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SUPERCONVERGENCE ANALYSIS OF FINITE ELEMENT METHODS FOR OPTIMAL CONTROL PROBLEMS OF THE STATIONARY BÉNARD TYPE*

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Abstract

In this paper, we consider the finite element approximation of the distributed optimal control problems of the stationary Bénard type under the pointwise control constraint. The states and the co-states are approximated by polynomial functions of lowest-order mixed finite element space or piecewise linear functions and the control is approximated by piecewise constant functions. We give the superconvergence analysis for the control; it is proved that the approximation has a second-order rate of convergence. We further give the superconvergence analysis for the states and the co-states. Then we derive error estimates in L^∞-norm and optimal error estimates in L^2-norm.

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1. Introduction

The control of viscous flow for the purpose of achieving some desired objective is crucial to many technological and scientific applications. The Boussinesq approximation of the Navier-Stokes system is frequently used as mathematical model for fluid flow in semiconductor melts. In many crystal growth technics, such as Czochralski growth and zone-melting technics, the behavior of the flow has considerable impact on the crystal quality. It is therefore quite natural to establish flow conditions that guarantee desired crystal properties. As control actions, they include distributed forcing, distributed heating, and others. For example, the control of vorticity has significant applications in science and engineering such as control of turbulence and control of crystal growth process.

Considerable progress has been made in mathematics, physics and computation of the optimal control problems for the viscous flow; see [1, 2, 9, 12, 14, 15] and references therein. Optimal control problems for the thermally coupled incompressible Navier-Stokes equation by Neumann and Dirichlet boundary heat controls were considered in [12, 15]. Also, the time dependent problems were considered in the literature. In this article, we consider the Bénard problem whose state is governed by the Boussinesq equations, which is crucial to many technological and scientific applications. Without the control constraint, the analysis of approximation about optimal control of the stationary Bénard problem was considered in [20], and it uses the

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gradient iterative method to solve the discretized equations. For the constrained control case, there seems to be little work on this problem. This paper is concerned with the finite element approximation and error analysis of the constrained optimal control problem of the stationary Bénard problem:

$$(\mathcal{P}) \quad \min_{Q\in K} \quad J(Q) = \{\frac{1}{2}\|\mathbf{u}-\mathbf{U}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\alpha}{2}\|Q\|^2_{0,\Omega}\},$$

subject to the Boussinesq system:

$$\begin{aligned} &(a) \quad -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = T\mathbf{g} + f \quad in\ \Omega,\\ &(b) \quad \nabla\cdot\mathbf{u} = 0 \quad in\ \Omega,\\ &(c) \quad -\kappa\Delta T + \mathbf{u}\cdot\nabla T = Q \quad in\ \Omega,\\ &(d) \quad \mathbf{u} = 0 \quad T = 0 \quad on\ \partial\Omega \end{aligned} \tag{1.1}$$

and subject to the control constraint

$$K = \{Q \in L^2(\Omega) : Q(x) \geq 0;\ a.e.\ x \in \Omega\}, \tag{1.2}$$

where Ω is a regular bounded and convex open set in $\mathbb{R}^n$ ($n = 2$, or 3), with $\partial\Omega \in C^{1,1}$. $\mathbf{u}, p, T$ denote the velocity, pressure and temperature fields, respectively, f is a body force, and control Q. The vector $\mathbf{g}$ is in the direction of gravitational acceleration and $\kappa > 0$ the thermal conductivity parameter. In this paper we only consider, for the simplicity, the case where κ is constant. Assume $\nu > 0$ is the kinematic viscosity.

The optimal control problems $(\mathcal{P})$ are to seek the state variables $(\mathbf{u}, p, T)$ and Q such that the functional J is minimized subject to (1.1) where $\mathbf{U}$ is some desired velocity fields. The physics objective of the minimization problem is to match a desired flow field by adjusting the distributed control Q.

Approximation properties of the optimal control problems have long been investigated in the past years. For some classic work, we refer to Falk [10], Geveci [11] and Malanowski [26]. Theory and numerical results for elliptic control problems have been known for a long time, and can be found, for example, in Casas, Mateos, and Tröltzsch [5] or Casas and Tröltzsch [7], [24] and [25]. However, new discretization concepts have been developed in recent years. The variational approach by Hinze [16] and the superconvergence approach of Meyer and Rösch [27] can achieve approximation order h^2 in the L^2-norm using the piecewise constant control approximation for some simpler linear optimal control problems. However there seems to exist few known result on the analysis of the above control problem, which is a coupled nonlinear control problem.

In this work we show that the method cited above can be adapted to the Boussinesq equations. Here the control is discretized by piecewise constant functions. Clearly, the optimal approximation order of the control is expected to be h. However, we will show a superconvergence result that improves the order to h^2 only assuming first order global regularity. We will show the state $\mathbf{u}$ and the related co-state have the approximation order of h^2 in the L^2-norm.

The paper is organized as follows. In Section 2, we give some notations and assumptions that will be used throughout the paper. In Section 3, we will discuss the finite element approximation of the optimal control problem. In Section 4, the main results will be given and the proof of the superconvergence results will be presented in Section 5.

2. Notations and Preliminaries

Using the classical techniques, it can be proved that the optimal control problem has at least one solution. The reader is referred to [19] for the details.

From [20], we have the following optimality conditions for the optimal control problem $(\mathcal{P})$:

$$\begin{aligned} &(a)\quad -\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=T\mathbf{g}+f \quad in\ \Omega,\\ &(b)\quad \nabla\cdot\mathbf{u}=0 \quad in\ \Omega,\\ &(c)\quad -\kappa\Delta T+\mathbf{u}\cdot\nabla T=Q \quad in\ \Omega,\\ &(d)\quad \mathbf{u}=0 \quad T=0 \quad on\ \partial\Omega \end{aligned} \tag{2.1}$$

coupled with the co-state equations and variational inequality:

$$\begin{aligned} &(a)\quad -\nu\Delta\mathbf{w}-(\mathbf{u}\cdot\nabla)\mathbf{w}+\nabla\mathbf{u}^{tr}\mathbf{w}-\nabla\sigma+\varphi\nabla T=\mathbf{u}-\mathbf{U} \quad in\ \Omega,\\ &(b)\quad \nabla\cdot\mathbf{w}=0 \quad in\ \Omega,\\ &(c)\quad -\kappa\Delta\phi-\mathbf{u}\cdot\nabla\varphi=\mathbf{w}\cdot\mathbf{g} \quad in\ \Omega,\\ &(d)\quad \mathbf{w}=0 \quad \varphi=0 \quad on\ \partial\Omega,\\ &(e)\quad \int_\Omega(\alpha Q+\varphi)(P-Q)\,dx\ge 0 \quad \forall\ P\in K. \end{aligned} \tag{2.2}$$

To consider the weak formulations of Eqs. (2.1) and (2.2), we need to introduce some function spaces and the bilinear and trilinear forms. In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the norm $\|\cdot\|_{m,q,\Omega}$ and the seminorm $|\cdot|_{m,q,\Omega}$. We denote $W^{m,2}(\Omega)$ $(W_0^{m,2}(\Omega))$ by $H^m(\Omega)$ $(H_0^m(\Omega))$ with the norm $\|\cdot\|_{m,\Omega}$ and the semi-norm $|\cdot|_{m,\Omega}$. For vector-valued functions and spaces of vector-valued functions, which are indicated by boldface, we define the Sobolev space $\mathbf{H}^m(\Omega)$,

$$\mathbf{H}^m(\Omega)=\{\mathbf{u}=(u_1,\cdots,u_n)|\ u_i\in H^m(\Omega),\ i=1,\cdots,n\},$$

and its associated norm $\|\cdot\|_{\mathbf{H}^m(\Omega)}$ is given by

$$\|\mathbf{u}\|^2_{\mathbf{H}^m(\Omega)}=\sum_{i=1}^n\|u_i\|^2_{H^m(\Omega)}.$$

We also define the subspaces

$$L_0^2(\Omega)=\Big\{f\in L^2(\Omega):\ \int_\Omega f\,dx=0\Big\},\quad \mathbf{H}_0^1(\Omega)=\{\mathbf{u}\in\mathbf{H}^1(\Omega);\ \mathbf{u}=0\ \ on\ \partial\Omega\}.$$

