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UNIFORMLY-STABLE FINITE ELEMENT METHODS FOR DARCY-STOKES-BRINKMAN MODELS*

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Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

In this paper, we consider 2D and 3D Darcy-Stokes interface problems. These equations are related to Brinkman model that treats both Darcy's law and Stokes equations in a single form of PDE but with strongly discontinuous viscosity coefficient and zeroth-order term coefficient. We present three different methods to construct uniformly stable finite element approximations. The first two methods are based on the original weak formulations of Darcy-Stokes-Brinkman equations. In the first method we consider the existing Stokes elements. We show that a stable Stokes element is also uniformly stable with respect to the coefficients and the jumps of Darcy-Stokes-Brinkman equations if and only if the discretely divergence-free velocity implies almost everywhere divergence-free one. In the second method we construct uniformly stable elements by modifying some well-known $H(\mathbf{div})$-conforming elements. We give some new 2D and 3D elements in a unified way. In the last method we modify the original weak formulation of Darcy-Stokes-Brinkman equations with a stabilization term. We show that all traditional stable Stokes elements are uniformly stable with respect to the coefficients and their jumps under this new formulation.

Mathematics subject classification: 65N12, 65N15, 65N22, 65N30.
Key words: Darcy-Stokes equation, Brinkman, Finite element, Uniformly stable.

1. Introduction

In this paper, we consider the following model equations on a bounded, connected, and polygonal domain $\Omega \subset R^d$ $(d = 2, 3)$ (Fig. 1.1 is an example of two dimensional domain). A velocity $\mathbf{u}$ and a pressure p satisfy

$$\begin{cases} -\nabla \cdot (\nu(x)\nabla \mathbf{u}) + \alpha(x)\mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = g & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

with piecewise-constant viscosity coefficient

$$\nu(x) = \nu_i > 0, \qquad x \in \Omega_i, \tag{1.2}$$

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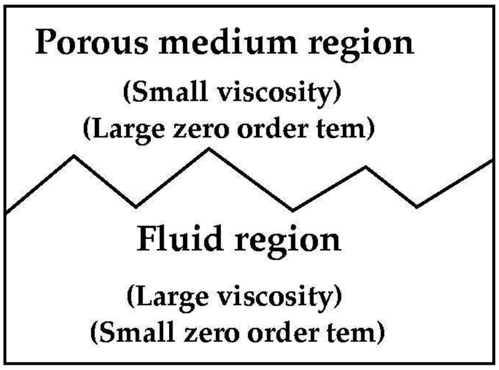

Fig. 1.1. Domain.

and piecewise-constant zeroth-order term coefficient

$$\alpha(x) = \alpha_i \geq 0, \qquad x \in \Omega_i. \tag{1.3}$$

The sub-domains Ω_i are assumed to be bounded connected polygonal domains such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $\overline{\Omega} = \bigcup_{i=1}^m \overline{\Omega}_i$. By Γ_{ij}, we denote the interface between two adjacent sub-domains Ω_i and Ω_j, namely, $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. For other notations: $\sigma(\mathbf{u}, p) = \nu(x)\nabla\mathbf{u} - p\mathbf{I}$ is a stress tensor; $\mathbf{n}$ is the unit normal vector to $\Gamma_{i,j}$;

$$[\mathbf{u}]\,|_{\Gamma_{ij}} = \mathbf{u}|_{\partial\Omega_i\cap\Gamma_{ij}} - \mathbf{u}|_{\partial\Omega_j\cap\Gamma_{ij}}; \quad [\sigma(\mathbf{u}, p)\mathbf{n}]\,|_{\Gamma_{ij}} = \sigma(\mathbf{u}, p)\mathbf{n}|_{\partial\Omega_i\cap\Gamma_{ij}} - \sigma(\mathbf{u}, p)\mathbf{n}|_{\partial\Omega_j\cap\Gamma_{ij}}.$$

For the interface boundary conditions, we have $[\sigma(\mathbf{u}, p)\mathbf{n}]\,|_{\Gamma_{ij}} = 0$, and $[\mathbf{u}]\,|_{\Gamma_{ij}} = 0$. In addition, the source term g is assumed to satisfy the solvability condition:

$$\int_\Omega g dx = 0. \tag{1.4}$$

When α_i is big and ν_i is small in some sub-domains, the equation is close to Darcy equation; in some sub-domain where ν_i is big and α_i is small together with $g = 0$, the equation is close to the Stokes equation. This Darcy-Stokes equation is called Brinkman equation [1], which models porous media flow coupled with viscous fluid flow in a single form of equation.

