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STRONG STABILITY PRESERVING PROPERTY OF THE DEFERRED CORRECTION TIME DISCRETIZATION*

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Abstract

In this paper, we study the strong stability preserving (SSP) property of a class of deferred correction time discretization methods, for solving the method-of-lines schemes approximating hyperbolic partial differential equations.

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1. Introduction

In this paper, we are interested in the numerical solutions of hyperbolic partial differential equations (PDEs). A typical example is the nonlinear conservation law

$$u_t = -f(u)_x. \tag{1.1}$$

A commonly used approach to design numerical schemes for approximating such PDEs is to first design a stable spatial discretization, obtaining the following method-of-lines ordinary differential equation (ODE) system,

$$u_t = L(u), \tag{1.2}$$

to approximate (1.1). Notice that even though we use the same letter u in (1.1) and (1.2), they have different meanings. In (1.1), $u = u(x,t)$ is a function of x and t, while in (1.2), $u = u(t)$ is a (vector) function of t only. Stable spatial discretization for (1.1) includes, for example, the total variation diminishing (TVD) methods [6], the weighted essentially non-oscillatory (WENO) methods [7], and the discontinuous Galerkin (DG) methods [1]. In this paper, we *assume* that the spatial discretization (1.2) is stable for the first-order Euler forward time discretization

$$u^{n+1} = u^n + \Delta t L(u^n) \tag{1.3}$$

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under a suitable time step restriction

$$\Delta t \le \Delta t_0. \tag{1.4}$$

This stability is given as

$$\|u^{n+1}\| \le \|u^n\| \tag{1.5}$$

for a suitable norm or semi-norm $\|\cdot\|$. For the TVD schemes [6], $\|\cdot\|$ is taken as the total variation semi-norm. For technical reasons, we would also need a different but closely related spatial discretization to (1.1):

$$u_t = \tilde{L}(u) \tag{1.6}$$

with the property that the first-order "backward" time discretization

$$u^{n+1} = u^n - \Delta t \tilde{L}(u^n) \tag{1.7}$$

is stable in the sense of (1.5) under the same time step restriction (1.4). For the conservation law (1.1), the operator $\tilde{L}$ can often be obtained simply by reversing the wind direction in the upwind approximation. We refer to, e.g., [1, 7, 11] for such implementation in ENO, WENO and DG methods.

Even though the fully discretized scheme (1.3) is assumed to be stable as in (1.5), it is only first-order accurate in time. For a high-order spatial discretization such as in the WENO and DG methods, we would certainly hope to have higher-order accuracy in time as well. A higher-order time discretization for (1.2) is called strong stability preserving (SSP) with a CFL coefficient c, if it is stable in the sense of (1.5) under a possibly modified time step restriction

$$\Delta t \le c\,\Delta t_0. \tag{1.8}$$

SSP time discretizations were first developed in [10] for multi-step methods and in [11] for Runge-Kutta methods. They were referred to as TVD time discretizations in these papers, since the semi-norm involved in the stability (1.5) was the total variation semi-norm. More general SSP time discretizations can be found in, e.g., [3, 4, 12, 13]. The review paper [5] summarizes the development of the SSP method until the time of its publication.

In this paper we study the SSP property of a newly developed time discretization technique, namely the (spectral) deferred correction (DC) method constructed in [2]. An advantage of this method is that it is a one step method (namely, to march to time level $n+1$ one would only need to store the value of the solution at time level n) and can be constructed easily and systematically for any order of accuracy. This is in contrast to Runge-Kutta methods which are more difficult to construct for higher order of accuracy, and to multi-step methods which need more storage space and are more difficult to restart with a different choice of the time step Δt. Linear stability, such as the A-stability, $A(\alpha)$-stability, or L-stability issues for the DC methods were studied in, e.g., [2, 8, 14]. However, for approximating hyperbolic equations such as (1.1) with discontinuous solutions, linear stability may not be enough and one would hope the time discretization to have the SSP property as well.

The $(s+1)$-th order DC time discretization to (1.2) that we consider in this paper can be formulated as follows. We first divide the time step $[t^n, t^{n+1}]$, where

$$t^{n+1} = t^n + \Delta t$$

into s subintervals by choosing the points $t^{(m)}$ for $m = 0, 1, \cdots, s$ such that

$$t^n = t^{(0)} < t^{(1)} < \cdots < t^{(m)} < \cdots < t^{(s)} = t^{n+1}.$$

We use

$$\Delta t^{(m)} = t^{(m+1)} - t^{(m)}$$

to denote the sub-time step and $u_k^{(m)}$ to denote the k-th order approximation to $u(t^{(m)})$. The nodes $t^{(m)}$ can be chosen equally spaced, or as the Chebyshev Gauss-Lobatto nodes on $[t^n, t^{n+1}]$ for high-order accurate DC schemes to avoid possible instability associated with interpolation on equally spaced points. Starting from u^n, the DC algorithm to calculate u^{n+1} is in the following.

Compute the initial approximation

$u_1^{(0)} = u^n$.

Use the forward Euler method to compute a first-order accurate approximate solution u_1 at the nodes $\{t^{(m)}\}_{m=1}^s$:

For $m = 0, \cdots, s-1$

$$u_1^{(m+1)} = u_1^{(m)} + \Delta t^{(m)} L(u_1^{(m)}). \tag{1.9}$$

Compute successive corrections

For $k = 1, \cdots, s$

$$u_{k+1}^{(0)} = u^n.$$

For $m = 0, \cdots, s-1$

$$u_{k+1}^{(m+1)} = u_{k+1}^{(m)} + \theta_k \Delta t^{(m)} (L(u_{k+1}^{(m)}) - L(u_k^{(m)})) + I_m^{m+1}(L(u_k)), \tag{1.10}$$

where

$$0 \leq \theta_k \leq 1 \tag{1.11}$$

and $I_m^{m+1}(L(u_k))$ is the integral of the s-th degree interpolating polynomial on the $s+1$ points $(t^{(\ell)}, L(u_k^{(\ell)}))_{\ell=0}^s$ over the subinterval $[t^{(m)}, t^{(m+1)}]$, which is the numerical quadrature approximation of

$$\int_{t^{(m)}}^{t^{(m+1)}} L(u(\tau)) d\tau. \tag{1.12}$$

Finally we have

$$u^{n+1} = u_{s+1}^{(s)}.$$

The scheme described above with $\theta_k = 1$ is the one discussed in [2, 8]. In [14], the scheme is also discussed with general $0 \leq \theta_k \leq 1$ to enhance linear stability. The term with the coefficient θ_k does not affect accuracy.

In the next three sections we will study the SSP properties of the DC time discretization for the second-, third-and fourth-order accuracy ($s = 1, 2, 3$), respectively. In Section 5 we will provide a numerical example of using the SSP DC time discretizations coupled with a WENO spatial discretization [7] to solve the Burgers equation. Concluding remarks are given in Section 6.

2. Second-Order Discretization

For the second-order ($s = 1$) DC time discretization, there is no subgrid point inside the interval $[t^n, t^{n+1}]$. We can easily work out the explicit form of the scheme

$$\begin{aligned} u_1^{(1)} &= u^n + \Delta t L(u^n), \\ u^{n+1} &= u^n + \frac{1}{2}\Delta t \left(L(u^n) + L(u_1^{(1)}) \right). \end{aligned} \tag{2.1}$$

Notice that this is exactly the optimal second-order SSP Runge-Kutta scheme originally given in [11] and proven optimal for the SSP property among all second-order Runge-Kutta schemes in [4]. The CFL coefficient c in (1.8) for this scheme is 1.

Even though the SSP property for the scheme (2.1) was already proven in [11, 4], we will prove it again here to illustrate the approach that we will use also for higher-order DC time discretizations. This approach was used in [12] to study SSP Runge-Kutta methods. The first equation in (2.1) is already in Euler forward format. The idea of the proof is to write the second equation in (2.1) as a convex combination of Euler forward steps. That is, for arbitrary α_1, α_2 satisfying

$$\alpha_1 \geq 0, \ \ \alpha_2 \geq 0, \qquad \alpha_1 + \alpha_2 = 1, \tag{2.2}$$

we rewrite the second equation in (2.1) as

$$u^{n+1} = \alpha_1 u^n + \frac{1}{2}\Delta t L(u^n) + \alpha_2 u^n + \frac{1}{2}\Delta t L(u_1^{(1)})$$

and substitute the first equation in (2.1) into the $\alpha_2 u^n$ term of the equation above to obtain

$$u^{n+1} = \alpha_1 \left(u^n + \frac{1 - 2\alpha_2}{2\alpha_1}\Delta t L(u^n) \right) + \alpha_2 \left(u_1^{(1)} + \frac{1}{2\alpha_2}\Delta t L(u_1^{(1)}) \right). \tag{2.3}$$

Clearly, this is a convex combination of two Euler forward steps. By assumption, the first-order Euler forward step (1.3) is stable in the sense of (1.5) under the time step restriction (1.4), hence it is clear that (2.3) is stable in the sense of (1.5) under the modified time step restriction

$$\frac{1 - 2\alpha_2}{2\alpha_1}\Delta t \leq \Delta t_0, \qquad \frac{1}{2\alpha_2}\Delta t \leq \Delta t_0.$$

Notice that α_1, α_2 are arbitrary subject to (2.2), hence the CFL coefficient c defined in (1.8) for the step (2.3), hence the scheme (2.1), to be SSP is

$$c = \max \min \left\{ \frac{2\alpha_1}{1 - 2\alpha_2}, \, 2\alpha_2 \right\}, \tag{2.4}$$

where the optimization is taken subject to the constraint (2.2). As in [12], we reformulate the optimization problem (2.4) as

$$c = \max_{\{\alpha_1, \alpha_2\}} z \tag{2.5}$$

subject to the constraint (2.2) and

$$2\alpha_1 \geq z(1 - 2\alpha_2), \qquad 2\alpha_2 \geq z. \tag{2.6}$$

We then use the Matlab routine "fminicon" to obtain the solution c. The Matlab routine produces the optimal solution $c = 1$ achieved at $\alpha_1 = \alpha_2 = 1/2$. This is the same result as the one already obtained in [4] theoretically. Of course, for this simple optimization problem, it is not necessary to use the Matlab routine. However for the more complicated optimization problems later associated with higher-order DC schemes, the usage of this Matlab routine will be helpful.

We remark that the sole purpose of writing the second equation of (2.1) into the mathematically equivalent but more complicated form (2.3) is to obtain the optimal CFL coefficient c in (1.8) for the provable SSP property of the scheme (2.1). In actual computation we would use (2.1) since it is simpler to implement.

3. Third-Order Discretization

For the third-order ($s = 2$) DC time discretization, there is only one subgrid point inside the interval $[t^n, t^{n+1}]$. By symmetry, this point should be placed in the middle, that is,

$$t^{(0)} = t^n, \quad t^{(1)} = t^n + \frac{1}{2}\Delta t, \quad t^{(2)} = t^{n+1}.$$

We can then easily write out the explicit form of the scheme:

$$\begin{aligned}
u_1^{(1)} &= u^n + \frac{1}{2}\Delta t L(u^n), \quad u_1^{(2)} = u_1^{(1)} + \frac{1}{2}\Delta t L(u_1^{(1)}),\\
u_2^{(1)} &= u^n + \frac{1}{2}\Delta t\left(\frac{5}{12}L(u^n) + \frac{2}{3}L(u_1^{(1)}) - \frac{1}{12}L(u_1^{(2)})\right),\\
u_2^{(2)} &= u_2^{(1)} + \frac{1}{2}\theta_1\Delta t\left(L(u_2^{(1)}) - L(u_1^{(1)})\right) + \frac{1}{2}\Delta t\left(-\frac{1}{12}L(u^n) + \frac{2}{3}L(u_1^{(1)}) + \frac{5}{12}L(u_1^{(2)})\right),\\
u_3^{(1)} &= u^n + \frac{1}{2}\Delta t\left(\frac{5}{12}L(u^n) + \frac{2}{3}L(u_2^{(1)}) - \frac{1}{12}L(u_2^{(2)})\right),\\
u^{n+1} &= u_3^{(1)} + \frac{1}{2}\theta_2\Delta t\left(L(u_3^{(1)}) - L(u_2^{(1)})\right) + \frac{1}{2}\Delta t\left(-\frac{1}{12}L(u^n) + \frac{2}{3}L(u_2^{(1)}) + \frac{5}{12}L(u_2^{(2)})\right).
\end{aligned} \tag{3.1}$$

For our analysis, the following equivalent form of the scheme is more convenient:

$$\begin{aligned}
u_1^{(1)} &= u^n + \frac{1}{2}\Delta t L(u^n), \quad u_1^{(2)} = u_1^{(1)} + \frac{1}{2}\Delta t L(u_1^{(1)}),\\
u_2^{(1)} &= u^n + \frac{1}{2}\Delta t\left(\frac{5}{12}L(u^n) + \frac{2}{3}L(u_1^{(1)}) - \frac{1}{12}L(u_1^{(2)})\right),\\
u_2^{(2)} &= u^n + \frac{1}{2}\theta_1\Delta t\left(L(u_2^{(1)}) - L(u_1^{(1)})\right) + \frac{1}{2}\Delta t\left(\frac{1}{3}L(u^n) + \frac{4}{3}L(u_1^{(1)}) + \frac{1}{3}L(u_1^{(2)})\right),\\
u_3^{(1)} &= u^n + \frac{1}{2}\Delta t\left(\frac{5}{12}L(u^n) + \frac{2}{3}L(u_2^{(1)}) - \frac{1}{12}L(u_2^{(2)})\right),\\
u^{n+1} &= u^n + \frac{1}{2}\theta_2\Delta t\left(L(u_3^{(1)}) - L(u_2^{(1)})\right) + \frac{1}{2}\Delta t\left(\frac{1}{3}L(u^n) + \frac{4}{3}L(u_2^{(1)}) + \frac{1}{3}L(u_2^{(2)})\right).
\end{aligned} \tag{3.2}$$

We now attempt to rewrite each equation in (3.2) as a convex combination of forward (or backward) Euler steps, as in the previous section. The first two equations are already of the forward Euler type and would be SSP for a CFL coefficient $c = 2$. We would need to write the remaining equations for $u_2^{(1)}$, $u_2^{(2)}$, $u_3^{(1)}$ and u^{n+1} into convex combinations of forward (or backward) Euler steps. We present the details of this procedure for the last equation involving u^{n+1} only, as the process is similar for the other equations.