Then we introduce the bilinear and trilinear forms, for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{H}^1(\Omega)$, $T,S\in H^1(\Omega)$ and $q\in L_0^2(\Omega)$,

$$\begin{aligned} a_0(\mathbf{u},\mathbf{v})&=\int_\Omega\nu\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx,\quad a_1(T,S)=\int_\Omega\kappa\nabla T\cdot\nabla S\,dx,\\ c_0(\mathbf{u},\mathbf{v},\mathbf{w})&=\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx,\quad c_1(\mathbf{u},T,S)=\int_\Omega\mathbf{u}\cdot\nabla TS\,dx, \end{aligned}$$

and

$$b(\mathbf{v},q)=-\int_\Omega q\nabla\cdot\mathbf{v}\,dx,\quad d(T,\mathbf{v})=\int_\Omega T\mathbf{g}\cdot\mathbf{v}\,dx.$$

Moreover we assume that $b(\mathbf{v}, q)$ satisfies the *inf-sup* condition, i.e., there exists a constant $\beta > 0$ such that

$$\inf_{0\neq q\in L_0^2(\Omega)} \sup_{\mathbf{0}\neq \mathbf{v}\in \mathbf{H}_0^1(\Omega)} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{\mathbf{H}^1}\|q\|_{L^2}} \geq \beta. \tag{2.3}$$

Then, we have the weak formulation: seek $(\mathbf{u}, p, T, \mathbf{w}, \sigma, \varphi, Q) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times K$ such that

$$\begin{aligned}
&(a)\quad a_0(\mathbf{u},\mathbf{v}) + c_0(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = d(T,\mathbf{v}) + (f,\mathbf{v}) \quad \forall\ \mathbf{v}\in \mathbf{H}_0^1(\Omega),\\
&(b)\quad b(\mathbf{u},q) = 0 \quad \forall\ \ q \in L_0^2(\Omega),\\
&(c)\quad a_1(T,S) + c_1(\mathbf{u},T,S) = (Q,S) \quad \forall\ S\in H_0^1(\Omega)
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
&(a)\quad a_0(\mathbf{w},\mathbf{v}) + c_0(\mathbf{v},\mathbf{u},\mathbf{w}) + c_0(\mathbf{u},\mathbf{v},\mathbf{w}) - b(\mathbf{v},\sigma) = (\mathbf{u}\text{-}\mathbf{U},\mathbf{v}) - c_1(\mathbf{v},T,\varphi) \quad \forall\ \mathbf{v}\in \mathbf{H}_0^1(\Omega),\\
&(b)\quad b(\mathbf{w},q) = 0 \quad \forall\ q\in L_0^2(\Omega),\\
&(c)\quad a_1(\varphi,S) + c_1(\mathbf{u},S,\varphi) = d(S,\mathbf{w}) \quad \forall\ S\in H_0^1(\Omega),\\
&(d)\quad (\alpha Q + \varphi, P - Q) \geq 0 \quad \forall\ P\in K.
\end{aligned} \tag{2.5}$$

From (2.5) (d), we can see that

$$Q = \max(0, -\frac{1}{\alpha}\varphi). \tag{2.6}$$

In the next section, we will discuss the finite element approximation of the problem (2.4) and (2.5).

3. Finite Element Approximation

We are now able to introduce a finite-element based approximation of the optimal control (1.1). To this end, we consider a family of triangulations $\mathcal{T}_h$, $h > 0$, of $\bar{\Omega}$. With each element $\mathcal{T} \in \mathcal{T}_h$, we associate two parameters $\rho(\mathcal{T})$ and $\sigma(\mathcal{T})$, where $\rho(\mathcal{T})$ denotes the diameter of the set $\mathcal{T}$ and $\sigma(\mathcal{T})$ is the diameter of the largest ball contained in $\mathcal{T}$. The mesh size of the grid is defined by $h = \max_{\mathcal{T}\in\mathcal{T}_h} \rho(\mathcal{T})$. We suppose that triangulations $\mathcal{T}_h$ satisfy the following regularity assumptions:

(H$_1$) There exist two positive constants ρ and σ such that

$$\frac{\rho(\mathcal{T})}{\sigma(\mathcal{T})} \leq \sigma, \qquad \frac{\sigma(\mathcal{T})}{\rho(\mathcal{T})} \leq \rho$$

hold for all $\mathcal{T} \in \mathcal{T}_h$ and all $0 < h \leq 1$.

(H$_2$) Define $\bar{\Omega}_h = \bigcup_{\mathcal{T}\in\mathcal{T}_h} \mathcal{T}$, and let Ω_h and Γ_h denote its interior and its boundary, respectively. We assume that $\bar{\Omega}_h$ is convex and that the vertices of $\mathcal{T}_h$ placed on the boundary of Γ_h are points of Γ. We also assume that

$$|\Omega\backslash\Omega_h| \leq Ch^2.$$

Next, to every boundary triangle $\mathcal{T}$ of $\mathcal{T}_h$ we associate another triangle $\hat{\mathcal{T}}$ with curved boundary, in which the edge between boundary nodes of $\mathcal{T}$ is substituted by the corresponding curved part of Γ. We denote by $\hat{\mathcal{T}}_h$ the union of these curved boundary triangles with interior triangles of $\mathcal{T}_h$, such that $\bar{\Omega} = \bigcup_{\hat{\mathcal{T}}\in\hat{\mathcal{T}}_h} \hat{\mathcal{T}}$.

Denote by P_k the function space of polynomials of degree less than or equal to k. Introduce finite element spaces as follows:

$$K_h = \{Q_h \in L^2(\Omega) : Q_h|_{\hat{\mathcal{T}}} = constant, \quad \hat{\mathcal{T}} \in \hat{\mathcal{T}}_h\}, \quad K_h^{ad} = K_h \cap K,$$
$$V_h = \{y_h \in C(\bar{\Omega}) : y_h|_{\mathcal{T}} \in P_1(\mathcal{T}), \quad \mathcal{T} \in \mathcal{T}_h; \quad y_h = 0 \ \ on \ \ \bar{\Omega}\backslash\Omega_h\}.$$

Next we introduce the order-one Raviart-Thomas mixed finite element spaces as in [29]: $\bar{\mathbf{V}}_h \times \bar{X}_h \subset \mathbf{H}_0^1 \times L_0^2$ such that for a positive constant β_0, the following *inf-sup* condition is satisfied:

$$\inf_{0\neq q_h\in\bar{X}_h} \sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{H}^1}\|q_h\|_{L^2}} \geq \beta_0. \tag{3.1}$$

Moreover, similarly to V_h, we define

$$\mathbf{V}_h = \{\mathbf{y}_h \in \bar{\mathbf{V}}_h \quad on\ \Omega_h; \quad \mathbf{y}_h = 0 \ \ on \ \ \bar{\Omega}\backslash\Omega_h\},$$
$$X_h = \{p_h \in \bar{X}_h : p_h|_{\hat{\mathcal{T}}} = constant, \quad \hat{\mathcal{T}} \in \hat{\mathcal{T}}_h\}.$$

Now, it is obvious that $\mathbf{V}_h \times X_h$ is defined on $\bar{\Omega}$, and then the finite-dimensional approximation of the optimal control problem is:

$$(\mathcal{P}_h) \quad \min_{Q_h\in K_h} \quad J_h(Q_h) = \{\frac{1}{2}\|\mathbf{u}_h - \mathbf{U}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\alpha}{2}\|Q_h\|^2_{0,\Omega}\} \tag{3.2}$$

subject to seek $(\mathbf{u}_h, p_h, T_h) \in \mathbf{V}_h \times X_h \times V_h$ such that

$$\begin{aligned}
&(a) \ \ a_0(\mathbf{u}_h, \mathbf{v}_h) + c_0(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = d(T_h, \mathbf{v}_h) + (f, \mathbf{v}_h) \quad \forall\ \mathbf{v}_h \in \mathbf{V}_h,\\
&(b) \ \ b(\mathbf{u}_h, q_h) = 0 \quad \forall\ q_h \in X^h,\\
&(c) \ \ a_1(T_h, S_h) + c_1(\mathbf{u}_h, T_h, S_h) = (Q_h, S_h) \quad \forall\ S_h \in V_h.
\end{aligned} \tag{3.3}$$

The optimal control problem $(\mathcal{P}_h)$ associated with state equations (3.3) is equivalent to optimality conditions as follows:

