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# A NOTE ON RICHARDSON EXTRAPOLATION OF GALERKIN METHODS FOR EIGENVALUE PROBLEMS OF FREDHOLM INTEGRAL EQUATIONS*

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**Abstract**

In this paper, we introduce a new extrapolation formula by combining Richardson extrapolation and Sloan iteration algorithms. Using this extrapolation formula, we obtain some asymptotic expansions of the Galerkin finite element method for semi-simple eigenvalue problems of Fredholm integral equations of the second kind and improve the accuracy of the numerical approximations of the corresponding eigenvalues. Some numerical experiments are carried out to demonstrate the effectiveness of our new method and to confirm our theoretical results.

*Mathematics subject classification:* 65N25, 65N30, 65R20.
*Key words:* Fredholm integral equations, Semi-simple eigenvalues, Asymptotic expansion, Galerkin method, Richardson extrapolation, Sloan iteration.

## 1. Introduction

In this paper, we consider the eigenvalue problem of the Fredholm integral equation of the second kind: Find an eigen-pair $(\lambda, u) \in R \times L^2(\Omega)$, such that

$$\int_\Omega k(t,s)u(s)ds = \lambda u(t), \quad t, s \in \Omega \subseteq R^n, \tag{1.1}$$

where $k(s,t)$ is a given smooth function in $D := \Omega \times \Omega$ satisfying $k(t,s) = k(s,t)$. Let $T$ be an integral operator defined by:

$$(Tu)(t) = \int_\Omega k(t,s)u(s)ds, \quad t, s \in \Omega.$$

The corresponding operator form of (1.1) is

$$(Tu)(t) = \lambda u(t), \quad t \in \Omega.$$

Then the integral operator $T$ is self-adjoint and compact. Thus the eigenvalues $\lambda$ of $T$ are semi-simple, i.e., the algebraic multiplicity of $\lambda$ equals the geometric multiplicity of $\lambda$.

Let the algebraic multiplicity of the semi-simple eigenvalue $\lambda$ be $r$. Then there are $r$ numerical eigenvalues approximating $\lambda$. The authors of [9] established asymptotic expansions for

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arithmetic mean of $r$ eigenvalues (approximating $\lambda$) in the Galerkin method. Lin, Sloan and Xie [12] proved similar results for the solution of Fredholm equations of the second kind. Mclean [14] also discussed asymptotic error expansions for the solution of Fredholm equations of the second kind.

There have been many attempts in improving the accuracy of numerical solutions. The most popular methods include the Sloan iteration method (iteration post-processing method), interpolation post-processing method and the Richardson extrapolation, see, e.g., [2, 5, 9, 11, 13, 15].

Suppose that the Galerkin eigen-pair $(\lambda_h, u_h)$ of degree $m-1$ approximates $(\lambda, u)$. In a recent work [17], the authors derived an asymptotic expansion of the eigenvalue approximation error for Problem (1.1) by means of iterated Galerkin finite element methods in certain piecewise polynomial spaces:

$$\begin{aligned}\lambda - \lambda_h &= (u, T(I-P_h)u) + \mathcal{O}(h^{3m})\\ &= \beta_u h^{2m} + \mathcal{O}(h^{2m+2}) + \mathcal{O}(h^{3m}),\end{aligned}$$

where $\beta_u$ depends only on the eigenfunction $u$. Replacing $h$ with $h/2$ for the above equation and extrapolating between $\lambda_h$ and $\lambda_{h/2}$, the authors obtained a higher order approximation for a simple eigenvalue $\lambda$:

$$\begin{aligned}&\frac{2^{2m}\lambda_{h/2} - \lambda_h}{2^{2m}-1} = \lambda + \mathcal{O}(h^{2m+2}),\ m \ge 2;\\ &\frac{4\lambda_{h/2} - \lambda_h}{3} = \lambda + \mathcal{O}(h^3),\ m = 1.\end{aligned}$$

In order to use the method in [17], it is crucial that the eigenvalue $\lambda$ is simple, so that the $u$ approximated by $u_h$ and the $u$ approximated by $u_{h/2}$ are the same. However, this is not the case for semi-simple eigenvalues, which occur when we solve boundary integral equations and high dimensional integral equations. Therefore, there is a need to develop a new method for higher order approximations of semi-simple eigenvalues.

In this note we propose the following procedure.

1) Calculate eigen-pair $(\lambda_h, u_h)$ by the Galerkin finite element method.
2) Apply the Sloan iteration to $u_h$ to obtain $u_h^s = \lambda_h^{-1} T u_h$ and the normalized $\widetilde{u_h^s} = u_h^s / \|u_h^s\|$.
3) Project $\widetilde{u_h^s}$ onto the bi-sectioned mesh and obtain $\overline{u}^s = P_{h/2}\widetilde{u_h^s}$.
4) Calculate another eigenvalue approximation $\lambda^s = (T\overline{u}^s, \widetilde{u_h^s})$.
5) Extrapolate between $\lambda_h$ and $\lambda^s$ to achieve higher-order eigenvalue approximation.

The main advantage of our new method is twofold. First, it is applicable to both simple and semi-simple eigenvalues as well as higher-dimensional cases. Second, comparing with the traditional Richardson extrapolation between $\lambda_h$ and $\lambda_{h/2}$, extrapolation between $\lambda_h$ and $\lambda^s$ is much cheaper. Note that the cost for $\lambda_{h/2}$ is $8n^3$ times of the cost for $\lambda_h$ in case of a typical $n$-dimensional eigenvalue problem.

Here is the outline of the remaining sections. In Section 2, we state our main results. To prove these results, we list some relevant lemmas in Section 3. Section 4 provides proofs of the main theorems. Finally, numerical results are presented in Section 5.

## 2. The Asymptotic Expansion

The equivalent form of (1.1) consists in finding $\lambda \in R, u \in L^2(\Omega)$ with $(u,u)=1$, such that

$$(Tu,v)=\lambda(u,v), \quad \forall v \in L^2(\Omega), \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the usual inner product in $L^2$- space.

Let $K_h : \{\sigma_k | \sigma_i \cap \sigma_j = \phi, \bigcup_{i=1}^n \overline{\sigma_i} = \overline{\Omega}\}$ be a partition for the domain $\Omega$. Let $h_k$ be the diameter of $\sigma_k$ and $h := \max_k\{h_k\}$. We assume that $h \to 0$ as $n \to \infty$ and that $K_h$ is quasi-uniform, i.e., there exists a constant $C$ independent of $n$, with the property:

$$\frac{h}{\min_k\{h_k\}} \le C \qquad \text{for all } \ k.$$

The corresponding finite element space is defined by

$$S_{m-1}^{(-1)}(K_h)=\{\varphi : \varphi|_{\sigma_k} \in P_{m-1}, 1 \le k \le n\}.$$

Where $P_{m-1}$ denotes the space of polynomials of degree not exceeding $m-1$. Here we use the superscript (-1) in the notation for the above finite element space to emphasize that it is not a subspace of $C(\Omega)$.

