Journal of Computational Mathematics, Vol.26, No.4, 2008, 488–510.

A POSTERIORI ENERGY-NORM ERROR ESTIMATES FOR ADVECTION-DIFFUSION EQUATIONS APPROXIMATED BY WEIGHTED INTERIOR PENALTY METHODS*

Alexandre Ern
Université Paris-Est, Cermics, Ecole des Ponts, 6 et 8 avenue Blaise Pascal, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France
Email: ern@cermics.enpc.fr
Annette F. Stephansen
Université Paris-Est, Cermics, Ecole des Ponts, 6 et 8 avenue Blaise Pascal, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France
Andra, Parc de la Croix-Blanche, 1-7 rue Jean Monnet, 92298 Châtenay-Malabry cedex, France
Email: stephansen@cermics.enpc.fr

Abstract

We propose and analyze a posteriori energy-norm error estimates for weighted interior penalty discontinuous Galerkin approximations of advection-diffusion-reaction equations with heterogeneous and anisotropic diffusion. The weights, which play a key role in the analysis, depend on the diffusion tensor and are used to formulate the consistency terms in the discontinuous Galerkin method. The error upper bounds, in which all the constants are specified, consist of three terms: a residual estimator which depends only on the elementwise fluctuation of the discrete solution residual, a diffusive flux estimator where the weights used in the method enter explicitly, and a non-conforming estimator which is nonzero because of the use of discontinuous finite element spaces. The three estimators can be bounded locally by the approximation error. A particular attention is given to the dependency on problem parameters of the constants in the local lower error bounds. For moderate advection, it is shown that full robustness with respect to diffusion heterogeneities is achieved owing to the specific design of the weights in the discontinuous Galerkin method, while diffusion anisotropies remain purely local and impact the constants through the square root of the condition number of the diffusion tensor. For dominant advection, it is shown, in the spirit of previous work by Verfürth on continuous finite elements, that the local lower error bounds can be written with constants involving a cut-off for the ratio of local mesh size to the reciprocal of the square root of the lowest local eignevalue of the diffusion tensor.

Mathematics subject classification: 65N30, 65N15, 76R99.
Key words: Discontinuous Galerkin, Weighted interior penalty, A posteriori error estimate, Heterogeneous diffusion, Advection-diffusion.

1. Introduction

In this work, we are interested in a posteriori energy-norm error estimates for a particular class of discontinuous Galerkin (dG) approximations of the advection-diffusion-reaction

* Received July 24, 2007 / Revised version received December 13, 2007 / Accepted January 25, 2008 /

equation

$$\begin{cases} -\nabla\cdot(K\nabla u) + \beta\cdot\nabla u + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where for simplicity homogeneous Dirichlet boundary conditions are considered. Here, Ω is a polygonal domain in $\mathbb{R}^d$ with boundary $\partial\Omega$, $\mu \in L^\infty(\Omega)$, $\beta \in [L^\infty(\Omega)]^d$ with $\nabla\cdot\beta \in L^\infty(\Omega)$, $\tilde{\mu} := \mu - \frac{1}{2}\nabla\cdot\beta$ is assumed to be nonnegative, the diffusion tensor K is a symmetric, uniformly positive definite field in $[L^\infty(\Omega)]^{d,d}$ and $f \in L^2(\Omega)$. Owing to the above assumptions, (1.1) is well-posed; see, e.g., [1].

DG methods received extensive interest in the past decade, in particular because of the flexibility they offer in the construction of approximation spaces using non-matching meshes and variable polynomial degrees. For diffusion problems, various DG methods have been analyzed, including the Symmetric Interior Penalty method [2, 3], the Nonsymmetric method with [4] or without [5] penalty, and the Local Discontinuous Galerkin method [6]; see [7] for a unified analysis. For linear hyperbolic problems (e.g., advection–reaction), one of the most common approaches is to use upwind fluxes to formulate the DG method [8, 9]. A unified theory of DG approximations encompassing elliptic and hyperbolic PDEs can be found in [10, 11]. The approximation of the advection-diffusion-reaction problem (1.1) using DG methods has been analyzed in [12] and more recently in [1] with a focus on the high Péclet regime with isotropic and uniform diffusion. The case of high contrasts in the diffusivity poses additional difficulties. Recently, a (Symmetric) Weighted Interior Penalty method has been proposed and analyzed to approximate satisfactorily (1.1) in this situation [13]. The key idea is to use weighted averages (depending on the normal diffusivities at the two mesh elements sharing a given interface) to formulate the consistency terms and to penalize the jumps of the discrete solution by a factor proportional to the harmonic mean of the neighboring normal diffusivities; the idea of using weighted interior penalties in this context can be traced back to [14]; see also [15].

The present paper addresses the a posteriori error analysis of the weighted interior penalty method. Many significant advances in the a posteriori error analysis of dG methods have been accomplished in the past few years. For energy-norm estimates, we refer to the pioneering work of Becker, Hansbo and Larson [16] and that of Karakashian and Pascal [17], while further developments can be found in the work of Ainsworth [18, 19] regarding robustness with respect to diffusivity and that of Houston, Schötzau and Wihler [20] regarding the *hp*-analysis; see also [21, 22]. Furthermore, for L^2-norm estimates, we mention the work of Becker, Hansbo and Stenberg [23], that of Rivière and Wheeler [24], and that of Castillo [25]. Broadly speaking, two approaches can be undertaken to derive a posteriori energy-norm error estimates; in [16, 18, 21], a Helmholtz decomposition of the error is used, following a technique introduced in [26, 27], while the analysis in [17, 20] relies more directly on identifying a conforming part in the discrete solution. The analysis presented herein will be closer to the latter approach. We also mention recent work relying on the reconstruction of a diffusive flux; see [28, 29].

This paper is organized as follows. §2 presents the discrete setting, including the weighted interior penalty bilinear form used to formulate the discrete problem. §3 contains the main results of this work. The starting point is the abstract framework for a posteriori error estimates presented in §3.1 and which is closely inspired by the work of Vohralík for mixed finite element discretizations [30]. Then, §3.2 addresses the case of pure diffusion with heterogeneous and possibly anisotropic diffusivity. We derive an upper bound for the error consisting of three error indicators, i.e. a residual, a diffusive flux and a non-conforming one. This form is similar

to that obtained in previous work. The key point however is that the diffusive flux error indicators also provide local lower error bounds that are fully robust with respect to diffusivity heterogeneities and that depend on the local degree of anisotropy via the elementwise condition number of the diffusion tensor; see Propositions 3.1 and 3.2. A key ingredient to obtain this result is the use of weighted averages in writing the consistency term. §3.3 extends the previous analysis to the advection-diffusion-reaction problem. Here, the focus is set on achieving a certain degree of robustness in the high Péclet regime, namely that achieved by Verfürth [31] for a posteriori energy-norm error estimates with conforming finite elements and streamline diffusion stabilization. Although these estimates are not independent of the Péclet number (see, e.g., [32] for fully robust estimates with suitable norm modification), their present extension to dG methods constitutes the first results of this type. Finally, numerical results are presented in §4.

2. The Discrete Setting

Let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of affine triangulations covering exactly the polygonal domain Ω. The meshes $\mathcal{T}_h$ may possess hanging nodes, as long as shape-regularity holds for a hierarchical refinement of the mesh without hanging nodes. This assumption is needed in the local lower error bounds when using the approximation properties of the Oswald interpolate (see (3.23)–(3.24) below) and when working with edge bubble functions (see, e.g., the proof of Proposition 3.2). A generic element in $\mathcal{T}_h$ is denoted by T, h_T denotes the diameter of T and n_T its outward unit normal. Let an integer $p \geq 1$. We consider the usual dG space

$$V_h = \{v_h \in L^2(\Omega); \forall T \in \mathcal{T}_h, v_h|_T \in \mathbb{P}_p(T)\}, \tag{2.1}$$

where $\mathbb{P}_p(T)$ is the set of polynomials of degree less than or equal to p on T. The L^2-scalar product and its associated norm on a region $R \subset \Omega$ are indicated by the subscript $0, R$. For $s \geq 1$, a norm (semi-norm) with the subscript s, R designates the usual norm (semi-norm) in $H^s(R)$. For $s \geq 1$, $H^s(\mathcal{T}_h)$ denotes the usual broken Sobolev space on $\mathcal{T}_h$ and for $v \in H^1(\mathcal{T}_h)$, $\nabla_h v$ denotes the piecewise gradient of v, that is, $\nabla_h v \in [L^2(\Omega)]^d$ and for all $T \in \mathcal{T}_h$, $(\nabla_h v)|_T = \nabla(v|_T)$.

We say that F is an interior face of the mesh if there are $T^-(F)$ and $T^+(F)$ in $\mathcal{T}_h$ such that $F = T^-(F) \cap T^+(F)$. We set $\mathcal{T}(F) = \{T^-(F), T^+(F)\}$ and let n_F be the unit normal vector to F pointing from $T^-(F)$ towards $T^+(F)$. The analysis hereafter does not depend on the arbitrariness of this choice. Similarly, we say that F is a boundary face of the mesh if there is $T^-(F) \in \mathcal{T}_h$ such that $F = T^-(F) \cap \partial\Omega$. We set $\mathcal{T}(F) = \{T^-(F)\}$ and let n_F coincide with the outward normal to $\partial\Omega$. All the interior (resp., boundary) faces of the mesh are collected into the set $\mathcal{F}_h^i$ (resp., $\mathcal{F}_h^{\partial\Omega}$) and we let $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^{\partial\Omega}$. For $T \in \mathcal{T}_h$, $\mathcal{F}_T$ denotes the set of its faces and $\widetilde{\mathcal{F}}_T$ the set of mesh faces that share at least a vertex with T. Henceforth, we shall often deal with functions that are double-valued on $\mathcal{F}_h^i$ and single-valued on $\mathcal{F}_h^{\partial\Omega}$. This is the case, for instance, for functions in V_h. On interior faces, when the two branches of the function in question, say v, are associated with restrictions to the neighboring elements $T^{\mp}(F)$, these branches are denoted by $v^{\mp}$ and the jump of v across F is defined as

$$[\![v]\!]_F = v^- - v^+. \tag{2.2}$$

We set $[\![v]\!]_F = v|_F$ on boundary faces. On an interior face $F \in \mathcal{F}_h^i$, we also define the standard (arithmetic) average as $\{v\}_F = \frac{1}{2}(v^- + v^+)$. The subscript F in the above jumps and averages

is omitted if there is no ambiguity. We define the weighted average of a two-valued function v on an interior face $F \in \mathcal{F}_h^i$ as

$$\{v\}_\omega = \omega_{T^-(F),F} v^- + \omega_{T^+(F),F} v^+, \tag{2.3}$$

where the weights are defined as

$$\omega_{T^-(F),F} = \frac{\delta_{K+}}{\delta_{K+} + \delta_{K-}}, \qquad \omega_{T^+(F),F} = \frac{\delta_{K-}}{\delta_{K+} + \delta_{K-}}, \tag{2.4}$$

with $\delta_{K\mp} = n_F^t (K|_{T^\mp(F)}) n_F$. We extend the above definitions to boundary faces by formally letting $\delta_{K+} = +\infty$ so that $\omega^- = 1$ and $\omega^+ = 0$. For the standard average, it is instead more convenient to set $\{v\}_F = \frac{1}{2} v|_F$ on boundary faces.