Among many applications to the Darcy-Stokes-Brinkman equations, our motivation comes from computational fuel cell dynamics [2–4]. A fuel cell is a clean chemical energy conversion device which has potential to replace the traditional combustion engine. In the fuel cell, there are porous gas diffusion layers and gas channels. The two-phase mixture flow in the porous media is modeled by Darcy's law and flow in the gas channel is modeled by Navier-Stokes equations [5–10]. Reviews for this area can be found in [11, 12].

It is so-called single-domain approach that models multi-domain problems using single set of equations with highly discontinuous coefficients $\nu(x)$ and $\alpha(x)$. In this approach, the internal interface conditions are straightforward (the velocity and normal component of stress tensor are continuous), compared to other types of multi-domain Darcy-Stokes models that couple through three interface conditions [13–21].

The goal of this paper is to explore finite element methods which behave uniformly with respect to the highly discontinuous coefficients, $\nu(x)$ and $\alpha(x)$, and their jumps. We present three different methods.

In the first two methods, we consider the original weak formulation. As discussed above, our model problems can be reduced to two extreme cases. One is standard Stokes equation.

The other one is Darcy's law (essentially a mixed form of an elliptic problem). For the Stokes equation, there are many stable elements available. But not all elements lead to uniformly stable approximations for the Darcy-Stokes-Brinkman problem (1.1). For the mixed form elliptic problem or Darcy's law, there are also stable $H(\mathbf{div})$-conforming elements available. However, none of them usually work for the original problem (1.1).

In the first method, we show that any stable Stokes element (i.e., satisfies (**H1**) in Section 3) leads to a uniformly stable approximation for the Darcy-Stokes-Brinkman problem if and only if the assumption (**H2**) (in Section 3) holds. Roughly speaking, this assumption says discretely divergence-free velocity implies the almost everywhere divergence-free one. The element satisfying (**H2**) is also stable for the limiting case, Darcy's law, of the equation.

On the other hand, in the second method, we consider the construction of uniformly stable elements based on some well-known $H(\mathbf{div})$-conforming elements. Under reasonable assumptions we find that $H(\mathbf{div})$ stable elements are also uniformly stable for the Darcy-Stokes-Brinkman problem. Considering the approximation property, we still need to add something to the $H(\mathbf{div})$ finite element space to approximate H^1 space. Based on the analysis, we construct some new uniformly stable elements for the Darcy-Stokes-Brinkman equations.

In the last method, we consider a modified equivalent formulation by adding a proper stabilization term. Brezzi, Fortin and Marini [22] presented a stabilization technique that allows the use of continuous finite element spaces. Their technique involves a modification of the usual mixed equations. We employ this technique to modify the Darcy-Stoke-Brinkman models. Under this modification, traditional stable Stokes elements are indeed uniformly stable with respect to the coefficients and their jumps. There are also other stabilized approaches (see, e.g., Franca and Hughes [23], Burman and Hansbo [21], and the references therein).

This paper is organized as follows. In Section 2, we describe the continuous and discrete weak formulations and discuss how to choose the appropriate norms. In Section 3, we investigate special stable stokes elements which lead to uniformly stable finite element approximations to our model problem (1.1). In Section 4, we start from standard $H(\mathbf{div})$-conforming elements to construct uniformly stable elements. In Section 5, we discuss the method to modify original weak formulation by adding a proper stabilization term. Finally in Section 6, we give concluding remarks.