To this purpose we take

$$\alpha_{3,1}^{(2)} \geq 0,\ \alpha_{3,2}^{(2)} \geq 0,\ \alpha_{3,3}^{(2)} \geq 0, \alpha_{3,4}^{(2)} \geq 0, \qquad \alpha_{3,1}^{(2)} + \alpha_{3,2}^{(2)} + \alpha_{3,3}^{(2)} + \alpha_{3,4}^{(2)} = 1, \tag{3.3}$$

and further

$$\beta_{3,1}^{(2)} \geq 0,\ \beta_{3,2}^{(2)} \geq 0,\ \beta_{3,3}^{(2)} \geq 0, \qquad \beta_{3,1}^{(2)} + \beta_{3,2}^{(2)} + \beta_{3,3}^{(2)} = \alpha_{3,1}^{(2)}, \tag{3.4}$$

and rewrite the first term u^n on the right-hand side of the last equation in (3.2) as

$$u^n = (\alpha_{3,1}^{(2)} + \alpha_{3,2}^{(2)} + \alpha_{3,3}^{(2)} + \alpha_{3,4}^{(2)})u^n = (\beta_{3,1}^{(2)} + \beta_{3,2}^{(2)} + \beta_{3,3}^{(2)} + \alpha_{3,2}^{(2)} + \alpha_{3,3}^{(2)} + \alpha_{3,4}^{(2)})u^n.$$

After a further algebraic manipulation using all the equations in (3.2), we can then rewrite the last equation in (3.2) into the form

$$\begin{aligned} u^{n+1} = &\left[\beta_{3,1}^{(2)} u^n + \left(\frac{1}{6} - \frac{5}{24}\alpha_{3,2}^{(2)} - \frac{1}{6}\alpha_{3,3}^{(2)} - \frac{5}{24}\alpha_{3,4}^{(2)} - \frac{1}{2}\beta_{3,2}^{(2)} - \frac{1}{2}\beta_{3,3}^{(2)}\right)\Delta t L(u^n)\right] \\ &+ \left[\alpha_{3,2}^{(2)} u_2^{(1)} + \left(\frac{2}{3} - \frac{1}{2}\theta_2 - \frac{1}{2}\theta_1\alpha_{3,3}^{(2)} - \frac{1}{3}\alpha_{3,4}^{(2)}\right)\Delta t L(u_2^{(1)})\right] \\ &+ \left[\alpha_{3,3}^{(2)} u_2^{(2)} + \left(\frac{1}{6} + \frac{1}{24}\alpha_{3,4}^{(2)}\right)\Delta t L(u_2^{(2)})\right] + \left[\alpha_{3,4}^{(2)} u_3^{(1)} + \frac{1}{2}\theta_2 \Delta t L(u_3^{(1)})\right] \\ &+ \left[\beta_{3,2}^{(2)} u_1^{(1)} + \left(\frac{1}{2}\theta_1\alpha_{3,3}^{(2)} - \frac{1}{3}\alpha_{3,2}^{(2)} - \frac{2}{3}\alpha_{3,3}^{(2)} - \frac{1}{2}\beta_{3,3}^{(2)}\right)\Delta t L(u_1^{(1)})\right] \\ &+ \left[\beta_{3,3}^{(2)} u_1^{(2)} + \left(\frac{1}{24}\alpha_{3,2}^{(2)} - \frac{1}{6}\alpha_{3,3}^{(2)}\right)\Delta t L(u_1^{(2)})\right]. \end{aligned} \tag{3.5}$$

To simplify and standardize the notations, we denote

$$a_{2,4}^{(1)} = \alpha_{3,2}^{(2)}, \quad a_{2,4}^{(2)} = \alpha_{3,3}^{(2)}, \quad a_{3,4}^{(1)} = \alpha_{3,4}^{(2)}, \quad a_{1,4}^{(0)} = \beta_{3,1}^{(2)}, \quad a_{1,4}^{(1)} = \beta_{3,2}^{(2)}, \quad a_{1,4}^{(2)} = \beta_{3,3}^{(2)} \tag{3.6}$$

and

$$\begin{aligned} &b_{2,4}^{(1)} = \frac{2}{3} - \frac{1}{2}\theta_2 - \frac{1}{2}\theta_1\alpha_{3,3}^{(2)} - \frac{1}{3}\alpha_{3,4}^{(2)}, \quad b_{2,4}^{(2)} = \frac{1}{6} + \frac{1}{24}\alpha_{3,4}^{(2)}, \\ &b_{3,4}^{(1)} = \frac{1}{2}\theta_2, \quad b_{1,4}^{(0)} = \frac{1}{6} - \frac{5}{24}\alpha_{3,2}^{(2)} - \frac{1}{6}\alpha_{3,3}^{(2)} - \frac{5}{24}\alpha_{3,4}^{(2)} - \frac{1}{2}\beta_{3,2}^{(2)} - \frac{1}{2}\beta_{3,3}^{(2)}, \\ &b_{1,4}^{(1)} = -\frac{1}{3}\alpha_{3,2}^{(2)} + \frac{1}{2}\theta_1\alpha_{3,3}^{(2)} - \frac{2}{3}\alpha_{3,3}^{(2)} - \frac{1}{2}\beta_{3,3}^{(2)}, \quad b_{1,4}^{(2)} = \frac{1}{24}\alpha_{3,2}^{(2)} - \frac{1}{6}\alpha_{3,3}^{(2)} \end{aligned} \tag{3.7}$$

and write (3.5) as

$$u^{n+1} = \sum_{i,j} \left(a_{j,4}^{(i)} u_j^{(i)} + b_{j,4}^{(i)} \Delta t L(u_j^{(i)})\right). \tag{3.8}$$

Similarly, we obtain

$$u_2^{(1)} = \sum_{i,j} \left(a_{j,1}^{(i)} u_j^{(i)} + b_{j,1}^{(i)} \Delta t L(u_j^{(i)})\right), \tag{3.9}$$

with

$$a_{1,1}^{(0)} = \alpha_{2,1}^{(1)}, \quad a_{1,1}^{(1)} = \alpha_{2,2}^{(1)}, \quad a_{1,1}^{(2)} = \alpha_{2,3}^{(1)},$$
$$b_{1,1}^{(0)} = \frac{5}{24} - \frac{1}{2}\alpha_{2,2}^{(1)} - \frac{1}{2}\alpha_{2,3}^{(1)}, \quad b_{1,1}^{(1)} = \frac{1}{3} - \frac{1}{2}\alpha_{2,3}^{(1)}, \quad b_{1,1}^{(2)} = -\frac{1}{24}, \tag{3.10}$$

where

$$\alpha_{2,1}^{(1)} \geq 0,\ \alpha_{2,2}^{(1)} \geq 0,\ \alpha_{2,3}^{(1)} \geq 0, \quad \alpha_{2,1}^{(1)} + \alpha_{2,2}^{(1)} + \alpha_{2,3}^{(1)} = 1. \tag{3.11}$$

Moreover,

$$u_2^{(2)} = \sum_{i,j} \left(a_{j,2}^{(i)} u_j^{(i)} + b_{j,2}^{(i)} \Delta t L(u_j^{(i)}) \right), \tag{3.12}$$

with

$$a_{1,2}^{(0)} = \alpha_{2,1}^{(2)}, \quad a_{1,2}^{(1)} = \alpha_{2,2}^{(2)}, \quad a_{1,2}^{(2)} = \alpha_{2,3}^{(2)}, \quad a_{2,2}^{(1)} = \alpha_{2,4}^{(2)},$$
$$b_{1,2}^{(0)} = \frac{1}{6} - \frac{1}{2}\alpha_{2,2}^{(2)} - \frac{1}{2}\alpha_{2,3}^{(2)} - \frac{5}{24}\alpha_{2,4}^{(2)}, \quad b_{1,2}^{(1)} = \frac{2}{3} - \frac{1}{2}\theta_1 - \frac{1}{2}\alpha_{2,3}^{(2)} - \frac{1}{3}\alpha_{2,4}^{(2)}, \tag{3.13}$$
$$b_{1,2}^{(2)} = \frac{1}{6} + \frac{1}{24}\alpha_{2,4}^{(2)}, \quad b_{2,2}^{(1)} = \frac{1}{2}\theta_1,$$

where

$$\alpha_{2,1}^{(2)} \geq 0,\ \alpha_{2,2}^{(2)} \geq 0,\ \alpha_{2,3}^{(2)} \geq 0, \alpha_{2,4}^{(2)} \geq 0, \quad \alpha_{2,1}^{(2)} + \alpha_{2,2}^{(2)} + \alpha_{2,3}^{(2)} + \alpha_{2,4}^{(2)} = 1. \tag{3.14}$$

And finally,

$$u_3^{(1)} = \sum_{i,j} \left(a_{j,3}^{(i)} u_j^{(i)} + b_{j,3}^{(i)} \Delta t L(u_j^{(i)}) \right), \tag{3.15}$$

with

$$a_{1,3}^{(0)} = \beta_{3,1}^{(1)}, \quad a_{1,3}^{(1)} = \beta_{3,2}^{(1)}, \quad a_{1,3}^{(2)} = \beta_{3,3}^{(1)}, \quad a_{2,3}^{(1)} = \alpha_{3,2}^{(1)}, \quad a_{2,3}^{(2)} = \alpha_{3,3}^{(1)},$$
$$b_{1,3}^{(0)} = \frac{5}{24} - \frac{5}{24}\alpha_{3,2}^{(1)} - \frac{1}{6}\alpha_{3,3}^{(1)} - \frac{1}{2}\beta_{3,2}^{(1)} - \frac{1}{2}\beta_{3,3}^{(1)}, \quad b_{1,3}^{(1)} = \frac{1}{2}\theta_1 \alpha_{3,3}^{(1)} - \frac{1}{3}\alpha_{3,2}^{(1)} - \frac{2}{3}\alpha_{3,3}^{(1)} - \frac{1}{2}\beta_{3,3}^{(1)}, \tag{3.16}$$
$$b_{1,3}^{(2)} = \frac{1}{24}\alpha_{3,2}^{(1)} - \frac{1}{6}\alpha_{3,3}^{(1)}, \quad b_{2,3}^{(1)} = \frac{1}{3} - \frac{1}{2}\theta_1 \alpha_{3,3}^{(1)}, \quad b_{2,3}^{(2)} = -\frac{1}{24},$$

where

$$\alpha_{3,1}^{(1)} \geq 0,\ \alpha_{3,2}^{(1)} \geq 0,\ \alpha_{3,3}^{(1)} \geq 0, \quad \alpha_{3,1}^{(1)} + \alpha_{3,2}^{(1)} + \alpha_{3,3}^{(1)} = 1, \tag{3.17}$$

and further

$$\beta_{3,1}^{(1)} \geq 0,\ \beta_{3,2}^{(1)} \geq 0,\ \beta_{3,3}^{(1)} \geq 0, \quad \beta_{3,1}^{(1)} + \beta_{3,2}^{(1)} + \beta_{3,3}^{(1)} = \alpha_{3,1}^{(1)}. \tag{3.18}$$

We have now written all the equations in (3.2) as convex combinations of forward or backward Euler steps, depending on the signs of $b_{j,k}^{(i)}$, in (3.8), (3.9), (3.12) and (3.15). We notice, from their definitions in (3.7), (3.10), (3.13) and (3.16), that $b_{2,4}^{(2)}$, $b_{3,4}^{(1)}$, $b_{1,2}^{(2)}$ and $b_{2,2}^{(1)}$ are always non-negative, $b_{1,1}^{(2)}$ and $b_{2,3}^{(2)}$ are always non-positive, and the other $b_{j,k}^{(i)}$ could be either positive or negative, at least *a priori*. Because of our stability assumption (1.5) for the Euler forward step (1.3) and the Euler backward step (1.7), we would need to replace the operator $L(u_{j,k}^{(i)})$ by $\tilde{L}(u_{j,k}^{(i)})$ when the corresponding $b_{j,k}^{(i)}$ is negative. After this modification, the scheme (3.2) is clearly SSP under the modified time step restriction (1.8) with the choice of the CFL coefficient

$$c = \max \min_{i,j,k} \left\{ \frac{a_{j,k}^{(i)}}{|b_{j,k}^{(i)}|} \right\} \tag{3.19}$$

subject to the restrictions (1.11), (3.3), (3.4), (3.11), (3.14), (3.17) and (3.18).

As before, we optimize the equivalent problem:

$$c = \max_{\{\alpha_{j,k}^{(i)},\beta_{j,k}^{(i)}\}} z \tag{3.20}$$

subject to the restrictions (1.11), (3.3), (3.4), (3.11), (3.14), (3.17) and (3.18), and, for all the relevant i, j and k,

$$a_{j,k}^{(i)} \geq z|b_{j,k}^{(i)}| \tag{3.21}$$

by the Matlab routine "fminicon". As mentioned before, when the resulting $b_{j,k}^{(i)}$ is negative, we will change the relevant $L(u_{j,k}^{(i)})$ by $\tilde{L}(u_{j,k}^{(i)})$. The optimal scheme in terms of the CFL coefficient (1.8) is the following

$$\begin{aligned}
u_1^{(1)} =& u^n + \frac{1}{2}\Delta t L(u^n), \quad u_1^{(2)} = u_1^{(1)} + \frac{1}{2}\Delta t L(u_1^{(1)}),\\
u_2^{(1)} =& \left(a_{1,1}^{(0)}u^n + b_{1,1}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,1}^{(1)}u_1^{(1)} + b_{1,1}^{(1)}\Delta t L(u_1^{(1)})\right) + \left(a_{1,1}^{(2)}u_1^{(2)} + b_{1,1}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right),\\
u_2^{(2)} =& \left(a_{1,2}^{(0)}u^n + b_{1,2}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,2}^{(1)}u_1^{(1)} + b_{1,2}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right)\\
&+ \left(a_{1,2}^{(2)}u_1^{(2)} + b_{1,2}^{(2)}\Delta t L(u_1^{(2)})\right) + \left(a_{2,2}^{(1)}u_2^{(1)} + b_{2,2}^{(1)}\Delta t L(u_2^{(1)})\right),\\
u_3^{(1)} =& \left(a_{1,3}^{(0)}u^n + b_{1,3}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,3}^{(1)}u_1^{(1)} + b_{1,3}^{(1)}\Delta t L(u_1^{(1)})\right) + \left(a_{1,3}^{(2)}u_1^{(2)} + b_{1,3}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right)\\
&+ \left(a_{2,3}^{(1)}u_2^{(1)} + b_{2,3}^{(1)}\Delta t L(u_2^{(1)})\right) + \left(a_{2,3}^{(2)}u_2^{(2)} + b_{2,3}^{(2)}\Delta t\widetilde{L}(u_2^{(2)})\right),\\
u^{n+1} =& \left(a_{1,4}^{(0)}u^n + b_{1,4}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,4}^{(1)}u_1^{(1)} + b_{1,4}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right)\\
&+ \left(a_{1,4}^{(2)}u_1^{(2)} + b_{1,4}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) + \left(a_{2,4}^{(1)}u_2^{(1)} + b_{2,4}^{(1)}\Delta t L(u_2^{(1)})\right)\\
&+ \left(a_{2,4}^{(2)}u_2^{(2)} + b_{2,4}^{(2)}\Delta t L(u_2^{(2)})\right) + \left(a_{3,4}^{(1)}u_3^{(1)} + b_{3,4}^{(1)}\Delta t L(u_3^{(1)})\right),
\end{aligned} \tag{3.22}$$

with the coefficients $a_{j,k}^{(i)}$ and $b_{j,k}^{(i)}$ given by (3.6), (3.7), (3.10), (3.13) and (3.16), and

$$\begin{aligned}
&\alpha_{2,1}^{(1)} = 0.2912, \quad \alpha_{2,2}^{(1)} = 0.2911, \quad \alpha_{2,1}^{(2)} = 0.1374, \quad \alpha_{2,2}^{(2)} = 0.0736,\\
&\alpha_{2,3}^{(2)} = 0.2453, \quad \alpha_{3,1}^{(1)} = 0.5026, \quad \alpha_{3,2}^{(1)} = 0.3664, \quad \beta_{3,1}^{(1)} = 0.1284,\\
&\beta_{3,2}^{(1)} = 0.2686, \quad \alpha_{3,1}^{(2)} = 0.2457, \quad \alpha_{3,2}^{(2)} = 0.0000, \quad \alpha_{3,3}^{(2)} = 0.2435,\\
&\beta_{3,1}^{(2)} = 0.0811, \quad \beta_{3,2}^{(2)} = 0.1120, \quad \theta_1 = 0.8393, \quad \theta_2 = 0.7884.
\end{aligned} \tag{3.23}$$

The CFL coefficient for this scheme is $c = 1.2956$. Therefore, we have proved the following result.

Theorem 3.1. *The third-order DC scheme* (3.22)-(3.23) *is SSP under the time step restriction* (1.8) *with the CFL coefficient* $c = 1.2956$.

Even though the CFL coefficient for the scheme (3.22)-(3.23) is reasonably high, it requires 10 evaluations of L or $\widetilde{L}$. Comparing with the optimal SSP third-order Runge-Kutta method in [4, 11], which has a CFL coefficient 1 and requires only 3 evaluations of L, the third order SSP DC scheme (3.22)-(3.23) is much less efficient. Of course, since we have used an optimization routine to obtain the optimal value of c, we cannot guarantee that we have obtained the theoretical optimal value of this CFL coefficient. Theorem 3.1 provides therefore only a lower bound of the CFL coefficient to guarantee SSP. The actual DC scheme may be SSP for a larger value of the CFL coefficient.