The weak formulation of (1.1) consists of finding $u \in V := H_0^1(\Omega)$ such that

$$B(u, v) = (f, v)_{0,\Omega}, \qquad \forall v \in V, \tag{2.5}$$

with the bilinear form

$$B(v, w) = (K \nabla_h v, \nabla_h w)_{0,\Omega} + (\beta \cdot \nabla_h v, w)_{0,\Omega} + (\mu v, w)_{0,\Omega}. \tag{2.6}$$

Piecewise gradients are used so as to extend the domain of B to functions in $V + V_h$. The energy norm is

$$\|v\|_B^2 = \sum_{T \in \mathcal{T}_h} \|v\|_{B,T}^2, \qquad \|v\|_{B,T}^2 = (K \nabla_h v, \nabla_h v)_{0,T} + (\tilde{\mu} v, v)_{0,T}. \tag{2.7}$$

The discrete problem consists of finding $u_h \in V_h$ such that

$$B_h(u_h, v_h) = (f, v_h)_{0,\Omega}, \qquad \forall v_h \in V_h, \tag{2.8}$$

with the bilinear form

$$\begin{aligned} B_h(v, w) = {} & (K \nabla_h v, \nabla_h w)_{0,\Omega} + ((\mu - \nabla \cdot \beta) v, w)_{0,\Omega} - (v, \beta \cdot \nabla_h w)_{0,\Omega} \\ & + \sum_{F \in \mathcal{F}_h} \Big[(\gamma_F [\![v]\!], [\![w]\!])_{0,F} - (n_F^t \{K \nabla_h v\}_\omega, [\![w]\!])_{0,F} - \theta (n_F^t \{K \nabla_h w\}_\omega, [\![v]\!])_{0,F} \Big] \\ & + \sum_{F \in \mathcal{F}_h} (\beta \cdot n_F \{v\}, [\![w]\!])_{0,F}. \end{aligned} \tag{2.9}$$

The penalty parameter γ_F is defined for all $F \in \mathcal{F}_h$ as $\gamma_F = \sigma h_F^{-1} \gamma_{K,F} + \gamma_{\beta,F}$ with

$$\gamma_{K,F} = \frac{\delta_{K+} \delta_{K-}}{\delta_{K+} + \delta_{K-}}, \qquad \gamma_{\beta,F} = \tfrac{1}{2} |\beta \cdot n_F|, \tag{2.10}$$

where σ is a positive parameter (σ can also vary from face to face) to be taken large enough, the minimal value being independent of the mesh size and of the diffusion tensor; see [13]. Note that by the above convention, $\gamma_{K,F} = \delta_{K-}$ on boundary faces. Finally, the parameter θ can take values in $\{-1, 0, +1\}$. The particular value taken by θ plays no role in the subsequent analysis.

To avoid technicalities, the diffusion tensor K is assumed to be piecewise constant on $\mathcal{T}_h$ and its restriction to an element $T \in \mathcal{T}_h$ is denoted by K_T. We will indicate by $c_{K,T}$ and $C_{K,T}$ respectively the minimum and the maximum eigenvalue of K on T. The degree of diffusion anisotropy on an element T is evaluated by the condition number of K_T, namely $\Delta_T = \frac{C_{K,T}}{c_{K,T}}$. Furthermore, the minimum value of $\tilde{\mu}$ on T is indicated by $\tilde{\mu}_{m,T}$. We assume that if $\tilde{\mu}_{m,T} = 0$, then $\|\mu\|_{L^\infty(T)} = \|\nabla \cdot \beta\|_{L^\infty(T)} = 0$.

3. A Posteriori Error Analysis

3.1. Abstract setting

In this section we present the basic abstract framework for our a posteriori error estimates. The following result is directly inspired from the abstract framework introduced by Vohralík [30].

Lemma 3.1. *Let Z and Z_h be two vector spaces. Let A be a bilinear form defined on $Z^+ \times Z^+$ with $Z^+ := Z + Z_h$. Assume that A can be decomposed into the form $A = A_S + A_{SS}$ where A_S is symmetric and nonnegative on Z^+ and where A_{SS} is skew-symmetric on Z (but not necessarily on Z^+). Then, defining the semi-norm $|\cdot|_* := A_S(\cdot,\cdot)^{1/2}$, the following holds for all $u, s \in Z$ and $u_h \in Z_h$,*

$$|u - u_h|_* \le |s - u_h|_* + \sup_{\phi \in Z, |\phi|_* = 1} |A(u - u_h, \phi) + A_{SS}(u_h - s, \phi)|. \tag{3.1}$$

Proof. Let $u, s \in Z$ and $u_h \in Z_h$. Observe that if $u = s$, (3.1) obviously holds so that we may now suppose $u \neq s$. Suppose first that $|u - s|_* \le |u - u_h|_*$. Then,

$$\begin{aligned}
|u - u_h|_*^2 &= A(u - u_h, u - u_h) - A_{SS}(u - u_h, u - u_h) \\
&= A(u - u_h, u - s) + A(u - u_h, s - u_h) - A_{SS}(u - u_h, u - u_h) \\
&= A(u - u_h, u - s) + A_S(u - u_h, s - u_h) \\
&\quad + A_{SS}(u - u_h, s - u_h) - A_{SS}(u - u_h, u - u_h) \\
&= A(u - u_h, u - s) + A_S(u - u_h, s - u_h) + A_{SS}(u - u_h, s - u) \\
&= A(u - u_h, u - s) + A_S(u - u_h, s - u_h) + A_{SS}(u_h - s, u - s),
\end{aligned}$$

where we have used $A_{SS}(u - s, u - s) = 0$ since $(u - s) \in Z$. Introducing $\phi_s = (u - s)/|u - s|_*$ and using the fact that for all $v, w \in Z^+$, $A_S(v, w) \le |v|_* |w|_*$ since A_S is symmetric and nonnegative on Z^+ yields

$$|u - u_h|_*^2 \le |u - s|_* A(u - u_h, \phi_s) + |u - u_h|_* |s - u_h|_* + |u - s|_* A_{SS}(u_h - s, \phi_s). \tag{3.2}$$

Having hypothesized that $|u - s|_* \le |u - u_h|_*$, we infer

$$|u - u_h|_* \le |s - u_h|_* + |A(u - u_h, \phi_s) + A_{SS}(u_h - s, \phi_s)|, \tag{3.3}$$

whence (3.1) follows. Consider now the case $|u - u_h|_* \le |u - s|_*$. Since $A_{SS}(u - s, u - s) = 0$,

$$\begin{aligned}
|u - s|_*^2 &= A(u - s, u - s) = A(u - u_h, u - s) + A_S(u_h - s, u - s) + A_{SS}(u_h - s, u - s) \\
&\le |u - s|_* A(u - u_h, \phi_s) + |u_h - s|_* |u - s|_* + |u - s|_* A_{SS}(u_h - s, \phi_s).
\end{aligned}$$

Thus,

$$|u - u_h|_* \le |u - s|_* \le A(u - u_h, \phi_s) + |s - u_h|_* + A_{SS}(u_h - s, \phi_s). \tag{3.4}$$

Combining the results we obtain (3.1). □

3.2. Pure diffusion

Let $\beta = 0$ and $\mu = 0$ in (1.1), i.e., we consider a diffusion problem with anisotropic and heterogeneous diffusivity:

$$\begin{cases} -\nabla\cdot(K\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

The bilinear form B defined by (2.6) becomes

$$B(v,w) = (K\nabla_h v, \nabla_h w)_{0,\Omega}, \tag{3.6}$$

while the definition of the (semi-)norm $\|\cdot\|_B$ involves only the diffusive contribution, i.e., $\|v\|_{B,T}^2 = (K\nabla_h v, \nabla_h v)_{0,T}$. The discrete problem is still (2.8) with the bilinear form B_h defined by

$$\begin{aligned} B_h(v,w) = (K\nabla_h v, \nabla_h w)_{0,\Omega} + \sum_{F\in\mathcal{F}_h} [\sigma h_F^{-1}(\gamma_{K,F}[\![v]\!], [\![w]\!])_{0,F} \\ - (n_F^t\{K\nabla_h v\}_\omega, [\![w]\!])_{0,F} - \theta(n_F^t\{K\nabla_h w\}_\omega, [\![v]\!])_{0,F}]. \end{aligned} \tag{3.7}$$

Lemma 3.1 can be applied by letting $Z := V$, $Z_h := V_h$, $A = A_S := B$ and $A_{SS} := 0$. The semi-norm $|\cdot|_*$ coincides with $\|\cdot\|_B$. This yields

$$\|u - u_h\|_B \le \inf_{s\in V} \|u_h - s\|_B + \sup_{\phi\in V, \|\phi\|_B=1} |B(u-u_h,\phi)|. \tag{3.8}$$

We now proceed to estimate the second term in the right-hand side of (3.8). Let Π_h denote the L^2-orthogonal projection onto the vector space of piecewise constant functions on $\mathcal{T}_h$. It is well-known that for $v \in L^2(\Omega)$, $\Pi_h v$ coincides on each mesh element with the mean value of v on that element. The projector Π_h satisfies the following approximation properties: For all $T \in \mathcal{T}_h$ and for all $\phi \in H^1(T)$,

$$\|\phi - \Pi_h\phi\|_{0,T} \le C_p^{\frac{1}{2}} h_T \|\nabla\phi\|_{0,T} \le C_p^{\frac{1}{2}} h_T c_{K,T}^{-\frac{1}{2}} \|\phi\|_{B,T}, \tag{3.9}$$

$$\|\phi - \Pi_h\phi\|_{0,\partial T} \le C_T^{\frac{1}{2}} h_T^{\frac{1}{2}} \|\nabla\phi\|_{0,T} \le C_T^{\frac{1}{2}} h_T^{\frac{1}{2}} c_{K,T}^{-\frac{1}{2}} \|\phi\|_{B,T}. \tag{3.10}$$

The constant C_p in the Poincaré-type inequality (3.9) can be bounded for each convex T by π^{-2}, see [33,34], while it follows from [35] that the constant C_T in the trace inequality (3.10) is given by $C_T = 3d\rho_T$ with $\rho_T = h_T|\partial T|/|T|$ where $|\partial T|$ denotes the $(d-1)$-measure of ∂T and $|T|$ the d-measure of T; note that ρ_T is uniformly bounded owing to the shape-regularity of the mesh family. For all $T \in \mathcal{T}_h$, define on T the volumetric residual

$$R(u_h) = f + \nabla_h\cdot(K\nabla_h u_h), \tag{3.11}$$

and on ∂T the boundary residual such that for $F \subset \partial T$,

$$J_K(u_h)|_F = \bar{\omega}_{T,F} n_T^t [\![K\nabla_h u_h]\!] + \sigma h_F^{-1}\gamma_{K,F}[\![u_h]\!], \tag{3.12}$$

where

$$\bar{\omega}_{T,F} = 1 - \omega_{T,F}, \tag{3.13}$$

with $F \in \mathcal{F}_T$. Note that $\bar{\omega}_{T,F} = 0$ on boundary faces.