Scheme I. Seek $(\mathbf{u}_h, p_h, T_h) \in \mathbf{V}_h \times X_h \times V_h$ such that

$$\begin{aligned}
&(a) \ \ a_0(\mathbf{u}_h, \mathbf{v}_h) + c_0(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = d(T_h, \mathbf{v}_h) + (f, \mathbf{v}_h) \quad \forall\ \mathbf{v}_h \in \mathbf{V}_h,\\
&(b) \ \ b(\mathbf{u}_h, q_h) = 0 \quad \forall\ q_h \in X^h,\\
&(c) \ \ a_1(T_h, S_h) + c_1(\mathbf{u}_h, T_h, S_h) = (Q_h, S_h) \quad \forall\ S_h \in V_h.
\end{aligned} \tag{3.4}$$

couple with co-state system and inequality: seek $(\mathbf{w}_h, \sigma_h, \varphi_h, Q_h) \in \mathbf{V}_h \times X_h \times V_h \times K_h^{ad}$ such that

$$\begin{aligned}
&(a) \ \ a_0(\mathbf{w}_h, \mathbf{v}_h) + c_0(\mathbf{v}_h, \mathbf{u}_h, \mathbf{w}_h) + c_0(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) - b(\mathbf{v}_h, \sigma_h)\\
&\qquad = (\mathbf{u}_h - \mathbf{U}, \mathbf{v}_h) - c_1(\mathbf{v}_h, T_h, \varphi_h) \quad \forall\ \mathbf{v}_h \in \mathbf{V}_h,\\
&(b) \ \ b(\mathbf{w}_h, q_h) = 0 \quad \forall\ q_h \in X_h,\\
&(c) \ \ a_1(\varphi_h, S_h) + c_1(\mathbf{u}_h, S_h, \varphi_h) = d(S_h, \mathbf{w}_h) \quad \forall\ S_h \in V_h,\\
&(d) \ \ (\alpha Q_h + \varphi_h, \hat{P}_h - Q_h) \geq 0 \quad \forall\ \hat{P}_h \in K_h^{ad}.
\end{aligned} \tag{3.5}$$

Similarly, it follows from (3.5)(d) that

$$Q_h = \max(0, -\frac{1}{\alpha}\pi_h\varphi_h), \tag{3.6}$$

where π_h is defined by

$$\pi_h v|_{\hat{\mathcal{T}}} = \frac{1}{|\hat{\mathcal{T}}|}\int_{\hat{\mathcal{T}}} v\,dx \quad \forall\, \hat{\mathcal{T}} \in \hat{\mathcal{T}}_h.$$

On the other hand, define

$$\widetilde{\mathbf{V}} = \{\mathbf{v} \in \mathbf{H_0^1}; \quad \nabla\cdot\mathbf{v} = 0 \;\; on\ \Omega\}.$$

Then the weak form of the optimality conditions reads: seek $(\mathbf{u}, T) \in \widetilde{\mathbf{V}} \times H_0^1(\Omega)$ such that

$$\begin{aligned} &(a)\;\; a_0(\mathbf{u},\mathbf{v}) + c_0(\mathbf{u},\mathbf{u},\mathbf{v}) = d(T,\mathbf{v}) + (f,\mathbf{v}) \quad \forall\,\mathbf{v}\in\widetilde{\mathbf{V}},\\ &(b)\;\; a_1(T,S) + c_1(\mathbf{u},T,S) = (Q,S) \quad \forall\, S \in H_0^1(\Omega) \end{aligned} \tag{3.7}$$

and seek $(\mathbf{w}, \varphi, Q) \in \widetilde{\mathbf{V}} \times H_0^1(\Omega) \times K$ such that

$$\begin{aligned} &(a)\;\; a_0(\mathbf{w},\mathbf{v}) + c_0(\mathbf{v},\mathbf{u},\mathbf{w}) + c_0(\mathbf{u},\mathbf{v},\mathbf{w}) = (\mathbf{u}\text{-}\mathbf{U},\mathbf{v}) - c_1(\mathbf{v},T,\varphi) \quad \forall\,\mathbf{v}\in\widetilde{\mathbf{V}},\\ &(b)\;\; a_1(\varphi,S) + c_1(\mathbf{u},S,\varphi) = d(S,\mathbf{w}) \quad \forall\, S\in H_0^1(\Omega),\\ &(c)\;\; (\alpha Q + \varphi, P - Q) \ge 0 \quad \forall\, P \in K. \end{aligned} \tag{3.8}$$

Define the finite element space $\widetilde{\mathbf{V}}_h$ of the form:

$$\mathbf{Z}_h = \{\mathbf{v}_h \in \mathbf{V}_h; \quad \nabla\cdot\mathbf{v}_h = 0 \;\; on\ \Omega^h\} \qquad \widetilde{\mathbf{V}}_h = \mathbf{V}_h \cap \mathbf{Z}_h.$$

It is clear $\widetilde{\mathbf{V}}_h \in \widetilde{\mathbf{V}}$. The finite-dimensional approximation of the optimal control problem is defined by

$$(\widetilde{\mathcal{P}}_h)\;\; \min_{Q_h\in K_h} \quad J_h(Q_h) = \{\frac{1}{2}\|\mathbf{u}_h - \mathbf{U}\|_{0,\Omega}^2 + \frac{\alpha}{2}\|Q_h\|_{0,\Omega}^2\} \tag{3.9}$$

subject to seeking $(\mathbf{u}_h, T_h) \in \widetilde{\mathbf{V}}_h \times V_h$ such that

$$\begin{aligned} &(a)\;\; a_0(\mathbf{u}_h,\mathbf{v}_h) + c_0(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) = d(T_h,\mathbf{v}_h) + (f,\mathbf{v}_h) \quad \forall\,\mathbf{v}_h\in\widetilde{\mathbf{V}}_h,\\ &(b)\;\; a_1(T_h,S_h) + c_1(\mathbf{u}_h,T_h,S_h) = (Q_h,S_h) \quad \forall\, S_h\in V_h. \end{aligned} \tag{3.10}$$

This results in another finite element system.

Scheme II. seek $(\mathbf{u}_h, T_h) \in \widetilde{\mathbf{V}}_h \times V_h$ such that

$$\begin{aligned} &(a)\;\; a_0(\mathbf{u}_h,\mathbf{v}_h) + c_0(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) = d(T_h,\mathbf{v}_h) + (f,\mathbf{v}_h) \quad \forall\,\mathbf{v}_h\in\widetilde{\mathbf{V}}_h,\\ &(b)\;\; a_1(T_h,S_h) + c_1(\mathbf{u}_h,T_h,S_h) = (Q_h,S_h) \quad \forall\, S_h\in V_h \end{aligned} \tag{3.11}$$

coupled with seeking $(\mathbf{w}_h, \varphi_h, Q_h) \in \widetilde{\mathbf{V}}_h \times V_h \times K_h^{ad}$ such that

$$\begin{aligned} &(a)\; a_0(\mathbf{w}_h,\mathbf{v}_h) + c_0(\mathbf{v}_h,\mathbf{u}_h,\mathbf{w}_h) + c_0(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h) = (\mathbf{u}_h - \mathbf{U},\mathbf{v}_h) - c_1(\mathbf{v}_h,T_h,\varphi_h) \;\; \forall\,\mathbf{v}_h\in\widetilde{\mathbf{V}}_h,\\ &(b)\; a_1(\varphi_h,S_h) + c_1(\mathbf{u}_h,S_h,\varphi_h) = d(S_h,\mathbf{w}_h) \quad \forall\, S_h\in V_h,\\ &(c)\; (\alpha Q_h + \varphi_h, \hat{P}_h - Q_h) \ge 0 \quad \forall\, \hat{P}_h \in K_h^{ad}. \end{aligned} \tag{3.12}$$

The scheme II involves no pressure terms explicitly, they are easy to be solved. However, it is often difficult to construct divergence-free finite element spaces. The readers are referred to [8, 18] for the detailed discussion of divergence-free finite element spaces.

From the existing studies in [19, 20], we see the following convergence result.

Lemma 3.1. *Suppose* $(\mathbf{H}_1)$-$(\mathbf{H}_2)$ *hold. Let the sequence* Q_h *with* $h \to 0$ *be a sequence of solutions to* $(\mathcal{P}_h)$ *or* $(\widetilde{\mathcal{P}}_h)$. *Then there exist weakly-converged subsequences in* $L^2(\Omega)$. *If the subsequence* Q_h *(still indexed by* h*) weakly converges to* Q *in* $L^2(\Omega)$, *then* Q *is a solution of* $(\mathcal{P})$. *Moreover,*

$$\lim_{h\to 0} J_h(Q_h) = J(Q). \tag{3.13}$$

In the next sections we will discuss and prove better convergent orders under some reasonable conditions and assumptions. In addition, throughout the next sections, c or C denotes a general positive constant independent of the mesh size h.

