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ENERGY ESTIMATES FOR DELAY DIFFUSION-REACTION EQUATIONS*

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Abstract

In this paper we consider nonlinear delay diffusion-reaction equations with initial and Dirichlet boundary conditions. The behaviour and the stability of the solution of such initial boundary value problems (IBVPs) are studied using the energy method. Simple numerical methods are considered for the computation of numerical approximations to the solution of the nonlinear IBVPs. Using the discrete energy method we study the stability and convergence of the numerical approximations. Numerical experiments are carried out to illustrate our theoretical results.

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1. Introduction

Initial boundary value problems with memory defined by the nonlinear delay diffusion-reaction equation

$$\frac{\partial u}{\partial t}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t) + f(u(x,t), u(x,t-\tau)), \qquad (x,t) \in (a,b) \times (0,T], \tag{1.1}$$

where $\tau > 0$ is a delay parameter, $\alpha > 0$, and by the conditions

$$u(a,t) = u_a(t),\ u(b,t) = u_b(t),\ t \in (0,T], \tag{1.2}$$

$$u(x,t) = u_0(x,t),\ x \in (a,b),\ t \in [-\tau, 0], \tag{1.3}$$

or systems of delay diffusion-reaction equations of type (1.1), are largely used on the description of biological phenomena. The simplest model is the one obtained replacing the diffusion Verhulst equation by the logistic delay equation (1.1) with the reaction term

$$f(u(x,t), u(x,t-\tau)) = ru(x,t)\big(1 - \frac{u(x,t-\tau)}{\beta}\big),$$

where r and β are positive constants. Other versions of Eq. (1.1) are considered to model growth population phenomena. For instance, the x-independent version of Eq. (1.1) with

$$f(u(x,t), u(x,t-\tau)) = be^{-au(x,t-\tau)-d_1\tau} u(x,t-\tau) - du(x,t),$$

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where a, b, d and d_1 are positive parameters, is proposed in [5] to study a grow birth population. Eq. (1.1) with

$$f(u(x,t), u(x,t-\tau)) = bu(x,t-\tau)(1-u(x,t)) - cu(x,t),$$

where b and c are positive parameters, is considered in [11] independent of x, to model epidemic propagations phenomena.

Systems of delay partial differential equations of type (1.1) have been also used to describe mathematically biological phenomena. In [13], the x-independent version of the system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \alpha_1 \dfrac{\partial^2 u_1}{\partial x^2} - R_0 u_1(x,t) u_2(x,t-\tau) + u_2(x,t), \\ \dfrac{\partial u_2}{\partial t} = \alpha_2 \dfrac{\partial^2 u_2}{\partial x^2} + R_0 u_1(x,t) u_2(x,t-\tau) - u_2(x,t), \end{cases} \tag{1.4}$$

where u_1 and u_2 represent the ratio of susceptible and infected individuals and $\alpha_i, i = 1, 2$, R_0 are positive constants, was used to study an epidemic propagation.

All the models presented before have been based on the Fick's law for the flux combined with a mass conservation law. The memory in the mathematical model is introduced using the reaction. Recently, some new models have been proposed, where the memory phenomenon is taken into account by changing the Fick's law, see, e.g., [1, 2, 6].

On the context of biological phenomena, the qualitative properties of the solution of the nonlinear problem (1.1)-(1.3) have an important role on the description of the dynamic of the species that are being studied. Such qualitative properties depend on the behaviour of reaction terms.

It is known that in general the explicit expression for the solution of (1.1)-(1.3) is unavailable, numerical methods are the only way to get quantitative information to the nonlinear problem (1.1)-(1.3). The study of delay Cauchy or delay IBVPs has been very fruitful in the last twenty years, see, e.g., the books [3, 4, 14, 15] and the references therein. Moreover, the study of mathematical models containing delay equations continues to be a fruitful topic. We mention, without being exhaustive, the papers [7, 10, 12] contain the analysis of some biological systems, [9] presents a qualitative study of the solution of a hyperbolic delay equation. In [8], spectral collocation methods for a parabolic reaction-diffusion equation of type (1.1) are studied.

The characterization of the behaviour of the solution u of the IBVP (1.1)-(1.3) and the solution u_h^n of its discretization using the behaviour of the reaction term f is the aim of this paper. This characterization has an important role on the description of the behaviour of whole system.

Using energy method we establish estimates for u and u_h^n that depend on the derivatives of the reaction term f. As a consequence of these estimates, we reach conclusions concerning the stability of the solutions when the initial condition u_0 is perturbed.

The paper is organized as follows. In Section 2 we consider IBVPs (1.1)-(1.3) with the reaction term depending only on $u(x,t-\tau)$. In Section 2.1 the behaviour of the solution u and its stability are studied. In Section 2.2 a numerical method which can be seen as a combination of the spatial discretization defined by the centered finite difference operators and a time integration defined by the θ-method is considered. We study the behaviour of the finite difference solution and a discrete version of the result established in the continuous context is obtained. The stability and the convergence of the numerical method are also proved. The procedures used for the continuous and discrete models with a reaction term depending on $u(x,t-\tau)$ are easily extended in Section 3 to delay partial differential equations with a reaction term depending on $u(x,t)$ and $u(x,t-\tau)$. In Section 4 some extensions to systems of delay equations are considered. Numerical simulations illustrating the theoretical results obtained in this paper are included in Section 5.

2. Reaction Term Depending on $u(x,t-\tau)$

In this section we assume that the reaction term depends on $u(x,t-\tau), \tau>0$. Then, f is a single variable function.

2.1. Continuous models

The stability of the IBVP (1.1)-(1.3) is studied with respect to the L^2-norm. In what follows we use the following assumptions:

$$u(x,t)\in[c,d], \quad (x,t)\in[a,b]\times[-\tau,T], \tag{2.1}$$

for some c,d, and

$$f\in\mathbb{C}^1[c,d], \quad f(0)=0, \quad f'_{max}:=\max_{y\in[c,d]} f'(y). \tag{2.2}$$

We assume that $T=k\tau$, for some $k\in\mathbb{N}$. Let v be a function defined in $[a,b]\times[-\tau,T]$. Then, for each t, v is a function of x which is denote by $v(t)$.

Theorem 2.1. *Let u be a solution of the IBVP* (1.1)-(1.3) *with homogeneous boundary conditions. Let us suppose that u satisfies* (2.1) *and the reaction term f satisfies* (2.2)*. If $\partial u/\partial t$, $\partial^j u/\partial x^\ell\in L^2(a,b), \ell=1,2$, then, for $t\in[(m-1)\tau,m\tau]\subseteq[0,T]$, the following estimate holds*

$$\|u(t)\|^2_{L^2}\le e^{\gamma m\tau}\max_{s\in[-\tau,0]}\|u_0(s)\|^2_{L^2}, \tag{2.3}$$

where $\gamma=(f'_{max})^2(b-a)^2/2\alpha$.

Proof. Multiplying Eq. (1.1) by $u(t)$ gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2}=-\alpha\|\frac{\partial u}{\partial x}\|^2_{L^2}+(f(u(t-\tau)),u(t)). \tag{2.4}$$

As $(f(u(t-\tau)),u(t))=(f'(\mu(t))(u(t-\tau)-u(t)),u(t))$ with $\mu(t)$ in the segment defined by $u(t-\tau)$ and $u(t)$, we deduce that

$$(f(u(t-\tau)),u(t))\le\eta^2 f'^2_{max}\|u(t)\|^2_{L^2}+\frac{1}{4\eta^2}\|u(t-\tau)\|^2_{L^2}, \tag{2.5}$$

where $\eta\neq0$ is an arbitrary constant. Using (2.5) in (2.4) and considering the Poincaré-Friedrichs inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2}\le\big(-\frac{\alpha}{(b-a)^2}+\eta^2 f'^2_{max}\big)\|u(t)\|^2_{L^2}+\frac{1}{4\eta^2}\|u(t-\tau)\|^2_{L^2}. \tag{2.6}$$