If our objective is to have as few evaluations of L or $\widetilde{L}$ as possible, we may require as many $b_{j,k}^{(i)}$ to be positive as possible. A careful search reveals that we need at least 9 evaluations of L or $\widetilde{L}$ to obtain a SSP scheme. This leads to the following third-order DC scheme:

$$\begin{aligned}
u_1^{(1)} =& u^n + \frac{1}{2}\Delta t L(u^n), \quad u_1^{(2)} = u_1^{(1)} + \frac{1}{2}\Delta t L(u_1^{(1)}),\\
u_2^{(1)} =& \left(a_{1,1}^{(0)}u^n + b_{1,1}^{(0)}\Delta t L(u^n)\right) + \left(a_{1,1}^{(1)}u_1^{(1)} + b_{1,1}^{(1)}\Delta t L(u_1^{(1)})\right) + \left(a_{1,1}^{(2)}u_1^{(2)} + b_{1,1}^{(2)}\Delta t \widetilde{L}(u_1^{(2)})\right),\\
u_2^{(2)} =& \left(a_{1,2}^{(0)}u^n + b_{1,2}^{(0)}\Delta t L(u^n)\right) + \left(a_{1,2}^{(1)}u_1^{(1)} + b_{1,2}^{(1)}\Delta t L(u_1^{(1)})\right)\\
&+ \left(a_{1,2}^{(2)}u_1^{(2)} + b_{1,2}^{(2)}\Delta t L(u_1^{(2)})\right) + \left(a_{2,2}^{(1)}u_2^{(1)} + b_{2,2}^{(1)}\Delta t L(u_2^{(1)})\right),\\
u_3^{(1)} =& \left(a_{1,3}^{(0)}u^n + b_{1,3}^{(0)}\Delta t L(u^n)\right) + \left(a_{1,3}^{(1)}u_1^{(1)} + b_{1,3}^{(1)}\Delta t L(u_1^{(1)})\right) + \left(a_{1,3}^{(2)}u_1^{(2)} + b_{1,3}^{(2)}\Delta t L(u_1^{(2)})\right)\\
&+ \left(a_{2,3}^{(1)}u_2^{(1)} + b_{2,3}^{(1)}\Delta t L(u_2^{(1)})\right) + \left(a_{2,3}^{(2)}u_2^{(2)} + b_{2,3}^{(2)}\Delta t \widetilde{L}(u_2^{(2)})\right),\\
u^{n+1} =& \left(a_{1,4}^{(0)}u^n + b_{1,4}^{(0)}\Delta t L(u^n)\right) + \left(a_{1,4}^{(1)}u_1^{(1)} + b_{1,4}^{(1)}\Delta t \widetilde{L}(u_1^{(1)})\right)\\
&+ \left(a_{1,4}^{(2)}u_1^{(2)} + b_{1,4}^{(2)}\Delta t \widetilde{L}(u_1^{(2)})\right) + \left(a_{2,4}^{(1)}u_2^{(1)} + b_{2,4}^{(1)}\Delta t L(u_2^{(1)})\right)\\
&+ \left(a_{2,4}^{(2)}u_2^{(2)} + b_{2,4}^{(2)}\Delta t L(u_2^{(2)})\right) + \left(a_{3,4}^{(1)}u_3^{(1)} + b_{3,4}^{(1)}\Delta t L(u_3^{(1)})\right),
\end{aligned} \tag{3.24}$$

with the coefficients $a_{j,k}^{(i)}$ and $b_{j,k}^{(i)}$ given by (3.6), (3.7), (3.10), (3.13) and (3.16), and

$$\begin{aligned}
&\alpha_{2,1}^{(1)} = 0.5833, \quad \alpha_{2,2}^{(1)} = 0.2041, \quad \alpha_{2,1}^{(2)} = 0.4310, \quad \alpha_{2,2}^{(2)} = 0.0000,\\
&\alpha_{2,3}^{(2)} = 0.1650, \quad \alpha_{3,1}^{(1)} = 0.6266, \quad \alpha_{3,2}^{(1)} = 0.3065, \quad \beta_{3,1}^{(1)} = 0.3603,\\
&\beta_{3,2}^{(1)} = 0.1550, \quad \alpha_{3,1}^{(2)} = 0.3654, \quad \alpha_{3,2}^{(2)} = 0.0593, \quad \alpha_{3,3}^{(2)} = 0.1652,\\
&\beta_{3,1}^{(2)} = 0.2827, \quad \beta_{3,2}^{(2)} = 0.0602, \quad \theta_1 = 0.8990, \quad \theta_2 = 0.9115.
\end{aligned} \tag{3.25}$$

The CFL coefficient for this scheme is $c = 0.8990$. Apparently, this scheme has a much smaller CFL coefficient and only 1 fewer evaluation of L or $\widetilde{L}$ than that of the scheme (3.22)-(3.23), hence is much less efficient.

As indicated in the introduction, the original spectral deferred correction scheme in [2, 8] corresponds to $\theta_1 = \theta_2 = 1$. Within this subclass, we apply our optimization procedure above to obtain the following third-order DC scheme:

$$\begin{aligned}
u_1^{(1)} =& u^n + \frac{1}{2}\Delta t L(u^n), \quad u_1^{(2)} = u_1^{(1)} + \frac{1}{2}\Delta t L(u_1^{(1)}),\\
u_2^{(1)} =& \left(a_{1,1}^{(0)}u^n + b_{1,1}^{(0)}\Delta t \widetilde{L}(u^n)\right) + \left(a_{1,1}^{(1)}u_1^{(1)} + b_{1,1}^{(1)}\Delta t L(u_1^{(1)})\right) + \left(a_{1,1}^{(2)}u_1^{(2)} + b_{1,1}^{(2)}\Delta t \widetilde{L}(u_1^{(2)})\right),\\
u_2^{(2)} =& \left(a_{1,2}^{(0)}u^n + b_{1,2}^{(0)}\Delta t \widetilde{L}(u^n)\right) + \left(a_{1,2}^{(1)}u_1^{(1)} + b_{1,2}^{(1)}\Delta t \widetilde{L}(u_1^{(1)})\right)\\
&+ \left(a_{1,2}^{(2)}u_1^{(2)} + b_{1,2}^{(2)}\Delta t L(u_1^{(2)})\right) + \left(a_{2,2}^{(1)}u_2^{(1)} + b_{2,2}^{(1)}\Delta t L(u_2^{(1)})\right),\\
u_3^{(1)} =& \left(a_{1,3}^{(0)}u^n + b_{1,3}^{(0)}\Delta t \widetilde{L}(u^n)\right) + \left(a_{1,3}^{(1)}u_1^{(1)} + b_{1,3}^{(1)}\Delta t L(u_1^{(1)})\right) + \left(a_{1,3}^{(2)}u_1^{(2)} + b_{1,3}^{(2)}\Delta t \widetilde{L}(u_1^{(2)})\right)\\
&+ \left(a_{2,3}^{(1)}u_2^{(1)} + b_{2,3}^{(1)}\Delta t L(u_2^{(1)})\right) + \left(a_{2,3}^{(2)}u_2^{(2)} + b_{2,3}^{(2)}\Delta t \widetilde{L}(u_2^{(2)})\right),
\end{aligned} \tag{3.26}$$

$$\begin{aligned} u^{n+1} =& \left(a_{1,4}^{(0)}u^n + b_{1,4}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,4}^{(1)}u_1^{(1)} + b_{1,4}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) \\ &+ \left(a_{1,4}^{(2)}u_1^{(2)} + b_{1,4}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) + \left(a_{2,4}^{(1)}u_2^{(1)} + b_{2,4}^{(1)}\Delta t\widetilde{L}(u_2^{(1)})\right) \\ &+ \left(a_{2,4}^{(2)}u_2^{(2)} + b_{2,4}^{(2)}\Delta tL(u_2^{(2)})\right) + \left(a_{3,4}^{(1)}u_3^{(1)} + b_{3,4}^{(1)}\Delta tL(u_3^{(1)})\right), \end{aligned}$$

with the coefficients $a_{j,k}^{(i)}$ and $b_{j,k}^{(i)}$ given by (3.6), (3.7), (3.10), (3.13) and (3.16), and

$$\begin{aligned} &\alpha_{2,1}^{(1)} = 0.3333, \quad \alpha_{2,2}^{(1)} = 0.3333, \quad \alpha_{2,1}^{(2)} = 0.1405, \quad \alpha_{2,2}^{(2)} = 0.1405, \\ &\alpha_{2,3}^{(2)} = 0.1977, \quad \alpha_{3,1}^{(1)} = 0.5636, \quad \alpha_{3,2}^{(1)} = 0.2552, \quad \beta_{3,1}^{(1)} = 0.1814, \\ &\beta_{3,2}^{(1)} = 0.2124, \quad \alpha_{3,1}^{(2)} = 0.1742, \quad \alpha_{3,2}^{(2)} = 0.1092, \quad \alpha_{3,3}^{(2)} = 0.1961, \\ &\beta_{3,1}^{(2)} = 0.0577, \quad \beta_{3,2}^{(2)} = 0.0872, \quad \theta_1 = 1.0000, \quad \theta_2 = 1.0000. \end{aligned} \tag{3.27}$$

The CFL coefficient for this scheme is $c = 1.0411$. However, it needs 11 evaluations of L or $\widetilde{L}$. Therefore, it is much less efficient than the scheme (3.22)-(3.23).

Within the subclass of $\theta_1 = \theta_2 = 1$, we can also explore SSP schemes with as few evaluations of L or $\widetilde{L}$ as possible. We would still need at least 9 evaluations of L or $\widetilde{L}$ to obtain a SSP scheme, namely (3.24) with the coefficients $a_{j,k}^{(i)}$ and $b_{j,k}^{(i)}$ given by (3.6), (3.7), (3.10), (3.13) and (3.16), and

$$\begin{aligned} &\alpha_{2,1}^{(1)} = 0.5866, \quad \alpha_{2,2}^{(1)} = 0.2058, \quad \alpha_{2,1}^{(2)} = 0.4773, \quad \alpha_{2,2}^{(2)} = 0.0811, \\ &\alpha_{2,3}^{(2)} = 0.1170, \quad \alpha_{3,1}^{(1)} = 0.6133, \quad \alpha_{3,2}^{(1)} = 0.3112, \quad \beta_{3,1}^{(1)} = 0.3535, \\ &\beta_{3,2}^{(1)} = 0.1298, \quad \alpha_{3,1}^{(2)} = 0.3786, \quad \alpha_{3,2}^{(2)} = 0.1799, \quad \alpha_{3,3}^{(2)} = 0.1170, \\ &\beta_{3,1}^{(2)} = 0.2945, \quad \beta_{3,2}^{(2)} = 0.0599, \quad \theta_1 = 1.0000, \quad \theta_2 = 1.0000. \end{aligned} \tag{3.28}$$

The CFL coefficient for this scheme is $c = 0.6491$, which is not very impressive.

Finally, we consider a special class of the third-order DC scheme (3.1), in which $\theta_2 = 0$. In this subclass, we do not need to evaluate $u_3^{(1)}$, hence this may lead to a scheme with fewer evaluations of L or $\widetilde{L}$. After removing the constraints associated with the evaluation of $u_3^{(1)}$ and setting $\theta_2 = 0$, the optimization procedure described above yields the following scheme within this subclass:

$$\begin{aligned} u_1^{(1)} =& u^n + \frac{1}{2}\Delta tL(u^n), \quad u_1^{(2)} = u_1^{(1)} + \frac{1}{2}\Delta tL(u_1^{(1)}), \\ u_2^{(1)} =& \left(a_{1,1}^{(0)}u^n + b_{1,1}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,1}^{(1)}u_1^{(1)} + b_{1,1}^{(1)}\Delta tL(u_1^{(1)})\right) + \left(a_{1,1}^{(2)}u_1^{(2)} + b_{1,1}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right), \\ u_2^{(2)} =& \left(a_{1,2}^{(0)}u^n + b_{1,2}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,2}^{(1)}u_1^{(1)} + b_{1,2}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) \\ &+ \left(a_{1,2}^{(2)}u_1^{(2)} + b_{1,2}^{(2)}\Delta tL(u_1^{(2)})\right) + \left(a_{2,2}^{(1)}u_2^{(1)} + b_{2,2}^{(1)}\Delta tL(u_2^{(1)})\right), \\ u^{n+1} =& \left(a_{1,4}^{(0)}u^n + b_{1,4}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,4}^{(1)}u_1^{(1)} + b_{1,4}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,4}^{(2)}u_1^{(2)} + b_{1,4}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) \\ &+ \left(a_{2,4}^{(1)}u_2^{(1)} + b_{2,4}^{(1)}\Delta tL(u_2^{(1)})\right) + \left(a_{2,4}^{(2)}u_2^{(2)} + b_{2,4}^{(2)}\Delta tL(u_2^{(2)})\right), \end{aligned} \tag{3.29}$$

with the coefficients $a_{j,k}^{(i)}$ and $b_{j,k}^{(i)}$ given by (3.6), (3.7), (3.10) and (3.13), and

$$\begin{aligned} &\alpha_{2,1}^{(1)} = 0.3238, \quad \alpha_{2,2}^{(1)} = 0.3237, \quad \alpha_{2,1}^{(2)} = 0.1264, \quad \alpha_{2,2}^{(2)} = 0.2204, \\ &\alpha_{2,3}^{(2)} = 0.1774, \quad \alpha_{3,1}^{(2)} = 0.2825, \quad \alpha_{3,2}^{(2)} = 0.5589, \quad \alpha_{3,3}^{(2)} = 0.1586, \\ &\beta_{3,1}^{(2)} = 0.0757, \quad \beta_{3,2}^{(2)} = 0.2038, \quad \theta_1 = 1.0000, \quad \theta_2 = 0.0000. \end{aligned} \tag{3.30}$$

The CFL coefficient for this scheme is $c = 0.9515$. This scheme is still less efficient than the scheme (3.22)-(3.23), even though it has only 8 evaluations of L or $\widetilde{L}$, 2 fewer than the scheme (3.22)-(3.23) has.

We can further reduce the number of evaluations of L or $\widetilde{L}$ to 7 within this subclass, yielding the following scheme:

$$
\begin{aligned}
u_1^{(1)} =& u^n + \frac{1}{2}\Delta t L(u^n), \quad u_1^{(2)} = u_1^{(1)} + \frac{1}{2}\Delta t L(u_1^{(1)}),\\
u_2^{(1)} =& \left(a_{1,1}^{(0)}u^n + b_{1,1}^{(0)}\Delta t L(u^n)\right) + \left(a_{1,1}^{(1)}u_1^{(1)} + b_{1,1}^{(1)}\Delta t L(u_1^{(1)})\right) + \left(a_{1,1}^{(2)}u_1^{(2)} + b_{1,1}^{(2)}\Delta t \widetilde{L}(u_1^{(2)})\right),\\
u_2^{(2)} =& \left(a_{1,2}^{(0)}u^n + b_{1,2}^{(0)}\Delta t L(u^n)\right) + \left(a_{1,2}^{(1)}u_1^{(1)} + b_{1,2}^{(1)}\Delta t \widetilde{L}(u_1^{(1)})\right)\\
&+ \left(a_{1,2}^{(2)}u_1^{(2)} + b_{1,2}^{(2)}\Delta t L(u_1^{(2)})\right) + \left(a_{2,2}^{(1)}u_2^{(1)} + b_{2,2}^{(1)}\Delta t L(u_2^{(1)})\right),\\
u^{n+1} =& \left(a_{1,4}^{(0)}u^n + b_{1,4}^{(0)}\Delta t L(u^n)\right) + \left(a_{1,4}^{(1)}u_1^{(1)} + b_{1,4}^{(1)}\Delta t \widetilde{L}(u_1^{(1)})\right) + \left(a_{1,4}^{(2)}u_1^{(2)} + b_{1,4}^{(2)}\Delta t \widetilde{L}(u_1^{(2)})\right)\\
&+ \left(a_{2,4}^{(1)}u_2^{(1)} + b_{2,4}^{(1)}\Delta t L(u_2^{(1)})\right) + \left(a_{2,4}^{(2)}u_2^{(2)} + b_{2,4}^{(2)}\Delta t L(u_2^{(2)})\right),
\end{aligned}
\tag{3.31}
$$

with the coefficients $a_{j,k}^{(i)}$ and $b_{j,k}^{(i)}$ given by (3.6), (3.7), (3.10) and (3.13), and

$$
\begin{aligned}
&\alpha_{2,1}^{(1)} = 0.5862, \quad \alpha_{2,2}^{(1)} = 0.2481, \quad \alpha_{2,1}^{(2)} = 0.4252, \quad \alpha_{2,2}^{(2)} = 0.0293,\\
&\alpha_{2,3}^{(2)} = 0.1315, \quad \alpha_{3,1}^{(2)} = 0.4546, \quad \alpha_{3,2}^{(2)} = 0.4281, \quad \alpha_{3,3}^{(2)} = 0.1173,\\
&\beta_{3,1}^{(2)} = 0.3387, \quad \beta_{3,2}^{(2)} = 0.1147, \quad \theta_1 = 1.0000, \quad \theta_2 = 0.0000.
\end{aligned}
\tag{3.32}
$$

The CFL coefficient for this scheme is $c = 0.7040$. This scheme is slightly less efficient than the scheme (3.29)-(3.30).

4. Fourth-Order Discretization

For the fourth-order ($s = 3$) DC time discretization, there are two subgrid points inside the interval $[t^n, t^{n+1}]$. By symmetry, these two points should be placed at $t^{(1)} = t^n + a\Delta t$ and $t^{(2)} = t^n + (1-a)\Delta t$ respectively for $0 < a < \frac{1}{2}$. We will only consider the standard choice of the Chebyshev Gauss-Lobatto nodes with $a = (5-\sqrt{5})/10$, however, see Remark 4.1 for the general case of arbitrary a.

