous Galerkin method, respectively. Notice that it is impossible to evaluate $E(\|u - u_h^s\|)$ since it is impossible obtain an explicit expression for u.

Example 1. We test the performance of the finite element method and the discontinuous Galerkin method by solving the following problem on domain $\Omega = [0,1]^2$.

$$\begin{cases} \Delta u(x,y) + k^2 u(x,y) = (k^2 - 2\pi^2)\sin \pi x \sin \pi y + \dot{W}(x,y), & (x,y) \in \Omega, \\ u(x,y) = 0, & (x,y) \in \partial\Omega. \end{cases} \tag{4.1}$$

Table 4.1: FEM for (4.1) with $k = 1$ on unit square: Test 1.

h	e_1	rate	$E(\|u_h^s\|^2)$	e_2	rate
1/4	6.57e-2		0.18531	6.60e-2	
1/8	2.01e-2	1.71	0.23882	1.24e-2	2.41
1/16	5.79e-3	1.80	0.24883	2.40e-3	2.37
1/32	1.57e-3	1.88	0.25061	6.19e-3	1.95
1/64	4.09e-4	1.94	0.25143	2.01e-4	1.62

Table 4.2: FEM for (4.1) with $k = 1$ on unit square: Test 2.

h	e_1	rate	$E(\|u_h^s\|^2)$	e_2	rate
1/4	7.90e-2		0.18042	7.08e-2	
1/8	2.31e-2	1.77	0.23692	1.43e-2	2.31
1/16	6.62e-3	1.80	0.24919	2.04e-3	2.81
1/32	1.82e-3	1.86	0.25077	4.59e-3	2.15
1/64	4.81e-4	1.92	0.25109	1.39e-4	1.72

Table 4.3: DG for (4.1) with $k = 10$ on unit square: Test 3.

h	e_1	rate	$E(\|u_h^s\|^2)$	e_2	rate
1/4	7.34e-2		0.18931	6.20e-2	
1/8	2.08e-2	1.82	0.23811	1.31e-2	2.23
1/16	5.79e-3	1.84	0.24886	2.37e-3	2.46
1/32	1.57e-3	1.88	0.25064	5.89e-3	2.01
1/64	4.15e-4	1.92	0.25139	1.61e-4	1.87

Table 4.4: DG for (4.1) with $k = 10$ on unit square: Test 4.

h	e_1	rate	$E(\|u_h^s\|^2)$	e_2	rate
1/4	8.35e-2		0.18742	6.38e-2	
1/8	2.41e-2	1.79	0.23741	1.38e-2	2.21
1/16	6.62e-3	1.86	0.24770	3.53e-3	1.97
1/32	1.77e-3	1.90	0.25037	8.59e-3	2.03
1/64	4.61e-4	1.94	0.25095	2.79e-4	1.62

In the absence of the white noise, the exact solution of the above problem is $\bar{u} = \bar{u}(x, y) = \sin \pi x \sin \pi y$. Obviously

$$E(u) = \bar{u}, \quad \|E(u)\|^2 = \frac{1}{4}.$$

Recall that (cf. [6])

$$G(x, y; \xi, \eta) = \frac{4}{\pi^2} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{(p+q)^2} \sin p\pi x \sin p\pi \xi \sin q\pi y \sin q\pi \eta.$$

It is easy to see from Ito's isometry that

$$E(\|u\|^2) = \|E(u)\|^2 + \int_\Omega \int_\Omega G(x, y; \xi, \eta)^2 dx dy d\xi d\eta.$$