Let η be such that

$$-\frac{\alpha}{(b-a)^2}+\eta^2 f'^2_{max}=0. \tag{2.7}$$

Then, from (2.6) we get

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 \leq \frac{1}{2\eta^2}\|u(t-\tau)\|_{L^2}^2, \tag{2.8}$$

which is equivalent to

$$\frac{d}{dt}\Big(\|u(t)\|_{L^2}^2 - \frac{1}{2\eta^2}\int_0^t \|u(s-\tau)\|_{L^2}^2 ds\Big) \leq 0. \tag{2.9}$$

From (2.9) we conclude that, for $t \in [0, \tau]$,

$$\|u(t)\|_{L^2}^2 \leq \big(1 + \tau\frac{f_{max}^{'2}(b-a)^2}{2\alpha}\big) \max_{s\in[-\tau,0]} \|u_0(s)\|_{L^2}^2.$$

The above inequality can be easily extended for $t \in [(m-1)\tau, m\tau]$

$$\|u(t)\|_{L^2}^2 \leq \big(1 + \tau\frac{f_{max}^{'2}(b-a)^2}{2\alpha}\big)^m \max_{s\in[-\tau,0]} \|u_0(s)\|_{L^2}^2,$$

which allows us to conclude (2.3). □

Let u_1 and u_2 be solutions of the IBVP (1.1)-(1.3) with initial conditions $u_{0,1}$ and $u_{0,2}$, respectively. Then $w = u_1 - u_2$ satisfies the nonlinear delay equation

$$\frac{\partial w}{\partial t} = \alpha\frac{\partial^2 w}{\partial x^2} + f(u_1(t-\tau)) - f(u_2(t-\tau)).$$

Following the proof of Theorem 2.1 and noting that

$$f(u_1(t-\tau)) - f(u_2(t-\tau)) = f'(\xi)w(t-\tau),$$

where ξ in the segment with end points $u_1(t-\tau)$, $u_2(t-\tau)$, and w satisfies the homogeneous boundary conditions, the next stability result can be proved.

Theorem 2.2. *Let u_1, u_2 be solutions of the IBVP* (1.1)-(1.3)*, with initial conditions $u_{0,1}, u_{0,2}$ respectively. If u_1 and u_2 satisfy* (2.1)*, then, under the assumptions of Theorem* 2.1*, for $t \in [(m-1)\tau, m\tau] \subseteq [0, T]$, we have*

$$\|u_1(t) - u_2(t)\|_{L^2}^2 \leq e^{\gamma m\tau} \max_{s\in[-\tau,0]} \|u_{0,1}(s) - u_{0,2}(s)\|_{L^2}^2, \tag{2.10}$$

where $\gamma = (f'_{max})^2(b-a)^2/2\alpha$.

The behaviour of $u_1 - u_2$ is tremendously determined by the magnitude of $\|u_{0,1} - u_{0,2}\|_{L^2}$ and by the behaviour of the reaction term. Nevertheless, independently of $f_{max}^{'2}$, if $\|u_{0,1}(s) - u_{0,2}(s)\|_{L^2}$, for $s \in [0, \tau]$, is small enough, then $\|u_1(t) - u_2(t)\|_{L^2}$ is also small enough in $[0, T]$. As a consequence of Theorem 2.2, we conclude that, if u_1 and u_2 are solutions of the IBVP (1.1)- (1.3), then $u_1 = u_2$.

Finally, we compare the estimate (2.3) with the one obtained when the reaction term f depends also on $u(x, t)$. In this case, we obtain (2.3) with $\gamma = -\alpha(b-a)^{-2} + f'_{max}$ and $\max_{s\in[-\tau,0]} \|u_0(s)\|_{L^2}^2$ replaced by $\|u_0\|_{L^2}^2$. For instance, if $f' < 0$ then we conclude that $\|u(t)\|_{L^2}$ decreases in time. This behaviour can not be deduced from the energy estimate obtained for the solution of the IBVP when the reaction term depends on $u(x, t-\tau)$.

2.2. Discrete models

In this section, we study the behaviour of the finite difference approximations for the solutions of IBVPs considered in Section 2.1. The numerical approximation is defined by a spatial discretization using centered finite differencing and a time integration using the θ-method.

In $[a,b]$, we introduce the grid $I_h = \{x_i, i = 0, \cdots, N\}$ with $x_0 = a, x_N = b$ and $x_{i+1} = x_i + h, i = 0, \cdots, N-1$. Let Δt be the temporal stepsize and let $j \in \mathbb{N}$ be such that $j = \tau/\Delta t$. In $[-\tau, T]$, we consider the grid $\{t_\ell, \ell = -j, \cdots, M\}$ defined by

$$t_{-j} = -\tau, t_{\ell+1} = t_\ell + \Delta t, \ \ell = -j, \cdots, M-1.$$

Let $u_h^{n+1}(x_i)$ be the fully discrete approximation to $u(x_i, t_{n+1})$ defined by

$$u_h^{n+1}(x_i) = u_h^n(x_i) + \Delta t \alpha D_2 u_h^{n+1} + \Delta t(1-\theta) f(u_h^{n-j}(x_i)) + \Delta t \theta f(u_h^{n+1-j}(x_i)), \tag{2.11}$$

for $i = 1, \cdots, N-1, \ n = 1, \cdots, M-1$, and such that

$$u_h^n(x_0) = u_a(t_n), \ u_h^n(x_N) = u_b(t_n), \quad n = 1, \cdots, M, \tag{2.12}$$

$$u_h^n(x_i) = u_0(x_i, t_n), \quad i = 0, \cdots, N, \quad n = -j+1, \cdots, 0. \tag{2.13}$$

In (2.11), $\theta \in [0,1]$ and the difference operator D_2 is the usual second-order centered finite difference operator

$$D_2 v_h(x_i) = \frac{v_h(x_{i+1}) - 2v_h(x_i) + v_h(x_{i-1})}{h^2}, \quad i = 1, \cdots, N-1.$$

The stability and convergence analysis are established with respect to a L^2 discrete norm which is defined below. By $L^2(I_h)$ we denote the space of grid functions v_h such that $v_h(x_0) = v_h(x_N) = 0$. In $L^2(I_h)$, we introduce the inner product

$$(v_h, w_h)_h = h \sum_{i=1}^{N-1} v_h(x_i) w_h(x_i), \ v_h, w_h \in L^2(I_h). \tag{2.14}$$

By $\|.\|_{L^2(I_h)}$ we denote the norm induced by the inner product (2.14).

Let D_{-x} be the usual backward finite difference operator. The following relations

$$(D_2 v_h, w_h)_h = -h \sum_{i=1}^{N} D_{-x} v_h(x_i) D_{-x} w_h(x_i), \ v_h, w_h \in L^2(I_h), \tag{2.15}$$

$$\|v_h\|^2_{L^2(I_h)} \leq (b-a)^2 \sum_{i=1}^{N} h(D_{-x} v_h(x_i))^2, \ v_h \in L^2(I_h), \tag{2.16}$$

play a central role on the proof of the main result, i.e., Theorem 2.3. The identity (2.15) can be proved using summation by parts. The second relation is known as a discrete Poincaré-Friedrichs inequality.

The next result is a discrete version of Theorem 2.1 and establishes a characterization of the solution of (2.11), (2.13) when homogeneous boundary conditions are considered.