With the choice of the Chebyshev Gauss-Lobatto nodes, we can easily write out the fourth-order DC scheme

$$
\begin{aligned}
u_1^{(1)} =& u^n + \gamma_0\Delta t L(u^n), \quad u_1^{(2)} = u_1^{(1)} + \frac{\sqrt{5}}{5}\Delta t L(u_1^{(1)}), \quad u_1^{(3)} = u_1^{(2)} + \gamma_0\Delta t L(u_1^{(2)}),\\
u_2^{(1)} =& u^n + \frac{1}{2}\Delta t\left(\gamma_1^+ L(u^n) + \gamma_2^- L(u_1^{(1)}) + \gamma_3^- L(u_1^{(2)}) + \gamma_4^- L(u_1^{(3)})\right),\\
u_2^{(2)} =& u_2^{(1)} + \frac{\sqrt{5}}{5}\Delta t\theta_1\left(L(u_2^{(1)}) - L(u_1^{(1)})\right)\\
&+ \frac{1}{2}\Delta t\left(-\frac{\sqrt{5}}{30}L(u^n) + \frac{7\sqrt{5}}{30}L(u_1^{(1)}) + \frac{7\sqrt{5}}{30}L(u_1^{(2)}) - \frac{\sqrt{5}}{30}L(u_1^{(3)})\right),\\
u_2^{(3)} =& u_2^{(2)} + \gamma_0\Delta t\theta_2\left(L(u_2^{(2)}) - L(u_1^{(2)})\right)\\
&+ \frac{1}{2}\Delta t\left(\gamma_4^- L(u^n) + \gamma_3^- L(u_1^{(1)}) + \gamma_2^- L(u_1^{(2)}) + \gamma_1^+ L(u_1^{(3)})\right),
\end{aligned}
$$

$$
\begin{aligned}
u_3^{(1)} =& u^n + \frac{1}{2}\Delta t\left(\gamma_1^+ L(u^n) + \gamma_2^- L(u_2^{(1)}) + \gamma_3^- L(u_2^{(2)}) + \gamma_4^- L(u_2^{(3)})\right),\\
u_3^{(2)} =& u_3^{(1)} + \frac{\sqrt{5}}{5}\Delta t\theta_3\left(L(u_3^{(1)}) - L(u_2^{(1)})\right)\\
&+ \frac{1}{2}\Delta t\left(\frac{-\sqrt{5}}{30}L(u^n) + \frac{7\sqrt{5}}{30}L(u_2^{(1)}) + \frac{7\sqrt{5}}{30}L(u_2^{(2)}) - \frac{\sqrt{5}}{30}L(u_2^{(3)})\right),\\
u_3^{(3)} =& u_3^{(2)} + \gamma_0\Delta t\theta_4\left(L(u_3^{(2)}) - L(u_2^{(2)})\right)\\
&+ \frac{1}{2}\Delta t\left(\gamma_4^- L(u^n) + \gamma_3^- L(u_2^{(1)}) + \gamma_2^- L(u_2^{(2)}) + \gamma_1^+ L(u_2^{(3)})\right),\\
u_4^{(1)} =& u^n + \frac{1}{2}\Delta t\left(\gamma_1^+ L(u^n) + \gamma_2^- L(u_3^{(1)}) + \gamma_3^- L(u_3^{(2)}) + \gamma_4^- L(u_3^{(3)})\right),\\
u_4^{(2)} =& u_4^{(1)} + \frac{\sqrt{5}}{5}\Delta t\theta_5\left(L(u_4^{(1)}) - L(u_3^{(1)})\right)\\
&+ \frac{1}{2}\Delta t\left(-\frac{\sqrt{5}}{30}L(u^n) + \frac{7\sqrt{5}}{30}L(u_3^{(1)}) + \frac{7\sqrt{5}}{30}L(u_3^{(2)}) - \frac{\sqrt{5}}{30}L(u_3^{(3)})\right),\\
u^{n+1} =& u_4^{(2)} + \gamma_0\Delta t\theta_6\left(L(u_4^{(2)}) - L(u_3^{(2)})\right)\\
&+ \frac{1}{2}\Delta t\left(\gamma_4^- L(u^n) + \gamma_3^- L(u_3^{(1)}) + \gamma_2^- L(u_3^{(2)}) + \gamma_1^+ L(u_3^{(3)})\right),
\end{aligned}
\tag{4.1a}
$$

where

$$
\begin{aligned}
&\gamma_0 = \frac{5-\sqrt{5}}{10}, \quad \gamma_1^\pm = \frac{11\pm\sqrt{5}}{60}, \quad \gamma_2^\pm = \frac{25\pm\sqrt{5}}{60},\\
&\gamma_3^\pm = \frac{25\pm 13\sqrt{5}}{60}, \quad \gamma_4^\pm = \frac{\sqrt{5}\pm 1}{60}.
\end{aligned}
\tag{4.1b}
$$

We can rewrite (4.1) into an equivalent form similar to (3.2), then attempt to rewrite each equation as a convex combination of forward (or backward) Euler steps, as in the previous section. The first three equations are already of the forward Euler type and would be SSP for a CFL coefficient $c > 2$. As to the fourth equation, we can rewrite it as

$$
u_2^{(1)} = \sum_{i,j}\left(a_{j,1}^{(i)} u_j^{(i)} + b_{j,1}^{(i)}\Delta t L(u_j^{(i)})\right) \tag{4.2}
$$

with

$$
\begin{aligned}
&a_{1,1}^{(0)} = \alpha_{2,1}^{(1)}, \quad a_{1,1}^{(1)} = \alpha_{2,2}^{(1)}, \quad a_{1,1}^{(2)} = \alpha_{2,3}^{(1)}, \quad a_{1,1}^{(3)} = \alpha_{2,4}^{(1)},\\
&b_{1,1}^{(0)} = \frac{1}{2}\gamma_1^+ - \gamma_0\alpha_{2,2}^{(1)} - \gamma_0\alpha_{2,3}^{(1)} - \gamma_0\alpha_{2,4}^{(1)},\\
&b_{1,1}^{(1)} = \frac{1}{2}\gamma_2^- - \frac{\sqrt{5}}{5}\alpha_{2,3}^{(1)} - \frac{\sqrt{5}}{5}\alpha_{2,4}^{(1)},\\
&b_{1,1}^{(2)} = \frac{1}{2}\gamma_3^- - \gamma_0\alpha_{2,4}^{(1)}, \quad b_{1,1}^{(3)} = \frac{1}{2}\gamma_4^-,
\end{aligned}
\tag{4.3}
$$

where

$$
\alpha_{2,1}^{(1)} \ge 0,\ \alpha_{2,2}^{(1)} \ge 0,\ \alpha_{2,3}^{(1)} \ge 0,\ \alpha_{2,4}^{(1)} \ge 0, \quad \alpha_{2,1}^{(1)} + \alpha_{2,2}^{(1)} + \alpha_{2,3}^{(1)} + \alpha_{2,4}^{(1)} = 1. \tag{4.4}
$$

Similarly, the fifth equation in (4.1) can be rewritten as

$$
u_2^{(2)} = \sum_{i,j}\left(a_{j,2}^{(i)} u_j^{(i)} + b_{j,2}^{(i)}\Delta t L(u_j^{(i)})\right) \tag{4.5}
$$

with

$$\begin{aligned}
&a_{1,2}^{(0)}=\alpha_{2,1}^{(2)}, \quad a_{1,2}^{(1)}=\alpha_{2,2}^{(2)}, \quad a_{1,2}^{(2)}=\alpha_{2,3}^{(2)}, \quad a_{1,2}^{(3)}=\alpha_{2,4}^{(2)}, \quad a_{2,2}^{(1)}=\alpha_{2,5}^{(2)},\\
&b_{1,2}^{(0)}=\frac{1}{2}\gamma_1^- -\gamma_0\alpha_{2,2}^{(2)}-\gamma_0\alpha_{2,3}^{(2)}-\gamma_0\alpha_{2,4}^{(2)}-\frac{1}{2}\gamma_1^+\alpha_{2,5}^{(2)},\\
&b_{1,2}^{(1)}=\frac{1}{2}\gamma_3^+ -\frac{\sqrt{5}}{5}\theta_1-\frac{\sqrt{5}}{5}\alpha_{2,3}^{(2)}-\frac{\sqrt{5}}{5}\alpha_{2,4}^{(2)}-\frac{1}{2}\gamma_2^-\alpha_{2,5}^{(2)},\\
&b_{1,2}^{(2)}=\frac{1}{2}\gamma_2^+ -\gamma_0\alpha_{2,4}^{(2)}-\frac{1}{2}\gamma_3^-\alpha_{2,5}^{(2)},\\
&b_{1,2}^{(3)}=-\frac{1}{2}\gamma_4^+ -\frac{1}{2}\gamma_4^-\alpha_{2,5}^{(2)}, \quad b_{2,2}^{(1)}=\frac{\sqrt{5}}{5}\theta_1,
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
&\alpha_{2,1}^{(2)}\ge 0, \ \ \alpha_{2,2}^{(2)}\ge 0, \ \ \alpha_{2,3}^{(2)}\ge 0, \ \ \alpha_{2,4}^{(2)}\ge 0, \ \ \alpha_{2,5}^{(2)}\ge 0,\\
&\alpha_{2,1}^{(2)}+\alpha_{2,2}^{(2)}+\alpha_{2,3}^{(2)}+\alpha_{2,4}^{(2)}+\alpha_{2,5}^{(2)}=1.
\end{aligned} \tag{4.7}$$

The sixth equation in (4.1) can be rewritten as

$$u_2^{(3)}=\sum_{i,j}\left(a_{j,3}^{(i)}u_j^{(i)}+b_{j,3}^{(i)}\Delta tL(u_j^{(i)})\right) \tag{4.8}$$

with

$$\begin{aligned}
&a_{1,3}^{(0)}=\alpha_{2,1}^{(3)}, \quad a_{1,3}^{(1)}=\alpha_{2,2}^{(3)}, \quad a_{1,3}^{(2)}=\alpha_{2,3}^{(3)}, \quad a_{1,3}^{(3)}=\alpha_{2,4}^{(3)}, \quad a_{2,3}^{(1)}=\alpha_{2,5}^{(3)}, \quad a_{2,3}^{(2)}=\alpha_{2,6}^{(3)},\\
&b_{1,3}^{(0)}=\frac{1}{12}-\gamma_0\alpha_{2,2}^{(3)}-\gamma_0\alpha_{2,3}^{(3)}-\gamma_0\alpha_{2,4}^{(3)}-\frac{1}{2}\gamma_1^+\alpha_{2,5}^{(3)}-\frac{1}{2}\gamma_1^-\alpha_{2,6}^{(3)},\\
&b_{1,3}^{(1)}=\frac{5}{12}-\frac{\sqrt{5}}{5}\theta_1-\frac{\sqrt{5}}{5}\alpha_{2,3}^{(3)}-\frac{\sqrt{5}}{5}\alpha_{2,4}^{(3)}-\frac{1}{2}\gamma_2^-\alpha_{2,5}^{(3)}+\frac{\sqrt{5}}{5}\theta_1\alpha_{2,6}^{(3)}-\frac{1}{2}\gamma_3^+\alpha_{2,6}^{(3)},\\
&b_{1,3}^{(2)}=\frac{5}{12}-\gamma_0\theta_2-\gamma_0\alpha_{2,4}^{(3)}-\frac{1}{2}\gamma_3^-\alpha_{2,5}^{(3)}-\frac{1}{2}\gamma_2^+\alpha_{2,6}^{(3)},\\
&b_{1,3}^{(3)}=\frac{1}{12}-\frac{1}{2}\gamma_4^-\alpha_{2,5}^{(3)}+\frac{1}{2}\gamma_4^+\alpha_{2,6}^{(3)},\\
&b_{2,3}^{(1)}=\frac{\sqrt{5}}{5}\theta_1-\frac{\sqrt{5}}{5}\theta_1\alpha_{2,6}^{(3)}, \quad b_{2,3}^{(2)}=\gamma_0\theta_2,
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
&\alpha_{2,1}^{(3)}\ge 0, \ \ \alpha_{2,2}^{(3)}\ge 0, \ \ \alpha_{2,3}^{(3)}\ge 0, \ \ \alpha_{2,4}^{(3)}\ge 0, \ \ \alpha_{2,5}^{(3)}\ge 0, \ \ \alpha_{2,6}^{(3)}\ge 0,\\
&\alpha_{2,1}^{(3)}+\alpha_{2,2}^{(3)}+\alpha_{2,3}^{(3)}+\alpha_{2,4}^{(3)}+\alpha_{2,5}^{(3)}+\alpha_{2,6}^{(3)}=1.
\end{aligned} \tag{4.10}$$

The seventh equation in (4.1) can be rewritten as

$$u_3^{(1)}=\sum_{i,j}\left(a_{j,4}^{(i)}u_j^{(i)}+b_{j,4}^{(i)}\Delta tL(u_j^{(i)})\right) \tag{4.11}$$

with

$$\begin{aligned}
&a_{1,4}^{(0)} = \beta_{3,1}^{(1)},\ a_{1,4}^{(1)} = \beta_{3,2}^{(1)},\ a_{1,4}^{(2)} = \beta_{3,3}^{(1)},\ a_{1,4}^{(3)} = \beta_{3,4}^{(1)},\ a_{2,4}^{(1)} = \alpha_{3,2}^{(1)},\ a_{2,4}^{(2)} = \alpha_{3,3}^{(1)},\ a_{2,4}^{(3)} = \alpha_{3,4}^{(1)},\\
&b_{1,4}^{(0)} = \frac{1}{2}\gamma_1^+ - \frac{1}{2}\gamma_1^+\alpha_{3,2}^{(1)} - \frac{1}{2}\gamma_1^- a_{3,3}^{(1)} - \frac{1}{12}\alpha_{3,4}^{(1)} - \gamma_0\beta_{3,2}^{(1)} - \gamma_0\beta_{3,3}^{(1)} - \gamma_0\beta_{3,4}^{(1)},\\
&b_{1,4}^{(1)} = -\frac{1}{2}\gamma_2^-\alpha_{3,2}^{(1)} + \frac{\sqrt{5}}{5}\theta_1\alpha_{3,3}^{(1)} - \frac{1}{2}\gamma_3^+\alpha_{3,3}^{(1)} + \frac{\sqrt{5}}{5}\theta_1\alpha_{3,4}^{(1)} - \frac{5}{12}\alpha_{3,4}^{(1)} - \frac{\sqrt{5}}{5}\beta_{3,3}^{(1)} - \frac{\sqrt{5}}{5}\beta_{3,4}^{(1)},\\
&b_{1,4}^{(2)} = -\frac{1}{2}\gamma_3^-\alpha_{3,2}^{(1)} - \frac{1}{2}\gamma_2^+\alpha_{3,3}^{(1)} + \gamma_0\theta_2\alpha_{3,4}^{(1)} - \frac{5}{12}\alpha_{3,4}^{(1)} - \gamma_0\beta_{3,4}^{(1)},\\
&b_{1,4}^{(3)} = -\frac{1}{2}\gamma_4^-\alpha_{3,2}^{(1)} + \frac{1}{2}\gamma_4^+\alpha_{3,3}^{(1)} - \frac{1}{12}\alpha_{3,4}^{(1)},\\
&b_{2,4}^{(1)} = \frac{1}{2}\gamma_2^- - \frac{\sqrt{5}}{5}\theta_1\alpha_{3,3}^{(1)} - \frac{\sqrt{5}}{5}\theta_1\alpha_{3,4}^{(1)},\ b_{2,4}^{(2)} = \frac{1}{2}\gamma_3^- - \gamma_0\theta_2\alpha_{3,4}^{(1)},\ b_{2,4}^{(3)} = \frac{1}{2}\gamma_4^-,
\end{aligned} \tag{4.12}$$

where

$$\alpha_{3,1}^{(1)} \geq 0,\ \alpha_{3,2}^{(1)} \geq 0,\ \alpha_{3,3}^{(1)} \geq 0,\ \alpha_{3,4}^{(1)} \geq 0, \quad \alpha_{3,1}^{(1)} + \alpha_{3,2}^{(1)} + \alpha_{3,3}^{(1)} + \alpha_{3,4}^{(1)} = 1, \tag{4.13}$$

and further

$$\beta_{3,1}^{(1)} \geq 0,\ \beta_{3,2}^{(1)} \geq 0,\ \beta_{3,3}^{(1)} \geq 0,\ \beta_{3,4}^{1)} \geq 0, \quad \beta_{3,1}^{(1)} + \beta_{3,2}^{(1)} + \beta_{3,3}^{(1)} + \beta_{3,4}^{(1)} = \alpha_{3,1}^{(1)}. \tag{4.14}$$