Lemma 3.2. *The following holds:*

$$\sup_{\phi\in V,\|\phi\|_B=1} |B(u-u_h,\phi)| \leq \left(\sum_{T\in\mathcal{T}_h} (\eta_T+\zeta_T)^2\right)^{\frac{1}{2}}, \tag{3.14}$$

where the residual error indicator η_T is

$$\eta_T = C_p^{\frac{1}{2}} h_T c_{K,T}^{-\frac{1}{2}} \|(I-\Pi_h)R(u_h)\|_{0,T}, \tag{3.15}$$

and the diffusive flux error indicator is

$$\zeta_T = C_T^{\frac{1}{2}} h_T^{\frac{1}{2}} c_{K,T}^{-\frac{1}{2}} \|J_K(u_h)\|_{0,\partial T}. \tag{3.16}$$

Proof. Let $\phi\in V$ such that $\|\phi\|_B=1$. Using $B(u,\phi)=(f,\phi)_{0,\Omega}$ and integrating by parts we obtain

$$B(u-u_h,\phi) = \sum_{T\in\mathcal{T}_h} (f+\nabla_h\cdot(K\nabla_h u_h),\phi)_{0,T} - \sum_{F\in\mathcal{F}_h^i} (n_F^t[\![K\nabla_h u_h]\!],\phi)_{0,F}$$

since $\phi\in V=H_0^1(\Omega)$. Testing the discrete equations with $\Pi_h\phi$ yields

$$\sum_{F\in\mathcal{F}_h} (\sigma h_F^{-1}\gamma_{K,F}[\![u_h]\!] - n_F^t\{K\nabla_h u_h\}_\omega, [\![\Pi_h\phi]\!])_{0,F} = (f,\Pi_h\phi)_{0,\Omega}.$$

On interior faces $F\in\mathcal{F}_h^i$, define the conjugate weighted average

$$\{v\}_{\bar\omega} = \omega_{T^+(F),F} v^- + \omega_{T^-(F),F} v^+,$$

so that $[\![vw]\!] = \{v\}_\omega [\![w]\!] + \{w\}_{\bar\omega}[\![v]\!]$ for any functions v and w which are (possibly) double-valued on F. Using this identity yields

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h} (\nabla_h\cdot(K\nabla_h u_h),\Pi_h\phi)_{0,T} \\
&= \sum_{F\in\mathcal{F}_h} (n_F^t\{K\nabla_h u_h\}_\omega,[\![\Pi_h\phi]\!])_{0,F} + \sum_{F\in\mathcal{F}_h^i} (n_F^t[\![K\nabla_h u_h]\!],\{\Pi_h\phi\}_{\bar\omega})_{0,F}.
\end{aligned}$$

Combining the above equations and using $[\![\phi]\!]=0$ leads to

$$\begin{aligned}
B(u-u_h,\phi) &= \sum_{T\in\mathcal{T}_h} (f+\nabla_h\cdot(K\nabla_h u_h),\phi-\Pi_h\phi)_{0,T} - \sum_{F\in\mathcal{F}_h} (\sigma h_F^{-1}\gamma_{K,F}[\![u_h]\!],[\![\phi-\Pi_h\phi]\!])_{0,F} \\
&\quad - \sum_{F\in\mathcal{F}_h^i} (n_F^t[\![K\nabla_h u_h]\!],\{\phi-\Pi_h\phi\}_{\bar\omega})_{0,F} \\
&= \sum_{T\in\mathcal{T}_h} (R(u_h),\phi-\Pi_h\phi)_{0,T} - \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T} n_T\cdot n_F (J_K(u_h),\phi-\Pi_h\phi|_T)_{0,F}.
\end{aligned}$$

The conclusion is straightforward using (3.9)–(3.10) and the fact that $\Pi_h(R(u_h))$ and $(\phi-\Pi_h\phi)$ are L^2-orthogonal on each $T\in\mathcal{T}_h$. □

Remark 3.1. Taking off the mean value of $R(u_h)$ in the residual error estimator is possible because the discrete space contains piecewise constant functions. This is a feature of dG approximations, but not, for instance, of continuous finite element approximations.

Theorem 3.1. *Pick any $s_h \in V$ and define the non-conforming error indicator ι_T as*

$$\iota_T = \|u_h - s_h\|_{B,T}. \tag{3.17}$$

Then, the following holds

$$\|u - u_h\|_B \leq \left(\sum_{T \in \mathcal{T}_h} (\eta_T + \zeta_T)^2 \right)^{\frac{1}{2}} + \left(\sum_{T \in \mathcal{T}_h} \iota_T^2 \right)^{\frac{1}{2}}. \tag{3.18}$$

Proof. Direct consequence of Lemma 3.2 and of (3.8). □

We now investigate the local efficiency of the above error indicators η_T, ζ_T and ι_T. More precisely, we derive a local upper bound for these indicators in terms of the approximation error $u - u_h$, this time measured in the full dG energy-norm, that is the energy semi-norm augmented by a term with jumps. Recall that the dG method converges optimally in this norm, as established in [13]. Furthermore, $x \lesssim y$ indicates the inequality $x \leq cy$ with positive c independent of the mesh and of the diffusion tensor. To simplify, the data f is assumed to be a polynomial; otherwise, the usual data oscillation term has to be added to the estimates. The following two propositions establish that the error indicators η_T and ζ_T are fully robust with respect to heterogeneities in the diffusion tensor, while the dependency on anisotropies remains local, i.e., only the square root of the diffusion condition numbers on T and neighboring elements appears in the local lower bounds, but not the ratios of two diffusion tensor eigenvalues from different elements.

Proposition 3.1. *For all $T \in \mathcal{T}_h$,*

$$\eta_T \lesssim \Delta_T^{\frac{1}{2}} \|u - u_h\|_{B,T}. \tag{3.19}$$

Proof. Since $\|(I - \Pi_h)R(u_h)\|_{0,T} \leq \|R(u_h)\|_{0,T}$, we simply bound $\|R(u_h)\|_{0,T}$. To this purpose, we use the technique of element bubble functions introduced by Verfürth [36, 37]; the arguments, which are fairly standard, are only briefly sketched. Let $T \in \mathcal{T}_h$, let b_T be a suitable local bubble function in T vanishing on ∂T and set $\nu_T = b_T R(u_h)$. Then,

$$\begin{aligned} \|R(u_h)\|_{0,T}^2 &\lesssim (R(u_h), \nu_T)_{0,T} = (K \nabla_h (u - u_h), \nabla \nu_T)_{0,T} \\ &\lesssim C_{K,T}^{\frac{1}{2}} h_T^{-1} \|u - u_h\|_{B,T} \|R(u_h)\|_{0,T}. \end{aligned}$$

Hence,

$$\eta_T \lesssim h_T c_{K,T}^{-\frac{1}{2}} \|R(u_h)\|_{0,T} \lesssim h_T c_{K,T}^{-\frac{1}{2}} C_{K,T}^{\frac{1}{2}} h_T^{-1} \|u - u_h\|_{B,T},$$

whence (3.19) follows. □

Proposition 3.2. *For all $v \in H^1(\mathcal{T}_h)$, define*

$$\|v\|_{B,*,\mathcal{F}}^2 = \sum_{F \in \mathcal{F}} \|\gamma_F^{\frac{1}{2}} [\![v]\!]\|_{0,F}^2, \tag{3.20}$$

where $\mathcal{F}$ is a subset of $\mathcal{F}_h$. Then, for all $T \in \mathcal{T}_h$,

$$\zeta_T \lesssim \Delta_T^{\frac{1}{2}} \left(\|u - u_h\|_{B,*,\mathcal{F}_T} + \sum_{\tilde{T} \in \mathcal{N}_T} \Delta_{\tilde{T}}^{\frac{1}{2}} \|u - u_h\|_{B,\tilde{T}} \right), \tag{3.21}$$

where $\mathcal{N}_T$ is the set of elements sharing at least a face with the element T.

Proof. Let $T \in \mathcal{T}_h$. Observe that

$$\begin{aligned}|\zeta_T| &\lesssim c_{K,T}^{-\frac{1}{2}} \sum_{F\in\mathcal{F}_T} \gamma_{K,F} h_F^{-\frac{1}{2}} \|[\![u_h]\!]\|_{0,F} + c_{K,T}^{-\frac{1}{2}} h_T^{\frac{1}{2}} \sum_{F\in\mathcal{F}_T} \bar{\omega}_{T,F} \|n_F^t [\![K\nabla_h u_h]\!]\|_{0,F} \\ &\equiv X + Y,\end{aligned}$$

and let us bound X and Y.
(i) Bound on X. It is clear that

$$X \lesssim \Delta_T^{\frac{1}{2}} \|u - u_h\|_{B,*,\mathcal{F}_T},$$

since $\gamma_{K,F} \le n_F^t K_T n_F \le C_{K,T}$.
(ii) Bound on Y. Let $F \in \mathcal{F}_T$. Using the technique of edge bubble functions introduced by Verfürth [36, 37], it is shown that

$$h_F^{\frac{1}{2}} \|n_F^t [\![K\nabla_h u_h]\!]\|_{0,F} \lesssim \sum_{T'\in\mathcal{T}(F)} C_{K,T'}^{\frac{1}{2}} \|u - u_h\|_{B,T'}.$$

Hence,

$$\begin{aligned}Y &\lesssim c_{K,T}^{-\frac{1}{2}} \sum_{F\in\mathcal{F}_T} \bar{\omega}_{T,F} \sum_{T'\in\mathcal{T}(F)} C_{K,T'}^{\frac{1}{2}} \|u - u_h\|_{B,T'} \\ &\lesssim \Delta_T^{\frac{1}{2}} \sum_{F\in\mathcal{F}_T} \sum_{T'\in\mathcal{T}(F)} (C_{K,T}^{-\frac{1}{2}} c_{K,T'}^{\frac{1}{2}} \bar{\omega}_{T,F}) \Delta_{T'}^{\frac{1}{2}} \|u - u_h\|_{B,T'} \\ &\lesssim \Delta_T^{\frac{1}{2}} \sum_{F\in\mathcal{F}_T} \sum_{T'\in\mathcal{T}(F)} \Delta_{T'}^{\frac{1}{2}} \|u - u_h\|_{B,T'},\end{aligned}$$

since

$$C_{K,T}^{-\frac{1}{2}} c_{K,T'}^{\frac{1}{2}} \bar{\omega}_{T,F} \le \frac{(n_F K_T n_F)^{\frac{1}{2}} (n_F K_{T'} n_F)^{\frac{1}{2}}}{(n_F K_T n_F) + (n_F K_{T'} n_F)} \le \frac{1}{2}.$$

The proof is complete. □

Remark 3.2. The local efficiency stated in Proposition 3.2 holds for the energy semi-norm augmented by the natural DG jump semi-norm. Owing to a recent result of Ainsworth [38], global efficiency of ζ_T in the energy semi-norm $\|\cdot\|_B$ holds for sufficiently large stabilization parameters σ in the case $d = 2$, $p = 1$, isotropic diffusion, and $\theta = 1$.