The Galerkin approximation of (2.1) is defined as: Find $\lambda_h \in R, u_h \in S_{m-1}^{(-1)}(K_h)$ with $(u_h,u_h)=1$, such that

$$(Tu_h,v)=\lambda_h(u_h,v), \quad \forall v \in S_{m-1}^{(-1)}(K_h). \tag{2.2}$$

Let the eigen-pair $(\lambda_h,u_h)$ approximate the eigen-pair $(\lambda,u)$ of (2.1), and let $u_h^s$ be the Sloan iteration of $u_h$, i.e., $\lambda_h u_h^s = Tu_h$. Then we have $P_h u_h^s = u_h$ and

$$u_h^s = T(\lambda_h^{-1}u_h), \tag{2.3}$$

where $P_h : L^2(\Omega) \to S_{m-1}^{(-1)}(K_h)$ is an $L^2$-projection operator defined by

$$(u,v)=(P_h u,v), \qquad \forall v \in S_{m-1}^{(-1)}(K_h). \tag{2.4}$$

Moreover, we assume that $\widetilde{u_h^s}$ is the unit function of $u_h^s$, i.e., $\widetilde{u_h^s}=u_h^s/\|u_h^s\|_0$. Let

$$\overline{u}^s = P_{\frac{h}{2}}\widetilde{u_h^s}, \quad \lambda^s = (T\overline{u}^s, \widetilde{u_h^s}). \tag{2.5}$$

The following are the main results of this paper.

**Theorem 2.1.** *Let* $(\lambda,u)$ *and* $(\lambda_h,u_h)$ *be solutions of* (2.1) *and* (2.2)*, respectively. Suppose that* $P_h : L^2(\Omega) \to S_{m-1}^{(-1)}(K_h)$ $(\Omega \subseteq R^n)$ *is the* $L^2$*-projection operator defined by* (2.4) *and that* $\lambda^s$ *is defined by* (2.5)*, then we have*

$$\lambda - \lambda_h = (u, T(I-P_h)u) + \mathcal{O}(h^{3m}), \tag{2.6}$$

$$\lambda - \lambda^s = (u, T(I-P_{\frac{h}{2}})u) + \mathcal{O}(h^{3m}). \tag{2.7}$$

We postpone the proof to Section 4.

To obtain asymptotic expansions of $\lambda-\lambda_h$ and $\lambda-\lambda^s$, we need to have asymptotic expansions of $(u, T(I-P_h)u)$ and $(u, T(I-P_{h/2})u)$, respectively. For the one-dimensional case, it is easy to do so (see the following Lemma 3.5). But for the two-dimensional or higher dimensional cases, we need to extend Theorem 2.2 in [4] (the following Lemma 3.5) from $I := [0,1]$ to $\Omega := [a,b]\times[c,d]$ (or $\Omega \subseteq R^n$).

Divide $[a,b]$ into $N$ sub-intervals, $a = t_0 < \cdots < t_N = b$. Let $h_{1k} = t_{k+1} - t_k$, $k = 0,1,\cdots,N-1$. Divide $[c,d]$ into $M$ sub-intervals, $c = s_0 < \cdots < s_M = d$. Let $h_{2l} = s_{l+1}-s_l$, $l = 0,1,\cdots,M-1$. We get $N\times M$ sub-domains $\sigma_{kl}$ $(k = 0,1,\cdots,N-1;\ l = 0,1,\cdots,M-1, n = N\times M)$ of $\Omega$, where

$$\sigma_{kl} = \begin{cases} [t_0,t_1]\times[s_0,s_1] & k = l = 0; \\ [t_0,t_1]\times(s_1,s_{l+1}] & k = 0,\ 1\le l\le M-1; \\ (t_k,t_{k+1}]\times[s_0,s_1] & l = 0,\ 1\le k\le N-1; \\ (t_k,t_{k+1}]\times(s_l,s_{l+1}] & 1\le k\le N-1,\ 1\le l\le M-1. \end{cases}$$

Let $K_h$ be the corresponding mesh for the domain $\Omega$, $h_{kl} = \text{diameter}(\sigma_{kl})$, $h = \max\limits_{k,l}\{h_{kl}\}$.

**Theorem 2.2.** *Let $\Omega \subseteq R^2$. Assume that $u, v \in C^{m+2}(\Omega)$. Then there exists a constant $\widetilde{c} = \widetilde{c}(m,i)$, independent of $h$, such that*

$$\begin{aligned} &\int\limits_{\sigma_{kl}} v(t,s)(u - i_h^{m-1}u)(t,s)dsdt \\ &= \sum_{i=0}^{m} \widetilde{c} h_{1k}^{2i} h_{2l}^{2m-2i} \int\limits_{\sigma_{kl}} \frac{\partial^m v}{\partial t^i \partial s^{m-i}} \frac{\partial^m u}{\partial t^i \partial s^{m-i}} dsdt + \mathcal{O}(h_{kl}^{2m+4}), \end{aligned} \tag{2.8}$$

*where $i_h^{m-1}$ is an interpolation operator defined in the next section.*

**Corollary 2.1.** *Under the same assumption as in Theorem 2.2, if $\Omega$ is a square, and if the mesh is refined uniformly with $M = N$ (which implies $h = h_{1k} = h_{2l}$), then (2.8) can be written as follows*

$$\begin{aligned} &\int\limits_{\sigma_{kl}} v(t,s)(u - i_h^{m-1}u)(t,s)dsdt \\ &= \widetilde{c} h^{2m} \int\limits_{\sigma_{kl}} v^{(m)}(t,s)\cdot u^{(m)}(t,s)dsdt + \mathcal{O}(h^{2m+4}), \end{aligned}$$

*where the dot denotes the inner product and*

$$f^{(m)}(t,s) = \left(\frac{\partial^m f}{\partial s^m}, \frac{\partial^m f}{\partial t \partial s^{m-1}}, \cdots, \frac{\partial^m f}{\partial t^i \partial s^{m-i}}, \cdots, \frac{\partial^m f}{\partial t^m}\right), \quad f = u, v.$$

We can also extend Theorem 2.2 from $\Omega \subseteq R^2$ to $\Omega := [a_1,b_1]\times\cdots\times[a_n,b_n] \subseteq R^n$. Divide $[a_i,b_i]$ into $M_i$ intervals, $a_i = t_{i0} < t_{i1} < \cdots < t_{iM_i} = b_i$ $(i = 1,2,\cdots,n)$. Then we have the following result.