The eighth equation in (4.1) can be rewritten as

$$u_3^{(2)} = \sum_{i,j}\left(a_{j,5}^{(i)}u_j^{(i)} + b_{j,5}^{(i)}\Delta t L(u_j^{(i)})\right) \tag{4.15}$$

with

$$\begin{aligned}
a_{1,5}^{(0)} &= \beta_{3,1}^{(2)}, \quad a_{1,5}^{(1)} = \beta_{3,2}^{(2)}, \quad a_{1,5}^{(2)} = \beta_{3,3}^{(2)}, \quad a_{1,5}^{(3)} = \beta_{3,4}^{(2)},\\
a_{2,5}^{(1)} &= \alpha_{3,2}^{(2)}, \quad a_{2,5}^{(2)} = \alpha_{3,3}^{(2)}, \quad a_{2,5}^{(3)} = \alpha_{3,4}^{(2)}, \quad a_{3,5}^{(1)} = \alpha_{3,5}^{(2)}\\
b_{1,5}^{(0)} &= \frac{1}{2}\gamma_1^- - \frac{1}{2}\gamma_1^+\alpha_{3,2}^{(2)} - \frac{1}{2}\gamma_1^-\alpha_{3,3}^{(2)} - \frac{1}{12}\alpha_{3,4}^{(2)} - \frac{1}{2}\gamma_1^+\alpha_{3,5}^{(2)} - \gamma_0\beta_{3,2}^{(2)} - \gamma_0\beta_{3,3}^{(2)} - \gamma_0\beta_{3,4}^{(2)},\\
b_{1,5}^{(1)} &= -\frac{1}{2}\gamma_2^-\alpha_{3,2}^{(2)} + \frac{\sqrt{5}}{5}\theta_1\alpha_{3,3}^{(2)} - \frac{1}{2}\gamma_3^+\alpha_{3,3}^{(2)} + \frac{\sqrt{5}}{5}\theta_1\alpha_{3,4}^{(2)} - \frac{5}{12}\alpha_{3,4}^{(2)} - \frac{\sqrt{5}}{5}\beta_{3,3}^{(2)} - \frac{\sqrt{5}}{5}\beta_{3,4}^{(2)},\\
b_{1,5}^{(2)} &= -\frac{1}{2}\gamma_3^-\alpha_{3,2}^{(2)} - \frac{1}{2}\gamma_2^+\alpha_{3,3}^{(2)} + \gamma_0\theta_2\alpha_{3,4}^{(2)} - \frac{5}{12}\alpha_{3,4}^{(2)} - \gamma_0\beta_{3,4}^{(2)},\\
b_{1,5}^{(3)} &= -\frac{1}{2}\gamma_4^-\alpha_{3,2}^{(2)} + \frac{1}{2}\gamma_4^+\alpha_{3,3}^{(2)} - \frac{1}{12}\alpha_{3,4}^{(2)},\\
b_{2,5}^{(1)} &= \frac{1}{2}\gamma_3^+ - \frac{\sqrt{5}}{5}\theta_3 - \frac{\sqrt{5}}{5}\theta_1\alpha_{3,3}^{(2)} - \frac{\sqrt{5}}{5}\theta_1\alpha_{3,4}^{(2)} - \frac{1}{2}\gamma_2^-\alpha_{3,5}^{(2)},\\
b_{2,5}^{(2)} &= \frac{1}{2}\gamma_2^+ - \gamma_0\theta_2\alpha_{3,4}^{(2)} - \frac{1}{2}\gamma_3^-\alpha_{3,5}^{(2)},\\
b_{2,5}^{(3)} &= -\frac{1}{2}\gamma_4^+ - \frac{1}{2}\gamma_4^-\alpha_{3,5}^{(2)}, \quad b_{3,5}^{(1)} = \frac{\sqrt{5}}{5}\theta_3,
\end{aligned} \tag{4.16}$$

where

$$\begin{aligned}
&\alpha_{3,1}^{(2)} \geq 0, \quad \alpha_{3,2}^{(2)} \geq 0, \quad \alpha_{3,3}^{(2)} \geq 0, \quad \alpha_{3,4}^{(2)} \geq 0, \quad \alpha_{3,5}^{(2)} \geq 0,\\
&\alpha_{3,1}^{(2)} + \alpha_{3,2}^{(2)} + \alpha_{3,3}^{(2)} + \alpha_{3,4}^{(2)} + \alpha_{3,5}^{(2)} = 1,
\end{aligned} \tag{4.17}$$

and further

$$\beta_{3,1}^{(2)} \geq 0,\ \beta_{3,2}^{(2)} \geq 0,\ \beta_{3,3}^{(2)} \geq 0,\ \beta_{3,4}^{(2)} \geq 0, \quad \beta_{3,1}^{(2)} + \beta_{3,2}^{(2)} + \beta_{3,3}^{(2)} + \beta_{3,4}^{(2)} = \alpha_{3,1}^{(2)}. \tag{4.18}$$

The ninth equation in (4.1) can be rewritten as

$$u_3^{(3)} = \sum_{i,j} \left(a_{j,6}^{(i)} u_j^{(i)} + b_{j,6}^{(i)} \Delta t L(u_j^{(i)}) \right) \tag{4.19}$$

with

$$\begin{aligned}
a_{1,6}^{(0)} =& \beta_{3,1}^{(3)}, \quad a_{1,6}^{(1)} = \beta_{3,2}^{(3)}, \quad a_{1,6}^{(2)} = \beta_{3,3}^{(3)}, \quad a_{1,6}^{(3)} = \beta_{3,4}^{(3)}, \\
a_{2,6}^{(1)} =& \alpha_{3,2}^{(3)}, \quad a_{2,6}^{(2)} = \alpha_{3,3}^{(3)}, \quad a_{2,6}^{(3)} = \alpha_{3,4}^{(3)}, \quad a_{3,6}^{(1)} = \alpha_{3,5}^{(3)}, \quad a_{3,6}^{(2)} = \alpha_{3,6}^{(3)}, \\
b_{1,6}^{(0)} =& \frac{1}{12} - \frac{1}{2}\gamma_1^+ \alpha_{3,2}^{(3)} - \frac{1}{2}\gamma_1^- \alpha_{3,3}^{(3)} - \frac{1}{12}\alpha_{3,4}^{(3)} - \frac{1}{2}\gamma_1^+ \alpha_{3,5}^{(3)} - \frac{1}{2}\gamma_1^- \alpha_{3,6}^{(3)} \\
& - \gamma_0 \beta_{3,2}^{(3)} - \gamma_0 \beta_{3,3}^{(3)} - \gamma_0 \beta_{3,4}^{(3)}, \\
b_{1,6}^{(1)} =& -\frac{1}{2}\gamma_2^- \alpha_{3,2}^{(3)} + \frac{\sqrt{5}}{5}\theta_1 \alpha_{3,3}^{(3)} - \frac{1}{2}\gamma_3^+ \alpha_{3,3}^{(3)} + \frac{\sqrt{5}}{5}\theta_1 \alpha_{3,4}^{(3)} - \frac{5}{12}\alpha_{3,4}^{(3)} - \frac{\sqrt{5}}{5}\beta_{3,3}^{(3)} - \frac{\sqrt{5}}{5}\beta_{3,4}^{(3)}, \\
b_{1,6}^{(2)} =& -\frac{1}{2}\gamma_3^- \alpha_{3,2}^{(3)} - \frac{1}{2}\gamma_2^+ \alpha_{3,3}^{(3)} + \gamma_0 \theta_2 \alpha_{3,4}^{(3)} - \frac{5}{12}\alpha_{3,4}^{(3)} - \gamma_0 \beta_{3,4}^{(3)}, \\
b_{1,6}^{(3)} =& -\frac{1}{2}\gamma_4^- \alpha_{3,2}^{(3)} + \frac{1}{2}\gamma_4^+ \alpha_{3,3}^{(3)} - \frac{1}{12}\alpha_{3,4}^{(3)}, \\
b_{2,6}^{(1)} =& \frac{5}{12} - \frac{\sqrt{5}}{5}\theta_3 - \frac{\sqrt{5}}{5}\theta_1 \alpha_{3,3}^{(3)} - \frac{\sqrt{5}}{5}\theta_1 \alpha_{3,4}^{(3)} - \frac{1}{2}\gamma_2^- \alpha_{3,5}^{(3)} + \frac{\sqrt{5}}{5}\theta_3 \alpha_{3,6}^{(3)} - \frac{1}{2}\gamma_3^+ \alpha_{3,6}^{(3)}, \\
b_{2,6}^{(2)} =& \frac{5}{12} - \gamma_0 \theta_4 - \gamma_0 \theta_2 \alpha_{3,4}^{(3)} - \frac{1}{2}\gamma_3^- \alpha_{3,5}^{(3)} - \frac{1}{2}\gamma_2^+ \alpha_{3,6}^{(3)}, \\
b_{2,6}^{(3)} =& \frac{1}{12} - \frac{1}{2}\gamma_4^- \alpha_{3,5}^{(3)} + \frac{1}{2}\gamma_4^+ \alpha_{3,6}^{(3)}, \\
b_{3,6}^{(1)} =& \frac{\sqrt{5}}{5}\theta_3 - \frac{\sqrt{5}}{5}\theta_3 \alpha_{3,6}^{(3)}, \quad b_{3,6}^{(2)} = \gamma_0 \theta_4,
\end{aligned} \tag{4.20}$$

where

$$\begin{aligned}
& \alpha_{3,1}^{(3)} \geq 0, \quad \alpha_{3,2}^{(3)} \geq 0, \quad \alpha_{3,3}^{(3)} \geq 0, \quad \alpha_{3,4}^{(3)} \geq 0, \quad \alpha_{3,5}^{(3)} \geq 0, \quad \alpha_{3,6}^{(3)} \geq 0, \\
& \alpha_{3,1}^{(3)} + \alpha_{3,2}^{(3)} + \alpha_{3,3}^{(3)} + \alpha_{3,4}^{(3)} + \alpha_{3,5}^{(3)} + \alpha_{3,6}^{(3)} = 1,
\end{aligned} \tag{4.21}$$

and further

$$\begin{aligned}
& \beta_{3,1}^{(3)} \geq 0, \quad \beta_{3,2}^{(3)} \geq 0, \quad \beta_{3,3}^{(3)} \geq 0, \quad \beta_{3,4}^{(3)} \geq 0, \\
& \beta_{3,1}^{(3)} + \beta_{3,2}^{(3)} + \beta_{3,3}^{(3)} + \beta_{3,4}^{(3)} = \alpha_{3,1}^{(3)}.
\end{aligned} \tag{4.22}$$

The tenth equation in (4.1) can be rewritten as

$$u_4^{(1)} = \sum_{i,j} \left(a_{j,7}^{(i)} u_j^{(i)} + b_{j,7}^{(i)} \Delta t L(u_j^{(i)}) \right) \tag{4.23}$$

with

$$
\begin{aligned}
a_{1,7}^{(0)} =&\gamma_{4,1}^{(1)}, \quad a_{1,7}^{(1)} = \gamma_{4,2}^{(1)}, \quad a_{1,7}^{(2)} = \gamma_{4,3}^{(1)}, \quad a_{1,7}^{(3)} = \gamma_{4,4}^{(1)}, \quad a_{2,7}^{(1)} = \beta_{4,2}^{(1)},\\
a_{2,7}^{(2)} =&\beta_{4,3}^{(1)}, \quad a_{2,7}^{(3)} = \beta_{4,4}^{(1)}, \quad a_{3,7}^{(1)} = \alpha_{4,2}^{(1)}, \quad a_{3,7}^{(2)} = \alpha_{4,3}^{(1)}, \quad a_{3,7}^{(3)} = \alpha_{4,4}^{(1)},\\
b_{1,7}^{(0)} =&\frac{1}{2}\gamma_1^+ - \frac{1}{2}\gamma_1^+\alpha_{4,2}^{(1)} - \frac{1}{2}\gamma_1^-\alpha_{4,3}^{(1)} - \frac{1}{12}\alpha_{4,4}^{(1)} - \frac{1}{2}\gamma_1^+\beta_{4,2}^{(1)} - \frac{1}{2}\gamma_1^-\beta_{4,3}^{(1)}\\
&- \frac{1}{12}\beta_{4,4}^{(1)} - \gamma_0\gamma_{4,2}^{(1)} - \gamma_0\gamma_{4,3}^{(1)} - \gamma_0\gamma_{4,4}^{(1)},\\
b_{1,7}^{(1)} =& -\frac{1}{2}\gamma_2^-\beta_{4,2}^{(1)} + \frac{\sqrt{5}}{5}\theta_1\beta_{4,3}^{(1)} - \frac{1}{2}\gamma_3^+\beta_{4,3}^{(1)} + \frac{\sqrt{5}}{5}\theta_1\beta_{4,4}^{(1)} - \frac{5}{12}\beta_{4,4}^{(1)} - \frac{\sqrt{5}}{5}\gamma_{4,3}^{(1)} - \frac{\sqrt{5}}{5}\gamma_{4,4}^{(1)},\\
b_{1,7}^{(2)} =& -\frac{1}{2}\gamma_3^-\beta_{4,2}^{(1)} - \frac{1}{2}\gamma_2^+\beta_{4,3}^{(1)} + \gamma_0\theta_2\beta_{4,4}^{(1)} - \frac{5}{12}\beta_{4,4}^{(1)} - \gamma_0\gamma_{4,4}^{(1)},\\
b_{1,7}^{(3)} =& -\frac{1}{2}\gamma_4^-\beta_{4,2}^{(1)} + \frac{1}{2}\gamma_4^+\beta_{4,3}^{(1)} - \frac{1}{12}\beta_{4,4}^{(1)},\\
b_{2,7}^{(1)} =& -\frac{1}{2}\gamma_2^-\alpha_{4,2}^{(1)} + \frac{\sqrt{5}}{5}\theta_3\alpha_{4,3}^{(1)} - \frac{1}{2}\gamma_3^+\alpha_{4,3}^{(1)} + \frac{\sqrt{5}}{5}\theta_3\alpha_{4,4}^{(1)} - \frac{5}{12}\alpha_{4,4}^{(1)} - \frac{\sqrt{5}}{5}\theta_1\beta_{4,3}^{(1)} - \frac{\sqrt{5}}{5}\theta_1\beta_{4,4}^{(1)},\\
b_{2,7}^{(2)} =& -\frac{1}{2}\gamma_3^-\alpha_{4,2}^{(1)} - \frac{1}{2}\gamma_2^+\alpha_{4,3}^{(1)} + \gamma_0\theta_4\alpha_{4,4}^{(1)} - \frac{5}{12}\alpha_{4,4}^{(1)} - \gamma_0\theta_2\beta_{4,4}^{(1)},\\
b_{2,7}^{(3)} =& -\frac{1}{2}\gamma_4^-\alpha_{4,2}^{(1)} + \frac{1}{2}\gamma_4^+\alpha_{4,3}^{(1)} - \frac{1}{12}\alpha_{4,4}^{(1)},\\
b_{3,7}^{(1)} =&\frac{1}{2}\gamma_2^- - \frac{\sqrt{5}}{5}\theta_3\alpha_{4,3}^{(1)} - \frac{\sqrt{5}}{5}\theta_3\alpha_{4,4}^{(1)},\\
b_{3,7}^{(2)} =&\frac{1}{2}\gamma_3^- - \gamma_0\theta_4\alpha_{4,4}^{(1)}, \quad b_{3,7}^{(3)} = \frac{1}{2}\gamma_4^-,
\end{aligned}
\tag{4.24}
$$

where

$$\alpha_{4,1}^{(1)} \geq 0,\ \alpha_{4,2}^{(1)} \geq 0,\ \alpha_{4,3}^{(1)} \geq 0,\ \alpha_{4,4}^{(1)} \geq 0, \quad \alpha_{4,1}^{(1)} + \alpha_{4,2}^{(1)} + \alpha_{4,3}^{(1)} + \alpha_{4,4}^{(1)} = 1, \tag{4.25}$$

$$\beta_{4,1}^{(1)} \geq 0,\ \beta_{4,2}^{(1)} \geq 0,\ \beta_{4,3}^{(1)} \geq 0,\ \beta_{4,4}^{(1)} \geq 0, \quad \beta_{4,1}^{(1)} + \beta_{4,2}^{(1)} + \beta_{4,3}^{(1)} + \beta_{4,4}^{(1)} = \alpha_{4,1}^{(1)}, \tag{4.26}$$

and further

$$\gamma_{4,1}^{(1)} \geq 0,\ \gamma_{4,2}^{(1)} \geq 0,\ \gamma_{4,3}^{(1)} \geq 0,\ \gamma_{4,4}^{(1)} \geq 0, \quad \gamma_{4,1}^{(1)} + \gamma_{4,2}^{(1)} + \gamma_{4,3}^{(1)} + \gamma_{4,4}^{(1)} = \beta_{4,1}^{(1)}. \tag{4.27}$$