Let us introduce some notations. In this paper, $H^k(\Omega)$ denotes the Sobolev space of scalar functions on Ω whose derivatives up to order k are square integrable, with the norm $\|\cdot\|_k$. The notation $|\cdot|_k$ denotes the semi-norm derived from the partial derivatives of order equal to k. Furthermore, $\|\cdot\|_{k,T}$ and $|\cdot|_{k,T}$ denote respectively the norm $\|\cdot\|_k$ and the semi-norm $|\cdot|_k$ restricted to the domain T. The notation $L_0^2(\Omega)$ denotes the space of L^2 functions with zero mean values. The space $H_0^k(\Omega)$ denotes the closure in $H^k(\Omega)$ of the set of infinitely differentiable functions with compact supports in Ω. For the corresponding d-dimensional vector spaces, we put superscript d on the scalar notation, such as, $H^k(\Omega)^d$ and $H_0^k(\Omega)^d$. We also denote

$$
\begin{aligned}
&H(\mathbf{div}) := H(\mathbf{div}, \Omega) := \{\mathbf{v} \in L^2(\Omega) | \mathbf{div}\mathbf{v} \in L^2(\Omega)\}, \\
&H_0(\mathbf{div}) := \{\mathbf{v} \in H(\mathbf{div}) | \mathbf{v} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega\}.
\end{aligned}
$$

Here $\mathbf{n}$ is the unit normal vector on $\partial\Omega$.

For simplicity, following Xu [24], we use $X \lesssim (\gtrsim)\ Y$ to denote that there exists a constant C such that $X \leq (\geq)\ CY$. Here, the constant C is independent of the mesh size h, the viscosity coefficient ν, and the zeroth-order term coefficient α.

2. Model Descriptions

2.1. Continuous problem

We introduce the variational formulation of the problem (1.1). Define the velocity and pressure spaces respectively as

$$V := H_0^1(\Omega)^d \quad \text{and} \quad W := L_0^2(\Omega).$$

Let V' and W' be the dual spaces of V and W respectively. Then, the variational formulation reads as follows: given $\mathbf{f} \in V'$ and $g \in W'$, find $\{\mathbf{u}, p\} \in V \times W$ such that

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) - (p, \mathbf{div}\mathbf{v}) =< \mathbf{f}, \mathbf{v} > & \forall \mathbf{v} \in V, \\ (\mathbf{div}\mathbf{u}, q) =< g, q > & \forall q \in W. \end{cases} \tag{2.1}$$

Here $a(\mathbf{u}, \mathbf{v}) = (\nu(x)\nabla \mathbf{u}, \nabla \mathbf{v}) + (\alpha(x)\mathbf{u}, \mathbf{v})$, $(\cdot, \cdot)$ denotes the L^2 inner product of a pair of functions on Ω, and $< \cdot, \cdot >$ denotes the duality pairing of the spaces.

In the limiting case of $\nu(x) \equiv 0$, the problem (2.1) reduces to a mixed form of an elliptic equation. Then, the space $H_0^1(\Omega)^d$ is no longer a proper function space for $\mathbf{u}$. Instead, the solution space is replaced by $H_0(\mathbf{div})$. For this consideration, we introduce the following parameter-dependent norms:

$$|||\mathbf{u}|||^2 := a(\mathbf{u}, \mathbf{u}) + M(\mathbf{div}\mathbf{u}, \mathbf{div}\mathbf{u}), \qquad \mathbf{u} \in V, \tag{2.2}$$

and

$$|||p||| = M^{-1/2}\|p\|_0, \quad p \in W. \tag{2.3}$$

Here

$$M = \max(\nu, \alpha, 1). \tag{2.4}$$

Under these norms, we shall show below the uniform stability conditions are straightforward. Firs of all, by definition we have

$$a(\mathbf{u}, \mathbf{v}) \le |||\mathbf{u}||| |||\mathbf{v}|||, \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{2.5}$$

$$a(\mathbf{v}, \mathbf{v}) = |||\mathbf{v}|||^2, \quad \forall \mathbf{v} \in Z, \tag{2.6}$$

where

$$Z = \{\mathbf{v} \in V : \mathbf{div}\mathbf{v} = 0\}. \tag{2.7}$$

Note that $\|\mathbf{div}\mathbf{v}\|_0 \le M^{-1/2}|||\mathbf{v}|||$, the continuity condition follows immediately:

$$(\mathbf{div}\mathbf{v}, q) \le |||\mathbf{v}||| |||q|||, \quad \forall \mathbf{v} \in V, \forall q \in W. \tag{2.8}$$