Theorem 2.3. *Let u_h^{n+1} be defined by* (2.11)-(2.13) *with homogeneous boundary conditions and such that $u_h^\ell(x_i) \in [c,d], i = 1, \cdots, N-1, \ell = -j+1, \cdots, M$. If the reaction term f satisfies* (2.2)*, then*

$$\|u_h^{mj+\ell}\|^2_{L^2(I_h)} \leq \mathbf{C} \max_{\mu=-j,\cdots,0} \|u_0(t_\mu)\|^2_{L^2(I_h)}, \quad n = 0, \cdots, M-1, \tag{2.17}$$

with $m \in \{1, \cdots, k-1\}, k\tau = T\,, \ell \in \{1, \cdots, j\}$,

$$\mathbf{C} = \big(1 + \gamma\tau\big)^{m+1}, \tag{2.18}$$

and

$$\gamma = \frac{(b-a)^2 f_{max}'^2}{2\alpha}. \tag{2.19}$$

Proof. Multiplying (2.11) by u_h^{n+1} with respect to the inner product $(.,.)_h$, we have

$$\begin{aligned}\|u_h^{n+1}\|^2_{L^2(I_h)} =&(u_h^n, u_h^{n+1})_h + \Delta t\alpha(D_2 u_h^{n+1}, u_h^{n+1})_h \\ &+ \Delta t\big((1-\theta)(f(u_h^{n-j}), u_h^{n+1})_h + \theta(f(u_h^{n+1-j}), u_h^{n+1})_h\big),\end{aligned} \tag{2.20}$$

where $f(u_h^p)(x_i) = f(u_h^p(x_i)), i = 1, \cdots, N-1$, for $p = n-j, n+1-j$. Considering in (2.20) the identity (2.15) with $v_h = w_h = u_h^{n+1}$, and using the Poincaré-Friedrichs inequality and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\big(\frac{1}{2} + \Delta t\frac{\alpha}{(b-a)^2}\big)\|u_h^{n+1}\|^2_{L^2(I_h)} \\ \leq&\frac{1}{2}\|u_h^n\|^2_{L^2(I_h)} + \Delta t\big((1-\theta)(f(u_h^{n-j}), u_h^{n+1})_h + \theta(f(u_h^{n+1-j}), u_h^{n+1})_h.\end{aligned} \tag{2.21}$$

Since

$$(f(u_h^{p-j}), u_h^{n+1})_h \leq \eta^2 f_{max}'^2 \|u_h^{n+1}\|^2_{L^2(I_h)} + \frac{1}{4\eta^2}\|u_h^{p-j}\|^2_{L^2(I_h)},$$

for $p = n, n+1$, where $\eta \neq 0$ is an arbitrary constant, from (2.21), we deduce

$$\begin{aligned}&\big(\frac{1}{2} + \Delta t\big(\frac{\alpha}{(b-a)^2} - \eta^2 f_{max}'^2\big)\big)\|u_h^{n+1}\|^2_{L^2(I_h)} \\ \leq&\frac{1}{2}\|u_h^n\|^2_{L^2(I_h)} + \Delta t\frac{1}{4\eta^2}\big((1-\theta)\|u_h^{n-j}\|^2_{L^2(I_h)} + \theta\|u_h^{n+1-j}\|^2_{L^2(I_h)}\big).\end{aligned} \tag{2.22}$$

Fixing η by $\eta^2 = \alpha(b-a)^{-2}(f'_{max})^{-2}$, we obtain

$$\|u_h^{n+1}\|^2_{L^2(I_h)} \leq \|u_h^n\|^2_{L^2(I_h)} + \Delta t\gamma\big((1-\theta)\|u_h^{n-j}\|^2_{L^2(I_h)} + \theta\|u_h^{n+1-j}\|^2_{L^2(I_h)}\big), \tag{2.23}$$

with γ defined by (2.19).

We remark that the grids considered in $[-\tau, T]$ are such that $j\Delta t = \tau$, $M\Delta t = T$ and $kj\Delta t = T$ (k is such that $k\tau = T$). In what follows we establish an estimate for $\|u_h^{mj+\ell}\|_{L^2(I_h)}$ with $m \in \{1, \cdots, k-1\}$.

From (2.23), it can be shown that

$$\|u_h^\ell\|^2_{L^2(I_h)} \leq \big(1 + \gamma\ell\Delta t\big) \max_{i=-j,\cdots,-j+\ell}\|u_h^i\|^2_{L^2(I_h)}, \quad \ell = 1, \cdots, j, \tag{2.24}$$

which implies that

$$\|u_h^\ell\|^2_{L^2(I_h)} \leq \big(1 + \gamma\tau\big) \max_{i=-j,\cdots,-j+\ell}\|u_h^i\|^2_{L^2(I_h)}, \quad \ell = 1, \cdots, j. \tag{2.25}$$

As we have

$$\|u_h^{j+\ell}\|^2_{L^2(I_h)} \leq \big(1 + \gamma\tau\big) \max_{i=0,\cdots,\ell,j}\|u_h^i\|^2_{L^2(I_h)}, \quad \ell = 1, \cdots, j, \tag{2.26}$$

from (2.25) we conclude the following estimate

$$\|u_h^{j+\ell}\|^2_{L^2(I_h)} \le \big(1+\gamma\tau\big)^2 \max_{i=-j,\cdots,0} \|u_h^i\|^2_{L^2(I_h)}, \quad \ell = 1,\cdots,j. \tag{2.27}$$

Following the previous steps, it can be shown that (2.17) holds. □

We establish in the next result the stability of the method (2.11)-(2.13).

Theorem 2.4. *Let $u_h^{n+1}, \tilde{u}_h^{n+1}$ be defined by* (2.11)-(2.13) *with initial conditions u_0 and $\tilde{u}_0$ respectively, and such that $u_h^\ell(x_i), \tilde{u}_h^\ell(x_i) \in [c,d], i = 1,\cdots,N-1, \ell = -j+1,\cdots,M$. If the reaction term f_i satisfies* (2.2), *then*

$$\|u_h^{n+1} - \tilde{u}_h^{mj+\ell}\|^2_{L^2(I_h)} \le \mathbf{C} \max_{\mu=-j,\cdots,0} \|u_0(t_\mu) - \tilde{u}_0(t_\mu)\|^2_{L^2(I_h)},\ n = 0,\cdots,M-1, \tag{2.28}$$

with $m \in \{1,\cdots,k-1\}, k\tau = T, \ell \in \{1,\cdots,j\}$, and $\mathbf{C}$ is defined by (2.18).

Proof. Let v_h^{n+1} be defined by $v_h^{n+1} = u_h^{n+1} - \tilde{u}_h^{n+1}$. We have

$$\begin{aligned} v_h^{n+1}(x_i) =& v_h^n(x_i) + \Delta t\alpha D_2 v_h^{n+1} + \Delta t\big((1-\theta)\big(f(u_h^{n-j}(x_i)) - f(\tilde{u}_h^{n-j}(x_i))\big) \\ &+\theta\big(f(u_h^{n+1-j}(x_i)) - f(\tilde{u}_h^{n+1-j}(x_i))\big). \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} &v_h^\ell(x_0) = v_h^\ell(x_N) = 0, \quad \ell = 0,\cdots,M, \\ &v_h^\ell(x_i) = u_0(x_i,t_\ell) - \tilde{u}_0(x_i,t_\ell), \quad i = 0,\cdots,N, \quad \ell = -j+1,\cdots,0. \end{aligned}$$

Using the fact that

$$f(u_h^{p-j}(x_i)) - f(\tilde{u}_h^{p-j}(x_i)) = f'(\xi_i) v_h^{p-j},$$

where ξ_i belongs to the segment with the end-points $u_h^{p-j}(x_i)$ and $\tilde{u}_h^{p-j}(x_i)$, for $p = n, n+1$, we can obtain (2.28) by following the proof of (2.17). □

Theorem 2.4 implies that the method (2.11) is unconditionally stable — stable without any condition on the step sizes Δt and h — with stability coefficient $\mathbf{C}$ defined by (2.18).

The convergence of the method (2.11) can be shown from the consistency and following the proof of Theorem 2.3. Let $e_h^{n+1}(x_i) = u(x_i,t_{n+1}) - u_h^{n+1}(x_i), i = 1,\cdots,N-1$, be the global error. This error satisfies the finite difference equation

$$\begin{aligned} e_h^{n+1}(x_i) =& e_h^n(x_i) + \Delta t\alpha D_2 e_h^{n+1} + \Delta t\big((1-\theta)\big(f(u(x_i,t_{n-j})) - f(u_h^{n-j}(x_i))\big) \\ &+\theta\big(f(u(x_i,t_{n+1-j})) - f(u_h^{n+1-j}(x_i))\big)\big) + \Delta t T_h^{n+1}(x_i), \quad 1 \le i \le N-1, \end{aligned} \tag{2.30}$$

and

$$e_h^\mu(x_i) = 0, \quad 0 \le i \le N, \quad -j+1 \le \mu \le 0, \quad e_h^n(x_0) = e_h^n(x_N) = 0, \quad 1 \le n \le M. \tag{2.31}$$

In (2.30), $T_h^{n+1}(x_i)$ denotes the truncation error at point (x_i,t_{n+1}). Under certain smoothness assumptions, this error behaves like $\mathcal{O}(\Delta t, h^2)$.