To analyze the local efficiency of the non-conforming error indicator ι_T, a particular choice must be made for $s_h \in V$. Presently, one of the state-of-the-art approaches consists of considering the so-called Oswald interpolate of the discrete solution u_h. For $v_h \in V_h$, its Oswald interpolate $\mathcal{I}_{Os}(v_h) \in V_h \cap V$ is defined by prescribing its values at the usual Lagrange interpolation nodes on each mesh element by taking the average of the values of v_h at the node,

$$\mathcal{I}_{Os}(v_h)(s) = \frac{1}{|\mathcal{T}_s|} \sum_{T\in\mathcal{T}_s} v_h|_T(s), \tag{3.22}$$

where $\mathcal{T}_s$ is the set of mesh elements that contain the node s and where $|\mathcal{T}_s|$ denotes the cardinal of that set. On boundary nodes, $\mathcal{I}_{Os}(v_h)(s)$ is set to zero. The Oswald interpolation operator

$\mathcal{I}_{Os}$ yields the following local approximation properties [17, 39]: For all $v_h \in V_h$ and for all $T \in \mathcal{T}_h$,

$$\|v_h - \mathcal{I}_{Os}(v_h)\|_{0,T}^2 \leq C \sum_{F \in \widetilde{\mathcal{F}}_T} h_F \|[\![v_h]\!]\|_{0,F}^2, \tag{3.23}$$

$$\|\nabla_h(v_h - \mathcal{I}_{Os}(v_h))\|_{0,T}^2 \leq C \sum_{F \in \widetilde{\mathcal{F}}_T} h_F^{-1} \|[\![v_h]\!]\|_{0,F}^2, \tag{3.24}$$

where the constant C depends on the space dimension, the polynomial degree p, and the shape-regularity parameter associated with the mesh $\mathcal{T}_h$; the dependency of the constant C on p has been recently explored in [40]. If the mesh possesses hanging nodes, a hierarchical refinement of the mesh without hanging nodes is constructed first. Setting $s_h := \mathcal{I}_{Os}(u_h)$ to evaluate ι_T, it is inferred using (3.24) that

$$\iota_T \lesssim \left(\frac{C_{K,T}}{c_{K,\mathcal{R}_T}}\right)^{\frac{1}{2}} \|u - u_h\|_{B,*,\widetilde{\mathcal{F}}_T}, \tag{3.25}$$

where

$$c_{K,\mathcal{R}_T} = \min_{T' \in \mathcal{R}_T} c_{K,T'}, \quad \mathcal{R}_T = \{T' \in \mathcal{T}_h; T \cap T' \neq \emptyset\}.$$

Clearly, the above estimate is not robust with respect to heterogeneities and/or anisotropies in the diffusion tensor. In the isotropic case, the result can be improved by using weighted averages in (3.22) to define the nodal values of the Oswald interpolate. The weights depend on the diffusivity and a robust bound can be inferred on ι_T when evaluated with this modified Oswald interpolate provided a monotonicity property of the diffusivity around vertices is assumed to hold; see [18, 41, 42]. To the authors' knowledge, no fully satisfactory result on a modified Oswald interpolation operator is yet available in the case of anisotropic diffusivity. We will not explore this issue further here. Finally, we point out that the error indicator ι_T can be readily sharpened by increasing the computational effort. Indeed, since any reconstructed function $s_h \in V$ can be chosen to evaluate it and since

$$\inf_{s_h \in V} \|u_h - s_h\|_{B,T} \leq \|u_h - u\|_{B,T}, \tag{3.26}$$

the local efficiency of ι_T can be improved simply by solving more detailed local problems, and full robustness with respect to the diffusion tensor can eventually be achieved.

Remark 3.3. Using a triangle inequality, the flux error indicator ζ_T can be split into two contributions, one associated with the jump of the diffusive flux and the other associated with the jump of the discrete solution itself, and the latter can be regrouped with the non-conforming error indicator ι_T. By proceeding this way, the error upper bound is somewhat less sharp because a triangle inequality has been used, but the final form of the a posteriori error estimate takes a more familiar form.

3.3. Advection-diffusion-reaction

In this section we turn to the general case of an advection-diffusion-reaction problem. Our purpose is to extend the a posteriori error indicators derived in Lemma 3.2 and in Theorem 3.1

to this situation, with a particular emphasis on the robustness of the estimates in the high-Péclet regime in the sense of Verfürth [31]. The starting point is again the abstract estimate derived in Lemma 3.1 which is now applied with $Z := V$, $Z_h := V_h$,

$$A_S(v,w) = (K\nabla_h v, \nabla_h w)_{0,\Omega} + (\tilde{\mu} v, w)_{0,\Omega}, \tag{3.27}$$

$$A_{SS}(v,w) = (\beta\cdot\nabla_h v, w)_{0,\Omega} + \tfrac{1}{2}((\nabla\cdot\beta)v, w)_{0,\Omega}, \tag{3.28}$$

and $A = A_S + A_{SS} = B$ as defined by (2.6). Observe that A_S is symmetric and nonnegative on $Z + Z_h$, that $|\cdot|_*$ coincides with $\|\cdot\|_B$ defined by (2.7), and that A_{SS} is skew-symmetric on Z (but not on $Z + Z_h$). As a first step, we rewrite the quantity $B(u - u_h, \phi) + A_{SS}(u_h - s, \phi)$ in a more convenient form.

Lemma 3.3. *Let $s \in V$. For all $T \in \mathcal{T}_h$, define on T the volumetric residual*

$$R(u_h) = f + \nabla_h\cdot(K\nabla_h u_h) - \beta\cdot\nabla_h u_h - \mu u_h, \tag{3.29}$$

let $J_K(u_h)$ be defined on ∂T by (3.12)*, and let $J_\beta(u_h - s)$ be defined such that for $F \in \mathcal{F}_T$,*

$$J_\beta(u_h - s)|_F = \langle \gamma_{\beta,F}[\![u_h - s]\!] + \beta\cdot n_F\{u_h - s\}\rangle_F, \tag{3.30}$$

where $\langle\cdot\rangle_F$ denotes the mean value over F. Then, for all $\phi \in V$,

$$B(u - u_h, \phi) + A_{SS}(u_h - s, \phi) = X_1 + X_2 + X_3, \tag{3.31}$$

with

$$X_1 = \sum_{T\in\mathcal{T}_h} ((I - \Pi_h)R(u_h), \phi - \Pi_h\phi)_{0,T}, \tag{3.32}$$

$$X_2 = -\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} n_T\cdot n_F(J_K(u_h), \phi - \Pi_h\phi|_T)_{0,F}, \tag{3.33}$$

$$X_3 = \sum_{T\in\mathcal{T}_h}\Big[((I - \Pi_h)(\beta\cdot\nabla_h(u_h - s)), \phi - \Pi_h\phi)_{0,T} + \tfrac{1}{2}((\nabla\cdot\beta)(u_h - s), \phi - 2\Pi_h\phi)_{0,T}\Big] + \sum_{F\in\mathcal{F}_h} (J_\beta(u_h - s), [\![\Pi_h\phi]\!])_{0,F}. \tag{3.34}$$

Proof. Let $\phi \in V$. Using $B(u,\phi) = (f,\phi)_{0,\Omega}$ and integrating by parts, we infer

$$B(u - u_h, \phi) = \sum_{T\in\mathcal{T}_h} (R(u_h), \phi)_{0,T} - \sum_{F\in\mathcal{F}_h^i} (n_F^t[\![K\nabla_h u_h]\!], \phi)_{0,F}.$$

Testing the discrete equations with $\Pi_h\phi$ yields

$$\sum_{F\in\mathcal{F}_h} (\gamma_F[\![u_h]\!] - n_F^t\{K\nabla_h u_h\}_\omega + \beta\cdot n_F\{u_h\}, [\![\Pi_h\phi]\!])_{0,F} + ((\mu - \nabla\cdot\beta)u_h, \Pi_h\phi)_{0,\Omega} = (f, \Pi_h\phi)_{0,\Omega}.$$

Combining the two above equations and proceeding as in the proof of Lemma 3.2 for the diffusive term leads to

$$B(u - u_h, \phi) = X_1 + X_2 + \sum_{F\in\mathcal{F}_h} (\gamma_{\beta,F}[\![u_h]\!], [\![\Pi_h\phi]\!])_{0,F} - \sum_{F\in\mathcal{F}_h} (\beta\cdot n_F[\![u_h]\!], \{\Pi_h\phi\})_{0,F}.$$

Using the relation

$$-\sum_{T\in\mathcal{T}_h}((\nabla\cdot\beta)(u_h-s),\Pi_h\phi)_{0,T}-\sum_{T\in\mathcal{T}_h}(\beta\cdot\nabla_h(u_h-s),\Pi_h\phi)_{0,T}$$
$$+\sum_{F\in\mathcal{F}_h}(\beta\cdot n_F[\![u_h]\!],\{\Pi_h\phi\})_{0,F}+\sum_{F\in\mathcal{F}_h}(\beta\cdot n_F\{u_h-s\},[\![\Pi_h\phi]\!])_{0,F}=0,$$

and adding $A_{SS}(u_h-s,\phi)$ as evaluated from (3.28), (3.31) is inferred. Note that the upwind related term $J_\beta(u_h-s)$ can be evaluated as a mean value over each face because it is tested against a piecewise constant function and that the mean value of $\beta\cdot\nabla_h(u_h-s)$ can be taken off on each element because it is tested against $\phi-\Pi_h\phi$. □

Remark 3.4. The idea of evaluating the upwind related term as a mean value over each face has been proposed by Vohralík [43]. Since for any function $\psi\in L^2(F)$, $\|\langle\psi\rangle_F\|_{0,F}\le\|\psi\|_{0,F}$, this modification can only sharpen the a posteriori error estimate.

The next step is to control $\phi-\Pi_h\phi$ for $\phi\in V$ in terms of the energy norm $\|\phi\|_B$. To obtain bounds that behave satisfactorily when the Péclet number is large, a sharper version of inequalities (3.9)–(3.10) needs to be used. Observing that on all $T\in\mathcal{T}_h$, $\|\phi-\Pi_h\phi\|_{0,T}\le\|\phi\|_{0,T}$ and letting

$$m_T=\min\left(C_p^{\frac12}h_Tc_{K,T}^{-\frac12},\tilde\mu_{m,T}^{-\frac12}\right),\tag{3.35}$$

the bound (3.9) can be sharpened as follows:

$$\|\phi-\Pi_h\phi\|_{0,T}\le m_T\|\phi\|_{B,T}.\tag{3.36}$$

Furthermore, owing to the trace inequality [44, §3.6] (see also [45, 46])

$$\forall v\in H^1(T),\quad \|v\|_{0,\partial T}\le\rho_T^{\frac12}[h_T^{-\frac12}\|v\|_{0,T}+\|v\|_{0,T}^{\frac12}\|\nabla v\|_{0,T}^{\frac12}],\tag{3.37}$$

where we recall that $\rho_T=h_T|\partial T|/|T|$, (3.10) can be sharpened as follows:

$$\begin{aligned}\|\phi-\Pi_h\phi\|_{0,\partial T}&\le\rho_T^{\frac12}\left[h_T^{-\frac12}m_T+c_{K,T}^{-\frac14}m_T^{\frac12}\right]\|\phi\|_{B,T}\\&\le\tilde C_T^{\frac12}c_{K,T}^{-\frac14}m_T^{\frac12}\|\phi\|_{B,T},\end{aligned}\tag{3.38}$$

where we have set

$$\tilde C_T^{\frac12}=\rho_T^{\frac12}(1+C_p^{\frac14}).\tag{3.39}$$

Furthermore, we define for all $F\in\mathcal{F}_h$,

$$\tilde m_F^2=\min\left(\max_{T'\in\mathcal{T}(F)}(C_{T'}h_{T'}c_{K,T'}^{-1}),\max_{T'\in\mathcal{T}(F)}(\rho_{T'}h_{T'}^{-1}\tilde\mu_{m,T'}^{-1})\right),\tag{3.40}$$

recalling that $C_{T'}=3d\rho_{T'}$. Finally, let $\kappa_{\mu,\beta,T}=\frac12\|\nabla\cdot\beta\|_{L^\infty(T)}\tilde\mu_{m,T}^{-\frac12}$. If $\tilde\mu_{m,T}=0$, $\kappa_{\mu,\beta,T}$ should be evaluated as zero (recall that we have assumed $\|\nabla\cdot\beta\|_{L^\infty(T)}=0$ in this case). To simplify the notation, we will use the convention $0/0=0$ in the sequel.