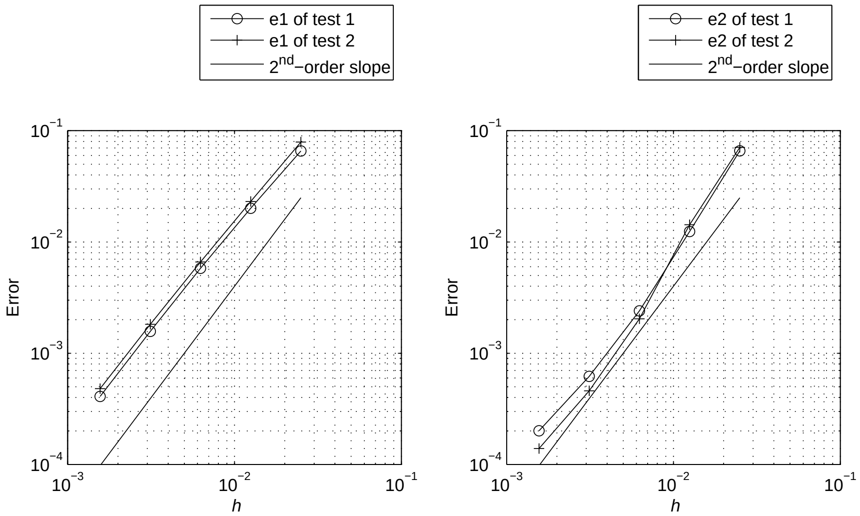

Fig. 4.1. Example 1: Convergence results of FE method for (4.1) when $k = 1$ on unit square.

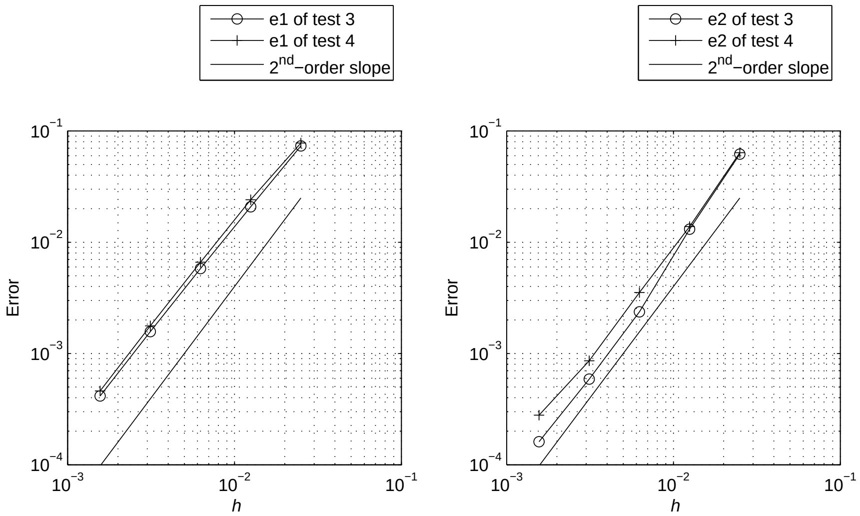

Fig. 4.2. Example 1: Convergence results of DG method for (4.1) when $k = 10$ on unit square.

A simple calculation gives

$$\begin{aligned} E(\|u\|^2) &= 0.25 + \frac{16}{\pi^4}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{1}{(p+q)^4}(\frac{1}{2})^4 \\ &= 0.25 + \frac{1}{\pi^4}\sum_{n=2}^{\infty}\sum_{p+q=n}\frac{1}{n^4} \\ &= 0.25 + \frac{1}{\pi^4}\sum_{n=2}^{\infty}\frac{n-1}{n^4} \approx 0.251229. \end{aligned}$$

The computational results of the finite element approximations for (4.1) with $k = 1$ on the unit square are displayed in Tables 4.1 and 4.2. Fig. 4.1 shows the corresponding convergence rates. The computational results of the discontinuous Galerkin method for (4.1) with $k = 10$ on the unit square are displayed in Tables 4.3 and 4.4 and Fig. 4.2 show the corresponding

Table 4.5: FEM for (4.2) with $k = 1$ on unit cube: Test 5.