4. Main Results of Superconvergence and Optimal Error Estimate

Let $(Q_h, \mathbf{u}_h, \mathbf{w}_h, T_h, \varphi_h)$ be a sequence of solutions of finite element approximation defined by the scheme I or scheme II with $h \to 0$. So, since $\mathbf{u}_h$ weakly converges to $\mathbf{u}$ in the H^1-norm, $\mathbf{u}_h$ strongly converges to $\mathbf{u}$ in L^2-norm by the embedding theory. Now from Lemma 3.1 and the definitions of $J_h(Q_h)$ and $J(Q)$, we can get that

$$\lim_{h\to 0} \|Q_h - Q\|_{L^2} = 0. \tag{4.1}$$

In this section, we discuss the superconvergence of the finite element approximation and the optimal error estimates in the L^2-norm and L^∞-norm. Some additional assumptions are needed. Firstly, we assume that the cost function J is strictly convex near the solutions Q, i.e.,

$(\mathbf{H}_3)$ For the solution Q there exists a neighborhood of Q in L^2 such that J is convex in the sense that there is a constant $c_* > 0$ satisfying:

$$c_*\|Q-P\|_{0,\Omega}^2 \le (J'(Q) - J'(P), Q-P), \tag{4.2}$$

for all P in this neighborhood of Q.

Secondly, we declare that the solution $(Q, \mathbf{u}, \mathbf{w}, T, \varphi)$ is nonsingular, if for the solution $\mathbf{u}$ of $(\mathcal{P})$, the linear co-state system

$$\begin{aligned}
&(a) \quad -\nu\Delta\mathbf{v} - (\mathbf{u}\cdot\nabla)\mathbf{v} + \nabla\mathbf{v}^{tr}\mathbf{u} - \nabla\zeta + \varrho\nabla T = \mathbf{r} \quad \textit{in } \Omega,\\
&(b) \quad \nabla\cdot\mathbf{w} = 0 \quad \textit{in } \Omega,\\
&(c) \quad -\kappa\Delta\varrho - \mathbf{u}\cdot\nabla\varrho - \mathbf{v}\cdot\mathbf{g} = g \quad \textit{in } \Omega,\\
&(d) \quad \mathbf{v} = 0, \quad \varrho = 0 \quad \textit{on } \partial\Omega
\end{aligned} \tag{4.3}$$

is well-posed, which means that:

$(\mathbf{R}_1)$ For each $(\mathbf{r}, g) \in [\mathbf{H}^{-1}(\Omega)]^n \times H^{-1}(\Omega)$, the system (4.3) has a unique solution and there holds the a priori estimate

$$\|\mathbf{v}\|_{\mathbf{H}^1} + \|\varrho\|_{1,\Omega} + \|\zeta\|_{0,\Omega} \le C\{\|\mathbf{r}\|_{\mathbf{H}^{-1}} + \|g\|_{-1,\Omega}\}. \tag{4.4}$$

It follows from the regularity theory of partial differential equations (see [4]) that

$$\|\mathbf{v}\|_{\mathbf{H}^2} + \|\varrho\|_{2,\Omega} + \|\zeta\|_{1,\Omega} \le C\{\|\mathbf{r}\|_{\mathbf{L}^2} + \|g\|_{0,\Omega}\}. \tag{4.5}$$

Thirdly, we need an additional assumption for the exact solution as following:

$(\mathbf{H}_4)$ $\mathbf{u}, \mathbf{w}, T, \varphi$ belong to a space $\mathbf{W}^{2,l}(\Omega)$ and $W^{2,l}(\Omega)$ respectively for some $l > n$.

From the theory of partial differential equations and the assumption $\partial\Omega \in C^{1,1}$, we can see $(\mathbf{H}_4)$ is reasonable.

The optimal control Q is obtained by the projection formula (2.6). Therefore, we can classify the triangles $\mathcal{T}$ into two sets Ω_1^h and Ω_2^h:

$$\Omega_1^h = \bigcup_{\mathcal{T}\in\mathcal{T}_h} \{\mathcal{T} : \ Q \textit{ is only Lipschitz contiuous on } \mathcal{T}\}, \quad \Omega_2^h = \bigcup_{\mathcal{T}\in\mathcal{T}_h} \{\mathcal{T} : \ Q \in W^{2,l}(\mathcal{T})\}.$$

This classification is correct because $W^{2,l}(\Omega)$ is embedded in $C^{0,1}(\bar{\Omega})$ and the projection operator *max* is continuous from $C^{0,1}(\bar{\Omega})$ and $C^{0,1}(\bar{\Omega})$.

Last, let $\Omega_0 = \{\mathbf{x} : Q(\mathbf{x}) = 0\}$. Then it is clear that the number of triangles in Ω_0 which are denoted by Ω_0^h grows as $h \to 0$. However, the following additional assumption is fulfilled in many practical case:

$(\mathbf{H}_5)$ Measure of Ω_1^h is bounded by $\mathcal{O}(h)$, i.e., there exists a constant C such that

$$|\Omega_1^h| \le Ch.$$

The condition $(\mathbf{H}_5)$ holds if $\partial\Omega_0$ has a finite length in the two-dimensional case, or area in the three-dimensional case.

In the sequel, we denote by $\mathcal{S}$ the centroid of the triangle $\mathcal{T}$. We define a piecewise constant function by the values of $Q(\mathcal{S})$ as in [27],

$$P_h(x) = Q(\mathcal{S}) \quad if\ x \in \mathcal{T}. \tag{4.6}$$

It is easy to verify that $P_h \in K_h^{ad}$.

Now we are able to state our superconvergence in the following theorem. To prove these superconvergent results, we need some auxiliary lemmas. In this section, we only state these auxiliary lemmas. The proofs of the lemmas are deferred to the next section for clearer presentations.

Theorem 4.1. *Suppose* $(\mathbf{H}_1)$-$(\mathbf{H}_5)$ *are fulfilled. Then the error estimates*

$$\|Q_h - P_h\|_{0,\Omega} \le Ch^2 \quad and \quad \|Q - Q_h\|_{0,\Omega} \le Ch \tag{4.7}$$

hold.

To prove Theorem 4.1, for any given $P \in L^2(\Omega)$, introduce auxiliary functions $(u_h(P)$, $p_h(P), T_h(P)$, $\mathbf{w}_h(P), \sigma(P), \varphi_h(P))$ satisfying the following problem:

$$\begin{aligned}
&(a)\ \ a_0(\mathbf{u}_h(P),\mathbf{v}_h) + c_0(\mathbf{u}_h(P),\mathbf{u}_h(P),\mathbf{v}_h) + b(\mathbf{v}_h, p_h(P)) = d(T_h(P),\mathbf{v}_h) + (f,\mathbf{v}_h)\ \forall\ \mathbf{v}_h \in \mathbf{V}_h,\\
&(b)\ \ b(\mathbf{u}_h(P), q_h) = 0 \quad \forall\ q_h \in X_h,\\
&(c)\ \ a_1(T_h(P), S_h) + c_1(\mathbf{u}_h(P), T_h(P), S_h) = (P, S_h) \quad \forall\ S_h \in V_h
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
&(a)\ \ a_0(\mathbf{w}_h(P),\mathbf{v}_h) + c_0(\mathbf{v}_h,\mathbf{u}_h(P),\mathbf{w}_h(P)) + c_0(\mathbf{u}_h(P),\mathbf{v}_h,\mathbf{w}_h(P)) - b(\mathbf{v}_h,\sigma_h(P))\\
&\qquad\quad = (\mathbf{u}_h(P) - \mathbf{U},\mathbf{v}_h) - c_1(\mathbf{v}_h, T_h(P),\varphi_h(P)) \quad \forall\ \mathbf{v}_h \in \mathbf{V}_h,\\
&(b)\ \ b(\mathbf{w}_h(P), q_h) = 0 \quad \forall\ q_h \in X_h,\\
&(c)\ \ a_1(\varphi_h(P), S_h) + c_1(\mathbf{u}_h(P), S_h, \varphi_h(P)) = d(S_h,\mathbf{w}_h(P)) \quad \forall\ S_h \in V_h.
\end{aligned} \tag{4.9}$$

The proof can be given by using the following four lemmas, whose proofs will be provided in the next section.