The eleventh equation in (4.1) can be rewritten as

$$u_4^{(2)} = \sum_{i,j}\left(a_{j,8}^{(i)}u_j^{(i)} + b_{j,8}^{(i)}\Delta t L(u_j^{(i)})\right) \tag{4.28}$$

with

$$
\begin{aligned}
a_{1,8}^{(0)} =&\gamma_{4,1}^{(2)}, \quad a_{1,8}^{(1)} = \gamma_{4,2}^{(2)}, \quad a_{1,8}^{(2)} = \gamma_{4,3}^{(2)}, \quad a_{1,8}^{(3)} = \gamma_{4,4}^{(2)},\\
a_{2,8}^{(1)} =&\beta_{4,2}^{(2)}, \quad a_{2,8}^{(2)} = \beta_{4,3}^{(2)}, \quad a_{2,8}^{(3)} = \beta_{4,4}^{(2)}, \quad a_{3,8}^{(1)} = \alpha_{4,2}^{(2)},\\
a_{3,8}^{(2)} =&\alpha_{4,3}^{(2)}, \quad a_{3,8}^{(3)} = \alpha_{4,4}^{(2)}, \quad a_{4,8}^{(1)} = \alpha_{4,5}^{(2)},
\end{aligned}
$$

$$
\begin{aligned}
b_{1,8}^{(0)} =& \frac{1}{2}\gamma_1^- - \frac{1}{2}\gamma_1^+\alpha_{4,2}^{(2)} - \frac{1}{2}\gamma_1^-\alpha_{4,3}^{(2)} - \frac{1}{12}\alpha_{4,4}^{(2)} - \frac{1}{2}\gamma_1^+\alpha_{4,5}^{(2)} \\
& - \frac{1}{2}\gamma_1^+\beta_{4,2}^{(2)} - \frac{1}{2}\gamma_1^-\beta_{4,3}^{(2)} - \frac{1}{12}\beta_{4,4}^{(2)} - \gamma_0\gamma_{4,2}^{(2)} - \gamma_0\gamma_{4,3}^{(2)} - \gamma_0\gamma_{4,4}^{(2)}, \\
b_{1,8}^{(1)} =& -\frac{1}{2}\gamma_2^-\beta_{4,2}^{(2)} + \frac{\sqrt{5}}{5}\theta_1\beta_{4,3}^{(2)} - \frac{1}{2}\gamma_3^+\beta_{4,3}^{(2)} + \frac{\sqrt{5}}{5}\theta_1\beta_{4,4}^{(2)} - \frac{5}{12}\beta_{4,4}^{(2)} - \frac{\sqrt{5}}{5}\gamma_{4,3}^{(2)} - \frac{\sqrt{5}}{5}\gamma_{4,4}^{(2)}, \\
b_{1,8}^{(2)} =& -\frac{1}{2}\gamma_3^-\beta_{4,2}^{(2)} - \frac{1}{2}\gamma_2^+\beta_{4,3}^{(2)} + \gamma_0\theta_2\beta_{4,4}^{(2)} - \frac{5}{12}\beta_{4,4}^{(2)} - \gamma_0\gamma_{4,4}^{(2)}, \\
b_{1,8}^{(3)} =& -\frac{1}{2}\gamma_4^-\beta_{4,2}^{(2)} + \frac{1}{2}\gamma_4^+\beta_{4,3}^{(2)} - \frac{1}{12}\beta_{4,4}^{(2)}, \qquad (4.29) \\
b_{2,8}^{(1)} =& -\frac{1}{2}\gamma_2^-\alpha_{4,2}^{(2)} + \frac{\sqrt{5}}{5}\theta_3\alpha_{4,3}^{(2)} - \frac{1}{2}\gamma_3^+\alpha_{4,3}^{(2)} + \frac{\sqrt{5}}{5}\theta_3\alpha_{4,4}^{(2)} - \frac{5}{12}\alpha_{4,4}^{(2)} - \frac{\sqrt{5}}{5}\theta_1\beta_{4,3}^{(2)} - \frac{\sqrt{5}}{5}\theta_1\beta_{4,4}^{(2)}, \\
b_{2,8}^{(2)} =& -\frac{1}{2}\gamma_3^-\alpha_{4,2}^{(2)} - \frac{1}{2}\gamma_2^+\alpha_{4,3}^{(2)} + \gamma_0\theta_4\alpha_{4,4}^{(2)} - \frac{5}{12}\alpha_{4,4}^{(2)} - \gamma_0\theta_2\beta_{4,4}^{(2)}, \\
b_{2,8}^{(3)} =& -\frac{1}{2}\gamma_4^-\alpha_{4,2}^{(2)} + \frac{1}{2}\gamma_4^+\alpha_{4,3}^{(2)} - \frac{1}{12}\alpha_{4,4}^{(2)}, \\
b_{3,8}^{(1)} =& \frac{1}{2}\gamma_3^+ - \frac{\sqrt{5}}{5}\theta_5 - \frac{\sqrt{5}}{5}\theta_3\alpha_{4,3}^{(2)} - \frac{\sqrt{5}}{5}\theta_3\alpha_{4,4}^{(2)} - \frac{1}{2}\gamma_2^-\alpha_{4,5}^{(2)}, \\
b_{3,8}^{(2)} =& \frac{1}{2}\gamma_2^+ - \gamma_0\theta_4\alpha_{4,4}^{(2)} - \frac{1}{2}\gamma_3^-\alpha_{4,5}^{(2)}, \\
b_{3,8}^{(3)} =& -\frac{1}{2}\gamma_4^+ - \frac{1}{2}\gamma_4^-\alpha_{4,5}^{(2)}, \quad b_{4,8}^{(1)} = \frac{\sqrt{5}}{5}\theta_5,
\end{aligned}
$$

where

$$
\begin{aligned}
&\alpha_{4,1}^{(2)} \ge 0, \quad \alpha_{4,2}^{(2)} \ge 0, \quad \alpha_{4,3}^{(2)} \ge 0, \quad \alpha_{4,4}^{(2)} \ge 0, \quad \alpha_{4,5}^{(2)} \ge 0, \\
&\alpha_{4,1}^{(2)} + \alpha_{4,2}^{(2)} + \alpha_{4,3}^{(2)} + \alpha_{4,4}^{(2)} + \alpha_{4,5}^{(2)} = 1,
\end{aligned} \qquad (4.30)
$$

and

$$
\beta_{4,1}^{(2)} \ge 0,\ \beta_{4,2}^{(2)} \ge 0,\ \beta_{4,3}^{(2)} \ge 0,\ \beta_{4,4}^{(2)} \ge 0, \quad \beta_{4,1}^{(2)} + \beta_{4,2}^{(2)} + \beta_{4,3}^{(2)} + \beta_{4,4}^{(2)} = \alpha_{4,1}^{(2)}, \qquad (4.31)
$$

and further

$$
\gamma_{4,1}^{(2)} \ge 0,\ \gamma_{4,2}^{(2)} \ge 0,\ \gamma_{4,3}^{(2)} \ge 0,\ \gamma_{4,4}^{(2)} \ge 0, \quad \gamma_{4,1}^{(2)} + \gamma_{4,2}^{(2)} + \gamma_{4,3}^{(2)} + \gamma_{4,4}^{(2)} = \beta_{4,1}^{(2)}. \qquad (4.32)
$$

Finally the twelfth equation in (4.1) can be rewritten as

$$
u^{n+1} = \sum_{i,j} \left(a_{j,9}^{(i)} u_j^{(i)} + b_{j,9}^{(i)} \Delta t L(u_j^{(i)}) \right) \qquad (4.33)
$$

with

$$
\begin{aligned}
a_{1,9}^{(0)} =& \gamma_{4,1}^{(3)}, \quad a_{1,9}^{(1)} = \gamma_{4,2}^{(3)}, \quad a_{1,9}^{(2)} = \gamma_{4,3}^{(3)}, \quad a_{1,9}^{(3)} = \gamma_{4,4}^{(3)}, \quad a_{2,9}^{(1)} = \beta_{4,2}^{(3)}, \quad a_{2,9}^{(2)} = \beta_{4,3}^{(3)}, \\
a_{2,9}^{(3)} =& \beta_{4,4}^{(3)}, \quad a_{3,9}^{(1)} = \alpha_{4,2}^{(3)}, \quad a_{3,9}^{(2)} = \alpha_{4,3}^{(3)}, \quad a_{3,9}^{(3)} = \alpha_{4,4}^{(3)}, \quad a_{4,9}^{(1)} = \alpha_{4,5}^{(3)}, \quad a_{4,9}^{(2)} = \alpha_{4,6}^{(3)}, \\
b_{1,9}^{(0)} =& \frac{1}{12} - \frac{1}{2}\gamma_1^+\alpha_{4,2}^{(3)} - \frac{1}{2}\gamma_1^-\alpha_{4,3}^{(3)} - \frac{1}{12}\alpha_{4,4}^{(3)} - \frac{1}{2}\gamma_1^+\alpha_{4,5}^{(3)} - \frac{1}{2}\gamma_1^-\alpha_{4,6}^{(3)} \\
& - \frac{1}{2}\gamma_1^+\beta_{4,2}^{(3)} - \frac{1}{2}\gamma_1^-\beta_{4,3}^{(3)} - \frac{1}{12}\beta_{4,4}^{(3)} - \gamma_0\gamma_{4,2}^{(3)} - \gamma_0\gamma_{4,3}^{(3)} - \gamma_0\gamma_{4,4}^{(3)},
\end{aligned}
$$

$$
\begin{aligned}
b_{1,9}^{(1)} =& -\frac{1}{2}\gamma_2^-\beta_{4,2}^{(3)} + \frac{\sqrt{5}}{5}\theta_1\beta_{4,3}^{(3)} - \frac{1}{2}\gamma_3^+\beta_{4,3}^{(3)} + \frac{\sqrt{5}}{5}\theta_1\beta_{4,4}^{(3)} - \frac{5}{12}\beta_{4,4}^{(3)} - \frac{\sqrt{5}}{5}\gamma_{4,3}^{(3)} - \frac{\sqrt{5}}{5}\gamma_{4,4}^{(3)}, \\
b_{1,9}^{(2)} =& -\frac{1}{2}\gamma_3^-\beta_{4,2}^{(3)} - \frac{1}{2}\gamma_2^+\beta_{4,3}^{(3)} + \gamma_0\theta_2\beta_{4,4}^{(3)} - \frac{5}{12}\beta_{4,4}^{(3)} - \gamma_0\gamma_{4,4}^{(3)}, \\
b_{1,9}^{(3)} =& -\frac{1}{2}\gamma_4^-\beta_{4,2}^{(3)} + \frac{1}{2}\gamma_4^+\beta_{4,3}^{(3)} - \frac{1}{12}\beta_{4,4}^{(3)}, \\
b_{2,9}^{(1)} =& -\frac{1}{2}\gamma_2^-\alpha_{4,2}^{(3)} + \frac{\sqrt{5}}{5}\theta_3\alpha_{4,3}^{(3)} - \frac{1}{2}\gamma_3^+\alpha_{4,3}^{(3)} + \frac{\sqrt{5}}{5}\theta_3\alpha_{4,4}^{(3)} - \frac{5}{12}\alpha_{4,4}^{(3)} - \frac{\sqrt{5}}{5}\theta_1\beta_{4,3}^{(3)} - \frac{\sqrt{5}}{5}\theta_1\beta_{4,4}^{(3)}, \\
b_{2,9}^{(2)} =& -\frac{1}{2}\gamma_3^-\alpha_{4,2}^{(3)} - \frac{1}{2}\gamma_2^+\alpha_{4,3}^{(3)} + \gamma_0\theta_4\alpha_{4,4}^{(3)} - \frac{5}{12}\alpha_{4,4}^{(3)} - \gamma_0\theta_2\beta_{4,4}^{(3)}, \qquad (4.34) \\
b_{2,9}^{(3)} =& -\frac{1}{2}\gamma_4^-\alpha_{4,2}^{(3)} + \frac{1}{2}\gamma_4^+\alpha_{4,3}^{(3)} - \frac{1}{12}\alpha_{4,4}^{(3)}, \\
b_{3,9}^{(1)} =& \frac{5}{12} - \frac{\sqrt{5}}{5}\theta_5 - \frac{\sqrt{5}}{5}\theta_3\alpha_{4,3}^{(3)} - \frac{\sqrt{5}}{5}\theta_3\alpha_{4,4}^{(3)} - \frac{1}{2}\gamma_2^-\alpha_{4,5}^{(3)} + \frac{\sqrt{5}}{5}\theta_5\alpha_{4,6}^{(3)} - \frac{1}{2}\gamma_3^+\alpha_{4,6}^{(3)}, \\
b_{3,9}^{(2)} =& \frac{5}{12} - \gamma_0\theta_6 - \gamma_0\theta_4\alpha_{4,4}^{(3)} - \frac{1}{2}\gamma_3^-\alpha_{4,5}^{(3)} - \frac{1}{2}\gamma_2^+\alpha_{4,6}^{(3)}, \\
b_{3,9}^{(3)} =& \frac{1}{12} - \frac{1}{2}\gamma_4^-\alpha_{4,5}^{(3)} + \frac{1}{2}\gamma_4^+\alpha_{4,6}^{(3)}, \quad b_{4,9}^{(1)} = \frac{\sqrt{5}}{5}\theta_5 - \frac{\sqrt{5}}{5}\theta_5\alpha_{4,6}^{(3)}, \quad b_{4,9}^{(2)} = \gamma_0\theta_6,
\end{aligned}
$$

where

$$
\begin{aligned}
&\alpha_{4,1}^{(3)} \ge 0, \ \ \alpha_{4,2}^{(3)} \ge 0, \ \ \alpha_{4,3}^{(3)} \ge 0, \ \ \alpha_{4,4}^{(3)} \ge 0, \ \ \alpha_{4,5}^{(3)} \ge 0, \ \ \alpha_{4,6}^{(3)} \ge 0, \\
&\alpha_{4,1}^{(3)} + \alpha_{4,2}^{(3)} + \alpha_{4,3}^{(3)} + \alpha_{4,4}^{(3)} + \alpha_{4,5}^{(3)} + \alpha_{4,6}^{(3)} = 1,
\end{aligned} \tag{4.35}
$$

and

$$
\beta_{4,1}^{(3)} \ge 0, \ \beta_{4,2}^{(3)} \ge 0, \ \beta_{4,3}^{(3)} \ge 0, \ \beta_{4,4}^{(3)} \ge 0, \quad \beta_{4,1}^{(3)} + \beta_{4,2}^{(3)} + \beta_{4,3}^{(3)} + \beta_{4,4}^{(3)} = \alpha_{4,1}^{(3)}, \tag{4.36}
$$

and further

$$
\gamma_{4,1}^{(3)} \ge 0, \ \gamma_{4,2}^{(3)} \ge 0, \ \gamma_{4,3}^{(3)} \ge 0, \ \gamma_{4,4}^{(3)} \ge 0, \quad \gamma_{4,1}^{(3)} + \gamma_{4,2}^{(3)} + \gamma_{4,3}^{(3)} + \gamma_{4,4}^{(3)} = \beta_{4,1}^{(3)}. \tag{4.37}
$$