**Corollary 2.2.** *Let $\Omega \subseteq R^n$. Assume that $u, v \in C^{m+2}(\Omega)$. If $\Omega$ is a hypercube and if the* *mesh is refined uniformly with* $M_1 = M_2 = \cdots = M_n$*, then* (2.8) *can be written as follows*

$$\int_{\sigma_{k_1\cdots k_n}} v(t)(u - i_h^{m-1}u)(t)dt$$
$$= \widetilde{c}h^{2m} \int_{\sigma_{k_1\cdots k_n}} v^{(m)}(t) \cdot u^{(m)}(t)dt + \mathcal{O}(h^{2m+n+2}),$$

*where* $h = diameter(\sigma_{k_1\cdots k_n})$ *and*

$$f^{(m)}(t) = \left(\frac{\partial^m f}{\partial t_1^m}, \cdots, \frac{\partial^m f}{\partial t_1^{i_1} \partial t_2^{i_2} \cdots \partial t_n^{i_n}}, \cdots, \frac{\partial^m f}{\partial t_n^m}\right), \quad f = u, v,$$

*with* $i_1 + i_2 + \cdots + i_n = m$*, "* $\cdot$ *" is the inner product symbol.*

**Theorem 2.3.** *Let* $\lambda$ *and* $\lambda_h$ *be eigenvalues of* (2.1) *and* (2.2)*, respectively. Assume that* $k(s,t) \in C^{m+2}(\Omega \times \Omega)$ $(\Omega \subseteq R^2)$ *in* (1.1) *and that* $\lambda^s$ *is defined by* (2.5)*. Then we have the following extrapolation*

$$\frac{2^{2m}\lambda^s - \lambda_h}{2^{2m} - 1} = \lambda + \mathcal{O}(h^{2m+2}) \quad m \geq 2, \tag{2.9}$$
$$\frac{4\lambda^s - \lambda_h}{3} = \lambda + \mathcal{O}(h^3) \qquad m = 1. \tag{2.10}$$

**Remark 2.1.** In the one-dimensional case, suppose that $P_{m-1} = span\{l_1, l_2, \cdots, l_m\}$. From the local property of $P_{h/2}$, we see that

$$P_{\frac{h}{2}}\widetilde{u_h^s} = \sum_{i=1}^m a_i l_i$$

at each element of $K_{h/2}$, where the coefficients $a_i$ $(i = 1, \cdots, m)$ are determined by the following local equations

$$\sum_{i=1}^m a_i(l_i, l_j) = (\widetilde{u_h^s}, l_j), \quad j = 1, \cdots, m.$$

Therefore, the calculation is relatively simple (in the two-dimensional case, $m$ should be replaced by $m^2$).

## 3. Some Relevant Lemmas

To prove our main results, we first establish some relevant lemmas. In this section, we assume that $\Omega \subseteq R^n$.

**Lemma 3.1.** *Let* $(\lambda, u)$ *and* $(\lambda_h, u_h)$ *be solutions of* (2.1) *and* (2.2)*, respectively. Assume that* $k(s,t)$ *associated with* $T$ *is a smooth function in* $D = \Omega \times \Omega$*. Then we have*

$$\|u - u_h\|_0 = \mathcal{O}(h^m); \tag{3.1}$$
$$|\lambda - \lambda_h| = \mathcal{O}(h^{2m}). \tag{3.2}$$

The results are well known [1, 5, 16]. We now define an interpolation operator $i_h^{m-1}: L^2(\Omega) \to S_{m-1}^{(-1)}(K_h)$ of degree $m-1$ as

$$i_h^{m-1} u|_{\sigma_k} \in P_{m-1},$$

with

$$\int_{\sigma_k} v i_h^{m-1} u dt = \int_{\sigma_k} vudt, \quad \forall v \in P_{m-1}. \tag{3.3}$$

**Lemma 3.2.** *Let $P_h : L^2(\Omega) \to S_{m-1}^{(-1)}(K_h)$ be an $L^2$-projection operator defined by* (2.4) *and $u$ be the eigenfunction of* (2.1)*. Then we have*

$$\|P_h u - u\|_0 = \mathcal{O}(h^m); \tag{3.4}$$

$$\|i_h^{m-1} u - u\|_{0,\infty,\sigma_k} \le c \cdot h^m \|u\|_{m,\infty}. \tag{3.5}$$

It is evident that (3.4) holds. The proof of (3.5) can be found in [7]. Since $S_{m-1}^{(-1)}(K_h)$ is a discontinuous piecewise polynomial space and $P_h$ possesses localization, we have

$$i_h^{m-1} = P_h, \quad \int_{\sigma_k} v P_h u dt = \int_{\sigma_k} vudt, \quad \forall v \in P_{m-1}.$$

**Lemma 3.3.** *Suppose that $u_h$ is the eigenfunction of* (2.2) *and $u_h^s$ is the Sloan iteration of $u_h$. Then $u_h$ and $u_h^s$ all approximate to $u$, with the following error estimate:*

$$\|u_h^s - u\|_0 = \mathcal{O}(h^{2m}). \tag{3.6}$$

The proof of this result can be seen in [5, 15].

**Lemma 3.4.** *Assume that $u$ is the eigenfunction of* (2.1) *and $\overline{u}^s$ is defined by* (2.5)*. Then there follows*

$$\|\overline{u}^s - u\|_0 = \mathcal{O}(h^m). \tag{3.7}$$

*Proof.* It follows from $\overline{u}^s = P_{\frac{h}{2}} \widetilde{u_h^s}$ and Lemma 3.2 that

$$\|\widetilde{u_h^s} - \overline{u}^s\|_0 = \mathcal{O}(h^m). \tag{3.8}$$

Note that $(u,u) = 1$ and $\widetilde{u_h^s} = u_h^s / \|u_h^s\|_0$. From Lemma 3.3, we have

$$\begin{aligned} \|\widetilde{u_h^s} - u\|_0 &= \|\frac{u_h^s - u\|u_h^s\|_0}{\|u_h^s\|_0}\|_0 = \frac{\|u_h^s - u + u\|u\|_0 - u\|u_h^s\|_0\|_0}{\|u_h^s\|_0} \\ &\le \frac{1}{\|u_h^s\|_0}(\|u_h^s - u\|_0 + \|u - u_h^s\|_0) = \mathcal{O}(h^{2m}). \end{aligned} \tag{3.9}$$

Combine (3.8) with (3.9), we obtain

$$\|\overline{u}^s - u\|_0 \le \|\overline{u}^s - \widetilde{u_h^s}\|_0 + \|\widetilde{u_h^s} - u\|_0 = \mathcal{O}(h^m).$$

Then (3.7) follows.