Next, it is well-known that the following inf-sup condition holds [25],

$$\sup_{\mathbf{v} \in V} \frac{(\mathbf{div}\mathbf{v}, q)}{\|\mathbf{v}\|_1} \gtrsim \|q\|_0, \quad \forall q \in W. \tag{2.9}$$

Since $|||\mathbf{v}||| \lesssim M^{1/2}\|\mathbf{v}\|_1$, we have the uniform inf-sup condition

$$\sup_{\mathbf{v} \in V} \frac{(\mathbf{div}\mathbf{v}, q)}{|||\mathbf{v}|||} \gtrsim |||q|||, \quad \forall q \in W. \tag{2.10}$$

By the Brezzi theory for saddle-point problems [26, 27], the problem (2.1) has a unique solution and the following estimate holds uniformly with respect to ν and α:

$$\|\mathbf{u}\| + \|p\| \lesssim \|\mathbf{f}\|_{V'} + \|g\|_{W'}. \tag{2.11}$$

Here the norms on V' and W' are defined by

$$\|\mathbf{f}\|_{V'} := \sup_{\mathbf{v}\in V} \frac{< \mathbf{f}, \mathbf{v} >}{\|\mathbf{v}\|} \quad \text{and} \quad \|g\|_{W'} := \sup_{q\in W} \frac{< g, q >}{\|q\|}. \tag{2.12}$$

Remark 2.1. Let us now take a closer look at the norms defined in (2.12) when $\mathbf{f}, g \in L^2$. Obviously, we always have $\|g\|_{W'} = M^{1/2}\|g\|_0$. For $\|\mathbf{f}\|_{V'}$, if $\alpha \geq \alpha_0 > 0$, we can easily see that $\|\mathbf{f}\|_{V'} \leq \alpha_0^{-\frac{1}{2}}\|\mathbf{f}\|_0$, since $\|\mathbf{v}\|_0 \leq \alpha_0^{-\frac{1}{2}}\|\mathbf{v}\|$. If $\alpha \equiv 0$, Olshanskii and Reusken [28] proved that $(\nu\mathbf{v}, \mathbf{v}) \lesssim (\nu\nabla\mathbf{v}, \nabla\mathbf{v})$ if $k = 2$ and one of the following assumptions is satisfied: $\text{meas}(\partial\Omega_i \cap \partial\Omega) > 0 \quad \text{for} \quad i = 1, 2$, or $\text{meas}(\partial\Omega_1 \cap \partial\Omega) > 0 \quad \text{and} \quad \nu_2 \lesssim \nu_1$. As a result, $\|\mathbf{f}\|_{V'} \lesssim \|\nu^{-\frac{1}{2}}\mathbf{f}\|_0$. For the general case of k sub-domains, similar results also hold if one of the following assumptions is satisfied: $\text{meas}(\partial\Omega_i \cap \partial\Omega) > 0 \quad \text{for} \quad i = 1, 2, \cdots, k$, or $\text{meas}(\partial\Omega_i \cap \partial\Omega) > 0$ for $i \in S_1$ and $\nu_j \lesssim \nu_{j_n}$ for $j_n \in N_j$ and $j \in S_2$. Here N_j denotes the set that consists of sub-domain indices for the neighbors of j sharing the same $d-1$ dimensional simplex. The sets $S_1, S_2 \subseteq \{1, 2, \cdots, n\}$, $S_1 \cup S_2 = \{1, 2, \cdots, n\}$, and $S_1 \cap S_2 = \emptyset$.

2.2. Discrete problem

Let $\mathcal{T}_h$ be a shape-regular simplicial triangulation of the domain Ω, where the edges or faces of any element lie on the interfaces. In the simplicial triangulation, the mesh parameter h of $\mathcal{T}_h$ is given by $h = \max_{T\in\mathcal{T}_h}\{\text{diameter of } T\}$, where T denotes triangle in 2D and tetrahedron in 3D. For 2D mesh, let $\mathcal{E}(T)$ denotes the set of all edges in T; for 3D mesh, let $\mathcal{F}(T)$ denotes the set of all faces in T. $P_k^d(\Omega)$ denotes the d-dimensional polynomial space on Ω. When $d = 1$, we drop the superscript. Let V_h ($\subset$ or $\not\subset V$) and $W_h \subset W$ denote velocity and pressure finite dimensional spaces respectively. The discrete weak formulation of the problem (2.1) reads as: Find $\{\mathbf{u}_h, p_h\} \in V_h \times W_h$ such that