Similar to the third-order case, we can formulate the optimization problem (3.20), subject to the restriction (1.11), (4.4), (4.7), (4.10), (4.13), (4.14), (4.17), (4.18), (4.21), (4.22), (4.25)-(4.27), (4.30)-(4.32), (4.35)-(4.37), and (3.21), and solve it using the Matlab routine "fminicon". We obtain the following optimal scheme:

$$
\begin{aligned}
u_1^{(1)} =& u^n + \gamma_0\Delta t L(u^n), \quad u_1^{(2)} = u_1^{(1)} + \frac{\sqrt{5}}{5}\Delta t L(u_1^{(1)}), \quad u_1^{(3)} = u_1^{(2)} + \gamma_0\Delta t L(u_1^{(2)}), \\
u_2^{(1)} =& \left(a_{1,1}^{(0)}u^n + b_{1,1}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,1}^{(1)}u_1^{(1)} + b_{1,1}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,1}^{(2)}u_1^{(2)} + b_{1,1}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) \\
&+ \left(a_{1,1}^{(3)}u_1^{(3)} + b_{1,1}^{(3)}\Delta t L(u_1^{(3)})\right), \\
u_2^{(2)} =& \left(a_{1,2}^{(0)}u^n + b_{1,2}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,2}^{(1)}u_1^{(1)} + b_{1,2}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,2}^{(2)}u_1^{(2)} + b_{1,2}^{(2)}\Delta t L(u_1^{(2)})\right) \\
&+ \left(a_{1,2}^{(3)}u_1^{(3)} + b_{1,2}^{(3)}\Delta t\widetilde{L}(u_1^{(3)})\right) + \left(a_{2,2}^{(1)}u_2^{(1)} + b_{2,2}^{(1)}\Delta t L(u_2^{(1)})\right), \\
u_2^{(3)} =& \left(a_{1,3}^{(0)}u^n + b_{1,3}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,3}^{(1)}u_1^{(1)} + b_{1,3}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,3}^{(2)}u_1^{(2)} + b_{1,3}^{(2)}\Delta t L(u_1^{(2)})\right) \\
&+ \left(a_{1,3}^{(3)}u_1^{(3)} + b_{1,3}^{(3)}\Delta t L(u_1^{(3)})\right) + \left(a_{2,3}^{(1)}u_2^{(1)} + b_{2,3}^{(1)}\Delta t L(u_2^{(1)})\right) + \left(a_{2,3}^{(2)}u_2^{(2)} + b_{2,3}^{(2)}\Delta t L(u_2^{(2)})\right),
\end{aligned}
$$

$$
\begin{aligned}
u_3^{(1)} =& \left(a_{1,4}^{(0)}u^n + b_{1,4}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,4}^{(1)}u_1^{(1)} + b_{1,4}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,4}^{(2)}u_1^{(2)} + b_{1,4}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) \\
&+ \left(a_{1,4}^{(3)}u_1^{(3)} + b_{1,4}^{(3)}\Delta t\widetilde{L}(u_1^{(3)})\right) + \left(a_{2,4}^{(1)}u_2^{(1)} + b_{2,4}^{(1)}\Delta tL(u_2^{(1)})\right) + \left(a_{2,4}^{(2)}u_2^{(2)} + b_{2,4}^{(2)}\Delta t\widetilde{L}(u_2^{(2)})\right) \\
&+ \left(a_{2,4}^{(3)}u_2^{(3)} + b_{2,4}^{(3)}\Delta tL(u_2^{(3)})\right), \\
u_3^{(2)} =& \left(a_{1,5}^{(0)}u^n + b_{1,5}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,5}^{(1)}u_1^{(1)} + b_{1,5}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,5}^{(2)}u_1^{(2)} + b_{1,5}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) \\
&+ \left(a_{1,5}^{(3)}u_1^{(3)} + b_{1,5}^{(3)}\Delta tL(u_1^{(3)})\right) + \left(a_{2,5}^{(1)}u_2^{(1)} + b_{2,5}^{(1)}\Delta t\widetilde{L}(u_2^{(1)})\right) + \left(a_{2,5}^{(2)}u_2^{(2)} + b_{2,5}^{(2)}\Delta tL(u_2^{(2)})\right) \\
&+ \left(a_{2,5}^{(3)}u_2^{(3)} + b_{2,5}^{(3)}\Delta t\widetilde{L}(u_2^{(3)})\right) + \left(a_{3,5}^{(1)}u_3^{(1)} + b_{3,5}^{(1)}\Delta tL(u_3^{(1)})\right), \qquad (4.38) \\
u_3^{(3)} =& \left(a_{1,6}^{(0)}u^n + b_{1,6}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,6}^{(1)}u_1^{(1)} + b_{1,6}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,6}^{(2)}u_1^{(2)} + b_{1,6}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) \\
&+ \left(a_{1,6}^{(3)}u_1^{(3)} + b_{1,6}^{(3)}\Delta t\widetilde{L}(u_1^{(3)})\right) + \left(a_{2,6}^{(1)}u_2^{(1)} + b_{2,6}^{(1)}\Delta t\widetilde{L}(u_2^{(1)})\right) + \left(a_{2,6}^{(2)}u_2^{(2)} + b_{2,6}^{(2)}\Delta tL(u_2^{(2)})\right) \\
&+ \left(a_{2,6}^{(3)}u_2^{(3)} + b_{2,6}^{(3)}\Delta tL(u_2^{(3)})\right) + \left(a_{3,6}^{(1)}u_3^{(1)} + b_{3,6}^{(1)}\Delta tL(u_3^{(1)})\right) + \left(a_{3,6}^{(2)}u_3^{(2)} + b_{3,6}^{(2)}\Delta tL(u_3^{(2)})\right), \\
u_4^{(1)} =& \left(a_{1,7}^{(0)}u^n + b_{1,7}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,7}^{(1)}u_1^{(1)} + b_{1,7}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,7}^{(2)}u_1^{(2)} + b_{1,7}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) \\
&+ \left(a_{1,7}^{(3)}u_1^{(3)} + b_{1,7}^{(3)}\Delta t\widetilde{L}(u_1^{(3)})\right) + \left(a_{2,7}^{(1)}u_2^{(1)} + b_{2,7}^{(1)}\Delta t\widetilde{L}(u_2^{(1)})\right) + \left(a_{2,7}^{(2)}u_2^{(2)} + b_{2,7}^{(2)}\Delta t\widetilde{L}(u_2^{(2)})\right) \\
&+ \left(a_{2,7}^{(3)}u_2^{(3)} + b_{2,7}^{(3)}\Delta t\widetilde{L}(u_2^{(3)})\right) + \left(a_{3,7}^{(1)}u_3^{(1)} + b_{3,7}^{(1)}\Delta tL(u_3^{(1)})\right) + \left(a_{3,7}^{(2)}u_3^{(2)} + b_{3,7}^{(2)}\Delta tL(u_3^{(2)})\right) \\
&+ \left(a_{3,7}^{(3)}u_3^{(3)} + b_{3,7}^{(3)}\Delta tL(u_3^{(3)})\right), \\
u_4^{(2)} =& \left(a_{1,8}^{(0)}u^n + b_{1,8}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,8}^{(1)}u_1^{(1)} + b_{1,8}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,8}^{(2)}u_1^{(2)} + b_{1,8}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) \\
&+ \left(a_{1,8}^{(3)}u_1^{(3)} + b_{1,8}^{(3)}\Delta tL(u_1^{(3)})\right) + \left(a_{2,8}^{(1)}u_2^{(1)} + b_{2,8}^{(1)}\Delta t\widetilde{L}(u_2^{(1)})\right) + \left(a_{2,8}^{(2)}u_2^{(2)} + b_{2,8}^{(2)}\Delta t\widetilde{L}(u_2^{(2)})\right) \\
&+ \left(a_{2,8}^{(3)}u_2^{(3)} + b_{2,8}^{(3)}\Delta tL(u_2^{(3)})\right) + \left(a_{3,8}^{(1)}u_3^{(1)} + b_{3,8}^{(1)}\Delta tL(u_3^{(1)})\right) + \left(a_{3,8}^{(2)}u_3^{(2)} + b_{3,8}^{(2)}\Delta tL(u_3^{(2)})\right) \\
&+ \left(a_{3,8}^{(3)}u_3^{(3)} + b_{3,8}^{(3)}\Delta t\widetilde{L}(u_3^{(3)})\right) + \left(a_{4,8}^{(1)}u_4^{(1)} + b_{4,8}^{(1)}\Delta tL(u_4^{(1)})\right), \\
u^{n+1} =& \left(a_{1,9}^{(0)}u^n + b_{1,9}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,9}^{(1)}u_1^{(1)} + b_{1,9}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,9}^{(2)}u_1^{(2)} + b_{1,9}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) \\
&+ \left(a_{1,9}^{(3)}u_1^{(3)} + b_{1,9}^{(3)}\Delta tL(u_1^{(3)})\right) + \left(a_{2,9}^{(1)}u_2^{(1)} + b_{2,9}^{(1)}\Delta t\widetilde{L}(u_2^{(1)})\right) + \left(a_{2,9}^{(2)}u_2^{(2)} + b_{2,9}^{(2)}\Delta t\widetilde{L}(u_2^{(2)})\right) \\
&+ \left(a_{2,9}^{(3)}u_2^{(3)} + b_{2,9}^{(3)}\Delta t\widetilde{L}(u_2^{(3)})\right) + \left(a_{3,9}^{(1)}u_3^{(1)} + b_{3,9}^{(1)}\Delta t\widetilde{L}(u_3^{(1)})\right) + \left(a_{3,9}^{(2)}u_3^{(2)} + b_{3,9}^{(2)}\Delta tL(u_3^{(2)})\right) \\
&+ \left(a_{3,9}^{(3)}u_3^{(3)} + b_{3,9}^{(3)}\Delta tL(u_3^{(3)})\right) + \left(a_{4,9}^{(1)}u_4^{(1)} + b_{4,9}^{(1)}\Delta tL(u_4^{(1)})\right) + \left(a_{4,9}^{(2)}u_4^{(2)} + b_{4,9}^{(2)}\Delta tL(u_4^{(2)})\right),
\end{aligned}
$$

with the coefficients $a_{j,k}^{(i)}$ and $b_{j,k}^{(i)}$ given by (4.3), (4.6), (4.9), (4.12), (4.16), (4.20), (4.24), (4.29), (4.34), and

$$
\begin{aligned}
&\alpha_{2,1}^{(1)} = 0.2505, \quad \alpha_{2,2}^{(1)} = 0.2506, \quad \alpha_{2,3}^{(1)} = 0.2505, \quad \alpha_{2,1}^{(2)} = 0.1288, \\
&\alpha_{2,2}^{(2)} = 0.1206, \quad \alpha_{2,3}^{(2)} = 0.2805, \quad \alpha_{2,4}^{(2)} = 0.0650, \quad \alpha_{2,1}^{(3)} = 0.0721, \\
&\alpha_{2,2}^{(3)} = 0.1273, \quad \alpha_{2,3}^{(3)} = 0.0642, \quad \alpha_{2,4}^{(3)} = 0.1162, \quad \alpha_{2,5}^{(3)} = 0.2689, \\
&\alpha_{3,1}^{(1)} = 0.5507, \quad \alpha_{3,2}^{(1)} = 0.1061, \quad \alpha_{3,3}^{(1)} = 0.2015, \quad \beta_{3,1}^{(1)} = 0.0751, \qquad (4.39) \\
&\beta_{3,2}^{(1)} = 0.2270, \quad \beta_{3,3}^{(1)} = 0.1554, \quad \alpha_{3,1}^{(2)} = 0.2758, \quad \alpha_{3,2}^{(2)} = 0.0243, \\
&\alpha_{3,3}^{(2)} = 0.2882, \quad \alpha_{3,4}^{(2)} = 0.0388, \quad \beta_{3,1}^{(2)} = 0.0666, \quad \beta_{3,2}^{(2)} = 0.1137,
\end{aligned}
$$

$$
\begin{aligned}
&\beta_{3,3}^{(2)}=0.0901, \quad \alpha_{3,1}^{(3)}=0.1384, \quad \alpha_{3,2}^{(3)}=0.0464, \quad \alpha_{3,3}^{(3)}=0.0160,\\
&\alpha_{3,4}^{(3)}=0.1188, \quad \alpha_{3,5}^{(3)}=0.2048, \quad \beta_{3,1}^{(3)}=0.0312, \quad \beta_{3,2}^{(3)}=0.0589,\\
&\beta_{3,3}^{(3)}=0.0356, \quad \alpha_{4,1}^{(1)}=0.7232, \quad \alpha_{4,2}^{(1)}=0.2164, \quad \alpha_{4,3}^{(1)}=0.0473,\\
&\beta_{4,1}^{(1)}=0.1778, \quad \beta_{4,2}^{(1)}=0.4305, \quad \beta_{4,3}^{(1)}=0.0348, \quad \gamma_{4,1}^{(1)}=0.0224,\\
&\gamma_{4,2}^{(1)}=0.1321, \quad \gamma_{4,3}^{(1)}=0.0104, \quad \alpha_{4,1}^{(2)}=0.3139, \quad \alpha_{4,2}^{(2)}=0.0000,\\
&\alpha_{4,3}^{(2)}=0.2877, \quad \alpha_{4,4}^{(2)}=0.0386, \quad \beta_{4,1}^{(2)}=0.1169, \quad \beta_{4,2}^{(2)}=0.1002,\\
&\beta_{4,3}^{(2)}=0.0910, \quad \gamma_{4,1}^{(2)}=0.0387, \quad \gamma_{4,2}^{(2)}=0.0539, \quad \gamma_{4,3}^{(2)}=0.0232,\\
&\alpha_{4,1}^{(3)}=0.2040, \quad \alpha_{4,2}^{(3)}=0.0315, \quad \alpha_{4,3}^{(3)}=0.0673, \quad \alpha_{4,4}^{(3)}=0.1130,\\
&\alpha_{4,5}^{(3)}=0.2507, \quad \beta_{4,1}^{(3)}=0.0691, \quad \beta_{4,2}^{(3)}=0.0653, \quad \beta_{4,3}^{(3)}=0.0595,\\
&\gamma_{4,1}^{(3)}=0.0168, \quad \gamma_{4,2}^{(3)}=0.0362, \quad \gamma_{4,3}^{(3)}=0.0160, \quad \theta_1=0.7043,\\
&\theta_2=1.0000, \quad \theta_3=0.6622, \quad \theta_4=1.0000, \quad \theta_5=0.6388, \quad \theta_6=0.9581.
\end{aligned}
$$

The CFL coefficient for this scheme is $c = 1.2592$. Therefore, we have proved the following result.

Theorem 4.1. *The fourth-order DC scheme* (4.38)-(4.39) *is SSP under the time step restriction* (1.8) *with the CFL coefficient* $c = 1.2592$.

The optimal scheme (4.38)-(4.39) needs 21 evaluations of L or $\widetilde{L}$. We can also obtain a fourth-order SSP DC scheme with 19 evaluations of L or $\widetilde{L}$, however the CFL coefficient is only $c = 0.6775$, hence it is much less efficient than the scheme (4.38)-(4.39). For the original spectral deferred correction scheme in [2, 8] corresponding to $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = 1$, we can obtain a SSP scheme (4.38) with different choices of parameters than those in (4.39), with 21 evaluations of L or $\widetilde{L}$ and a CFL coefficient of $c = 0.9463$. This is again much less efficient than the scheme (4.38)-(4.39). We do not list the details of these schemes to save space. Finally, when $\theta_5 = \theta_6 = 0$, we do not need to evaluate $u_4^{(1)}$ and $u_4^{(2)}$, leading to the following scheme:

$$
\begin{aligned}
u_1^{(1)} =& u^n + \gamma_0 \Delta t L(u^n), \quad u_1^{(2)} = u_1^{(1)} + \frac{\sqrt{5}}{5}\Delta t L(u_1^{(1)}), \quad u_1^{(3)} = u_1^{(2)} + \gamma_0 \Delta t L(u_1^{(2)}),\\
u_2^{(1)} =& \left(a_{1,1}^{(0)} u^n + b_{1,1}^{(0)} \Delta t \widetilde{L}(u^n)\right) + \left(a_{1,1}^{(1)} u_1^{(1)} + b_{1,1}^{(1)} \Delta t \widetilde{L}(u_1^{(1)})\right) + \left(a_{1,1}^{(2)} u_1^{(2)} + b_{1,1}^{(2)} \Delta t \widetilde{L}(u_1^{(2)})\right)\\
&+ \left(a_{1,1}^{(3)} u_1^{(3)} + b_{1,1}^{(3)} \Delta t L(u_1^{(3)})\right),\\
u_2^{(2)} =& \left(a_{1,2}^{(0)} u^n + b_{1,2}^{(0)} \Delta t \widetilde{L}(u^n)\right) + \left(a_{1,2}^{(1)} u_1^{(1)} + b_{1,2}^{(1)} \Delta t \widetilde{L}(u_1^{(1)})\right) + \left(a_{1,2}^{(2)} u_1^{(2)} + b_{1,2}^{(2)} \Delta t L(u_1^{(2)})\right)\\
&+ \left(a_{1,2}^{(3)} u_1^{(3)} + b_{1,2}^{(3)} \Delta t \widetilde{L}(u_1^{(3)})\right) + \left(a_{2,2}^{(1)} u_2^{(1)} + b_{2,2}^{(1)} \Delta t L(u_2^{(1)})\right),\\
u_2^{(3)} =& \left(a_{1,3}^{(0)} u^n + b_{1,3}^{(0)} \Delta t \widetilde{L}(u^n)\right) + \left(a_{1,3}^{(1)} u_1^{(1)} + b_{1,3}^{(1)} \Delta t \widetilde{L}(u_1^{(1)})\right) + \left(a_{1,3}^{(2)} u_1^{(2)} + b_{1,3}^{(2)} \Delta t L(u_1^{(2)})\right)\\
&+ \left(a_{1,3}^{(3)} u_1^{(3)} + b_{1,3}^{(3)} \Delta t L(u_1^{(3)})\right) + \left(a_{2,3}^{(1)} u_2^{(1)} + b_{2,3}^{(1)} \Delta t L(u_2^{(1)})\right) + \left(a_{2,3}^{(2)} u_2^{(2)} + b_{2,3}^{(2)} \Delta t L(u_2^{(2)})\right),\\
u_3^{(1)} =& \left(a_{1,4}^{(0)} u^n + b_{1,4}^{(0)} \Delta t \widetilde{L}(u^n)\right) + \left(a_{1,4}^{(1)} u_1^{(1)} + b_{1,4}^{(1)} \Delta t \widetilde{L}(u_1^{(1)})\right) + \left(a_{1,4}^{(2)} u_1^{(2)} + b_{1,4}^{(2)} \Delta t \widetilde{L}(u_1^{(2)})\right)\\
&+ \left(a_{1,4}^{(3)} u_1^{(3)} + b_{1,4}^{(3)} \Delta t \widetilde{L}(u_1^{(3)})\right) + \left(a_{2,4}^{(1)} u_2^{(1)} + b_{2,4}^{(1)} \Delta t L(u_2^{(1)})\right) + \left(a_{2,4}^{(2)} u_2^{(2)} + b_{2,4}^{(2)} \Delta t \widetilde{L}(u_2^{(2)})\right)\\
&+ \left(a_{2,4}^{(3)} u_2^{(3)} + b_{2,4}^{(3)} \Delta t L(u_2^{(3)})\right),
\end{aligned}
$$