h	e_1	rate	$E(\|u_h^s\|^2)$	e_2	rate
1/4	6.57e-2		0.07531	5.07e-2	
1/8	1.81e-2	1.86	0.10982	1.62e-2	1.64
1/16	4.78e-3	1.92	0.11983	6.21e-3	1.38
1/32	1.24e-3	1.94	0.12341	2.63e-3	1.24

Table 4.6: DG method for (4.2) with $k = 10$ on unit cube: Test 6.

h	e_1	rate	$E(\|u_h^s\|^2)$	e_2	rate
1/4	7.14e-2		0.08141	4.46e-2	
1/8	1.91e-2	1.90	0.10881	1.72e-2	1.37
1/16	4.98e-3	1.92	0.11855	7.49e-3	1.20
1/32	1.29e-3	1.95	0.12895	2.91e-3	1.36

convergence rates. The the second and third columns of the tables show that the rate of convergence for $E(u_h^s)$ is of order 2 as expected, which implies that our sample sizes are good enough to ensure the accuracy of the Monte Carlo method. We point out that the numerical results of using the finite element method for $k = 10$ are not feasible.

Example 2. In this example, we also test the performance of the finite element method and the discontinuous Galerkin method for solving the following problem on unit cube $\Omega = [0, 1]^3$,

$$\begin{cases} \Delta u(x,y,z) + k^2 u(x,y,z) = (k^2 - 3\pi^2)\sin \pi x \sin \pi y \sin \pi z + \dot{W}(x,y,z), & (x,y,z) \in \Omega, \\ u(x,y,z) = 0, & (x,y,z) \in \partial\Omega. \end{cases} \tag{4.2}$$

The exact solution of the above problem is $u(x,y,z) = \sin \pi x \sin \pi y \sin \pi z$ in the absence of the white noise. We have that $E(u) = u$, $\|E(u)\|^2 = \frac{1}{8}$. Recall that (cf. [6])

$$G(x,y,z;\xi,\eta,\zeta) = \frac{8}{\pi^2} \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{(k+p+q)^2} \sin k\pi x \sin k\pi\xi \sin p\pi y \sin p\pi\eta \sin q\pi z \sin q\pi\zeta.$$

It is easy to see from Ito's isometry that

$$E(\|u\|^2) = \|E(u)\|^2 + \int_\Omega \int_\Omega G(x,y,z;\xi,\eta,\zeta)^2 dxdydzd\xi d\eta d\zeta.$$

By a simple calculation, we obtain

$$\begin{aligned} E(\|u\|^2) &= 0.125 + \frac{64}{\pi^4} \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{(k+p+q)^4} (\frac{1}{2})^6 \\ &= 0.125 + \frac{1}{\pi^4} \sum_{n=3}^{\infty} \sum_{k+p+q=n} \frac{1}{n^4} \\ &= 0.125 + \frac{1}{\pi^4} \sum_{n=3}^{\infty} \frac{(n-1)(n-2)}{2n^4} \approx 0.1260438. \end{aligned}$$

The computational results of the finite element and the discontinuous Galerkin method for (4.2) with $k = 1$ and $k = 10$ on the unit cubic are displayed in Tables 4.5 and 4.6, respectively.

5. Conclusion

We have constructed numerical solutions for the stochastic Helmholtz equation driven by white noise forcing terms using the finite element method and the discontinuous Galerkin method in $\mathbb{R}^d$ $(d = 2, 3)$. We obtained error estimates under the assumptions that the domain is bounded and convex with smooth boundary, not just a rectangle, which is the main advantage of the finite element and discontinuous Galerkin method over other methods such as finite difference methods and spectral finite element methods. Results of the numerical experiments are provided to validate our theoretical analysis.

Acknowledgments. The first author is supported by NSF under grant number 0609918 and AFOSR under grant numbers FA9550-06-1-0234 and FA9550-07-1-0154. The second author is supported by NSFC (10671082, 10626026, 10471054). The third author is supported by NNSF (No. 10701039 of China) and 985 program of Jilin University.