Lemma 3.4. *Let $s \in V$. The following holds*

$$\sup_{\phi\in V,\|\phi\|_B=1} |B(u-u_h,\phi)+A_{SS}(u_h-s,\phi)| \le \left(\sum_{T\in\mathcal{T}_h}(\eta_T+\zeta_T+\iota'_T)^2\right)^{\frac{1}{2}}, \tag{3.41}$$

where the residual error indicator η_T is

$$\eta_T = m_T\|(I-\Pi_h)R(u_h)\|_{0,T}, \tag{3.42}$$

the diffusive flux error indicator ζ_T is

$$\zeta_T = \tilde{C}_T^{\frac{1}{2}} c_{K,T}^{-\frac{1}{4}} m_T^{\frac{1}{2}} \|J_K(u_h)\|_{0,\partial T}, \tag{3.43}$$

and the non-conforming error indicator ι'_T is

$$\begin{aligned}\iota'_T = {} & m_T\|(I-\Pi_h)(\beta\cdot\nabla_h(u_h-s))\|_{0,T} + \kappa_{\mu,\beta,T}\|u_h-s\|_{0,T} \\ & + \sum_{F\in\mathcal{F}_T} \tilde{m}_F\|J_\beta(u_h-s)\|_{0,F}.\end{aligned} \tag{3.44}$$

Proof. Let $\phi \in V$ such that $\|\phi\|_B = 1$. We bound the three terms X_1, X_2 and X_3 introduced in Lemma 3.3. Owing to (3.36) and (3.38), it is clear that

$$|X_1+X_2| \le \sum_{T\in\mathcal{T}_h}(\eta_T+\zeta_T)\|\phi\|_{B,T}.$$

Decompose X_3 into $X_3 = X_{3,1}+X_{3,2}$ where $X_{3,1}$ denotes the sum over elements and where $X_{3,2}$ denotes the sum over faces. Observing that $\|\phi - 2\Pi_h\phi\|_{0,T} = \|\phi\|_{0,T}$ and using again (3.36), we obtain

$$|X_{3,1}| \le \sum_{T\in\mathcal{T}_h}(m_T\|(I-\Pi_h)(\beta\cdot\nabla_h(u_h-s))\|_{0,T} + \kappa_{\mu,\beta,T}\|u_h-s\|_{0,T})\|\phi\|_{B,T}.$$

To bound $X_{3,2}$, let $F \in \mathcal{F}_h$. On the one hand, owing to (3.10),

$$\begin{aligned}|(J_\beta(u_h-s),[\![\Pi_h\phi]\!])_{0,F}| &= |(J_\beta(u_h-s),[\![\Pi_h\phi-\phi]\!])_{0,F}| \\ &\le \sum_{T'\in\mathcal{T}(F)} |(J_\beta(u_h-s),\Pi_h\phi|_{T'}-\phi)_{0,F}| \\ &\le \|J_\beta(u_h-s)\|_{0,F} \max_{T'\in\mathcal{T}(F)}(C_{T'}^{\frac{1}{2}} h_{T'}^{\frac{1}{2}} c_{K,T'}^{-\frac{1}{2}}) \sum_{T'\in\mathcal{T}(F)}\|\phi\|_{B,T'}.\end{aligned}$$

On the other hand, owing to (3.37) and since $\Pi_h\phi$ is piecewise constant,

$$\begin{aligned}|(J_\beta(u_h-s),[\![\Pi_h\phi]\!])_{0,F}| &\le \sum_{T'\in\mathcal{T}(F)} |(J_\beta(u_h-s),\Pi_h\phi|_{T'})_{0,F}| \\ &\le \|J_\beta(u_h-s)\|_{0,F} \max_{T'\in\mathcal{T}(F)}(\rho_{T'}^{\frac{1}{2}} h_{T'}^{-\frac{1}{2}} \tilde{\mu}_{m,T'}^{-\frac{1}{2}}) \sum_{T'\in\mathcal{T}(F)}\|\phi\|_{B,T'}.\end{aligned}$$

Hence,

$$|(J_\beta(u_h-s),[\![\Pi_h\phi]\!])_{0,F}| \le \tilde{m}_F\|J_\beta(u_h-s)\|_{0,F} \sum_{T'\in\mathcal{T}(F)}\|\phi\|_{B,T'},$$

and therefore,

$$|X_{3,2}| \leq \sum_{T\in\mathcal{T}_h} \left(\sum_{F\in\mathcal{F}_T} \tilde{m}_F \|J_\beta(u_h - s)\|_{0,F} \right) \|\phi\|_{B,T}.$$

The conclusion is straightforward. □

Theorem 3.2. *Pick any $s_h \in V$ and define the non-conforming error indicator ι''_T as*

$$\iota''_T = \|u_h - s_h\|_{B,T}, \tag{3.45}$$

and let ι'_T be evaluated from (3.44) *using s_h. Then,*

$$\|u - u_h\|_B \leq \left(\sum_{T\in\mathcal{T}_h} (\eta_T + \zeta_T + \iota'_T)^2 \right)^{\frac{1}{2}} + \left(\sum_{T\in\mathcal{T}_h} (\iota''_T)^2 \right)^{\frac{1}{2}}. \tag{3.46}$$

Proof. Applying Lemmas 3.1 and 3.4 yields the desired results. □

Remark 3.5. The non-conforming error indicators ι'_T and ι''_T can be regrouped into a single non-conforming error indicator ι_T by setting

$$\iota_T^2 = 4(\iota'_T)^2 + 2(\iota''_T)^2. \tag{3.47}$$

Then, (3.46) becomes

$$\|u - u_h\|_B \leq \left(2 \sum_{T\in\mathcal{T}_h} (\eta_T + \zeta_T)^2 \right)^{\frac{1}{2}} + \left(\sum_{T\in\mathcal{T}_h} \iota_T^2 \right)^{\frac{1}{2}}, \tag{3.48}$$

which is less sharp but has a more familiar form.

We now investigate the local efficiency of the above error indicators η_T, ζ_T and ι_T. Here, $x \lesssim y$ indicates the inequality $x \leq cy$ with positive c independent of the mesh and of the parameters K, β, and μ. Again, the data f is assumed to be a polynomial; otherwise, the usual data oscillation term has to be added to the estimates. As in the pure diffusion case, we will not take advantage of the presence of the operator $(I - \Pi_h)$ in η_T and in the first term of ι'_T to derive the bounds below.

Proposition 3.3. *For all $T \in \mathcal{T}_h$,*

$$\eta_T \lesssim m_T[C_{K,T}^{\frac{1}{2}} h_T^{-1} + \min(\alpha_{1,T}, \alpha_{2,T})]\|u - u_h\|_{B,T}, \tag{3.49}$$

where

$$\alpha_{1,T} = \|\mu\|_{L^\infty(T)} \tilde{\mu}_{m,T}^{-\frac{1}{2}} + \|\beta\|_{L^\infty(T)} c_{K,T}^{-\frac{1}{2}},$$
$$\alpha_{2,T} = (\|\mu - \nabla\cdot\beta\|_{L^\infty(T)} + \|\beta\|_{L^\infty(T)} h_T^{-1}) \tilde{\mu}_{m,T}^{-\frac{1}{2}}.$$

Proof. Let $T \in \mathcal{T}_h$, let b_T be a suitable local bubble function in T vanishing on ∂T and set $\nu_T = b_T R(u_h)$. Then,

$$\begin{aligned}\|R(u_h)\|_{0,T}^2 &\lesssim (R(u_h),\nu_T)_{0,T} = B(u-u_h,\nu_T)\\ &= (K\nabla_h(u-u_h),\nabla_h\nu_T)_{0,T} + (\mu(u-u_h),\nu_T)_{0,T} + (\beta\cdot\nabla_h(u-u_h),\nu_T)_{0,T}\\ &\lesssim [C_{K,T}^{\frac12}h_T^{-1} + \min(\alpha_{1,T},\alpha_{2,T})]\|u-u_h\|_{B,T}\|R(u_h)\|_{0,T},\end{aligned}$$

where the min is obtained by integrating by parts or not the advective derivative. The conclusion is straightforward. □

Proposition 3.4. *For all $T \in \mathcal{T}_h$,*

$$\zeta_T \lesssim \Delta_T^{\frac12}\|u-u_h\|_{B,*,\mathcal{F}_T} + \Delta_T^{\frac12}c_{K,T}^{\frac14}m_T^{\frac12}\sum_{\tilde T\in\mathcal{N}_T} m_{\tilde T}^{-\frac12}c_{K,\tilde T}^{-\frac14}(\Delta_{\tilde T}^{\frac12} + m_{\tilde T}\alpha_{1,\tilde T})\|u-u_h\|_{B,\tilde T}. \quad (3.50)$$

Proof. Let $T \in \mathcal{T}_h$. Observe that

$$\begin{aligned}|\zeta_T| &\lesssim c_{K,T}^{-\frac14}m_T^{\frac12}\sum_{F\in\mathcal{F}_T}\gamma_{K,F}h_F^{-1}\|[\![u_h]\!]\|_{0,F} + c_{K,T}^{-\frac14}m_T^{\frac12}\sum_{F\in\mathcal{F}_T}\bar\omega_{T,F}\|n_F^t[\![K\nabla_h u_h]\!]\|_{0,F}\\ &\equiv X + Y,\end{aligned}$$

and let us bound X and Y by the right-hand side of (3.50).
(i) Bound on X. Since $\gamma_{K,F} \le C_{K,T}$,

$$\begin{aligned}X &\lesssim c_{K,T}^{-\frac14}m_T^{\frac12}h_T^{-\frac12}C_{K,T}^{\frac12}\sum_{F\in\mathcal{F}_T}\gamma_{K,F}^{\frac12}h_F^{-\frac12}\|[\![u_h]\!]\|_{0,F}\\ &\lesssim \Delta_T^{\frac12}c_{K,T}^{\frac14}m_T^{\frac12}h_T^{-\frac12}\|u-u_h\|_{B,*,\mathcal{F}_T}.\end{aligned}$$