**Lemma 3.5.** ([4]) *Assume that $P_h : L^2(I) \to S_{m-1}^{(-1)}(K_h)$ is an $L^2$-projection operator defined by* (2.4) *and $u, v \in C^{m+2}(I)$ with $I = [0,1]$. Then there exists a constant $\widetilde{c} = \widetilde{c}(m)$, independent of the mesh $K_h$, such that*

$$\int_{\sigma_k} v(t)(u - P_h u)(t) dt = \widetilde{c} h_k^{2m} \int_{\sigma_k} u^{(m)}(t) v^{(m)}(t) dt + \mathcal{O}(h_k^{2m+3}). \tag{3.10}$$

# 4. Proof of Theorems

## 4.1. Proof of Theorem 2.1

Since $\lambda(u, u_h^s) = (\lambda u, u_h^s) = (Tu, u_h^s) = (u, Tu_h^s)$, we have

$$\lambda = \frac{(u, Tu_h^s)}{(u, u_h^s)}. \tag{4.1}$$

Further, using $\lambda_h(u, u_h^s) = (u, \lambda_h u_h^s) = (u, Tu_h)$ gives

$$\lambda_h = \frac{(u, Tu_h)}{(u, u_h^s)}. \tag{4.2}$$

Since $u_h = P_h u_h^s$, from (4.1) and (4.2), we have

$$\begin{aligned}\lambda - \lambda_h &= \frac{1}{(u, u_h^s)}(u, T(u_h^s - u_h)) = \frac{1}{(u, u_h^s)}(u, T(I - P_h)u_h^s) \\ &= \frac{1}{(u, u_h^s)}(u, T(I - P_h)u) + \frac{1}{(u, u_h^s)}(u, T(I - P_h)(u_h^s - u)) \\ &= \frac{1}{(u, u_h^s)}(u, T(I - P_h)u) + \frac{1}{(u, u_h^s)}((I - P_h)Tu, u_h^s - u). \end{aligned} \tag{4.3}$$

Using $(u, u) = 1$ and (3.6) yields

$$\begin{aligned}\frac{1}{(u, u_h^s)} &= \frac{(u, u) + (u, u_h^s - u) - (u, u_h^s - u)}{(u, u) + (u, u_h^s - u)} \\ &= 1 - \frac{(u, u_h^s - u)}{1 + (u, u_h^s - u)} = 1 + \mathcal{O}(h^{2m}). \end{aligned} \tag{4.4}$$

Substituting (4.4) into (4.3) and using Lemmas 3.2 and 3.3, we obtain

$$\lambda - \lambda_h = (u, T(I - P_h)u) + \mathcal{O}(h^{3m}).$$

Then (2.6) holds. Using the fact that

$$\lambda(u, \widetilde{u_h^s}) = (\lambda u, \widetilde{u_h^s}) = (Tu, \widetilde{u_h^s}) = (u, T\widetilde{u_h^s})$$

gives $\lambda = (u, T\widetilde{u_h^s})/(u, \widetilde{u_h^s})$. As $\lambda^s = (T\overline{u}^s, \widetilde{u_h^s})$, we have

$$\begin{aligned}\lambda - \lambda^s &= \frac{1}{(u, \widetilde{u_h^s})}[(u, T\widetilde{u_h^s}) - (T\overline{u}^s, \widetilde{u_h^s})(u, \widetilde{u_h^s})] \\ &= \frac{1}{(u, \widetilde{u_h^s})}(u, T\widetilde{u_h^s} - (T\overline{u}^s, \widetilde{u_h^s})\widetilde{u_h^s}) \\ &= \frac{1}{(u, \widetilde{u_h^s})}(u, T\widetilde{u_h^s} - T\overline{u}^s) + \frac{1}{(u, \widetilde{u_h^s})}(u, T\overline{u}^s - (T\overline{u}^s, \widetilde{u_h^s})\widetilde{u_h^s}). \end{aligned}$$

Note that $(T\overline{u}^s, \widetilde{u_h^s})\widetilde{u_h^s}$ is the projection of $T\overline{u}^s$ on $span\{\widetilde{u_h^s}\}$. Then

$$(\widetilde{u_h^s}, T\overline{u}^s - (T\overline{u}^s, \widetilde{u_h^s})\widetilde{u_h^s}) = 0. \tag{4.5}$$

From Lemma 3.2, we get

$$\|T\overline{u}^s - (T\overline{u}^s, \widetilde{u_h^s})\widetilde{u_h^s}\|_0 = \mathcal{O}(h^m). \tag{4.6}$$

Combining (4.5), (4.6), (3.9) and (2.5) gives

$$\begin{aligned}\lambda-\lambda^s &= \frac{1}{(u,\widetilde{u_h^s})}(u,T\widetilde{u_h^s}-T\overline{u}^s)+\frac{1}{(u,\widetilde{u_h^s})}(u-\widetilde{u_h^s},T\overline{u}^s-(T\overline{u}^s,\widetilde{u_h^s})\widetilde{u_h^s})\\ &= \frac{1}{(u,\widetilde{u_h^s})}(u,T\widetilde{u_h^s}-T\overline{u}^s)+\mathcal{O}(h^{3m})\\ &= \frac{1}{(u,\widetilde{u_h^s})}(u,T(I-P_{\frac{h}{2}})\widetilde{u_h^s})+\mathcal{O}(h^{3m})\\ &= \frac{1}{(u,\widetilde{u_h^s})}(u,T(I-P_{\frac{h}{2}})u)+\frac{1}{(u,\widetilde{u_h^s})}(u,T(I-P_{\frac{h}{2}})(\widetilde{u_h^s}-u))+\mathcal{O}(h^{3m})\\ &= \frac{1}{(u,\widetilde{u_h^s})}(u,T(I-P_{\frac{h}{2}})u)+\frac{1}{(u,\widetilde{u_h^s})}((I-P_{\frac{h}{2}})Tu,\widetilde{u_h^s}-u)+\mathcal{O}(h^{3m})\end{aligned}$$