Following the procedure used in the proof of Theorem 2.3 and noting that

$$(T_h^{n+1}, e_h^{n+1})_h \le \eta^2 \|e_h^{n+1}\|^2_{L^2(I_h)} + \frac{1}{4\eta^2}\|T_h^{n+1}\|^2_{L^2(I_h)},$$

where $\eta \neq 0$ is an arbitrary constant, it can be shown that

$$\begin{aligned}&\Big(1+\Delta t\big(\frac{\alpha}{(b-a)^2}-\eta^2(f_{max}^{\prime 2}+1)\big)\Big)\|e_h^{n+1}\|^2_{L^2(I_h)}\\ \leq&\|e_h^n\|^2_{L^2(I_h)}+\Delta t\frac{1}{2\eta^2}\big((1-\theta)\|e_h^{n-j}\|^2_{L^2(I_h)}+\theta\|e_h^{n+1-j}\|^2_{L^2(I_h)}+\|T_h^{n+1}\|^2_{L^2(I_h)}\big).\end{aligned} \tag{2.32}$$

Fixing in (2.32) η by

$$\eta^2=\frac{\alpha}{(b-a)^2(f_{max}^{\prime 2}+1)},$$

we obtain

$$\|e_h^{n+1}\|^2_{L^2(I_h)}\leq\|e_h^n\|^2_{L^2(I_h)}+\Delta t\gamma\big((1-\theta)\|e_h^{n-j}\|^2_{L^2(I_h)}+\theta\|e_h^{n+1-j}\|^2_{L^2(I_h)}+\|T_h^{n+1}\|^2_{L^2(I_h)}\big). \tag{2.33}$$

with γ defined now by

$$\gamma=\frac{(b-a)^2(f_{max}^{\prime 2}+1)}{2\alpha}. \tag{2.34}$$

We establish in what follows an estimate for $\|e_h^{mj+\ell}\|_{L^2(I_h)}$, with $1\leq m\leq k-1$ and $1\leq\ell\leq j$. Noting that

$$\|e_h^\ell\|^2_{L^2(I_h)}\leq(1+\gamma\tau)\max_{i=-j,\cdots,-j+\ell}\|e_h^i\|^2_{L^2(I_h)}+\Delta t\gamma\sum_{i=1}^{\ell}\|T_h^i\|^2_{L^2(I_h)},$$

and $\|e_h^i\|^2_{L^2(I_h)}=0$ for $i=-j,\cdots,-j+\ell$, we deduce

$$\|e_h^\ell\|^2_{L^2(I_h)}\leq\Delta t\gamma\sum_{i=1}^{\ell}\|T_h^i\|^2_{L^2(I_h)}. \tag{2.35}$$

We also have

$$\|e_h^{j+\ell}\|^2_{L^2(I_h)}\leq(1+\gamma\tau)\max_{i=0,\cdots,\ell,j}\|e_h^i\|^2_{L^2(I_h)}+\Delta t\gamma\sum_{i=j+1}^{j+\ell}\|T_h^i\|^2_{L^2(I_h)}.$$

By using the estimate (2.35) we conclude

$$\|e_h^{j+\ell}\|^2_{L^2(I_h)}\leq\Delta t\gamma(1+\gamma\tau)\sum_{i=1}^{j}\|T_h^i\|^2_{L^2(I_h)}+\Delta t\gamma\sum_{i=j+1}^{j+\ell}\|T_h^i\|^2_{L^2(I_h)}. \tag{2.36}$$

It is now a simple task to prove the following general estimate

$$\|e_h^{mj+\ell}\|^2_{L^2(I_h)}\leq\gamma\tau\max_{i=0,\cdots,mj+\ell}\|T_h^i\|^2_{L^2(I_h)}\sum_{i=0}^{m}(1+\gamma\tau)^i \tag{2.37}$$

for $m\in\{1,\cdots,k-1\},\ell\in\{1,\cdots,j\}$.

Theorem 2.5. *Under the assumptions of Theorems* 2.1 *and* 2.4 *for the error* e_h^{n+1}*, we have*

$$\|e_h^{n+1}\|^2_{L^2(I_h)}\leq\gamma\tau\mathbf{C}_e\max_{\mu=1,\cdots,M}\|T_h^\mu\|^2_{L^2(I_h)},\ n=0,\cdots,M-1, \tag{2.38}$$

with γ *defined by* (2.34) *and*

$$\mathbf{C}_e=\sum_{m=1}^{k}(1+\gamma\tau)^m. \tag{2.39}$$

For $\theta \in [0,1]$ and $\theta \neq 1/2$, the truncation error is of order $\mathcal{O}(\Delta t, h^2)$. By Theorem 2.5, we conclude that

$$\max_{n=0,\cdots,M-1} \|e_h^{n+1}\|_{L^2(I_h)} = \mathcal{O}(\Delta t, h^2).$$

For $\theta = 1/2$, the truncation error is of order $\mathcal{O}(\Delta t^2, h^2)$, we conclude, by using Theorem 2.5, that holds

$$\max_{n=0,\cdots,M-1} \|e_h^{n+1}\|_{L^2(I_h)} = \mathcal{O}(\Delta t^2, h^2).$$

Remark 2.1. The numerical method (2.11) is obtained using the second-order centered finite difference operator D_2 followed by a time integration. We point out that the semi-discrete system was integrated implicitly on the diffusion term and by using the θ-method in the reaction term. The stability of the method, with respect to $\|.\|_{L^2(I_h)}$ norm, for all $\theta \in [0,1]$, without any condition on the temporal stepsize, is a consequence of the stability inequality (2.22) and from the explicitly character of the discretization of the reaction term.

Let us consider the implicit-Euler method corresponding to (2.11). In this case, the reaction terms depending on $u(x,t)$ are being regarded. As a result, (2.23) is replaced by

$$\|u_h^{n+1}\|^2_{L^2(I_h)} \leq \gamma \|u_h^n\|^2_{L^2(I_h)}, \tag{2.40}$$

with

$$\gamma = \frac{1}{1 - 2\Delta t\big(\alpha(b-a)^{-2} + f'_{max}\big)}, \tag{2.41}$$

provided that

$$1 - 2\Delta t\Big(\frac{\alpha}{(b-a)^2} + f'_{max}\Big) > 0.$$

In this case we easily get the estimate

$$\|u_h^{n+1}\|^2_{L^2(I_h)} \leq e^{2(n+1)\Delta t\big(\alpha(b-a)^{-2} + f'_{max}\big)\gamma} \|u_h^0\|^2_{L^2(I_h)},$$

where γ is defined by (2.41). When the reaction term depends on $u(x, t-\tau)$, the estimate for $\|u_h^{n+1}\|_{L^2(I_h)}$ depends on $\|u_h^{n+1-j}\|_{L^2(I_h)}$ (see(2.23)). Such dependence does not allow us to obtain easily an estimate for the discrete solution as we can see in the proof of Theorem 2.3.

3. Reaction Term Depending on $u(x,t)$ and $u(x,t-\tau)$

In this section we consider the IBVP (1.1)-(1.3) with the reaction term depending on $u(x,t)$ and $u(x,t-\tau)$. Our aim is to extend the results obtained in the previous section.