Owing to the obvious bound $h_T^{-\frac12} \lesssim m_T^{-\frac12}c_{K,T}^{-\frac14}$, it is inferred that X is bounded by the first term on the right-hand side of (3.50).
(ii) Bound on Y. Let $F \in \mathcal{F}_T$. Following the ideas of Verfürth [31], let b_F be a suitable bubble function with support in F and let ℓ_F be the lifting of $(n_F^t[\![K\nabla_h u_h]\!])b_F$ in $\mathcal{T}(F)$ with cut-off parameter

$$\theta_{T'} = m_{T'}C_p^{-\frac12}h_{T'}^{-1}c_{K,T'}^{\frac12} \le 1,$$

on each $T' \in \mathcal{T}(F)$. Then,

$$\begin{aligned}&\|n_F^t[\![K\nabla_h u_h]\!]\|_{0,F}^2 \lesssim (n_F^t[\![K\nabla_h u_h]\!],\ell_F)_{0,F},\\ &\|\ell_F\|_{0,T'} \lesssim h_{T'}^{\frac12}\theta_{T'}^{\frac12}\|n_F^t[\![K\nabla_h u_h]\!]\|_{0,F} \lesssim m_{T'}^{\frac12}c_{K,T'}^{\frac14}\|n_F^t[\![K\nabla_h u_h]\!]\|_{0,F},\\ &\|\nabla\ell_F\|_{0,T'} \lesssim h_{T'}^{-\frac12}\theta_{T'}^{-\frac12}\|n_F^t[\![K\nabla_h u_h]\!]\|_{0,F} \lesssim m_{T'}^{-\frac12}c_{K,T'}^{-\frac14}\|n_F^t[\![K\nabla_h u_h]\!]\|_{0,F}.\end{aligned}$$

Observe that

$$B(u-u_h,\ell_F) = (R(u_h),\ell_F)_{0,\mathcal{T}(\mathcal{F})} + (n_F^t[\![K\nabla_h u_h]\!],\ell_F)_{0,F},$$

and that

$$\begin{aligned}&|B(u-u_h,\ell_F)|\\ &\lesssim \sum_{T'\in\mathcal{T}(F)}(C_{K,T'}^{\frac12}m_{T'}^{-\frac12}c_{K,T'}^{-\frac14}+m_{T'}^{\frac12}c_{K,T'}^{\frac14}\alpha_{1,T'})\|u-u_h\|_{B,T'}\|n_F^t[\![K\nabla_h u_h]\!]\|_{0,F}.\end{aligned}$$

Furthermore, since

$$\begin{aligned}|(R(u_h),\ell_F)_{0,\mathcal{T}(\mathcal{F})}| &\le \sum_{T'\in\mathcal{T}(F)}\|R(u_h)\|_{0,T'}\|\ell_F\|_{0,T'}\\ &\lesssim \sum_{T'\in\mathcal{T}(F)}[C_{K,T'}^{\frac12}h_{T'}^{-1}+\min(\alpha_{1,T'},\alpha_{2,T'})]\|u-u_h\|_{B,T'}\|\ell_F\|_{0,T'}\\ &\lesssim \sum_{T'\in\mathcal{T}(F)}[C_{K,T'}^{\frac12}h_{T'}^{-1}+\alpha_{1,T'}]m_{T'}^{\frac12}c_{K,T'}^{\frac14}\|u-u_h\|_{B,T'}\|n_F^t[\![K\nabla_h u_h]\!]\|_{0,F},\end{aligned}$$

and since $h_{T'}^{-1}m_{T'}^{\frac12}c_{K,T'}^{\frac14}\le m_{T'}^{-\frac12}c_{K,T'}^{-\frac14}$, it is inferred that $|(R(u_h),\ell_F)_{0,\mathcal{T}(\mathcal{F})}|$ can be bounded as $|B(u-u_h,\ell_F)|$, whence

$$\|n_F^t[\![K\nabla_h u_h]\!]\|_{0,F}\lesssim\sum_{T'\in\mathcal{T}(F)}(C_{K,T'}^{\frac12}m_{T'}^{-\frac12}c_{K,T'}^{-\frac14}+m_{T'}^{\frac12}c_{K,T'}^{\frac14}\alpha_{1,T'})\|u-u_h\|_{B,T'}.$$

As a result,

$$\begin{aligned}Y&\lesssim c_{K,T}^{-\frac14}m_T^{\frac12}\sum_{F\in\mathcal{F}_T}\sum_{T'\in\mathcal{T}(F)}\bar\omega_{T,F}(C_{K,T'}^{\frac12}m_{T'}^{-\frac12}c_{K,T'}^{-\frac14}+m_{T'}^{\frac12}c_{K,T'}^{\frac14}\alpha_{1,T'})\|u-u_h\|_{B,T'}\\ &\lesssim \Delta_T^{\frac12}c_{K,T}^{\frac14}m_T^{\frac12}\sum_{F\in\mathcal{F}_T}\sum_{T'\in\mathcal{T}(F)}(C_{K,T}^{-\frac12}c_{K,T'}^{\frac12}\bar\omega_{T,F})(\Delta_{T'}^{\frac12}+m_{T'}\alpha_{1,T'})m_{T'}^{-\frac12}c_{K,T'}^{-\frac14}\|u-u_h\|_{B,T'}\\ &\lesssim \Delta_T^{\frac12}c_{K,T}^{\frac14}m_T^{\frac12}\sum_{\tilde T\in\mathcal{N}_T}(\Delta_{\tilde T}^{\frac12}+m_{\tilde T}\alpha_{1,\tilde T})m_{\tilde T}^{-\frac12}c_{K,\tilde T}^{-\frac14}\|u-u_h\|_{B,\tilde T}.\end{aligned}$$

The conclusion is straightforward. □

Finally, we investigate the local efficiency of the non-conforming error estimator ι_T. To this purpose, we pick $s_h=\mathcal{I}_{Os}(u_h)$. As discussed at the end of §3.2, a modified Oswald interpolation operator can be considered in the case of isotropic and heterogeneous diffusivity with a monotonicity property around vertices to sharpen the result.

Proposition 3.5. *Set $s_h=\mathcal{I}_{Os}(u_h)$. Let $T\in\mathcal{T}_h$. Define $c_{\beta,\tilde{\mathcal{F}}_T}=\min_{F\in\tilde{\mathcal{F}}_T}\gamma_{\beta,F}$. Then,*

$$\begin{aligned}\iota_T\lesssim&\left(C_{K,T}^{\frac12}h_T^{-1}+\|\tilde\mu\|_{L^\infty(T)}^{\frac12}+m_T\|\beta\|_{L^\infty(T)}h_T^{-1}+\kappa_{\mu,\beta,T}+\sum_{F\in\mathcal{F}_T}\tilde m_F\|\beta\cdot n_F\|_{L^\infty(F)}h_T^{-\frac12}\right)\\ &\times\min\left(h_Tc_{K,\mathcal{R}_T}^{-\frac12},h_T^{\frac12}c_{\beta,\tilde{\mathcal{F}}_T}^{-\frac12}\right)\|u-u_h\|_{B,*,\tilde{\mathcal{F}}_T},\end{aligned}\tag{3.51}$$

recalling that $\mathcal{R}_T=\{T'\in\mathcal{T}_h;T\cap T'\neq\emptyset\}$.

Proof. Let $T \in \mathcal{T}_h$. Observe first that

$$\sum_{F\in\widetilde{\mathcal{F}}_T} \|[\![u_h]\!]\|_{0,F} \leq \min\left(h_T c_{K,\mathcal{R}_T}^{-\frac{1}{2}}, h_T^{\frac{1}{2}} c_{\beta,\widetilde{\mathcal{F}}_T}^{-\frac{1}{2}}\right) h_T^{-\frac{1}{2}} \|u-u_h\|_{B,*,\widetilde{\mathcal{F}}_T}.$$

Hence, using (3.23)–(3.24),

$$\iota_T'' = \|u_h - s_h\|_{B,T} \lesssim \left(C_{K,T}^{\frac{1}{2}} h_T^{-1} + \|\tilde{\mu}\|_{L^\infty(T)}^{\frac{1}{2}}\right) \min\left(h_T c_{K,\mathcal{R}_T}^{-\frac{1}{2}}, h_T^{\frac{1}{2}} c_{\beta,\widetilde{\mathcal{F}}_T}^{-\frac{1}{2}}\right) \|u-u_h\|_{B,*,\widetilde{\mathcal{F}}_T}.$$

Let us now consider ι_T'. Still using (3.23)–(3.24), the first two terms in ι_T' (see (3.44)) are bounded by

$$\left(m_T \|\beta\|_{L^\infty(T)} h_T^{-1} + \kappa_{\mu,\beta,T}\right) \min\left(h_T c_{K,\mathcal{R}_T}^{-\frac{1}{2}}, h_T^{\frac{1}{2}} c_{\beta,\widetilde{\mathcal{F}}_T}^{-\frac{1}{2}}\right) \|u-u_h\|_{B,*,\widetilde{\mathcal{F}}_T},$$

and it remains to bound the last term, namely $\sum_{F\in\mathcal{F}_T} \tilde{m}_F \|J_\beta(u_h - s_h)\|_{0,F}$. For all $T \in \mathcal{T}_h$, it can be shown by proceeding as in the proof of (3.23)–(3.24) that

$$\|\{u_h - \mathcal{I}_{Os}(u_h)\}\|_{0,\partial T} \lesssim \sum_{F\in\widetilde{\mathcal{F}}_T} \|[\![u_h]\!]\|_{0,F},$$

whence the conclusion is straightforward. □

To illustrate by a simple example, assume that β and μ are of order unity, that β is solenoidal (or that its divergence is uniformly bounded by $\tilde{\mu}$ locally), and that the diffusion is homogeneous and isotropic, i.e., $K = \epsilon I_d$ with real parameter $0 < \epsilon \leq 1$ and I_d the identity matrix in $\mathbb{R}^{d,d}$. Then, $m_T \simeq \min(h_T\epsilon^{-\frac{1}{2}}, 1)$, $\alpha_{1,T} \simeq 1 + \epsilon^{-\frac{1}{2}}$, $\alpha_{2,T} \simeq 1 + h_T^{-1}$, and it is readily verified that all the constants appearing in the local estimates for η_T, ζ_T, and ι_T are bounded by $(1 + \epsilon^{-\frac{1}{2}} \min(h_T\epsilon^{-\frac{1}{2}}, 1))$, which corresponds to the result derived in [31] for continuous finite elements.

4. Numerical Results

In this section, the present a posteriori error estimators are assessed on two test cases. The first one is a pure diffusion problem with heterogeneous isotropic diffusion; its aim is to verify numerically the sharpness of the diffusion flux error indicator ζ_T when evaluated with the proper weights. The second test case is an advection–diffusion-reaction problem with homogeneous diffusion; its aim is to verify the behavior of the a posteriori error estimates in the low- and high-Péclet number regimes. We have always taken $\sigma = 4$ and $\theta = 1$ in (2.10) and (2.9), respectively. The corresponding dG method is the so-called Symmetric Weighted Interior Penalty method analyzed recently in [13]. Moreover, we have set $p = 1$, i.e., used piecewise linears. In all cases, the non-conforming error indicators have been evaluated using the standard Oswald interpolate of the discrete solution; see (3.22).