It follows from (4.4), (3.4) and (3.9) that

$$\lambda-\lambda^s=(u,T(I-P_{\frac{h}{2}})u)+\mathcal{O}(h^{3m}).$$

Then (2.7) follows. We complete the proof of Theorem 2.1. □

### 4.2. Proof of Theorem 2.2

Denote the centroid point of the sub-domain $\sigma_{kl}$ as $(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})$. Using Taylor expansion of $v(t,s)$ at $(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})$ gives

$$\begin{aligned}v(t,s) \;=\; & v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})+\left[(t-t_{k+\frac{1}{2}})\frac{\partial}{\partial t}+(s-s_{l+\frac{1}{2}})\frac{\partial}{\partial s}\right]v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})+\cdots\\ &+\frac{1}{(m+1)!}\left[(t-t_{k+\frac{1}{2}})\frac{\partial}{\partial t}+(s-s_{l+\frac{1}{2}})\frac{\partial}{\partial s}\right]^{m+1}v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})+\mathcal{O}(h_{kl}^{m+2}).\end{aligned}$$

This, together with (3.3) and (3.5), yields

$$\begin{aligned}&\int_{\sigma_{kl}} v(t,s)(u-i_h^{m-1}u)(t,s)dsdt\\ =&\int_{\sigma_{kl}}\frac{1}{m!}\left[(t-t_{k+\frac{1}{2}})\frac{\partial}{\partial t}+(s-s_{l+\frac{1}{2}})\frac{\partial}{\partial s}\right]^{m}v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})(u-i_h^{m-1}u)(t,s)dsdt\\ &+\int_{\sigma_{kl}}\frac{1}{(m+1)!}\left[(t-t_{k+\frac{1}{2}})\frac{\partial}{\partial t}+(s-s_{l+\frac{1}{2}})\frac{\partial}{\partial s}\right]^{m+1}v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})\cdot(u-i_h^{m-1}u)(t,s)dsdt\\ &+\int_{\sigma_{kl}}\mathcal{O}(h_{kl}^{m+2})(u-i_h^{m-1}u)(t,s)dsdt:=I_1+I_2+\mathcal{O}(h_{kl}^{2m+4}).\end{aligned}\tag{4.7}$$

Set

$$E_1(t)=\frac{1}{2}\Big[(t-t_{k+\frac{1}{2}})^2-(\frac{h_{1k}}{2})^2\Big],\quad E_2(s)=\frac{1}{2}\Big[(s-s_{l+\frac{1}{2}})^2-(\frac{h_{2l}}{2})^2\Big].$$

It is easy to verify that

$$\frac{1}{m!}(t-t_{k+\frac{1}{2}})^m=\frac{2^m}{(2m)!}(E_1^m)^{(m)}+F_{m-2}(t),\tag{4.8}$$

$$\frac{1}{m!}(s-s_{l+\frac{1}{2}})^m=\frac{2^m}{(2m)!}(E_2^m)^{(m)}+F_{m-2}(s),\tag{4.9}$$

where $F_{m-2} \in P_{m-2}$. Note that $(E_i^m)^{(r)}(i=1,2)$ vanishes at the edges of $\sigma_{kl}$ when $r \le m-1$. Combining (4.8), (4.9), (3.3) and the binomial theorem, and repeating integration by parts with respect to $t$ and $s$ respectively, we have

$$\begin{aligned}
I_1 :=& \int_{\sigma_{kl}} \frac{1}{m!}\left[(t-t_{k+\frac{1}{2}})\frac{\partial}{\partial t}+(s-s_{l+\frac{1}{2}})\frac{\partial}{\partial s}\right]^m v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})(u-i_h^{m-1}u)(t,s)dsdt \\
=& \int_{\sigma_{kl}} \frac{1}{m!}\sum_{i=0}^{m} C_m^i \frac{\partial^m v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})}{\partial t^i \partial s^{m-i}}(t-t_{k+\frac{1}{2}})^i(s-s_{l+\frac{1}{2}})^{m-i}\cdot(u-i_h^{m-1}u)(t,s)dsdt \\
=& \sum_{i=0}^{m}\frac{\partial^m v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})}{\partial t^i \partial s^{m-i}}\int_{\sigma_{kl}}\frac{2^i}{(2i)!}(E_1^i(t))^{(i)}\frac{2^{m-i}}{(2m-2i)!}(E_2^{m-i}(s))^{(m-i)}\cdot(u-i_h^{m-1}u)(t,s)dsdt \\
=& \sum_{i=0}^{m}\frac{\partial^m v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})}{\partial t^i \partial s^{m-i}}\int_{\sigma_{kl}}(-1)^m\frac{2^m}{(2i)!(2m-2i)!}E_1^i(t)E_2^{m-i}(s)\cdot\frac{\partial^m(u-i_h^{m-1}u)(t,s)}{\partial t^i\partial s^{m-i}}dsdt \\
=& (-1)^m\frac{2^m}{(2m)!}\sum_{i=0}^{m}C_{2m}^{2i}\int_{\sigma_{kl}}\frac{\partial^m v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})}{\partial t^i \partial s^{m-i}}E_1^i(t)E_2^{m-i}(s)\frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}dsdt. \qquad (4.10)
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
I_2 =& (-1)^{m+1}\frac{2^{m+1}}{(2m+2)!}\sum_{i=0}^{m+1}C_{2m+2}^{2i}\frac{\partial^{m+1}v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})}{\partial t^i\partial s^{m+1-i}} \\
&\cdot\int_{\sigma_{kl}}E_1^i(t)E_2^{m+1-i}(s)\frac{\partial^{m+1}u(t,s)}{\partial t^i\partial s^{m+1-i}}dsdt.
\end{aligned}$$

Using the facts that

$$E_1^i(t)=\mathcal{O}(h_{kl}^{2i}), \qquad E_2^{m+1-i}(s)=\mathcal{O}(h_{kl}^{2m+2-2i}), \qquad (4.11)$$

and $u \in C^{m+2}(\Omega)$, we get

$$I_2 = C\cdot\mathcal{O}(k_{kl}^{2i})\cdot\mathcal{O}(h_{kl}^{2m+2-2i})\int_{\sigma_{kl}}dsdt=\mathcal{O}(h_{kl}^{2m+4}). \qquad (4.12)$$