3.1. Continuous models

We suppose that the solution of (1.1)-(1.3) satisfies the assumption (2.1), but the assumption (2.2) is replaced by the following one:

$$\begin{aligned} & f \in \mathbb{C}^1([c,d] \times [c,d]), \quad f(0,0) = 0, \\ & (f_x)_{max} := \max_{(x,y)\in[c,d]^2} \frac{\partial f}{\partial x}, \quad (f_y)_{max} := \max_{(x,y)\in[c,d]^2} \frac{\partial f}{\partial y}. \end{aligned} \tag{3.1}$$

We will prove that for the solution of the IBVP (1.1)-(1.3) with the reaction term $f(u(t), u(t-\tau))$ an extension of Theorem 2.1 is possible. We start by mentioning that the following inequality holds:

$$(f(u(t), u(t-\tau)), u(t)) \le \Big((f_x)_{max} + \eta^2 (f_y)^2_{max}\Big)\|u(t)\|^2_{L^2} + \frac{1}{4\eta^2}\|u(t-\tau)\|^2_{L^2},$$

where $\eta \neq 0$ is an arbitrary constant. If the reaction term and the diffusion coefficient satisfy

$$-(f_x)_{max} + \frac{\alpha}{(b-a)^2} > 0\,, \tag{3.2}$$

then fixing η by

$$\eta^2 = \Big(-(f_x)_{max} + \alpha(b-a)^{-2}\Big)\Big/(f_y)^2_{max},$$

and following the proof of Theorem 2.1 we conclude (2.8). Otherwise, it can be shown that

$$\frac{d}{dt}\|u(t)\|^2_{L^2} \le 2\gamma\|u(t)\|^2_{L^2} + \|u(t-\tau)\|^2_{L^2}, \tag{3.3}$$

with

$$\gamma = \frac{1}{2}(f_y)^2_{max} - \frac{\alpha}{(b-a)^2} + (f_x)_{max}.$$

For (3.3) we obtain, for $t \in [(m-1)\tau, m\tau] \subseteq [0, T]$, the estimate

$$\|u(t)\|^2_{L^2} \le (1+\tau)^m e^{2\gamma m\tau} \max_{s\in[-\tau,0]} \|u_0(s)\|^2_{L^2}. \tag{3.4}$$

We have proved the following extension of Theorem 2.1 which establishes an estimate for the total energy of the solution of the IBVP (1.1)-(1.3) when the reaction term f depends on $u(t)$ and $u(t-\tau)$.

Theorem 3.1. *Let u be a solution of the IBVP (1.1)-(1.3) with homogeneous boundary conditions and assume that (2.1) holds and $\partial u/\partial t$, $\partial^\ell u/\partial x^\ell \in L^2(a,b), \ell = 1, 2$. If the reaction term f satisfies (3.1), then for $t \in [(m-1)\tau, m\tau] \subseteq [0, T]$, we have*

$$\|u(t)\|^2_{L^2} \le e^{\gamma m\tau} \max_{s\in[-\tau,0]} \|u_0(s)\|^2_{L^2}, \tag{3.5}$$

where

$$\gamma = \frac{1}{2}\frac{(b-a)^2(f_y)^2_{max}}{\alpha - (b-a)^2(f_x)_{max}}$$

provided that the diffusion coefficient and the f satisfy (3.2); and

$$\|u(t)\|^2_{L^2} \le e^{(1+2\gamma)m\tau} \max_{s\in[-\tau,0]} \|u_0(s)\|^2_{L^2}, \tag{3.6}$$

where

$$\gamma = \frac{1}{2}(f_y)^2_{max} - \frac{\alpha}{(b-a)^2} + (f_x)_{max},$$

provided that f does not satisfy (3.2).

We point out that, for the IBVP (1.1)-(1.3) with a reaction term f depending on $u(x,t)$ and $u(x, t-\tau)$, a stability result analogous to Theorem 2.2 also holds.

3.2. Discrete model

Consider the numerical method (2.11) where $f(u_h^{p-j}(x_i))$ is replaced by $f(u_h^p(x_i), u_h^{p-j}(x_i))$ with $p = n, n+1$. A discrete version of Theorem 3.1, which can be regarded as an extension of Theorem 2.3, may be proved. In fact, for homogeneous boundary conditions and the condition

$$\frac{\alpha}{(b-a)^2} - \theta (f_x)_{max} > 0 \tag{3.7}$$

which replaces (3.2), it can be shown by following the proof of Theorem 2.3 that the inequality (2.23) is replaced by

$$\|u_h^{n+1}\|^2_{L^2(I_h)} \leq (1+\gamma(1-\theta)\Delta t)\|u_h^n\|^2_{L^2(I_h)} + \Delta t \gamma \Big((1-\theta)\|u_h^{n-j}\|^2_{L^2(I_h)} + \theta \|u_h^{n+1-j}\|^2_{L^2(I_h)}\Big), \tag{3.8}$$

with γ is given by

$$\gamma = \frac{(1-\theta)(f_x)^2_{max} + (f_y)^2_{max}}{2\big(\alpha(b-a)^{-2} - \theta (f_x)_{max}\big).} \tag{3.9}$$

Noting that $\ell \Delta t \leq \tau$, the inequality (3.8) allows us to conclude the estimate (2.17) with

$$\mathbf{C} = \begin{cases} e^{\gamma(m+1)\tau}\Big(\dfrac{2-\theta}{1-\theta}\Big)^{m+1}, & \theta \neq 1, \\ (1+\gamma\tau)^{m+1}, \quad \theta = 1, \end{cases} \tag{3.10}$$

where γ is defined by (3.9).

If the diffusion coefficient and the reaction term do not satisfy (3.7), then the inequality (2.23) is replaced by

$$\begin{aligned} \|u_h^{n+1}\|^2_{L^2(I_h)} \leq & \frac{1}{1-2\Delta t\gamma}\Big((1+\Delta t(1-\theta)\|u_h^n\|^2_{L^2(I_h)} \\ & + \Delta t(1-\theta)\|u_h^{n-j}\|^2_{L^2(I_h)} + \Delta t \theta \|u_h^{n+1-j}\|^2_{L^2(I_h)}\Big), \end{aligned} \tag{3.11}$$

provided that Δt is such that

$$1 - 2\gamma \Delta t > 0, \tag{3.12}$$

where γ is defined by

$$\gamma = \frac{1}{2}\Big((1-\theta)(f_x)^2_{max} + (f_y)^2_{max}\Big) + \theta (f_x)_{max} - \frac{\alpha}{(b-a)^2}. \tag{3.13}$$

From inequality (3.11) it can be shown that

$$\|u_h^{mj+\ell}\|^2_{L^2(I_h)} \leq e^{(m+1)\tau\left(\frac{2\gamma}{1-2\gamma\Delta t}+1-\theta\right)}\Big(\frac{2(\gamma+1)-\theta}{2\gamma+1-\theta}\Big)^{m+1} \max_{\mu=-j,\cdots,0} \|u_0(t_\mu)\|^2_{L^2(I_h)}, \tag{3.14}$$

for $m \in \{0, \cdots, k-1\}$.

We have proved the next result:

Theorem 3.2. *Let* u_h^{n+1} *be defined by* (2.11)-(2.13) *where* $f(u_h^{p-j}(x_i))$ *is replaced by* $f(u_h^p(x_i),$ $u_h^{p-j}(x_i))$, *for* $p = n, n+1$, *and with homogeneous boundary conditions and such that* $u_h^\ell(x_i) \in [c,d], i = 1, \cdots, N-1, \ell = -j+1, \cdots, M$. *If the diffusion coefficient and the reaction term satisfy* (3.7), *then it holds* (2.17) *with* γ *defined by* (3.10). *Else, it holds* (2.17) *with*

$$\mathbf{C} = e^{(m+1)\tau\left(\frac{2\gamma}{1-2\gamma\Delta t}+1-\theta\right)}\Big(\frac{2(\gamma+1)-\theta}{2\gamma+1-\theta}\Big)^{m+1},$$

provided that Δt *satisfies* (3.12).

The stability of the method (2.11) with $f(u_h^{p-j}(x_i)), p = n, n+1$, replaced by $f(u_h^p(x_i), u_h^{p-j}(x_i)), p = n, n+1$, can be also established.

Remark 3.1. We remark that if (3.7) holds which means that the diffusion-reaction is dominated by diffusion when $(f_x)_{max}$ is positive, the stability can be established without requiring any conditions on the time step-size. Otherwise, if the problem is dominated by the reaction then the stability is observed provided that (3.12) holds. In this case the restriction is severe when the constant γ defined by (3.13) increases which happens when θ decreases.