4.1. Heterogeneous diffusion

We consider the following test problem proposed in [47]. The domain $\Omega = (-1,1)\times(-1,1)$ is split into four subregions: $\Omega_1 = (0,1)\times(0,1)$, $\Omega_2 = (-1,0)\times(0,1)$, $\Omega_3 = (-1,0)\times(-1,0)$, and $\Omega_4 = (0,1)\times(-1,0)$. The source term f is zero. The diffusion tensor is isotropic, i.e.,

Table 4.1: Heterogeneous diffusion with parameter $\alpha = 0.13$.

N	$\|u-u_h\|_{0,\Omega}$	$\|u-u_h\|_B$	$(\sum_{T\in\mathcal{T}_h}\zeta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h}\iota_T^2)^{\frac{1}{2}}$	est.	eff.
112	8.48e-2	3.27	15.62	11.84	27.45	8.4
448	7.16e-2	3.11	10.92	11.35	22.26	7.2
1792	6.19e-2	2.93	9.89	10.82	20.72	7.1
7168	5.37e-2	2.75	9.16	10.27	19.43	7.1
order	0.21	0.09	0.11	0.08	0.09	–

of the form $K|_{\Omega_i} = \epsilon_i I$ with constant value within each subregion for $i \in \{1,2,3,4\}$. Letting $\epsilon_1 = \epsilon_3 = 100$ and $\epsilon_2 = \epsilon_4 = 1$, the exact solution written in polar coordinates is

$$u|_{\Omega_i} = r^\alpha \left(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta)\right), \tag{4.1}$$

with $\alpha = 0.12690207$ and

$$\begin{aligned} a_1 &= 0.100000000, & b_1 &= 1.000000000,\\ a_2 &= -9.603960396, & b_2 &= 2.960396040,\\ a_3 &= -0.480354867, & b_3 &= -0.882756593,\\ a_4 &= 7.701564882, & b_4 &= -6.456461752. \end{aligned}$$

Non-homogeneous Dirichlet boundary conditions as given by (4.1) are enforced on $\partial\Omega$. The exact solution possesses a singularity at the origin, and its regularity depends on the constant α, namely $u \in H^{1+\alpha}(\Omega)$; see [48] for further regularity results for this type of problems. The expected convergence order of the error in the L^2-norm is 2α, while the expected convergence order in the energy semi-norm is α. Table 4.1 presents the results on a series of quasi-uniform unstructured triangulations with N mesh elements. All the meshes are compatible with the above partition of the domain Ω. The last line of the table displays the convergence orders evaluated on the last two meshes. The convergence orders for the error both in the L^2-norm and in the energy semi-norm are in good agreement with the theoretical predictions. The same conclusion is reached for the a posteriori error estimators based on ζ_T and ι_T (observe that in the present case, $\eta_T = 0$ because $f = 0$ and $p = 1$). The column labelled "est" reports the total a posteriori error estimator derived in Theorem 3.1, and the column labelled "eff" reports the effectivity index of the estimator, namely the ratio of the a posteriori error estimator to the actual approximation error. The effectivity index is about 7 on all meshes. Notice that all the constants in the estimators are explicitly evaluated. To compare, using the more conventional dG method based on arithmetic averages (i.e., weights equal to $\frac{1}{2}$ on all faces) and a penalty term $\gamma_{K,F}$ equal to the arithmetic mean of the normal diffusivities on each face, the effectivity indices are about 5 times larger.

We have also examined a similar test case with a less singular solution corresponding to milder contrasts in the diffusion, namely $\epsilon_1 = \epsilon_3 = 5$ and $\epsilon_2 = \epsilon_4 = 1$. In this case, the exact solution is still given by (4.1) with $\alpha = 0.53544095$ and

$$\begin{aligned} a_1 &= 0.44721360, & b_1 &= 1.00000000,\\ a_2 &= -0.74535599, & b_2 &= 2.33333333,\\ a_3 &= -0.94411759, & b_3 &= 0.55555556,\\ a_4 &= -2.40170264, & b_4 &= -0.48148148. \end{aligned}$$

Table 4.2: Heterogeneous diffusion with parameter $\alpha = 0.54$.

N	$\|u-u_h\|_{0,\Omega}$	$\|u-u_h\|_B$	$(\sum_{T\in\mathcal{T}_h}\zeta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h}\iota_T^2)^{\frac{1}{2}}$	est.	eff.
112	2.66e-2	6.11e-1	4.82	8.70e-1	5.69	9.32
448	1.13e-2	4.28e-1	2.49	6.09e-1	3.10	7.23
1792	4.98e-3	2.97e-1	1.66	4.23e-1	2.08	7.00
7168	2.26e-3	2.06e-1	1.13	2.92e-1	1.42	6.90
order	1.14	0.53	0.56	0.53	0.55	–

Table 4.3: Error as a function of mesh elements on adaptive meshes.

N	$\|u-u_h\|_B$	$(\sum_{T\in\mathcal{T}_h}\zeta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h}\iota_T^2)^{\frac{1}{2}}$	eff.
112	6.11e-1	5.69	8.70e-1	9.3
148	4.58e-1	2.53	6.17e-1	5.5
200	3.51e-1	2.20	4.40e-1	6.3
288	2.86e-1	2.00	3.17e-1	7.0
394	2.69e-1	1.72	3.13e-1	6.4

Table 4.2 presents the results. The conclusions are similar to those reached with the previous test case. The effectivity index is approximately equal to 7 (except on the coarsest mesh), and thus takes comparable values to those in the previous test case, confirming the robustness of the estimates with respect to diffusion heterogeneities.

We conclude this section by an example on how the error estimator can be used to adapt the mesh. We consider the test case with parameter $\alpha = 0.54$. Starting from the quasi-uniform mesh with $N = 112$ considered previously, the adaptive mesh refinement procedure flags 5% of the mesh elements yielding the largest error indicators. Results are reported in Table 4.3. The efficiency of the procedure can be seen for instance by observing that the error in the energy semi-norm is approximately 0.29 on an adaptive mesh with $N = 288$ elements, while $N = 1792$ elements are needed in a quasi-uniform mesh to achieve the same target. Fig. 4.1 presents two meshes obtained with the adaptive refinement procedure, one with 148 elements and one with 200 elements. We see that the adaptive refinement correctly aims at capturing the singularity at the origin.

4.2. Advection-diffusion-reaction

Consider the domain $\Omega = (0,1)\times(0,1)$, the advection field $\beta = (1,0)^t$, the reaction coefficient $\mu = 1$, and an isotropic homogeneous diffusion tensor $K = \epsilon I$. We run tests with $\epsilon = 1$ and $\epsilon = 10^{-4}$ to examine the difference between dominant diffusion and dominant advection regimes. Since the diffusion is homogeneous and isotropic, the SWIP method coincides with the more conventional Interior Penalty dG method. The source term f is designed so that the exact solution is

$$u(x,y) = 0.5x(1-x)y(1-y)\left(1-\tanh\left(\frac{0.5-x}{\gamma}\right)\right). \tag{4.2}$$

Here, the parameter $\gamma = 0.05$ controls the thickness of the internal layer at $x = 0.5$. Homogeneous Dirichlet boundary conditions are enforced.

In Table 4.4 we present the results for the dominant diffusion regime. The estimator and the error converge at the same order, and the effectivity index is comparable with that obtained

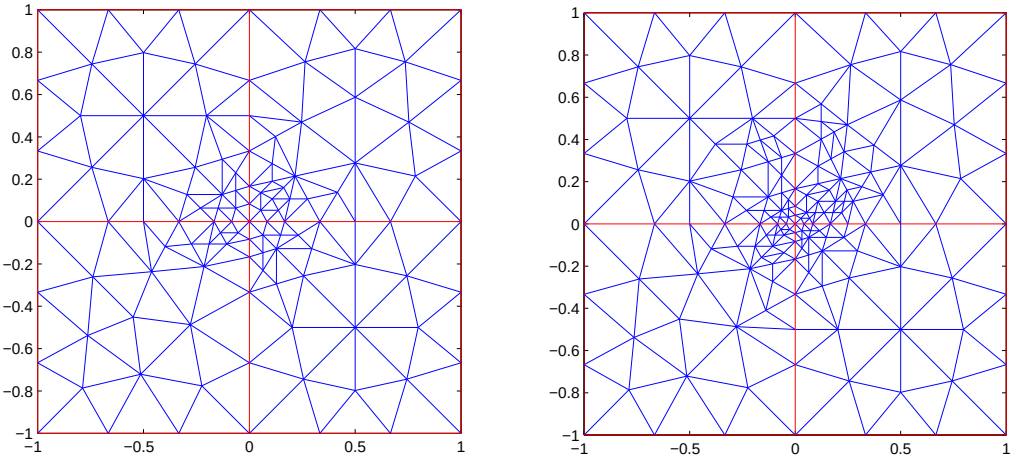

Fig. 4.1. Two adaptive meshes derived from the error estimator: 148 elements (left) and 200 elements (right)

for a pure diffusion problem. The dominant contribution to the total a posteriori error estimate is the diffusive flux error indicator. When the advection becomes dominant (Table 4.5), the main contribution is the non-conforming error indicator ι'_T and, marginally, the residual error indicator (which converges to second-order owing to the subtraction of the elementwise mean-value of the residue). Owing to the approrpiate use of cut-off functions, the effectivity index is only twenty times larger than in the dominant diffusion regime. The last line of Table 4.5 reports the convergence orders evaluated using the last two meshes. It can be observed that the last mesh is sufficiently fine, leading to first-order convergence of $\|u - u_h\|_B$, whereas this quantity converges to order 1.5 on coarser meshes where the L^2-contribution dominates. Finally, adaptive meshes can be generated using the above error indicators (not shown). As expected, the adaptive refinement occurs in the vicinity of the internal layer.

Table 4.4: Advection-diffusion with $\epsilon = 1$.

N	$\|u - u_h\|_B$	$(\sum_{T\in\mathcal{T}_h} \eta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h} \zeta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h} \iota_T'^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h} \iota_T''^2)^{\frac{1}{2}}$	est.	eff.
256	4.39e-2	3.48e-2	3.13e-1	1.13e-2	1.90e-2	3.74e-1	8.5
1024	2.28e-2	1.03e-2	1.74e-1	2.61e-3	1.11e-2	1.96e-1	8.6
4096	1.15e-2	2.68e-3	9.16e-2	5.51e-4	5.45e-3	9.71e-2	8.4
16384	5.71e-3	6.76e-4	4.65e-2	1.28e-4	2.61e-3	4.97e-2	8.7
order	1.01	1.98	0.99	2.10	1.07	0.97	–

Table 4.5: Advection-diffusion with $\epsilon = 10^{-4}$.