References

[1] E.J. Allen, S.J. Novosel and Z. Zhang, Difference approximation of some linear stochastic partial differential equations, *Stochastics and Stochastics Reports,* **64** (1998), 117-142.

[2] G.B. Alvarez, A.F.D. Loula, E.G. Dutra and F.A. Rochinha, A discontinuous finite element formulation for Helmholtz equation, *Comput. Method. Appl.,* **195**:33-36 (2006), 4018-4035.

[3] S. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods, Springer, Berlin, Heidelberg, New York, 1994.

[4] I. Babuska, R. Tempone and G. Zouraris, Galerkin fnite element approximations of stochastic elliptic partial differential equations, *SIAM J. Numer. Anal.*, **42** (2004), 800-825.

[5] R. Buckdahn and E. Pardoux, Monotonicity methods for white noise driven quasi-linear PDEs, *Progr. Probab.,* **22** (1990), 219-233.

[6] R. Courant and D. Hilbert, Methods of Mathematical Physics, Volume 1, Interscience, New York, 1953.

[7] Y.Z. Cao, H.T. Yang and L. Yin, Finite element methods for semilinear elliptic stochastic partial differential equations, *Numer. Math.,* **106** (2007), 181-198.

[8] Y.Z. Cao, R. Zhang and K. Zhang, Finite element method and discontinuous Galerkin method for stochastic scattering problem of Helmholtz type in $\mathbb{R}^d$ $(d = 2, 3)$. Preprint.

[9] Q. Du and T.Y. Zhang, Numerical approximation of some linear stochastic partial differential equations driven by special additive noise, *SIAM J. Numer. Anal.,* **4** (2002), 1421-1445.

[10] J.-F. Feng, G.-Y. Lei and M.-P. Qian, Second-order methods for solving stochastic differential equations, *J. Comput. Math.,* **10**:4 (1992), 376-387.

[11] I. Gyöngy and T. Martinez, On the approximation of solutions of stochastic partial differential equations of elliptic type, *Stochastics,* **78**:4 (2006), 213-231.

[12] A. Keese and H. Matthies, Parallel computation of stochastic groundwater flow, NIC Symposium 2004, Proceedings, John von Neumann Institute for Computing, Julich, NIC series, **20** (2003), 399-408.

[13] P. Kloeden and E. Platen, Numerical Solution of Stochastic Differential Equations, Springer-Verlag, New York, 1992.

[14] J.M. Melenk, On Generalized Finite Element Methods, PhD Thesis, University of Maryland, College Park, 1995.

[15] F. Natterer and O. Klyubina, Initial value techniques for the Helmholtz and Maxwell equations, *J. Comput. Math.*, **25**:3 (2007), 368-373.

[16] I. Perugia and D. Schötzau, An hp-analysis of the local discontinuous Galerkin method for diffusion problems, *J. Sci. Comput.,* **17** (2002), 561-571.

[17] W.J. Tang, H.Y. Fu and L.J. Shen, The numerical solution of first kind integral equation for the Helmholtz equation on smooth open arcs, *J. Comput. Math.,* **19** (2001), 489-500.

[18] J.B. Walsh, An Introduction to Stochastic Partial Differential Equations, Lecture Notes in Mathematics 1180, Springer-Verlag, (1986), 265-439.

[19] Y. Xu and C.-W. Shu, Local discontinuous Galerkin methods for three classes of nonlinear wave equations, *J. Comput. Math.,* **22**:2 (2004), 250-274.

[20] J.M. Yang and Y.P. Chen, A unified a posteriori error analysis for discontinuous Galerkin approximations of reactive transport equations, *J. Comput. Math.,* **24**:3 (2006), 425-434.

[21] L.A. Ying, Infinite element method for the exterior problems of the Helmholtz equations, *J. Comput. Math.,* **18**:6 (2000), 657-672.

[22] K. Zhang, R. Zhang, Y.G Yin, S. Yu, Domain decomposition methods for linear and semilinear elliptic stochastic partial differential equations, To appear in *Appl. Math. Comput.*