As far as the convergence is concerned, if (3.7) holds, then the global error $e_h^n, n = 1, \cdots, M$, satisfies the inequality (2.38) with $\mathbf{C}_e$ and γ defined, respectively, by

$$\mathbf{C}_e = \begin{cases} e^{(m+1)\tau\gamma(1-\theta)} \dfrac{1}{\gamma(1-\theta)} \displaystyle\sum_{i=0}^{m} \Big(\frac{2-\theta}{1-\theta}\Big)^i, & \theta \neq 1 \\ \tau \displaystyle\sum_{i=0}^{m} (1+\gamma\tau)^i, \quad \theta = 1, \end{cases} \tag{3.15}$$

and

$$\gamma = \frac{(1-\theta)(f_x)_{max}^2 + (f_y)_{max}^2 + 1}{2\big(\alpha(b-a)^{-2} - \theta(f_x)_{max}\big)}. \tag{3.16}$$

If the diffusion coefficient and the reaction term do not satisfy (3.2), then the global error satisfies the following inequality

$$\begin{aligned} \|e_h^{n+1}\|_{L^2(I_h)}^2 \le \frac{1}{1-2\gamma\Delta t}\Big(& (1+(1-\theta)\Delta t)\|e_h^n\|_{L^2(I_h)}^2 + (1-\theta)\Delta t\|e_h^{n-j}\|_{L^2(I_h)}^2 \\ & + \theta\Delta t\|e_h^{n+1-j}\|_{L^2(I_h)}^2 + \Delta t\|T_h^{n+1}\|_{L^2(I_h)}^2\Big), \end{aligned} \tag{3.17}$$

for $n = 0, \cdots, M-1$, provided that Δt satisfies (3.12) with γ defined by

$$\gamma = \frac{1}{2}\Big[(1-\theta)(f_x)_{max}^2 + (f_y)_{max}^2 + 1\Big] + \theta(f_x)_{max} - \frac{\alpha}{(b-a)^2}. \tag{3.18}$$

In (3.17), $T_h^{n+1}(x_i)$ denotes the truncation error at (x_i, t_{n+1})). The inequality (3.17) implies for the global error the estimate

$$\|e_h^{n+1}\|_{L^2(I_h)}^2 \le \max_{\mu=1,\cdots,n+1} \|T_h^\mu\|_{L^2(I_h)}^2 \frac{1}{2\gamma+1-\theta} e^{(m+1)\tau(1-\theta+\frac{2\gamma}{1-2\gamma\Delta t})} \sum_{\ell=0}^{m} \Big(\frac{2(\gamma+1)-\theta}{2\gamma+1-\theta}\Big)^\ell. \tag{3.19}$$

We have proved the following convergence result.

Theorem 3.3. *Let u be a solution of the IBVP* (1.1)-(1.3) *satisfying* (2.1). *Let u_h^{n+1} be the fully discrete approximation defined by* (2.11) *with $f(u_h^{n+1-j}(x_i))$ replaced by $f(u_h^{n+1}(x_i), u_h^{n+1-j}(x_i))$, where the reaction term f satisfies* (3.1). *If* (3.2) *holds, then the global error $e_h^\ell, \ell = 1, \cdots, M$, satisfies the inequality* (2.38) *with γ defined by* (3.16). *Else, the global error satisfies* (3.19) *provided that the time-step size satisfies* (3.12) *with γ defined by* (3.18).

4. Systems of Delay Equations

In this section we extend the results presented in the last two sections to systems of partial differential equations. Let $u = (u_1, u_2)$ be a solution of the partial differential problem

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \alpha_1 \dfrac{\partial^2 u_1}{\partial x^2} + f_1(u_1(t), u_2(t), u_1(t-\tau), u_2(t-\tau)), \\ \dfrac{\partial u_2}{\partial t} = \alpha_2 \dfrac{\partial^2 u_2}{\partial x^2} + f_2(u_1(t), u_2(t), u_1(t-\tau), u_2(t-\tau)), \quad \text{in } (a,b) \times (0,T], \end{cases} \tag{4.1}$$

with

$$\begin{array}{lll} u_1(a,t) = u_{1,a}(t), & u_1(b,t) = u_{1,b}(t), & t \in (0,T], \\ u_2(a,t) = u_{2,a}(t), & u_2(b,t) = u_{2,b}(t), & t \in (0,T], \end{array} \tag{4.2}$$

and

$$\begin{array}{lll} u_1(x,t) = u_{1,0}(x,t), & x \in (a,b), & t \in [-\tau, 0], \\ u_2(x,t) = u_{2,0}(x,t), & x \in (a,b), & t \in [-\tau, 0]. \end{array} \tag{4.3}$$

We suppose that the components of the solution, $u = (u_1, u_2)$, of this problem satisfy (2.1) and the reaction terms $f_i, i = 1, 2$, verify the following:

$$f_i \in C^1([c,d]^4), \quad f(0,0,0,0) = 0, \quad \Big(\frac{\partial f_i}{\partial x_\ell}\Big)_{max} = \max_{x \in [c,d]^4} \frac{\partial f_i}{\partial x_\ell}, \quad i = 1,2, \quad \ell = 1,2,3,4. \tag{4.4}$$

In (4.4) we use the notation $[c,d]^4 = \{x = (x_1, x_2, x_3, x_4) : x_i \in [c,d], i = 1,2,3,4\}$. If $v = (v_1, v_2)$ is such that $v_i \in L^2(a,b)$, then $\|v\|_{L^2}$ is defined by

$$\|v\|_{L^2}^2 = \|v_1\|_{L^2}^2 + \|v_2\|_{L^2}^2.$$

Theorem 4.1. *Let $u = (u_1, u_2)$ be a solution of the IBVP (4.1)-(4.3) with homogeneous boundary conditions and such that $u_i, i = 1, 2$, satisfy (2.1). If the reaction terms $f_i, i = 1, 2$, satisfy (4.4), then for $t \in [(m-1)\tau, m\tau] \subseteq [0,T]$, we have*

$$\|u(t)\|_{L^2}^2 \le e^{\gamma m \tau} \max_{s \in [-\tau, 0]} \|u_0(s)\|_{L^2}^2, \tag{4.5}$$

with

$$\gamma = \Big(\max_{i=1,2} \big(\frac{\partial f_i}{\partial x_3}\big)_{max}^2 + \big(\frac{\partial f_i}{\partial x_4}\big)_{max}^2\Big) \Big/ \gamma_1, \tag{4.6}$$

provided that

$$\gamma_1 := \frac{\min_{i=1,2} \alpha_i}{(b-a)^2} - \max_{i=1,2} \big(\frac{\partial f_i}{\partial x_i}\big)_{max} - \frac{1}{2}\Big(|\big(\frac{\partial f_1}{\partial x_2}\big)_{max}| + |\big(\frac{\partial f_2}{\partial x_1}\big)_{max}|\Big) > 0. \tag{4.7}$$

If $\gamma_1 < 0$, for $t \in [(m-1)\tau, m\tau] \subseteq [0,T]$, we have

$$\|u(t)\|_{L^2}^2 \le e^{(1+2\gamma) m \tau} \max_{s \in [-\tau, 0]} \|u_0(s)\|_{L^2}^2, \tag{4.8}$$

with

$$\gamma = \max_{i=1,2} \Big(\big(\frac{\partial f_i}{\partial x_3}\big)_{max}^2 + \big(\frac{\partial f_i}{\partial x_4}\big)_{max}^2\Big) - \gamma_1. \tag{4.9}$$

The proof of this result follows the proof of Theorem 3.1 . The stability of u, when the initial conditions $u_{1,0}, u_{2,0}$ are perturbed, can also be established. As a consequence of such stability result, we conclude that, if u and v are solutions of the IBVP (4.1)-(4.3), then $u = v$.

A stability result for the solution of (4.1)-(4.3) when the initial conditions are perturbed can also be established. Such stability result enable us to conclude that, if u_1 and u_2 are solutions of the IBVP under consideration, then $u_1 = u_2$.