From (4.7), (4.10) and (4.12), we get

$$\begin{aligned}
&\int_{\sigma_{kl}} v(t,s)(u-i_h^{m-1}u)(t,s)dsdt \qquad (4.13)\\
=&\ (-1)^m\frac{2^m}{(2m)!}\sum_{i=0}^{m}C_{2m}^{2i}\int_{\sigma_{kl}}\frac{\partial^m v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})}{\partial t^i\partial s^{m-i}}E_1^i(t)E_2^{m-i}(s)\frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}dsdt+\mathcal{O}(h_{kl}^{2m+4}).
\end{aligned}$$

By the Taylor expansion of the first derivative above at the point $v(t,s)$, we have

$$\begin{aligned}
&\frac{\partial^m v(t_{k+\frac{1}{2}},s_{l+\frac{1}{2}})}{\partial t^i\partial s^{m-i}} \\
=&\ \frac{\partial^m v(t,s)}{\partial t^i\partial s^{m-i}}+(t_{k+\frac{1}{2}}-t)\frac{\partial^{m+1}v(t,s)}{\partial t^{i+1}\partial s^{m-i}}+(s_{l+\frac{1}{2}}-s)\frac{\partial^{m+1}v(t,s)}{\partial t^i\partial s^{m+1-i}}+\mathcal{O}(h_{kl}^2). \qquad (4.14)
\end{aligned}$$

Substituting (4.14) into (4.13) and using (4.11) yield

$$\begin{aligned}
&\int_{\sigma_{kl}} v(t,s)(u-i_h^{m-1}u)(t,s)dsdt \\
&= (-1)^m \frac{2^m}{(2m)!}\sum_{i=0}^{m} C_{2m}^{2i}\Big[\int_{\sigma_{kl}} \frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} E_1^i(t)E_2^{m-i}(s)\frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}}dsdt \\
&\quad + \int_{\sigma_{kl}} \frac{\partial^{m+1} v(t,s)}{\partial t^{i+1}\partial s^{m-i}}(t_{k+\frac{1}{2}}-t)E_1^i(t)E_2^{m-i}(s)\cdot \frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}dsdt \\
&\quad + \int_{\sigma_{kl}} \frac{\partial^{m+1} v(t,s)}{\partial t^{i}\partial s^{m+1-i}}E_1^i(t)(s_{l+\frac{1}{2}}-s)E_2^{m-i}(s)\cdot \frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}dsdt\Big] + \mathcal{O}(h_{kl}^{2m+4}) \\
&= (-1)^m \frac{2^m}{(2m)!}\sum_{i=0}^{m} C_{2m}^{2i}\int_{\sigma_{kl}} \frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} E_1^i(t)E_2^{m-i}(s)\frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}}dsdt \\
&\quad + (-1)^m \frac{2^m}{(2m)!}\sum_{i=0}^{m} C_{2m}^{2i}(L_1+L_2) + \mathcal{O}(h_{kl}^{2m+4}). \qquad (4.15)
\end{aligned}$$

It can be verified that

$$(t_{k+\frac{1}{2}}-t)E_1^i(t) = -\frac{1}{i+1}(E_1^{i+1}(t))', \qquad (4.16)$$

$$(s_{l+\frac{1}{2}}-s)E_2^{m-i}(s) = -\frac{1}{m+1-i}(E_2^{m+1-i}(s))'. \qquad (4.17)$$

Using (4.11), (4.16) and integration by parts, we have

$$\begin{aligned}
L_1 &:= \int_{\sigma_{kl}} \frac{\partial^{m+1} v(t,s)}{\partial t^{i+1}\partial s^{m-i}}(t_{k+\frac{1}{2}}-t)E_1^i(t)E_2^{m-i}(s)\cdot \frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}dsdt \\
&= \int_{\sigma_{kl}} \frac{\partial^{m+1} v(t,s)}{\partial t^{i+1}\partial s^{m-i}}(-\frac{1}{i+1})[(E_1^{i+1}(t)]'E_2^{m-i}(s)\cdot \frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}dsdt \\
&= \int_{\sigma_{kl}} E_1^{i+1}(t)\frac{1}{i+1}E_2^{m-i}(s)\cdot\frac{\partial}{\partial t}(\frac{\partial^{m+1} v(t,s)}{\partial t^{i+1}\partial s^{m-i}}\frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}})dsdt = \mathcal{O}(h_{kl}^{2m+4}).
\end{aligned}$$

Similarly, using (4.11), (4.17) and integration by parts, we have

$$L_2 = \int_{\sigma_{kl}} E_1^{i}(t)\frac{1}{m+1-i}E_2^{m+1-i}(s)\cdot\frac{\partial}{\partial s}(\frac{\partial^{m+1} v(t,s)}{\partial t^{i}\partial s^{m+1-i}}\frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}})dsdt = \mathcal{O}(h_{kl}^{2m+4}).$$

Therefore,

$$\begin{aligned}
&\int_{\sigma_{kl}} v(t,s)(u-i_h^{m-1}u)(t,s)dsdt \\
&= (-1)^m \frac{2^m}{(2m)!}\sum_{i=0}^{m} C_{2m}^{2i}\int_{\sigma_{kl}} \frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} E_1^i(t)E_2^{m-i}(s)\frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}}dsdt + \mathcal{O}(h_{kl}^{2m+4}).
\end{aligned}$$

Using the binomial theorem, we have

$$E_1^i(t) = \frac{1}{2^i}\sum_{j=0}^{i}(-1)^{i-j}C_i^j\left(\frac{h_{1k}}{2}\right)^{2i-2j}(t-t_{k+\frac{1}{2}})^{2j},$$
$$E_2^{m-i}(s) = \frac{1}{2^{m-i}}\sum_{p=0}^{m-i}(-1)^{m-i-p}C_{m-i}^p\left(\frac{h_{2l}}{2}\right)^{2m-2i-2p}(s-s_{l+\frac{1}{2}})^{2p}.$$