$$\begin{cases} a_h(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \mathbf{div}\mathbf{v}_h) =< \mathbf{f}, \mathbf{v}_h > & \forall \mathbf{v}_h \in V_h, \\ (\mathbf{div}\mathbf{u}_h, q_h) =< g, q_h > & \forall q_h \in W_h. \end{cases} \tag{2.13}$$

Here $a_h(\mathbf{u}_h, \mathbf{v}_h)$ is defined by

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = \sum_{T\in\mathcal{T}_h} \left((\nu(x)\nabla\mathbf{u}_h, \nabla\mathbf{v}_h)_T + (\alpha(x)\mathbf{u}_h, \mathbf{v}_h)_T\right). \tag{2.14}$$

Here $(\cdot, \cdot)_T$ denotes the L^2 inner product on T.

Remark 2.2. If the space $V_h \not\subset H(\mathbf{div})$, throughout this paper, we view $\mathbf{div}\mathbf{v}_h$ as $\mathbf{div}_h\mathbf{v}_h$, $\mathbf{v}_h \in V_h$. The operator $\mathbf{div}_h$ denotes the piecewise divergence operator acting on element by element in simplicial triangulations.

Similar to the continuous problem, we define the discrete norm in V_h as follows: $\forall \mathbf{v}_h \in V_h$,

$$\|\mathbf{v}_h\|_h^2 := a_h(\mathbf{v}_h, \mathbf{v}_h) + M(\mathbf{div}\mathbf{v}_h, \mathbf{div}\mathbf{v}_h). \tag{2.15}$$

The discrete pressure norm is the same as the continuous one since $W_h \subset W$.

Denote the discretely divergence-free space Z_h as

$$Z_h := \{\mathbf{v}_h \in V_h : (\mathbf{div}\mathbf{v}_h, q_h) = 0, \quad \forall q_h \in W_h\}. \tag{2.16}$$

3. Application of Special Stable Stokes Elements

In the following, we shall show the critical conditions which lead to uniformly-stable finite element methods for the problem (1.1). First of all, one can expect that the elements are stable in the standard H^1 (or discrete H^1) norm for the velocity and L_0^2 norm for the pressure. Thus, the first assumption is that the following inf-sup condition holds:

$$(\mathbf{H1}) \qquad \sup_{\mathbf{v}_h\in V_h} \frac{(\mathbf{div}\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,h}} \gtrsim \|q_h\|_0 \qquad \forall q_h \in W_h.$$

Here discrete H^1 norm $\|\cdot\|_{1,h}$ is defined by $\|\mathbf{v}_h\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\mathbf{v}_h\|_{1,T,}^2$ $\forall \mathbf{v}_h \in V_h$. When $V_h \subset H_0^1(\Omega)^d$, the discrete norm recovers the standard H^1 norm.

The second assumption is:

$$(\mathbf{H2}) \qquad Z_h = \{\mathbf{v}_h \in V_h,\ \mathbf{div}\mathbf{v}_h = 0\}.$$

This assumption means that the discretely divergence-free velocity implies almost everywhere divergence-free one. If the pressure space contains divergence of velocity one, it yields the assumption (**H2**). We state this stronger assumption as

$$(\mathbf{H2'}) \qquad \mathbf{div}V_h \subseteq W_h.$$

Under the above two assumptions (**H1**) and (**H2**), we easily know the following uniform stability conditions hold.