Lemma 4.1. *Suppose* $(\mathbf{H}_1)$-$(\mathbf{H}_5)$ *are fulfilled. We have the following error estimate:*

$$\|Q_h - P_h\|_{0,\Omega} \leq C\{h^2 + \|\varphi - \varphi_h(Q)\|_{0,\Omega} + \|\varphi_h(Q) - \varphi_h(P_h)\|_{0,\Omega}\}. \tag{4.10}$$

Lemma 4.2. *Suppose* $(\mathbf{H}_1)$-$(\mathbf{H}_5)$ *are fulfilled. There hold the following error estimates:*

$$\|\boldsymbol{u} - \boldsymbol{u}_h(Q)\|_{\boldsymbol{L}^2(\Omega)} + \|T - T_h(Q)\|_{0,\Omega} \leq Ch^2, \tag{4.11}$$

$$\|\boldsymbol{w} - \boldsymbol{w}_h(Q)\|_{\boldsymbol{L}^2(\Omega)} + \|\varphi - \varphi_h(Q)\|_{0,\Omega} \leq Ch^2. \tag{4.12}$$

This lemma comes from the result of [20] based on the approximation theory of nonsingular solutions of coupled nonlinear systems such as [4].

Lemma 4.3. *Suppose that* $(\mathbf{H}_1)$-$(\mathbf{H}_5)$ *are valid. Then the following estimate holds:*

$$\|\boldsymbol{u}_h(Q) - \boldsymbol{u}_h(P_h)\|_{\boldsymbol{H}^1(\Omega)} + \|T_h(Q) - T_h(P_h)\|_{1,\Omega} + \|p_h(Q) - p_h(P_h)\|_{0,\Omega} \leq Ch^2. \tag{4.13}$$

Lemma 4.4. *Suppose that* $(\mathbf{H}_1)$-$(\mathbf{H}_5)$ *are valid. Then we have*

$$\|\boldsymbol{w}_h(Q) - \boldsymbol{w}_h(P_h)\|_{\boldsymbol{H}^1(\Omega)} + \|\varphi_h(Q) - \varphi_h(P_h)\|_{1,\Omega} + \|\sigma_h(Q) - \sigma_h(P_h)\|_{0,\Omega} \leq Ch^2. \tag{4.14}$$

Now it is clear that Theorem 4.1 is the direct consequence of Lemmas 4.1-4.4. Next, we consider the super-closing properties of the state and adjoint state.

Theorem 4.2. *Suppose* $(\mathbf{H}_1)$-$(\mathbf{H}_5)$ *are fulfilled. Then there hold the following super-closing properties:*

$$\|\boldsymbol{u}_h(Q) - \boldsymbol{u}_h\|_{\boldsymbol{H}^1(\Omega)} + \|T_h(Q) - T_h\|_{1,\Omega} + \|p_h(Q) - p_h\|_{0,\Omega} \leq Ch^2, \tag{4.15}$$

$$\|\boldsymbol{w}_h(Q) - \boldsymbol{w}_h\|_{\boldsymbol{H}^1(\Omega)} + \|\varphi_h(Q) - \varphi_h\|_{1,\Omega} + \|\sigma_h(Q) - \sigma_h\|_{0,\Omega} \leq Ch^2. \tag{4.16}$$

It is easy to see that these two results are the direct consequence of Theorem 4.1, Lemmas 4.3 and 4.4. Moreover, we have the following optimal a priori error estimates in the L^2-norm.

Theorem 4.3. *Assume that all the above conditions are valid. Then there holds the following a priori error estimate:*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^2(\Omega)} + \|T - T_h\|_{0,\Omega} + \|\boldsymbol{w} - \boldsymbol{w}_h\|_{\boldsymbol{L}^2(\Omega)} + \|\varphi - \varphi_h\|_{0,\Omega} \leq Ch^2. \tag{4.17}$$

Proof. It follows from the superconvergent results of [30] and Theorem 4.2 that

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} = \|\mathbf{u} - \mathbf{u}_h(Q)\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_h(Q) - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \leq Ch^2.$$

The other terms on the left-hand side of (4.17) can also be estimated similarly. This completes the proof of Theorem 4.3. □

Now, if we define $\hat{Q}_h = \max(0, -\frac{1}{\alpha}\varphi_h)$, then we have the superconvergence:

$$\|Q - \hat{Q}_h\|_{L^2} \leq Ch^2.$$

Furthermore, we have the following almost optimal a priori error estimates in the L^∞-norm.

Theorem 4.4. *Assume that all the above conditions are valid. In the 2D case, if* $(\boldsymbol{u}, \boldsymbol{w}) \in \boldsymbol{W}^{2,\infty}(\Omega), (T, \varphi) \in W^{2,\infty}(\Omega)$ *respectively, then there hold the following a priori error estimates:*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^\infty(\Omega)} + \|T - T_h\|_{\infty,\Omega} + \|\boldsymbol{w} - \boldsymbol{w}_h\|_{\boldsymbol{L}^\infty(\Omega)} + \|\varphi - \varphi_h\|_{\infty,\Omega} \le Ch^2 |\ln h|^{\frac{1}{2}}, \tag{4.18}$$

and

$$\|Q - Q_h\|_{\infty,\Omega} \le Ch. \tag{4.19}$$

Proof. In the 2D case, by using the known result

$$\|w_h\|_{L^\infty(\Omega)} \le C|\ln h|^{\frac{1}{2}} \|\nabla w_h\|_{L^2(\Omega)} \quad \forall\, w_h \in V_h$$

and the superconvergent result in Theorem 4.2, we have

$$\begin{aligned}
&\|\mathbf{u}(Q) - \mathbf{u}_h(Q_h)\|_{\mathbf{L}^\infty(\Omega)} \\
&\le \|\mathbf{u}(Q) - \mathbf{u}_h(Q)\|_{\mathbf{L}^\infty(\Omega)} + C|\ln h|^{\frac{1}{2}} (\|\mathbf{u}_h(Q) - \mathbf{u}_h(Q_h)\|_{\mathbf{H}^1(\Omega)}) \\
&\le C|\ln h|^{\frac{1}{2}} h^2.
\end{aligned}$$

and other terms follow similarly. On the other hand, by using the inverse property of finite element spaces

$$\|v_h\|_{\infty,\Omega} \le Ch^{-1} \|v_h\|_{0,\Omega} \quad \forall\, v_h \in K_h,$$

and the superconvergent results in Theorem 4.1, we have

$$\begin{aligned}
&\|Q - Q_h\|_{\infty,\Omega} \le \|Q - P_h\|_{\infty,\Omega} + \|P_h - Q_h\|_{\infty,\Omega} \\
&\le \|Q - P_h\|_{\infty,\Omega} + Ch^{-1} \|P_h - Q_h\|_{0,\Omega} \\
&\le Ch.
\end{aligned}$$

This completes the proof of Theorem 4.4. □

5. Proofs of the Lemmas

In this section we give the proofs of the lemmas in Section 4. In the sequel, we give the proofs of all the results for the scheme I. As scheme II does not involve pressure terms explicitly, similar conclusions can be easily established using a similar procedure.

5.1. Proof of Lemma 4.1

From the assumption (4.2), by the proof contained in [3], there exists a constant $c > 0$ satisfying

$$c\|Q_h - P_h\|_{0,\Omega}^2 \le (J_h'(Q_h) - J_h'(P_h), Q_h - P_h). \tag{5.1}$$

Then, noting that

$$(\alpha Q_h + \varphi_h(Q_h), S_h - Q_h) \ge 0 \quad \forall\, S_h \in K_h,$$

we have

$$\begin{aligned}
&c\|Q_h - P_h\|_{0,\Omega}^2 \le (J_h'(Q_h) - J_h'(P_h), Q_h - P_h) \\
&= (\alpha Q_h + \varphi_h(Q_h), Q_h - P_h) - (\alpha P_h + \varphi_h(P_h), Q_h - P_h) \\
&\le -(\alpha P_h + \varphi_h(P_h), Q_h - P_h).
\end{aligned} \tag{5.2}$$

Next, we derive a variational inequality for the function P_h. Since

$$(\alpha Q + \varphi(Q), P - Q) \geq 0 \quad \forall\, P \in K,$$

the result is also true pointwise, i.e.,

$$(\alpha Q(\mathbf{x}) + \varphi(\mathbf{x}))(P(\mathbf{x}) - Q(\mathbf{x})) \geq 0 \quad \forall\, P \geq 0 \;\; and \;\; \mathbf{x} \in \Omega.$$