Consequently,

$$\begin{aligned}
&\int_{\sigma_{kl}} v(t,s)(u-i_h^{m-1}u)(t,s)dsdt\\
=\ &(-1)^m\frac{2^m}{(2m)!}\sum_{i=0}^{m}C_{2m}^{2i}\int_{\sigma_{kl}}\frac{\partial^m v(t,s)}{\partial t^i\partial s^{m-i}}\frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}\\
&\cdot\left[\frac{1}{2^i}\sum_{j=0}^{i}(-1)^{i-j}C_i^j\left(\frac{h_{1k}}{2}\right)^{2i-2j}(t-t_{k+\frac{1}{2}})^{2j}\right]\\
&\cdot\left[\frac{1}{2^{m-i}}\sum_{p=0}^{m-i}(-1)^{m-i-p}C_{m-i}^p\left(\frac{h_{2l}}{2}\right)^{2m-2i-2p}(s-s_{l+\frac{1}{2}})^{2p}\right]dsdt+\mathcal{O}(h_{kl}^{2m+4})\\
=\ &\frac{1}{(2m)!}\sum_{i=0}^{m}C_{2m}^{2i}\left(\frac{h_{1k}}{2}\right)^{2i}\left(\frac{h_{2l}}{2}\right)^{2m-2i}\int_{\sigma_{kl}}\frac{\partial^m v(t,s)}{\partial t^i\partial s^{m-i}}\frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}dsdt\\
&-\frac{1}{(2m)!}\sum_{i=0}^{m}C_{2m}^{2i}i\left(\frac{h_{1k}}{2}\right)^{2i-2}\left(\frac{h_{2l}}{2}\right)^{2m-2i}\cdot\int_{\sigma_{kl}}\frac{\partial^m v(t,s)}{\partial t^i\partial s^{m-i}}\frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}(t-t_{k+\frac{1}{2}})^2dsdt\\
&-\frac{1}{(2m)!}\sum_{i=0}^{m}C_{2m}^{2i}(m-i)\left(\frac{h_{1k}}{2}\right)^{2i}\left(\frac{h_{2l}}{2}\right)^{2m-2i-2}\cdot\int_{\sigma_{kl}}\frac{\partial^m v(t,s)}{\partial t^i\partial s^{m-i}}\frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}(s-s_{l+\frac{1}{2}})^2dsdt\\
&+(-1)^m\frac{2^m}{(2m)!}\sum_{i=0}^{m}C_{2m}^{2i}\int_{\sigma_{kl}}\frac{\partial^m v(t,s)}{\partial t^i\partial s^{m-i}}\frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}\\
&\cdot\left[\frac{1}{2^i}\sum_{j=1}^{i}(-1)^{i-j}C_i^j\left(\frac{h_{1k}}{2}\right)^{2i-2j}(t-t_{k+\frac{1}{2}})^{2j}\right]\\
&\cdot\left[\frac{1}{2^{m-i}}\sum_{p=1}^{m-i}(-1)^{m-i-p}C_{m-i}^p\left(\frac{h_{2l}}{2}\right)^{2m-2i-2p}(s-s_{l+\frac{1}{2}})^{2p}\right]dsdt+\mathcal{O}(h_{kl}^{2m+4})\\
:=\ &J_1+J_2+J_3+J_4+\mathcal{O}(h_{kl}^{2m+4}).
\end{aligned}\tag{4.18}$$

Note that $J_1$ above is of the form of (2.8). Moreover,

$$(t-t_{k+\frac{1}{2}})^2 = \frac{1}{3}(E_1^2(t))'' + \frac{1}{3}\left(\frac{h_{1k}}{2}\right)^2. \tag{4.19}$$

Integration by parts one more time gives

$$\begin{aligned}
&\int\limits_{\sigma_{kl}} \frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} \frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}} (t-t_{k+\frac{1}{2}})^2 ds dt \\
=& \int\limits_{\sigma_{kl}} \frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} \frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}} \left[\frac{1}{3}(E_1^2(t))'' + \frac{1}{3}\left(\frac{h_{1k}}{2}\right)^2\right] ds dt \\
=& \frac{1}{3}\int\limits_{\sigma_{kl}} E_1^2(t) \frac{\partial^2}{\partial t^2}\left(\frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} \frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}}\right) ds dt + \frac{1}{3}\left(\frac{h_{1k}}{2}\right)^2 \int\limits_{\sigma_{kl}} \frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} \frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}} ds dt \\
=& \frac{1}{3}\left(\frac{h_{1k}}{2}\right)^2 \int\limits_{\sigma_{kl}} \frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} \frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}} ds dt + \mathcal{O}(h_{1k}^6).
\end{aligned} \tag{4.20}$$

Substituting (4.20) into $J_2$, we have

$$\begin{aligned}
J_2 :=& -\frac{1}{(2m)!}\sum_{i=0}^{m} C_{2m}^{2i}\, i \left(\frac{h_{1k}}{2}\right)^{2i-2} \left(\frac{h_{2l}}{2}\right)^{2m-2i} \cdot \int\limits_{\sigma_{kl}} \frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} \frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}} (t-t_{k+\frac{1}{2}})^2 ds dt \\
=& -\frac{1}{(2m)!}\sum_{i=0}^{m} C_{2m}^{2i}\, i \left(\frac{h_{1k}}{2}\right)^{2i-2} \left(\frac{h_{2l}}{2}\right)^{2m-2i} \\
&\cdot \left[\frac{1}{3}\left(\frac{h_{1k}}{2}\right)^2 \int\limits_{\sigma_{kl}} \frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} \frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}} ds dt + \mathcal{O}(h_{1k}^6)\right] \\
=& -\frac{1}{3(2m)!}\sum_{i=0}^{m} C_{2m}^{2i}\, i \frac{h_{1k}^{2i} h_{2l}^{2m-2i}}{2^{2m}} \int\limits_{\sigma_{kl}} \frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} \frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}} ds dt + \mathcal{O}(h_{kl}^{2m+4}).
\end{aligned}$$

In analogy with (4.19), using

$$(s-s_{l+\frac{1}{2}})^2 = \frac{1}{3}(E_2^2(s))'' + \frac{1}{3}\left(\frac{h_{2l}}{2}\right)^2, \tag{4.21}$$

we obtain

$$J_3 = -\frac{1}{3(2m)!}\sum_{i=0}^{m} C_{2m}^{2i}(m-i)\frac{h_{1k}^{2i} h_{2l}^{2m-2i}}{2^{2m}} \int\limits_{\sigma_{kl}} \frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} \frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}} ds dt + \mathcal{O}(h_{kl}^{2m+4}),$$

$$J_2 + J_3 = -\frac{m}{3(2m)!2^{2m}}\sum_{i=0}^{m} C_{2m}^{2i} h_{1k}^{2i} h_{2l}^{2m-2i} \int\limits_{\sigma_{kl}} \frac{\partial^m v(t,s)}{\partial t^i \partial s^{m-i}} \frac{\partial^m u(t,s)}{\partial t^i \partial s^{m-i}} ds dt + \mathcal{O}(h_{kl}^{2m+4}). \tag{4.22}$$