$$a_h(\mathbf{v}_h, \mathbf{v}_h) \gtrsim |||\mathbf{v}_h|||_h^2, \qquad \forall \mathbf{v}_h \in Z_h, \tag{3.1}$$

$$\sup_{\mathbf{v}_h\in V_h} \frac{(\mathbf{div}\mathbf{v}_h, q_h)}{|||\mathbf{v}_h|||_h} \gtrsim |||q_h|||, \qquad \forall q_h \in W_h, \tag{3.2}$$

$$a_h(\mathbf{u}_h, \mathbf{v}_h) \lesssim |||\mathbf{u}_h|||_h |||\mathbf{v}_h|||_h, \qquad \forall \mathbf{u}_h, \mathbf{v}_h \in V_h, \tag{3.3}$$

$$(\mathbf{div}\mathbf{v}_h, q_h) \lesssim |||\mathbf{v}_h|||_h |||q_h|||, \qquad \forall \mathbf{v}_h \in V_h, \quad \forall q_h \in W_h. \tag{3.4}$$

Theorem 3.1. *Traditional stable Stokes elements* (*i.e., satisfy the inf-sup condition* (**H1**)) *are also uniformly stable for the model problem* (1.1), *if and only if the assumption* (**H2**) *holds.*

Proof. It is easy to see that the assumption (**H2**) is sufficient. For the necessity of the assumption (**H2**), we consider the inf-sup condition

$$\sup_{\mathbf{u}_h\in Z_h} \frac{a_h(\mathbf{u}_h, \mathbf{v}_h)}{|||\mathbf{u}_h|||_h} \gtrsim |||\mathbf{v}_h|||_h, \quad \forall \mathbf{v}_h \in Z_h. \tag{3.5}$$

When $\alpha(x)$ and $\nu(x)$ both approach to zero, in order to have the uniform inf-sup condition (3.5), we must have $\mathbf{div}\mathbf{v}_h = 0$, $\forall \mathbf{v}_h \in Z_h$. □

For nonconforming finite element methods, multiplying $\mathbf{v}_h \in V_h$ to the first equation of (1.1) and integrating by parts, we have

$$a_h(\mathbf{u}, \mathbf{v}_h) - (\mathbf{div}\mathbf{v}_h, p) =< \mathbf{f}, \mathbf{v}_h > + E_h(\mathbf{u}, p, \mathbf{v}_h), \tag{3.6}$$

where the consistency error term is defined by

$$E_h(\mathbf{u}, p, \mathbf{v}_h) = \sum_{T\in\mathcal{T}_h} \int_{\partial T} (\nu(x)\nabla\mathbf{u} - p\mathbf{I})\,\mathbf{n}\cdot\mathbf{v}_h ds = \sum_{T\in\mathcal{T}_h} \int_{\partial T} \sigma(\mathbf{u}, p)\mathbf{n}\cdot\mathbf{v}_h ds. \tag{3.7}$$

We are now in a position to state the following quasi-optimal approximation property. For completeness, we give a proof by following similar arguments in [25–27, 29].

Theorem 3.2. *Assume that* (**H1**) *and* (**H2**) *are satisfied. Then the problem* (2.13) *admits a unique solution* $\{\mathbf{u}_h, p_h\} \in V_h \times W_h$*, such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_h \lesssim \inf_{\mathbf{w}_h \in Z_h(g)} \|\mathbf{u} - \mathbf{w}_h\|_h + \sup_{\mathbf{v}_h \in Z_h} \frac{|E_h(\mathbf{u}, p, \mathbf{v}_h)|}{\|\mathbf{v}_h\|_h}, \tag{3.8}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_h \lesssim \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_h + \sup_{\mathbf{v}_h \in Z_h} \frac{|E_h(\mathbf{u}, p, \mathbf{v}_h)|}{\|\mathbf{v}_h\|_h}, \tag{3.9}$$

$$\|p - p_h\| \lesssim \inf_{q_h \in W_h} \|p - q_h\| + \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_h + \sup_{\mathbf{v}_h \in V_h} \frac{|E_h(\mathbf{u}, p, \mathbf{v}_h)|}{\|\mathbf{v}_h\|_h}. \tag{3.10}$$

Here,

$$Z_h(g) := \{\mathbf{v}_h \in V_h \ |(\mathbf{div}\mathbf{v}_h, q_h) =< g, q_h >, \ \forall q_h \in W_h\}. \tag{3.11}$$