Applying this formula at $\mathbf{x} = \mathcal{S}$ and $P = Q_h$, we have

$$(\alpha Q(\mathcal{S}) + \varphi(\mathcal{S}))(Q_h(\mathcal{S}) - Q(\mathcal{S})) \geq 0 \quad \forall\, \mathcal{S}.$$

Note that $P_h(\mathcal{S}) = Q(\mathcal{S})$ and let $\hat{\varphi}(\mathbf{x}) = \varphi(\mathcal{S})$ if $\mathbf{x} \in \mathcal{T}$ for each $\mathcal{T} \in \mathcal{T}_h$. It gives that

$$(\alpha P_h(\mathcal{S}) + \varphi(\mathcal{S}))(Q_h(\mathcal{S}) - P_h(\mathcal{S})) \geq 0 \quad \forall\, \mathcal{S}.$$

By integrating this formula over $\mathcal{T}$ and summing the resulting inequalities up, we have

$$(\alpha P_h + \hat{\varphi}, Q_h - P_h) \geq 0. \tag{5.3}$$

Substituting this into (5.2) leads to

$$\begin{aligned} &c\|Q_h - P_h\|_{0,\Omega}^2 \leq (\hat{\varphi} - \varphi_h(P_h), Q_h - P_h) \\ &= (\hat{\varphi} - \varphi, Q_h - P_h) + (\varphi - \varphi_h(Q), Q_h - P_h) \\ &\quad + (\varphi_h(Q) - \varphi_h(P_h), Q_h - P_h). \end{aligned} \tag{5.4}$$

We estimate the three terms on the right-hand side of (5.4). The first term represents a formula for the numerical integration. Let f be a function belonging to $H^2(\mathcal{T})$. The estimate

$$\left| \int_{\mathcal{T}} f(\mathbf{x}) - f(\mathcal{S})\, d\mathbf{x} \right| \leq ch^2 \sqrt{|\mathcal{T}|} |f|_{H^2(\mathcal{T})} \tag{5.5}$$

is valid. By using this estimate, we obtain

$$\begin{aligned} (\hat{\varphi} - \varphi, Q_h - P_h) &= \sum_{\mathcal{T} \in \mathcal{T}_h} \int_{\mathcal{T}} (\hat{\varphi} - \varphi)(Q_h - P_h)\, dx \\ &= \sum_{\mathcal{T} \in \mathcal{T}_h} (Q_h - P_h) \int_{\mathcal{T}} (\varphi(\mathcal{S}) - \varphi(\mathbf{x}))\, d\mathbf{x} \\ &\leq \sum_{\mathcal{T} \in \mathcal{T}_h} Ch^2\, |Q_h - P_h|\, \sqrt{|\mathcal{T}|} (|\varphi|_{2,\mathcal{T}}) \\ &\leq Ch^2 \|Q_h - P_h\|_{0,\Omega}. \end{aligned} \tag{5.6}$$

For the second and third terms, we have

$$\begin{aligned} &|(\varphi - \varphi_h(Q), Q_h - P_h)| + |(\varphi_h(Q) - \varphi_h(P_h), Q_h - P_h)| \\ &\leq \{\|\varphi - \varphi_h(Q)\|_{0,\Omega} + \|\varphi_h(Q) - \varphi_h(P_h)\|_{0,\Omega}\} \|Q_h - P_h\|_{0,\Omega}. \end{aligned} \tag{5.7}$$

Substituting (5.6) and (5.7) into (5.4) leads to (4.10). The proof of Lemma 4.1 is thereby complete. □

5.2. Proof of Lemma 4.3

To prove Lemma 4.3, we need some basic properties of bilinear forms and trilinear forms in the weak formulation. It is also clear that the bilinear forms $a_0(\cdot,\cdot)$, $a_1(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are continuous, i.e.,

$$|a_0(\mathbf{f},\mathbf{g})| \le C_1\|\mathbf{f}\|_{\mathbf{H}^1(\Omega)}\|\mathbf{g}\|_{\mathbf{H}^1(\Omega)}, \tag{5.8}$$

$$|a_1(F,G)| \le C_2\|F\|_{1,\Omega}\|G\|_{1,\Omega}, \tag{5.9}$$

$$|d(F,\mathbf{f})| \le C_3\|F\|_{0,\Omega}\|\mathbf{f}\|_{\mathbf{H}^1(\Omega)}. \tag{5.10}$$

We have the coercivity relations associated with $a_0(\cdot,\cdot)$ and $a_1(\cdot,\cdot)$:

$$|a_0(\mathbf{g},\mathbf{g})| \ge \nu\|\mathbf{g}\|^2_{\mathbf{H}^1(\Omega)}, \tag{5.11}$$

$$|a_1(G,G)| \ge \kappa\|G\|^2_{1,\Omega}. \tag{5.12}$$

For trilinear forms $c_0(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ and $c_1(\,\cdot\,,\,\cdot\,,\,\cdot\,)$, we have the estimates

$$c_0(\mathbf{f},\mathbf{g},\mathbf{v}) \le C_4\|\mathbf{f}\|_{\mathbf{H}^1(\Omega)}\|\mathbf{g}\|_{\mathbf{H}^1(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}, \tag{5.13}$$

$$c_0(\mathbf{v},\mathbf{f},\mathbf{f}) = 0 \quad if\ \mathbf{v}\in\mathbf{V}, \tag{5.14}$$

$$c_1(\mathbf{v},F,G) \le C_2\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}\|F\|_{1,\Omega}\|G\|_{1,\Omega} \quad if\ \mathbf{v}\in\mathbf{V}, \tag{5.15}$$

$$c_1(\mathbf{v},F,F) = 0 \quad if\ \mathbf{v}\in\mathbf{V}. \tag{5.16}$$

These estimates follow from the Poincaré inequality, the Cauchy-Schwarz inequality, the Hölder inequality and various embedding results.

Now we can prove the error estimate (4.13) in Lemma 4.3. The proof is divided into two steps. In the first step, we prove

$$\begin{aligned}&\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|_{\mathbf{H}^1(\Omega)} + \|T_h(Q)-T_h(P_h)\|_{1,\Omega}\\ &\le C\{h^2 + \|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|_{\mathbf{L}^2(\Omega)} + \|T_h(Q)-T_h(P_h)\|_{0,\Omega}\},\end{aligned} \tag{5.17}$$

which shows that the H^1-norm of the error functions is controlled by its L^2-norm. In the second step, we prove the L^2-estimate:

$$\begin{aligned}&\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|_{\mathbf{L}^2(\Omega)} + \|T_h(Q)-T_h(P_h)\|_{0,\Omega}\\ &\le C\{h^2 + h(\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|_{\mathbf{H}^1(\Omega)} + \|T_h(Q)-T_h(P_h)\|_{1,\Omega})\}.\end{aligned} \tag{5.18}$$

Substituting (5.18) into (5.17) leads to the estimates we needed in (4.13).

5.3. Proof of (5.17)

It follows from (4.8) that

$$\begin{aligned}
(a)\ \ & a_0(\mathbf{u}_h(Q)-\mathbf{u}_h(P_h),\mathbf{v}_h) + c_0(\mathbf{u}_h(Q)-\mathbf{u}_h(P_h),\mathbf{u}_h(Q),\mathbf{v}_h)\\
&\quad + c_0(\mathbf{u}_h(P_h),\mathbf{u}_h(Q)-\mathbf{u}_h(P_h),\mathbf{v}_h) + b(\mathbf{v}_h,p_h(Q)-p_h(P_h))\\
&= d(T_h(Q)-T_h(P_h),\mathbf{v}_h) \quad \forall\ \mathbf{v}_h\in\mathbf{V}_h,\\
(b)\ \ & b(\mathbf{u}_h(Q)-\mathbf{u}_h(P_h),q_h) = 0 \quad \forall\ q_h\in X_h,\\
(c)\ \ & a_1(T_h(Q)-T_h(P_h),S_h) + c_1(\mathbf{u}_h(Q)-\mathbf{u}_h(P_h),T_h(Q),S_h)\\
&\quad + c_1(u_h(P_h),T_h(Q)-T_h(P_h),S_h)\\
&= (Q-P_h,S_h) \quad \forall\ S_h\in V_h.
\end{aligned} \tag{5.19}$$