The last equation shows that $J_2 + J_3$ is of the form of (2.8). Using (4.8), (4.9), (4.19), (4.21) and repeating integration by parts, we find that $J_4$ is also of the form (2.8). These results, together with (4.18), complete the proof of Theorem 2.2. □

### 4.3. Proof of Theorem 2.3

We discuss the one-dimensional case first. Substituting (3.10) into (2.6), we have

$$\lambda - \lambda_h = \beta_u h^{2m} + \mathcal{O}(h^{2m+2}) \qquad m \geq 2; \tag{4.23}$$

$$\lambda - \lambda_h = \beta_u h^2 + \mathcal{O}(h^3) \qquad m = 1. \tag{4.24}$$

Then we divide every sub-interval into two equal parts. Replacing $P_h$ with $P_{h/2}$ in (3.10) and using (2.7) yield

$$\lambda - \lambda^s = \beta_u \left(\frac{h}{2}\right)^{2m} + \mathcal{O}(h^{2m+2}) \qquad m \geq 2; \tag{4.25}$$

$$\lambda - \lambda^s = \beta_u \left(\frac{h}{2}\right)^{2} + \mathcal{O}(h^{3}) \qquad m = 1. \tag{4.26}$$

For the two-dimensional case, we only consider the rectangular domain $\Omega$. We set $v = Tu$ and derive from Theorem 2.2 that

$$\begin{aligned}
&(u, T(I-P_h)u) = (Tu, (I-P_h)u)\\
=& \sum_{k=0}^{N-1}\sum_{l=0}^{M-1}\int_{\sigma_{kl}} v(t,s)(I-P_h)u(t,s)dsdt\\
=& \sum_{k=0}^{N-1}\sum_{l=0}^{M-1}\left[\sum_{i=0}^{m}\widetilde{c}h_{1k}^{2i}h_{2l}^{2m-2i}\int_{\sigma_{kl}}\frac{\partial^m v(t,s)}{\partial t^i\partial s^{m-i}}\frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}dsdt + \mathcal{O}(h_{kl}^{2m+4})\right]\\
:=& \alpha_u h^{2m} + \mathcal{O}(h^{2m+2}),
\end{aligned} \tag{4.27}$$

where

$$\alpha_u = \sum_{k=0}^{N-1}\sum_{l=0}^{M-1}\sum_{i=0}^{m}\widetilde{c}\left(\frac{h_{1k}}{h}\right)^{2i}\left(\frac{h_{2l}}{h}\right)^{2m-2i}\int_{\sigma_{kl}}\frac{\partial^m v(t,s)}{\partial t^i\partial s^{m-i}}\frac{\partial^m u(t,s)}{\partial t^i\partial s^{m-i}}dsdt,$$

which does not change with uniform mesh refinement.

Combining (2.6) in Theorem 2.1 and (4.27) yields

$$\begin{aligned}
\lambda - \lambda_h &= (u, T(I-P_h)u) + \mathcal{O}(h^{3m})\\
&= \alpha_u h^{2m} + \mathcal{O}(h^{2m+2}) + \mathcal{O}(h^{3m}).
\end{aligned}$$

Similarly, we divide each 2-dimensional sub-domain into four equal parts. Substituting $P_{h/2}$ for $P_h$ in (4.27) and using (2.7) in Theorem 2.1, we have

$$\lambda - \lambda^s = \alpha_u \left(\frac{h}{2}\right)^{2m} + \mathcal{O}(h^{2m+2}) + \mathcal{O}(h^{3m}).$$

Therefore, (4.23)-(4.26) are also valid for the two-dimensional eigenvalue problem of the Fredholm integral equation of the second kind. Extrapolating between $\lambda_h$ in (4.23) and $\lambda^s$ in (4.25), we get a new approximation using higher accuracy:

$$\frac{2^{2m}\lambda^s - \lambda_h}{2^{2m}-1} = \lambda + \mathcal{O}(h^{2m+2}) \qquad m \geq 2.$$

Extrapolating between $\lambda_h$ in (4.24) and $\lambda^s$ in (4.26), gives

$$\frac{4\lambda^s - \lambda_h}{3} = \lambda + \mathcal{O}(h^{3}) \qquad m = 1.$$

This completes the proof of Theorem 2.3. $\square$

## 5. Numerical Experiments

**Example 5.1.** Consider the eigenvalue problem of the Fredholm integral equation of the second kind with a polynomial kernel,

$$\int_0^1 \left(st - \frac{s^3t^3}{6}\right) u(s)ds = \lambda u(t), \quad 0 \le s, \quad t \le 1. \tag{5.1}$$

The eigenvalue of the largest modulus for the solution of Eq. (5.1) is $\lambda_1 = 0.31357339186336$. Let $m = 1$, i.e., $S_{m-1}^{(-1)}$ is the piecewise constant space. We introduce some notations for this example:

$$\text{err}_h = \lambda_1 - \lambda_{1,h}, \quad \text{err}_h^{\text{extra}} = \frac{4\lambda^s - \lambda_{1,h}}{3} - \lambda_1,$$
$$R_h = \log_2(\text{err}_\text{h}/\text{err}_{\text{h}/2}), \quad R_h^{\text{extra}} = \log_2(\text{err}_\text{h}^{\text{extra}}/\text{err}_{\text{h}/2}^{\text{extra}}).$$

The results are listed in the following table.

Table 5.1: Numerical errors and rate of convergence.

| $h$ | $\lambda_{1,h}$ | $\text{err}_h$ | $R_h$ | $\lambda^s$ | $\text{err}_h^{\text{extra}}$ | $R_h^{\text{extra}}$ |
|---|---|---|---|---|---|---|
| $4^{-1}$ | 0.309032 | 4.541212e-003 | 2.00 | 0.312439 | 8.568029e-007 | 4.01 |
| $8^{-1}$ | 0.312439 | 1.134393e-003 | 2.00 | 0.313290 | 5.309702e-008 | 4.00 |
| $16^{-1}$ | 0.313290 | 2.835419e-004 | 2.00 | 0.313503 | 3.311407e-009 | 4.00 |
| $32^{-1}$ | 0.313503 | 7.088195e-005 | 2.00 | 0.313556 | 2.068434e-010 | 4.00 |
| $64^{-1}$ | 0.313556 | 1.772027e-005 | | 0.313569 | 1.292333e-011 | |

We see from Table 5.1 that

$$\lambda_1 - \lambda_h = \mathcal{O}(h^2), \quad \frac{4\lambda^s - \lambda_{1,h}}{3} - \lambda_1 = \mathcal{O}(h^4).$$

The results demonstrated the efficiency of our method, and also confirm our theoretical results.

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