Proof. Applying Lemma I.4.1 in [25], the inf-sup condition (3.2) implies $Z_h(g)$ is not empty. Choose $\mathbf{u}_h^0 \in Z_h(g)$. By the conditions (3.1) and (3.3), there exists a unique solution $\mathbf{s}_h \in Z_h$, such that

$$a_h(\mathbf{s}_h, \mathbf{v}_h) =< \mathbf{f}, \mathbf{v}_h > -a_h(\mathbf{u}_h^0, \mathbf{v}_h), \ \forall \mathbf{v}_h \in Z_h.$$

Let $\mathbf{u}_h = \mathbf{s}_h + \mathbf{u}_h^0$. Furthermore, it follows from Corollary I.4.1 in [25] that there exists a unique p_h in W_h such that the pair $\{\mathbf{u}_h, p_h\}$ is the only solution of Problem (2.13).

For $\forall \mathbf{w}_h \in Z_h(g)$, $\mathbf{u}_h - \mathbf{w}_h \in Z_h$. By assumption (**H2**), $\mathbf{div}(\mathbf{u}_h - \mathbf{w}_h) = 0$. Thus, it yields the following identity:

$$a_h(\mathbf{u}_h - \mathbf{w}_h, \mathbf{u}_h - \mathbf{w}_h) = a_h(\mathbf{u} - \mathbf{w}_h, \mathbf{u}_h - \mathbf{w}_h) + E_h(\mathbf{u}, \mathbf{u}_h - \mathbf{w}_h).$$

From the coercivity condition (3.1) and continuity condition (3.3), we get

$$\|\mathbf{u}_h - \mathbf{w}_h\|_h \lesssim \|\mathbf{u} - \mathbf{w}_h\|_h + \frac{|E_h(\mathbf{u}, p, \mathbf{u}_h - \mathbf{w}_h)|}{\|\mathbf{u}_h - \mathbf{w}_h\|_h}. \tag{3.12}$$

Taking infimum of $\mathbf{w}_h$ and using triangle inequality, we obtain (3.8).

For $\forall \mathbf{v}_h \in V_h$, by Lemma I.4.1 in [25], inf-sup condition (3.2) implies: there exists a unique $\mathbf{r}_h \in Z_h^{\perp}$ such that $(\mathbf{div}\mathbf{r}_h, q_h) = (\mathbf{div}(\mathbf{u} - \mathbf{v}_h), q_h)$, $\forall q_h \in W_h$, and $\|\mathbf{r}_h\|_h \lesssim \|\mathbf{u} - \mathbf{v}_h\|_h$. Let $\mathbf{w}_h = \mathbf{v}_h + \mathbf{r}_h$, then $\mathbf{w}_h \in Z_h(g)$. Furthermore

$$\|\mathbf{u} - \mathbf{w}_h\|_h \leq \|\mathbf{u} - \mathbf{v}_h\|_h + \|\mathbf{r}_h\|_h \lesssim \|\mathbf{u} - \mathbf{v}_h\|_h. \tag{3.13}$$

(3.12), (3.13), and triangle inequality imply (3.9).

It remains to estimate $\|p - p_h\|_0$. From (3.6) and (2.13), we derive that

$$a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) - (\mathbf{div}\mathbf{v}_h, p - p_h) = E_h(\mathbf{u}, p, \mathbf{v}_h), \qquad \forall \mathbf{v}_h \in V_h. \tag{3.14}$$

Further, we can get

$$(\mathbf{div}\mathbf{v}_h, p_h - q_h) = a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) - E_h(\mathbf{u}, p, \mathbf{v}_h) + (\mathbf{div}\mathbf{v}_h, p - q_h). \tag{3.15}$$

By inf-sup condition (3.2),

$$\|p_h - q_h\| \lesssim \sup_{\mathbf{v}_h \in V_h} \frac{(\mathbf{div}\mathbf{v}_h, p_h - q_h)}{\|\mathbf{v}_h\|_h}. \tag{3.16}$$

From (3.15), continuity conditions (3.3) and (3.4), further we can get

$$\|p_h - q_h\| \lesssim \|\mathbf{u} - \mathbf{u}_h\|_h + \|p - q_h\| + \frac{E_h(\mathbf{u}, p, \mathbf{v}_h)}{\|\mathbf{v}_h\|_h}.$$