$$
\begin{aligned}
u_3^{(2)} =& \left(a_{1,5}^{(0)}u^n + b_{1,5}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,5}^{(1)}u_1^{(1)} + b_{1,5}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,5}^{(2)}u_1^{(2)} + b_{1,5}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) \\
&+ \left(a_{1,5}^{(3)}u_1^{(3)} + b_{1,5}^{(3)}\Delta tL(u_1^{(3)})\right) + \left(a_{2,5}^{(1)}u_2^{(1)} + b_{2,5}^{(1)}\Delta t\widetilde{L}(u_2^{(1)})\right) + \left(a_{2,5}^{(2)}u_2^{(2)} + b_{2,5}^{(2)}\Delta tL(u_2^{(2)})\right) \\
&+ \left(a_{2,5}^{(3)}u_2^{(3)} + b_{2,5}^{(3)}\Delta t\widetilde{L}(u_2^{(3)})\right) + \left(a_{3,5}^{(1)}u_3^{(1)} + b_{3,5}^{(1)}\Delta tL(u_3^{(1)})\right), \qquad (4.40) \\
u_3^{(3)} =& \left(a_{1,6}^{(0)}u^n + b_{1,6}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,6}^{(1)}u_1^{(1)} + b_{1,6}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,6}^{(2)}u_1^{(2)} + b_{1,6}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) \\
&+ \left(a_{1,6}^{(3)}u_1^{(3)} + b_{1,6}^{(3)}\Delta t\widetilde{L}(u_1^{(3)})\right) + \left(a_{2,6}^{(1)}u_2^{(1)} + b_{2,6}^{(1)}\Delta t\widetilde{L}(u_2^{(1)})\right) + \left(a_{2,6}^{(2)}u_2^{(2)} + b_{2,6}^{(2)}\Delta tL(u_2^{(2)})\right) \\
&+ \left(a_{2,6}^{(3)}u_2^{(3)} + b_{2,6}^{(3)}\Delta tL(u_2^{(3)})\right) + \left(a_{3,6}^{(1)}u_3^{(1)} + b_{3,6}^{(1)}\Delta tL(u_3^{(1)})\right) + \left(a_{3,6}^{(2)}u_3^{(2)} + b_{3,6}^{(2)}\Delta tL(u_3^{(2)})\right), \\
u^{n+1} =& \left(a_{1,9}^{(0)}u^n + b_{1,9}^{(0)}\Delta t\widetilde{L}(u^n)\right) + \left(a_{1,9}^{(1)}u_1^{(1)} + b_{1,9}^{(1)}\Delta t\widetilde{L}(u_1^{(1)})\right) + \left(a_{1,9}^{(2)}u_1^{(2)} + b_{1,9}^{(2)}\Delta t\widetilde{L}(u_1^{(2)})\right) \\
&+ \left(a_{1,9}^{(3)}u_1^{(3)} + b_{1,9}^{(3)}\Delta tL(u_1^{(3)})\right) + \left(a_{2,9}^{(1)}u_2^{(1)} + b_{2,9}^{(1)}\Delta t\widetilde{L}(u_2^{(1)})\right) + \left(a_{2,9}^{(2)}u_2^{(2)} + b_{2,9}^{(2)}\Delta t\widetilde{L}(u_2^{(2)})\right) \\
&+ \left(a_{2,9}^{(3)}u_2^{(3)} + b_{2,9}^{(3)}\Delta tL(u_2^{(3)})\right) + \left(a_{3,9}^{(1)}u_3^{(1)} + b_{3,9}^{(1)}\Delta tL(u_3^{(1)})\right) + \left(a_{3,9}^{(2)}u_3^{(2)} + b_{3,9}^{(2)}\Delta tL(u_3^{(2)})\right) \\
&+ \left(a_{3,9}^{(3)}u_3^{(3)} + b_{3,9}^{(3)}\Delta tL(u_3^{(3)})\right),
\end{aligned}
$$

with the coefficients $a_{j,k}^{(i)}$ and $b_{j,k}^{(i)}$ given by (4.3), (4.6), (4.9), (4.12), (4.16), (4.20), (4.34), and

$$
\begin{aligned}
&\alpha_{2,1}^{(1)} = 0.2266, \quad \alpha_{2,2}^{(1)} = 0.2267, \quad \alpha_{2,3}^{(1)} = 0.2259, \quad \alpha_{2,1}^{(2)} = 0.1108, \\
&\alpha_{2,2}^{(2)} = 0.1602, \quad \alpha_{2,3}^{(2)} = 0.2130, \quad \alpha_{2,4}^{(2)} = 0.1227, \quad \alpha_{2,1}^{(3)} = 0.0705, \\
&\alpha_{2,2}^{(3)} = 0.1409, \quad \alpha_{2,3}^{(3)} = 0.1290, \quad \alpha_{2,4}^{(3)} = 0.0911, \quad \alpha_{2,5}^{(3)} = 0.2797, \\
&\alpha_{3,1}^{(1)} = 0.6771, \quad \alpha_{3,2}^{(1)} = 0.1652, \quad \alpha_{3,3}^{(1)} = 0.0625, \quad \beta_{3,1}^{(1)} = 0.0923, \\
&\beta_{3,2}^{(1)} = 0.2362, \quad \beta_{3,3}^{(1)} = 0.1739, \quad \alpha_{3,1}^{(2)} = 0.1772, \quad \alpha_{3,2}^{(2)} = 0.1370, \\
&\alpha_{3,3}^{(2)} = 0.2395, \quad \alpha_{3,4}^{(2)} = 0.0322, \quad \beta_{3,1}^{(2)} = 0.0457, \quad \beta_{3,2}^{(2)} = 0.0724, \\
&\beta_{3,3}^{(2)} = 0.0567, \quad \alpha_{3,1}^{(3)} = 0.1469, \quad \alpha_{3,2}^{(3)} = 0.1162, \quad \alpha_{3,3}^{(3)} = 0.0609, \\
&\alpha_{3,4}^{(3)} = 0.0910, \quad \alpha_{3,5}^{(3)} = 0.2932, \quad \beta_{3,1}^{(3)} = 0.0287, \quad \beta_{3,2}^{(3)} = 0.0583, \\
&\beta_{3,3}^{(3)} = 0.0319, \quad \alpha_{4,1}^{(3)} = 0.2821, \quad \alpha_{4,2}^{(3)} = 0.2265, \quad \alpha_{4,3}^{(3)} = 0.4054, \\
&\beta_{4,1}^{(3)} = 0.0745, \quad \beta_{4,2}^{(3)} = 0.1062, \quad \beta_{4,3}^{(3)} = 0.0999, \quad \gamma_{4,1}^{(3)} = 0.0146, \\
&\gamma_{4,2}^{(3)} = 0.0381, \qquad \gamma_{4,3}^{(3)} = 0.0203, \\
&\theta_1 = 0.8523, \quad \theta_2 = 1.0000, \quad \theta_3 = 0.8972, \quad \theta_4 = 1.0000.
\end{aligned} \qquad (4.41)
$$

The CFL coefficient for the SSP scheme (4.40)-(4.41) is $c = 1.0319$, and it has 17 evaluations of L or $\widetilde{L}$. It is therefore slightly more efficient than the scheme (4.38)-(4.39). We could also reduce the number of L or $\widetilde{L}$ evaluations to 16, however the CFL coefficient reduces to $c = 0.6775$, which is not impressive at all.

Remark 4.1. Our analysis is based on the choice of the Chebyshev Gauss-Lobatto nodes as the subgrid points inside the interval $[t^n, t^{n+1}]$. We could also perform an analysis for the more general class of the fourth-order DC scheme in which the subgrid points are placed arbitrarily subject to a symmetry constraint. We have performed this analysis for the simple case of $\theta_k = 0$ for all k, and have failed to find a better scheme in terms of the SSP property. We will not present the details here to save space.

5. A Numerical Example

In this section, we perform a numerical study to assess the performance of the DC time discretizations, coupled with the fifth-order weighted essentially non-oscillatory (WENO) finite difference spatial operator with a Lax-Friedrichs flux splitting [7], to solve the following Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad -1 \le x < 1, \tag{5.1}$$

with the initial condition

$$u(x,0) = \frac{1}{3} + \frac{2}{3}\sin(\pi x) \tag{5.2}$$

and a periodic boundary condition. The exact solution is smooth up to $t = 1.5/\pi$, then it develops a moving shock which interacts with the rarefaction waves. We use the WENO spatial operator, rather than the TVD spatial operator, since the former gives better accuracy and is used more often in applications, even though the latter fits better the theoretical framework of this paper, being rigorously satisfying the TVD property (1.5) for the total variation semi-norm.

In Table 5.1 we list the L^1 errors and the numerical orders of accuracy, at the time $t = 0.2$ when the solution is still smooth. We compute with the third- and fourth-order SSP DC schemes (3.22)-(3.23) and (4.38)-(4.39), with the correct incorporation of the operator $\tilde{L}$, and with the original third- and fourth-order DC schemes (3.1) and (4.1), using the same values of θ_k but without using the operator $\tilde{L}$. For this test we take the CFL number to be 0.6, that is

$$\max_j |u_j^n| \frac{\Delta t}{\Delta x} = 0.6. \tag{5.3}$$

This choice is based on the heuristic argument that the spatial WENO operator is a high-order generalization of the second-order generalized MUSCL scheme [9], which is TVD for first-order Euler forward time discretization under the CFL condition $\max_j |u_j^n|\Delta t/\Delta x = 0.5$. We clearly observe in Table 5.1 that the designed order of accuracy is achieved or exceeded. The other SSP schemes in Sections 3 and 4 yield similar errors. We do not present their results in order to save space.

Table 5.1: L^1 errors and numerical orders of accuracy. Burgers equation with the initial condition (5.2). $t = 0.2$.

Number of	DC3		SSP DC3		DC4		SSP DC4	
cells	L^1 error	order	L^1 error	order	L^1 error	order	L^1 error	order
20	9.36E-4	–	1.27E-3	–	9.20E-4	–	1.35E-3	–
40	4.78E-5	4.29	6.62E-5	4.26	4.27E-5	4.43	6.46E-5	4.39
80	2.16E-6	4.47	2.82E-6	4.55	1.29E-6	5.05	2.11E-6	4.94
160	1.81E-7	3.58	2.04E-7	3.79	5.38E-8	4.58	9.31E-8	4.50
320	2.02E-8	3.16	2.07E-8	3.30	1.81E-9	4.89	3.21E-9	4.86
640	2.48E-9	3.03	2.49E-9	3.06	4.40E-11	5.36	7.62E-11	5.40

When $t = 0.6$, the discontinuity has already appeared. We plot, in Fig. 5.1, the solution obtained with the third and fourth order regular DC schemes (3.1) and (4.1), and SSP DC schemes (3.22)-(3.23) and (4.38)-(4.39), using the CFL condition (5.3) with $N = 40$ equally spaced grid points. We can see that the numerical solutions are indeed non-oscillatory. It

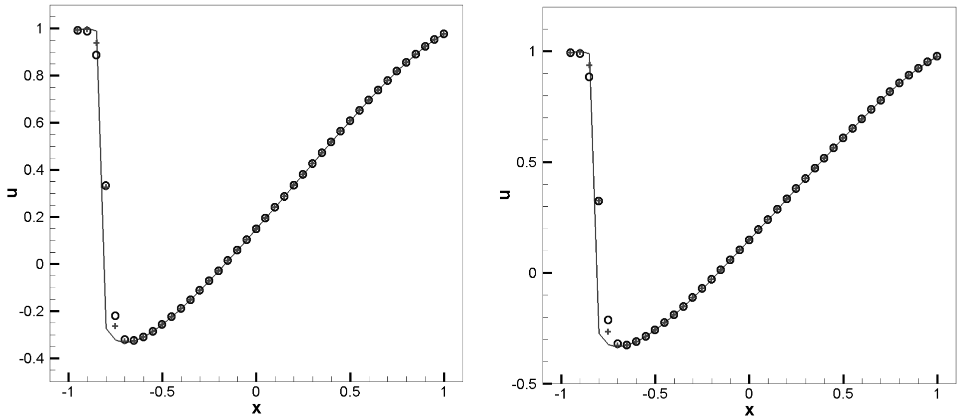

Fig. 5.1. Burgers equation with the initial condition (5.2). $t = 0.6$. $N = 40$ equally spaced grid points. CFL number 0.6. Left: third-order DC schemes; right: fourth-order DC schemes. Solid line: the exact solution. Circles: SSP DC schemes. Crosses: regular DC schemes.

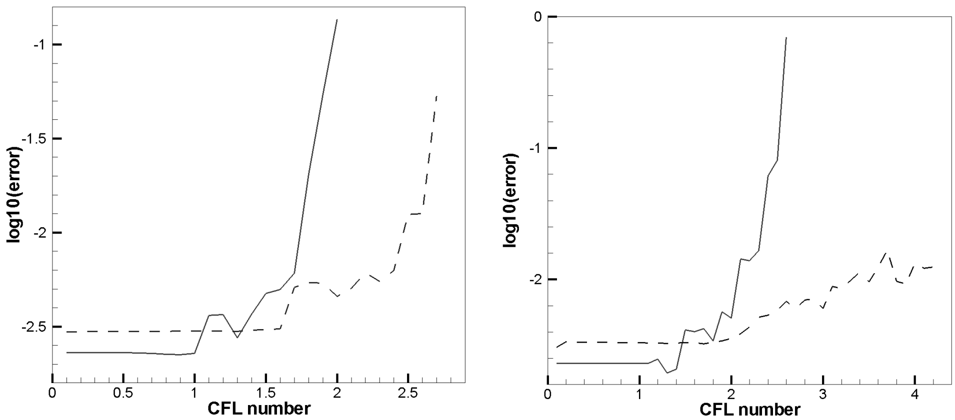

Fig. 5.2. Burgers equation with the initial condition (5.2). $t = 2.0$. $N = 160$ equally spaced grid points. L^1 errors (in logarithmic scale) versus the CFL number. Solid lines: regular DC schemes; dashed lines: SSP DC schemes. Left: third-order schemes; right: fourth-order schemes.

seems that for this test, the regular DC schemes without using the operator $\tilde{L}$ also produce non-oscillatory results for the CFL condition (5.3).

Finally, we would like to numerically assess how large the CFL number we can take and still maintain stability. We compute using both the third- and fourth-order SSP DC schemes (3.22)-(3.23) and (4.38)-(4.39), and the original third- and fourth-order DC schemes (3.1) and (4.1) using the same values of θ_k but without using the operator $\tilde{L}$, to $t = 2$, with $N = 160$ equally spaced grid points, with an ever increasing CFL number. In Fig. 5.2, we plot the L^1 errors of the numerical solution versus the CFL number for the third-order (left) and fourth-order (right) schemes. We observe that the SSP DC schemes are indeed stable for larger CFL numbers than the corresponding regular DC schemes, and the CFL numbers for stability are much larger than the theoretically predicted values in Theorems 3.1 and 4.1. This gap between the theoretically predicted bound for the CFL number and the numerically allowed value might become smaller for more demanding test cases, but we will not perform such exhaustive numerical tests in this paper. The theoretically predicted bound can serve as a safety net for guaranteed stability.

6. Concluding Remarks

We have studied the strong stability preserving (SSP) property of the second-, third- and fourth-order deferred correction (DC) time discretizations. The technique of the analysis can also be applied in principle to higher-order DC methods, although the algebra becomes more complicated. It seems that the DC methods do not have as large CFL coefficients as the Runge-Kutta methods for the SSP property. However, since the DC methods can be easily designed for arbitrary high-order accuracy, they have a good application potential and the analysis for their SSP property will be useful for their application to solve method of lines schemes for hyperbolic conservation laws.

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