In order to simplify the presentation we consider in what follows the implicit Euler's method. We denote by $u_n^{n+1}(x_i) = (u_{1,h}^{n+1}(x_i), u_{2,h}^{n+1}(x_i))$ the numerical approximation for the solution $u(x_i, t_{n+1}) = (u_1(x_i, t_{n+1}), u_2(x_i, t_{n+1}))$ of the IBVP (4.1)-(4.3) defined by

$$\begin{cases} \dfrac{u_{1,h}^{n+1}(x_i) - u_{1,h}^{n}(x_i)}{\Delta t} = \alpha_1 D_2 u_{1,h}^{n+1}(x_i) + f_1(u_{1,h}^{n+1}(x_i), u_{2,h}^{n+1}(x_i), u_{1,h}^{n+1-j}(x_i), u_{2,h}^{n+1-j}(x_i)), \\ \dfrac{u_{2,h}^{n+1}(x_i) - u_{2,h}^{n}(x_i)}{\Delta t} = \alpha_2 D_2 u_{2,h}^{n+1}(x_i) + f_2(u_{1,h}^{n+1}(x_i), u_{2,h}^{n+1}(x_i), u_{1,h}^{n+1-j}(x_i), u_{2,h}^{n+1-j}(x_i)), \end{cases} \tag{4.10}$$

for $i = 1, \cdots, N-1, n = 0, \cdots, M-1$, with

$$\begin{aligned} u_{1,h}^{\ell}(x_0) &= u_{1,a}(t_\ell), \quad u_{1,h}^{\ell}(x_N) = u_{1,b}(t_\ell), \\ u_{2,h}^{\ell}(x_0) &= u_{2,a}(t_\ell), \quad u_{2,h}^{\ell}(x_N) = u_{2,b}(t_\ell), \end{aligned} \tag{4.11}$$

for $\ell = 1, \cdots, M$, and

$$u_{1,h}^{\ell}(x_i) = u_{1,0}(x_i, t_\ell), \quad u_{2,h}^{\ell}(x_i) = u_{2,0}(x_i, t_\ell), \quad i = 1, \cdots, N-1, \quad \ell = -j, \cdots, 0. \tag{4.12}$$

We remark that, in this case, it can be shown that

$$\|u_h^{n+1}\|_{L^2(I_h)}^2 \leq \|u_h^n\|_{L^2(I_h)}^2 + \Delta t \gamma \|u_h^{n+1-j}\|_{L^2(I_h)}^2, \tag{4.13}$$

with γ defined by

$$\gamma = \max_{i=1,2} \Big((\frac{\partial f_i}{\partial x_3})_{max}^2 + (\frac{\partial f_i}{\partial x_4})_{max}^2 \Big) - \gamma_1, \tag{4.14}$$

provided homogeneous boundary conditions are considered.

In (4.13) we use the discrete $L^2 \times L^2$ norm

$$\|v_h\|_{L^2(I_h)}^2 = \|(v_{1,h}, v_{2,h})\|_{L^2(I_h)}^2 = \|v_{1,h}\|_{L^2(I_h)}^2 + \|v_{2,h}\|_{L^2(I_h)}^2.$$

Assuming that the time stepsize satisfies

$$1 - \Delta t \gamma > 0, \tag{4.15}$$

then the inequality (4.13) allows us to conclude that for the solution of (4.10)-(4.12) with homogeneous Dirichlet boundary conditions holds an extension of Theorem 3.2.

A convergence result analogous to Theorem 3.3 can be established for the solution of (4.10)-(4.12).

5. Numerical Results

In this section we consider some numerical experiments that illustrate the theoretical results presented in this paper. The numerical results were obtained by using a computer program developed by us by using MATLAB version 7.04. In the Examples 5.1 and 5.2 we illustrate

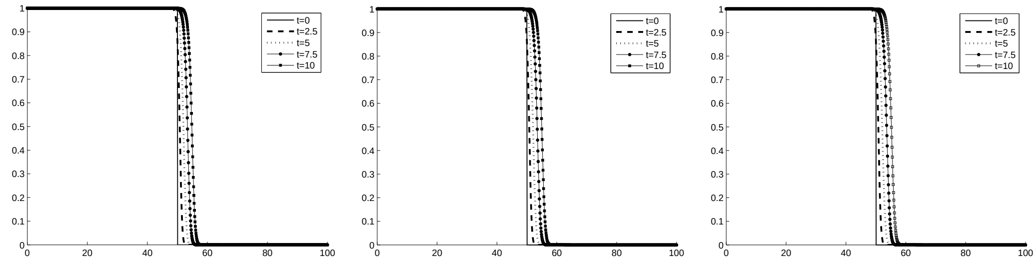

Fig. 5.1. Example 5.1: Numerical solutions obtained with $\Delta t = 0.01, 0.05, 0.1$.

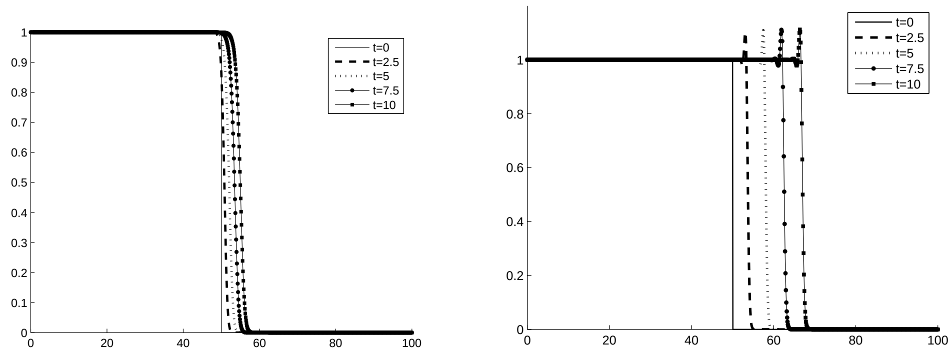

Fig. 5.2. Numerical solution obtained with $\Delta t = 0.05, r = 1$ (left) and $\Delta t = 0.05, r = 4.8$ (right).

the stability results obtained in Section 3.2. The convergence results obtained in this section are illustrated in Example 5.3. Finally, we consider the behaviour of the methods studied in Section 4 in Example 5.4. We point out that we took in all numerical experiments $\theta = 1$.

Example 5.1. *Consider Eq.* (1.1) *with the reaction term*

$$f(u(x,t), u(t-\tau)) = ru(x,t)\big(1 - u(x,t-\tau)\big),$$

$[a,b] = [0,100]$, *complemented with the initial condition*

$$u_0(x) = \begin{cases} 1, & x \le c, \\ 0, & x > c, \end{cases}$$

with $c = 50$ *and with the boundary conditions defined by*

$$u(a,t) = 1, \quad u(b,t) = 0, \quad t \ge 0.$$

The solution of this problem is a traveling wave connecting the stationary states $u = 0$ with $u = 1$. We start by considering α and r satisfying condition (3.2). In this case we have stability without any restriction to the stepsize Δt. In Fig. 5.1 we plot the results obtained with $\alpha = 0.1$, $r = 1$, $\tau = 0.2$ and $h = 0.1$ for different stepsizes.

Let us consider now α and r such that the condition (3.2) does not hold. In this case we use time stepsizes satisfying (3.12) with γ defined by (3.13). In Fig. 5.2 we plot the numerical results obtained using method (2.11) with $\alpha = 0.1$, $\tau = 0.2$, $h = 0.1$ and $\Delta t = 0.05, r = 1$, which satisfy the condition (3.12) and then presents a stable behaviour; $\Delta t = 0.05, r = 4.8$, which violate the mentioned condition and then presents an unstable behaviour.

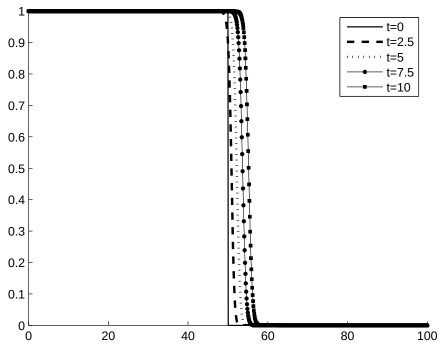

Fig. 5.3. Numerical solution obtained with $r = 1$, $\tau = 0.5$

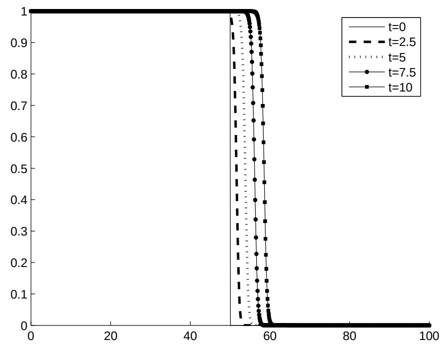

Fig. 5.4. Numerical solution obtained with $r = 2$, $\tau = 0.2$.