Using standard arguments, we have

$$\begin{aligned}&\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|^2_{\mathbf{H}^1(\Omega)}\\ \le& C\{\|T_h(Q)-T_h(P_h)\|_{0,\Omega}\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|_{\mathbf{L}^2(\Omega)}\\ &+\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|_{\mathbf{H}^1(\Omega)}\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|_{\mathbf{L}^2(\Omega)}\}\\ \le& C\{\|T_h(Q)-T_h(P_h)\|^2_{0,\Omega}+\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|^2_{\mathbf{L}^2(\Omega)}\}\\ &+\frac{C}{2\varepsilon}\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|^2_{\mathbf{L}^2(\Omega)}+\varepsilon\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|^2_{\mathbf{H}^1(\Omega)},\end{aligned}$$

where $1>\varepsilon>0$ is a suitable constant and

$$\|T_h(Q)-T_h(P_h)\|^2_{1,\Omega}\le C\{\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|^2_{\mathbf{H}^1(\Omega)}+|(Q-P_h,T_h(Q)-T_h(P_h))|\}.$$

So, we have

$$\begin{aligned}&\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|^2_{\mathbf{H}^1(\Omega)}+\|T_h(Q)-T_h(P_h)\|^2_{1,\Omega}\\ \le& C\{\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|^2_{\mathbf{L}^2(\Omega)}+\|T_h(Q)-T_h(P_h)\|^2_{0,\Omega}+|(Q-P_h,T_h(Q)-T_h(P_h))|\}.\end{aligned}\tag{5.20}$$

It remains to estimate the term $(Q-P_h,T_h(Q)-T_h(P_h))$. It follows that

$$\begin{aligned}&(Q-P_h,T_h(Q)-T_h(P_h))\\ =&(Q-P_h,T_h(Q)-T_h(P_h))_{\Omega_1^h}+(Q-P_h,T_h(Q)-T_h(P_h))_{\Omega_2^h}.\end{aligned}\tag{5.21}$$

Using the embedding theory of Sobolev space:

$$\|v\|_{L^1(\partial\Omega_0)}\le C\|v\|_{1,\Omega},$$

we have

$$\begin{aligned}&|(Q-P_h,T_h(Q)-T_h(P_h))_{\Omega_1^h}|\\ \le&|(Q-P_h,T_h(Q)-T_h(P_h)-\vartheta_h(T_h(Q)-T_h(P_h)))_{\Omega_1^h}|+|(Q-P_h,\vartheta_h(T_h(Q)-T_h(P_h))_{\Omega_1^h}|\\ \le& Ch^2\{\|Q\|_{1,\Omega}\|T_h(Q)-T_h(P_h)\|_{1,\Omega}+\|\nabla Q\|_{\infty,\Omega}\|T_h(Q)-T_h(P_h)\|_{L^1(\partial\Omega_0)}]\\ \le& Ch^2\|Q\|_{C^{0,1}(\bar\Omega)}\|T_h(Q)-T_h(P_h)\|_{1,\Omega},\end{aligned}\tag{5.22}$$

where ϑ_h is an orthogonal projection from $H^1(\Omega_1^h)$ onto $L^1(\partial\Omega_0)$. On the other hand, similar to (5.6), we have

$$\begin{aligned}&|(Q-P_h,T_h(Q)-T_h(P_h))_{\Omega_2^h}|\\ \le&|(Q-P_h,T_h(Q)-T_h(P_h)-\pi_h(T_h(Q)-T_h(P_h)))_{\Omega_2^h}|+|(Q-P_h,\pi_h(T_h(Q)-T_h(P_h)))_{\Omega_2^h}|\\ \le& C\{h^2\|T_h(Q)-T_h(P_h)\|_{1,\Omega}\|Q\|_{1,\Omega}+\sum_{\mathcal{T}\in\Omega_2^h}|\pi_h(T_h(Q)-T_h(P_h))|\left|\int_{\mathcal{T}}(Q(\mathbf{x})-Q(\mathcal{S}))d\mathbf{x})_{\Omega_2^h}\right|\\ \le& Ch^2\{\|Q\|_{1,\Omega}\|T_h(Q)-T_h(P_h)\|_{1,\Omega}+\|\varphi\|_{2,\Omega_h^2}\|T_h(Q)-T_h(P_h)\|_{0,\Omega}\}.\end{aligned}\tag{5.23}$$

Substituting (5.21)-(5.23) into (5.20) leads to (5.17).

5.4. Proof of (5.18)

For the analysis of the convergence rate in the L^2-norm, we introduce the auxiliary problem

$$\begin{aligned}
&(a) \quad -\nu\Delta\mathbf{R} - (\mathbf{u}(Q)\cdot\nabla)\mathbf{R} + \nabla\mathbf{u}(Q)^{tr}\mathbf{R} - \nabla\lambda + \phi\nabla T(Q) = \mathbf{u}_h(Q) - \mathbf{u}_h(P_h) \quad in\ \Omega,\\
&(b) \quad \nabla\cdot\mathbf{R} = 0 \quad in\ \Omega,\\
&(c) \quad -\kappa\Delta\phi - \mathbf{u}(Q)\cdot\nabla\phi - \mathbf{R}\cdot\mathrm{g} = T_h(Q) - T_h(P_h) \quad in\ \Omega,\\
&(d) \quad \mathbf{R} = 0 \quad \phi = 0 \quad on\ \partial\Omega.
\end{aligned} \tag{5.24}$$

Since we assume the solution $(\mathbf{u}, p, T)$ is nonsingular, the linear system (5.24) is uniquely solvable and satisfies the a priori estimate

$$\|\mathbf{R}\|_{\mathbf{H}^2(\Omega)} + \|\lambda\|_{1,\Omega} + \|\phi\|_{2,\Omega} \le C\{\|\mathbf{u}_h(Q) - \mathbf{u}_h(P_h)\|_{\mathbf{L}^2(\Omega)} + \|T_h(Q) - T_h(P_h)\|_{0,\Omega}\}. \tag{5.25}$$

Using the Aubin-Nietzchte technique, we have

$$\begin{aligned}
&(\mathbf{u}_h(Q) - \mathbf{u}_h(P_h), \mathbf{u}_h(Q) - \mathbf{u}_h(P_h)) + (T_h(Q) - T_h(P_h), T_h(Q) - T_h(P_h))\\
&= a_0(\mathbf{u}_h(Q) - \mathbf{u}_h(P_h), \mathbf{R}) + c_0(\mathbf{u}_h(Q) - \mathbf{u}_h(P_h), \mathbf{u}(Q), \mathbf{R})\\
&\quad + c_0(\mathbf{u}(P_h), \mathbf{u}_h(Q) - \mathbf{u}_h(P_h), \mathbf{R}) + c_0(\mathbf{u}(P_h) - \mathbf{u}(Q), \mathbf{R}, \mathbf{u}_h(Q) - \mathbf{u}_h(P_h))\\
&\quad - b(\mathbf{u}_h(Q) - \mathbf{u}_h(P_h), \lambda) + c_1(\mathbf{u}_h(Q) - \mathbf{u}_h(P_h), T(Q), \phi) + a_1(\phi, T_h(Q) - T_h(P_h))\\
&\quad + c_1(\mathbf{u}(P_h), T_h(Q) - T_h(P_h), \phi) - d(T_h(Q) - T_h(P_h), \mathbf{R})\\
&\quad + c_1(\mathbf{u}(P_h) - \mathbf{u}(Q), \phi, T_h(Q) - T_h(P_h)).
\end{aligned} \tag{5.26}$$

Here we define the L^2-orthogonal projection $\lambda_h \in X_h$, which satisfies: for any $q_h \in X_h$

$$(\lambda - \lambda_h, q_h) = 0, \quad \|\lambda - \lambda_h\|_{L^2} \le Ch\|\lambda\|_{H^1}.$$

Next, let us define the $(\mathbf{R}_h, \phi_h)$ are the Lagrange interpolations of $(\mathbf{R}, \phi)$ in the finite element spaces $(\mathbf{V}_h, V_h)$ respectively. Then we know that the following approximation properties:

$$\begin{aligned}
&\|\phi - \phi_h\|_{L^2} + h\|\phi - \phi_h\|_{H^1} \le Ch^2\|\phi\|_{H^2},\\
&\|\mathbf{R} - \mathbf{R}_h\|_{\mathbf{L}^2} + h\|\mathbf{R} - \mathbf{R}_h\|_{\mathbf{H}^1} \le Ch^2\|\mathbf{R}\|_{\mathbf{H}^2}.
\end{aligned}$$