N	$\|u - u_h\|_B$	$(\sum_{T\in\mathcal{T}_h} \eta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h} \zeta_T^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h} \iota_T'^2)^{\frac{1}{2}}$	$(\sum_{T\in\mathcal{T}_h} \iota_T''^2)^{\frac{1}{2}}$	est.	eff.
256	7.85e-4	3.85e-2	2.43e-3	6.70e-2	9.33e-4	1.05e-1	134
1024	2.98e-4	2.00e-2	1.88e-3	3.48e-2	2.47e-4	5.56e-2	186
4096	1.32e-4	8.40e-3	1.24e-3	1.74e-2	8.80e-5	2.67e-2	201
16384	6.40e-5	2.18e-3	6.23e-4	8.53e-3	3.90e-5	1.13e-2	177
order	1.06	1.95	0.99	1.03	1.17	1.25	–

5. Conclusions

In this work, we have proposed and analyzed a posteriori energy-norm error estimates for weighted interior penalty dG approximations of advection-diffusion-reaction equations with heterogeneous and anisotropic diffusion. All the constants in the error upper bounds have been specified, so that the present estimates can be used for actual error control in practical simulations. Local lower error bounds in which all the dependencies on model parameters are explicitly stated have been derived as well. In the case of pure diffusion, full robustness is achieved with respect to diffusion heterogeneities owing to the use of suitable diffusion-dependent weights to formulate the consistency terms in the dG method. This feature has been verified numerically and stands in contrast to the results obtained with more conventional interior penalty dG methods. Furthermore, diffusion anisotropies enter the lower error bounds only through the square root of the condition number of the diffusion tensor on a given mesh cell and its neighbors. In the presence of advection, we have shown, in the spirit of the work of Verfürth for continuous finite element methods with streamline diffusion stabilization, that the lower error bounds can be written with constants involving a cut-off for the ratio of local mesh size to the reciprocal of the square root of the lowest local eignevalue of the diffusion tensor.

Acknowledgment. This work was partially supported by GdR MoMaS (PACEN/CNRS, ANDRA, BRGM, CEA, EDF, IRSN). The authors are grateful to M. Vohralík (University Paris 6, France) for stimulating discussions. D. Di Pietro (IFP, France) developed the dG-C++ kernel in which the present a posteriori error estimates have been implemented.

References

[1] A. Ern and J.L. Guermond, Discontinuous Galerkin methods for Friedrichs' systems. III. Multifield theories with partial coercivity, *SIAM J. Numer. Anal.*, **46**:2 (2008), 805–831.

[2] D. Arnold, An interior penalty finite element method with discontinuous elements, *SIAM J. Numer. Anal.*, **19** (1982), 742–760.

[3] G. Baker, Finite element methods for elliptic equations using nonconforming elements, *Math. Comput.*, **31**:137 (1977), 45–59.

[4] B. Rivière, M.F. Wheeler and V. Girault, A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems, *SIAM J. Numer. Anal.*, **39**:3 (2001), 902–931.

[5] J.T. Oden, I. Babuška and C.E. Baumann, A discontinuous *hp* finite element method for diffusion problems, *J. Comput. Phys.*, **146**:2 (1998), 491–519.

[6] B. Cockburn and C. Shu, The local discontinuous Galerkin method for time-dependent convection-diffusion systems, *SIAM, J. Numer. Anal.*, **35** (1998), 2440–2463.

[7] D.N. Arnold, F. Brezzi, B. Cockburn and L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Numer. Anal*, **39**:5 (2001/02), 1749–1779.

[8] C. Johnson and J. Pitkäranta, An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation, *Math. Comput.*, **46**:173 (1986), 1–26.

[9] P. Lesaint, Sur la résolution des systèmes hyperboliques du premier ordre par des méthodes d'éléments finis, PhD thesis, University of Paris VI, 1975.

[10] A. Ern and J.L. Guermond, Discontinuous Galerkin methods for Friedrichs' systems. I. General theory, *SIAM J. Numer. Anal.*, **44**:2 (2006), 753–778.

[11] A. Ern and J.L. Guermond, Discontinuous Galerkin methods for Friedrichs' systems. II. Second-order elliptic PDEs, *SIAM J. Numer. Anal.*, **44**:6 (2006), 2363–2388 (electronic).

[12] P. Houston, C. Schwab and E. Süli, Discontinuous *hp*-finite element methods for advection-diffusion-reaction problems, *SIAM J. Numer. Anal.*, **39**:6 (2002), 2133–2163.
[13] A. Ern, A.F. Stephansen and P. Zunino, A discontinuous Galerkin method with weighted averages for advection-diffusion equations with locally small and anisotropic diffusivity, *IMA J. Numer. Anal.*, (2008), to appear.
[14] E. Burman and P. Zunino, A domain decomposition method based on weighted interior penalties for advection-diffusion-reaction problems, *SIAM J. Numer. Anal.*, **44**:4 (2006), 1612–1638.
[15] D. Di Pietro, A. Ern and J.L. Guermond, Discontinuous Galerkin methods for anisotropic semidefinite diffusion with advection, *SIAM J. Numer. Anal.*, **46**:2 (2008), 805–831.
[16] R. Becker, P. Hansbo and M.G. Larson, Energy norm a posteriori error estimation for discontinuous Galerkin methods, *Comput. Method. Appl. M.*, **192**:5-6 (2003), 723–733.
[17] O.A. Karakashian and F. Pascal, A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems, *SIAM J. Numer. Anal.*, **41**:6 (2003), 2374–2399.
[18] M. Ainsworth, Robust a posteriori error estimation for nonconforming finite element approximation, *SIAM J. Numer. Anal.*, **42**:6 (2005), 2320–2341.
[19] M. Ainsworth, A synthesis of a posteriori error estimation techniques for conforming, nonconforming and discontinuous Galerkin finite element methods, Recent advances in adaptive computation, volume 383 of *Contemp. Math.*, pages 1–14, Amer. Math. Soc., Providence, RI, 2005.
[20] P. Houston, D. Schötzau and T.P. Wihler, Energy norm a posteriori error estimation of *hp*-adaptive discontinuous Galerkin methods for elliptic problems, *Math. Mod. Meth. Appl. S.*, **17**:1 (2007), 33–62.
[21] R. Bustinza, G.N. Gatica and B. Cockburn, An a posteriori error estimate for the local discontinuous Galerkin method applied to linear and nonlinear diffusion problems, *J. Sci. Comput.*, **22/23** (2005), 147–185.
[22] R. Schneider, Y. Xu and A. Zhou, An analysis of discontinuous Galerkin methods for elliptic problems, *Adv. Comput. Math.*, **25**:1-3 (2006), 259–286.
[23] R. Becker, P. Hansbo and R. Stenberg, A finite element method for domain decomposition with non-matching grids, *M2AN Math. Model. Numer. Anal.*, **37**:2 (2003), 209–225.
[24] B. Rivière and M.F. Wheeler, A posteriori error estimates for a discontinuous Galerkin method applied to elliptic problems, *Comput. Math. Appl.*, **46**:1 (2003), 141–163.
[25] P. Castillo, An a posteriori error estimate for the local discontinuous Galerkin method, *J. Sci. Comput.*, **22/23** (2005), 187–204.
[26] E. Dari, R. Duran, C. Padra and V. Vampa, A posteriori error estimators for nonconforming finite element methods, *RAIRO Modél. Math. Anal. Numér.*, **30**:4 (1996), 385–400.
[27] C. Carstensen, S. Bartels and S. Jansche, A posteriori error estimates for nonconforming finite element methods, *Numer. Math.*, **92**:2 (2002), 233–256.
[28] A. Ern, A.F. Stephansen and M. Vohralík, Improved energy norm a posteriori error estimation based on flux reconstruction for discontinuous Galerkin methods, Technical Report 370, CERMICS/ENPC, 2007.
[29] R. Lazarov, S. Repin and S. Tomar, Functional a posteriori error estimates for discontinuous Galerkin approximations of elliptic problems, Technical report, 2006.
[30] M. Vohralík, A posteriori error estimates for lowest-order mixed finite element discretizations of convection-diffusion-reaction equations, *SIAM J. Numer. Anal.*, **45**:4 (2007), 1570–1599.
[31] R. Verfürth, A posteriori error estimators for convection-diffusion equations, *Numer. Math.*, **80**:4 (1998), 641–663.
[32] R. Verfürth, Robust a posteriori error estimates for stationary convection–diffusion equations, *SIAM J. Numer. Anal.*, **43**:5 (2005), 1783–1802.
[33] M. Bebendorf, A note on the Poincaré inequality for convex domains, *Z. Anal. Anwend.*, **22**:4 (2003), 751–756.

[34] L.E. Payne and H.F. Weinberger, An optimal Poincaré inequality for convex domains, *Arch. Ration. Mech. An.*, **5** (1960), 286–292.

[35] M. Vohralík, On the discrete Poincaré-Friedrichs inequalities for nonconforming approximations of the Sobolev space H^1, *Numer. Func. Anal. Opt.*, **26**:7-8 (2005), 925–952.

[36] R. Verfürth, A posteriori error estimations and adaptative mesh-refinement techniques, *J. Comput. Appl. Math.*, **50** (1994), 67–83.

[37] R. Verfürth, A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley, Chichester, UK, 1996.

[38] M. Ainsworth, A posteriori error estimation for discontinuous Galerkin finite element approximation, *SIAM J. Numer. Anal.*, **45**:4 (2007), 1777–1798.

[39] Y. Achdou, C. Bernardi and F. Coquel, A priori and a posteriori analysis of finite volume discretizations of Darcy's equations, *Numer. Math.*, **96**:1 (2003), 17–42.

[40] E. Burman and A. Ern, Continuous interior penalty *hp*-finite element methods for advection and advection–diffusion equations, *Math. Comput.*, **76**:259 (2007), 1119–1140.

[41] C. Bernardi and R. Verfürth, Adaptive finite element methods for elliptic equations with non-smooth coefficients, *Numer. Math.*, **85**:4 (2000), 579–608.

[42] L. El Alaoui and A. Ern, Residual and hierarchical a posteriori error estimates for nonconforming mixed finite element methods, *M2AN Math. Model. Numer. Anal.*, **38**:6 (2004), 903–929.

[43] M. Vohralík, Residual flux-based a posteriori error estimates for finite volume discretizations of inhomogeneous, anisotropic, and convection-dominated problems, *Submitted*, (2006).

[44] A.F. Stephansen, Méthodes de Galerkine discontinues et analyse d'erreur a posteriori pour les problèmes de diffusion hétérogène, PhD thesis, Ecole Nationale des Ponts et Chaussées, 2007.

[45] C. Carstensen and S.A. Funken, Constants in Clément-interpolation error and residual based a posteriori error estimates in finite element methods, *East-West J. Numer. Math.*, **8**:3 (2000), 153–175.

[46] P. Monk and E. Süli, The adaptive computation of far-field patterns by a posteriori error estimation of linear functionals, *SIAM J. Numer. Anal.*, **36**:1 (1999), 251–274.

[47] B. Rivière, M.F. Wheeler and K. Banas, Part II. Discontinuous Galerkin method applied to single phase flow in porous media, *Comput. Geosci-UK.*, **4**:4 (2000), 337–349.

[48] M. Petzoldt, Regularity results for Laplace interface problems in two dimensions, *Z. Anal. Anwend.*, **20**:2 (2001), 431–455.