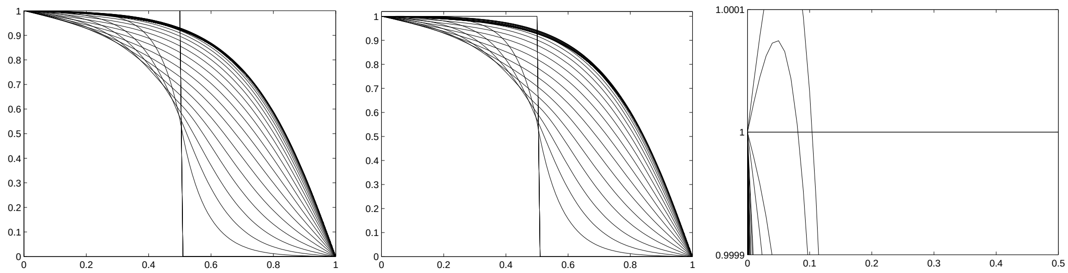

Fig. 5.5. Example 5.2: Numerical solution obtained with $r = 3.6$ (left) $r = 3.8$ (middle) and the zoom of the solution obtained with $r = 3.8$ (right).

The behaviour of the solution when the delay parameter increases is illustrated in Fig. 5.3. We observe that an increasing of τ implies a decreasing on the propagation speed of the front.

An increasing of the reaction parameter r implies an increasing of the propagation speed of the front. This is illustrated in Fig. 5.4.

Example 5.2. *Consider now* u_0 *as in the previous example, but with* $[a, b] = [0, 1]$ *and* $c = 0.5$. *In order to illustrate the sharpness of the estimate* (3.12) *with* γ *defined by* (3.13) *we remark that, in this case, we have*

$$\Delta t < \frac{1}{r^2 + 2r - 2\alpha(b-a)^{-2}}. \tag{5.1}$$

We consider $\tau = 0.2$ and $h = 0.01$. Consequently the method (2.11) should fail when r violates condition (5.1). For $\Delta t = 0.05$, $\alpha = 0.1$, the condition (5.1) is violated for

$$r = -1 + \sqrt{21 + 2\alpha(b-a)^{-2}}.$$

In Fig. 5.5 we plot the numerical results obtained with $r = 3.6$ (this value of r does not violates (5.1) and method (2.11) presents a stable behaviour), $r = 3.8$ (this value of r violates condition (5.1) and method (2.11) presents a unstable behaviour.)

We point out that $r = 3.7$ violates the condition (5.1) but the method (2.11) presents a stable behaviour.

Example 5.3. *Consider the initial boundary value problem of Example* 5.2 *with* $T = 1$, $\alpha = 0.1, r = 1, \tau = 0.2$.

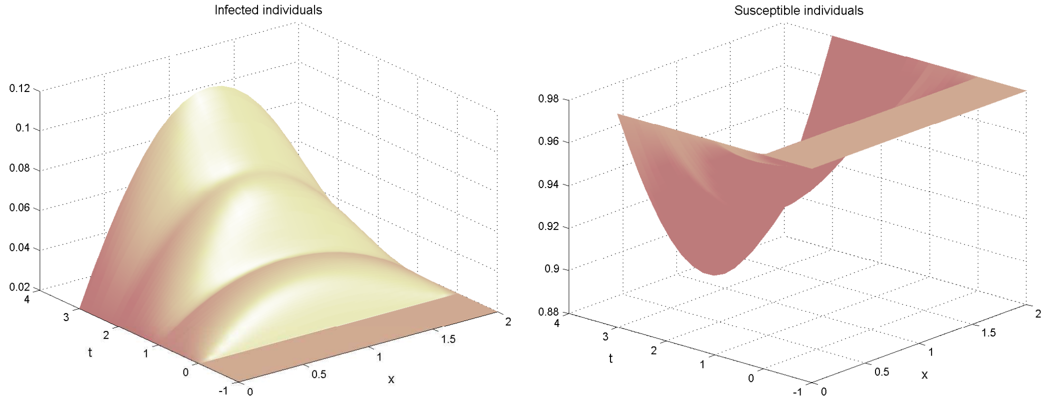

Fig. 5.6. Example 5.4: Numerical approximation for the infected individuals and for the susceptible individuals with $\alpha = 1$.

As such initial boundary value problem does not have a close form for the solution u we compute the "exact solution" using method (2.11) with $h = \Delta t = 0.0005$. In order to illustrate the convergence of method (2.11) we compute the error $\max_{n\in\{1,\cdots,M\}} \|u(t_n) - u_h^n\|_{L^2(I_h)}$ for $h = \Delta t = 0.1, 0.01, 0.001$. The numerical results obtained are presented in Table 5.1.

Table 5.1: Example 5.3: Numerical errors for the numerical results obtained by using method (2.11).

	$h = \Delta t = 0.1$	$h = \Delta t = 0.01$	$h = \Delta t = 0.001$
$\max\limits_{n\in\{1,\cdots,M\}} \|u(t_n) - u_h^n\|_{L^2(I_h)}$	8×10^{-2}	2×10^{-2}	4×10^{-3}

Example 5.4. *Consider the system* (1.4) *with* $[a, b] = [0, 2]$*,* $R_0 = 5$*, and* $\alpha_1 = \alpha_2 = \alpha$*. The boundary conditions are given by*

$$u_1(0, t) = u_1(2, t) = 0.98, \quad t > 0, \quad u_2(0, t) = u_2(2, t) = 0.02, \quad t > 0,$$

and with the initial conditions

$$u_1(x, 0) = 0.98, \quad x \in [0, 2], \quad u_2(x, 0) = 0.02, \quad x \in [0, 2].$$

The dependent variable u_1 represents the infected individuals being the susceptible individuals represented by u_2.

We consider the method (4.10)-(4.12) with

$$f_1(u_{1,h}^{n+1}(x_i), u_{2,h}^{n+1}(x_i), u_{1,h}^{n+1-j}(x_i), u_{2,h}^{n+1-j}(x_i)) = -R_0 u_{1,h}^{n+1}(x_i) u_{2,h}^{n+1-j}(x_i) + u_{2,h}^{n+1}(x_i),$$
$$f_2(u_{1,h}^{n+1}(x_i), u_{2,h}^{n+1}(x_i), u_{1,h}^{n+1-j}(x_i), u_{2,h}^{n+1-j}(x_i)) = R_0 u_{1,h}^{n+1}(x_i) u_{2,h}^{n+1-j}(x_i) - u_{2,h}^{n+1}(x_i).$$

In the numerical experiments we consider $h = 0.1$ and $\Delta = 0.05$. For these stepsizes the condition (4.15) holds with γ defined by (4.14). Figs. 5.6 illustrates the behaviour of the infected and susceptible individuals when the diffusion of all individuals is equal to one.

The influence of the diffusion coefficient on the dynamics of infected individuals is illustrated in Fig. 5.7.

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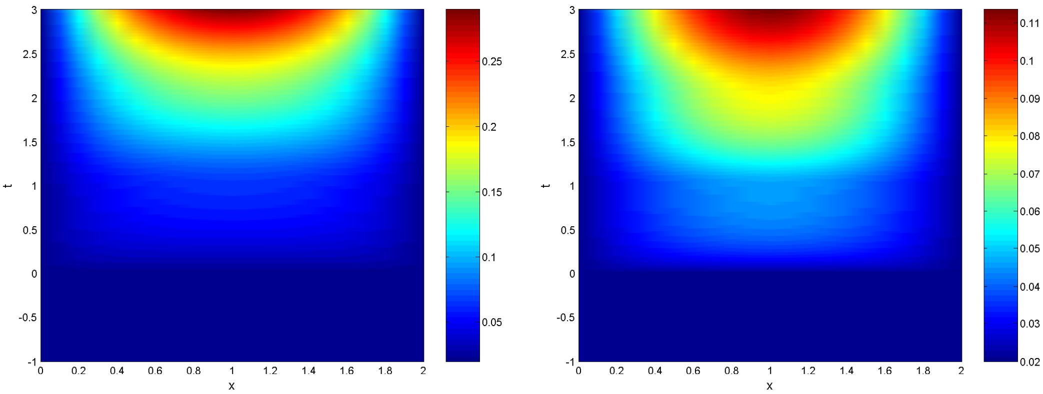

Fig. 5.7. Example 5.4: Numerical approximation for the infected individuals for $\alpha = 0.2, 1$.

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