We have

$$\begin{aligned}
&(\mathbf{u}_h(Q) - \mathbf{u}_h(P_h), \mathbf{u}_h(Q) - \mathbf{u}_h(P_h)) + (T_h(Q) - T_h(P_h), T_h(Q) - T_h(P_h))\\
&= a_0(\mathbf{u}_h(Q) - \mathbf{u}_h(P_h), \mathbf{R} - \mathbf{R}_h) + c_0(\mathbf{u}_h(Q) - \mathbf{u}_h(P_h), \mathbf{u}(Q), \mathbf{R} - \mathbf{R}_h)\\
&\quad + c_0(\mathbf{u}(P_h), \mathbf{u}_h(Q) - \mathbf{u}_h(P_h), \mathbf{R} - \mathbf{R}_h) + c_1(\mathbf{u}_h(Q) - \mathbf{u}_h(P_h), T(Q), \phi - \phi_h)\\
&\quad + c_1(\mathbf{u}(P_h), T_h(Q) - T_h(P_h), \phi - \phi_h) + a_1(\phi - \phi_h, T_h(Q) - T_h(P_h))\\
&\quad - b(\mathbf{u}_h(Q) - \mathbf{u}_h(P_h), \lambda - \lambda_h) - d(T_h(Q) - T_h(P_h), \mathbf{R} - \mathbf{R}_h) + (Q - P_h, \phi_h)\\
&\quad + c_0(\mathbf{u}_h(Q) - \mathbf{u}_h(P_h), \mathbf{u}(Q) - \mathbf{u}_h(Q), \mathbf{R}_h) + c_0(\mathbf{u}(P_h) - \mathbf{u}_h(P_h), \mathbf{u}_h(Q) - \mathbf{u}_h(P_h), \mathbf{R}_h)\\
&\quad + c_1(\mathbf{u}_h(Q) - \mathbf{u}_h(P_h), T(Q) - T_h(Q), \phi_h) + c_1(\mathbf{u}(P_h) - \mathbf{u}_h(P_h), T_h(Q) - T_h(P_h), \phi_h)\\
&\quad + c_0(\mathbf{u}(P_h) - \mathbf{u}(Q), \mathbf{R}, \mathbf{u}_h(Q) - \mathbf{u}_h(P_h)) + c_1(\mathbf{u}(P_h) - \mathbf{u}(Q), \phi, T_h(Q) - T_h(P_h))\\
&\le C\{[\ h(\|\mathbf{u}_h(Q) - \mathbf{u}_h(P_h)\|_{\mathbf{H}^1(\Omega)} + \|T_h(Q) - T_h(P_h)\|_{1,\Omega})\\
&\quad + \|\mathbf{u}(Q) - \mathbf{u}(P_h)\|_{\mathbf{H}^1(\Omega)}\|T_h(Q) - T_h(P_h)\|_{0,\Omega}\\
&\quad + \|\mathbf{u}(Q) - \mathbf{u}(P_h)\|_{\mathbf{H}^1(\Omega)}\|\mathbf{u}_h(Q) - \mathbf{u}_h(P_h)\|_{\mathbf{L}^2(\Omega)}][\ \|\mathbf{R}\|_{\mathbf{H}^2(\Omega)}\\
&\quad + \|\lambda\|_{1,\Omega} + \|\phi\|_{2,\Omega}\ ] + |(Q - P_h, \phi_h)|\},
\end{aligned} \tag{5.27}$$

where we have used the result of [30]. Since we assume that $(\mathbf{u}, T, Q)$ is one fixed branch of the nonsingular solutions and the above estimates are carried out on a fixed subsequence which converges to this fixed branch, it follows from the general theory of nonsingular solution [4] that

$$\|\mathbf{u}(Q)-\mathbf{u}(P_h)\|_{\mathbf{H}^1(\Omega)}+\|T(Q)-T(P_h)\|_{1,\Omega}\le C\|Q-P_h\|_{0,\Omega}\le Ch.$$

If h is sufficient small, using the same procedure of (5.21)-(5.23) to estimate the last term on the r.h.s of (5.27) gives

$$\begin{aligned}&\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|_{\mathbf{L}^2(\Omega)}+\|T_h(Q)-T_h(P_h)\|_{0,\Omega}\\ &\le C\Big(h^2+h(\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|_{\mathbf{H}^1(\Omega)}+\|T_h(Q)-T_h(P_h)\|_{1,\Omega})\Big).\end{aligned}\tag{5.28}$$

Consequently,

$$\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|_{\mathbf{H}^1(\Omega)}+\|T_h(Q)-T_h(P_h)\|_{1,\Omega}\le Ch^2.\tag{5.29}$$

Now, from (5.19), it is obvious that

$$\begin{aligned}&b(\mathbf{v}_h,p_h(Q)-p_h(P_h))\\ &=a_0(\mathbf{u}_h(Q)-\mathbf{u}_h(P_h),\mathbf{v}_h)+c_0(\mathbf{u}_h(Q)-\mathbf{u}_h(P_h),\mathbf{u}_h(Q),\mathbf{v}_h)\\ &\quad+c_0(\mathbf{u}_h(P_h),\mathbf{u}_h(Q)-\mathbf{u}_h(P_h),\mathbf{v}_h)-d(T_h(Q)-T_h(P_h),\mathbf{v}_h).\end{aligned}\tag{5.30}$$

Moreover, it follows from the *inf-sup* condition that

$$\begin{aligned}&\|p_h(Q)-p_h(P_h)\|_{0,\Omega}\\ &\le\frac{1}{\beta_0}\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{V}_h}\frac{b(\mathbf{v}_h,p_h(Q)-p_h(P_h))}{\|\mathbf{v}_h\|_{\mathbf{H}^1}}\\ &\le C\{\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|_{\mathbf{H}^1(\Omega)}+\|T_h(Q)-T_h(P_h)\|_{1,\Omega}\}\le Ch^2.\end{aligned}\tag{5.31}$$

5.5. Proof of Lemma 4.4

From (4.9), we have

$$\begin{aligned}&a_0(\mathbf{w}_h(Q)-\mathbf{w}_h(P_h),v_h)+c_0(v_h,\mathbf{u}_h(Q),\mathbf{w}_h(Q)-\mathbf{w}_h(P_h))\\ &\qquad+c_0(v_h,\mathbf{u}_h(Q)-\mathbf{u}_h(P_h),\mathbf{w}_h(P_h))+c_0(\mathbf{u}_h(Q),v_h,\mathbf{w}_h(Q)-\mathbf{w}_h(P_h))\\ &\qquad+c_0(\mathbf{u}_h(Q)-\mathbf{u}_h(P_h),v_h,\mathbf{w}_h(P_h))+c_1(v_h,T_h(Q)-T_h(P_h),\varphi_h(Q))\\ &\qquad+c_1(v_h,T_h(P_h),\varphi_h(Q)-\varphi_h(P_h))\\ &\quad=(\mathbf{u}_h(Q)-\mathbf{u}_h(P_h),v_h)+b(v_h,\sigma_h(Q)-\sigma_h(P_h))\qquad\forall\ v_h\in\mathbf{V}_h,\\ &b(\mathbf{w}_h(Q)-\mathbf{w}_h(P_h),q_h)=0\qquad\forall\ q_h\in X_h,\\ &a_1(\varphi_h(Q)-\varphi_h(P_h),S_h)+c_1(\mathbf{u}_h(Q),S_h,\varphi_h(Q)-\varphi_h(P_h))\\ &\qquad+c_1(\mathbf{u}_h(Q)-\mathbf{u}_h(P_h),S_h,\varphi_h(P_h))\\ &\quad=d(S_h,\mathbf{w}_h(Q)-\mathbf{w}_h(P_h))\qquad\forall\ S_h\in V_h.\end{aligned}\tag{5.32}$$

Using standard arguments, it is easy to get that

$$\begin{aligned}&\|\mathbf{w}_h(Q)-\mathbf{w}_h(P_h)\|^2_{\mathbf{H}^1(\Omega)}+\|\varphi_h(Q)-\varphi_h(P_h)\|^2_{1,\Omega}\\ &\le C\{\|\mathbf{u}_h(Q)-\mathbf{u}_h(P_h)\|^2_{\mathbf{H}^1(\Omega)}+\|T_h(Q)-T_h(P_h)\|^2_{1,\Omega}\}\le Ch^4.\end{aligned}\tag{5.33}$$

Hence, using the same discussion as in the proof of (5.30) and (5.31) yields

$$\|\sigma_h(Q) - \sigma_h(P_h)\|_{0,\Omega} \leq Ch^2.$$

This completes the proof of Lemma 4.4.

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