Thus (3.10) follows immediately by using triangle inequality. □

Remark 3.1. Under a stronger assumption (**H2′**), from (2.1) and (2.13), we have $Q_h \mathbf{div} u = \mathbf{div}\mathbf{u}_h$. Here $Q_h : W \to W_h$ is an L^2-orthogonal projection. Thus we get

$$\|\mathbf{div}\mathbf{u} - \mathbf{div}\mathbf{u}_h\|_0 = \|(I - Q_h)\mathbf{div}\mathbf{u}\|_0. \tag{3.17}$$

Examples (P_k^d-P_{k-1} type elements). In all these methods, we approximate the velocity by the continuous piecewise polynomials of order k and the pressure by the discontinuous piecewise polynomials of order $k-1$. The methods are all conforming in the sense that $V_h \subset V$ and $W_h \subset W$.

1. Scott and Vogelius [30] proposed 2D family of $P_k^2 - P_{k-1}$ type triangular elements for any $k \geq 4$, on singular-vertex free mesh. An internal vertex in 2D is said to be singular if edges meeting at the point fall into two straight lines.
2. Arnold and Qin [31] proposed a 2D finite element of $P_2^2 - P_1$ type on macro square meshes where each big square is subdivided into four triangles by connecting the square's vertices to the point midway between the center of the square and its bottom edge.
3. Qin [32] proposed 2D finite elements of $P_k^2 - P_{k-1}$ type, for $k = 2$ and $k = 3$, on macro triangular meshes where each big triangle is subdivided into three triangles by connecting the barycenter with three vertices.
4. Zhang [33] proposed 3D finite elements of $P_k^3 - P_{k-1}$ type, for $k \geq 3$, on macro tetrahedron meshes where each big tetrahedron is subdivided into four subtetrahedra by connecting the barycenter with four vertices.

For these finite element spaces, the assumption (**H2′**) is trivially satisfied by the definition of V_h and W_h.

4. Application of Modified $H(\mathbf{div})$-conforming Elements

The second method is to consider the construction of uniformly stable elements for the Darcy-Stokes-Brinkman equations on the basis of $H(\mathbf{div})$-conforming elements. We first consider the following mixed formulation of an elliptic problem, which can be viewed as one limiting case of the problem (2.13). Find $\mathbf{u}_h \in V_h^0$ and $p_h \in W_h^0$, such that

$$\begin{cases} (\mathbf{u}_h, \mathbf{v}_h) - (p_h, \mathbf{div}\mathbf{v}_h) = < \mathbf{f}, \mathbf{v}_h > & \forall \mathbf{v}_h \in V_h^0, \\ (\mathbf{div}\mathbf{u}_h, q_h) = < g, q_h > & \forall q_h \in W_h^0, \end{cases} \tag{4.1}$$

Here, $V_h^0 \subset V^0$ and $W_h^0 \subset L_0^2$ are two finite element spaces, and $V^0 := H_0(\mathbf{div})$, with the norm

$$\|\mathbf{v}\|_{H(\mathbf{div})}^2 := (\mathbf{div}\mathbf{v}, \mathbf{div}\mathbf{v}) + (\mathbf{v}, \mathbf{v}), \quad \mathbf{v} \in V^0.$$

The stability conditions for this problem are

$$(\mathbf{v}_h,\mathbf{v}_h) \gtrsim \|\mathbf{v}_h\|^2_{H(\mathbf{div})}, \quad \forall \mathbf{v}_h \in Z_h^0 := \{\mathbf{v}_h \in V_h^0 : (\mathbf{div}\mathbf{v}_h, q_h) = 0,\ \forall q_h \in W_h^0\}, \tag{4.2}$$

$$\sup_{\mathbf{v}_h\in V_h} \frac{(\mathbf{div}\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{H(\mathbf{div})}} \gtrsim \|q_h\|_0, \quad \forall q_h \in W_h^0, \tag{4.3}$$

$$(\mathbf{u}_h,\mathbf{v}_h) \lesssim \|\mathbf{u}_h\|_{H(\mathbf{div})}\|\mathbf{v}_h\|_{H(\mathbf{div})}, \quad \mathbf{u}_h,\mathbf{v}_h \in V_h^0, \tag{4.4}$$

$$(\mathbf{div}\mathbf{v}_h, q_h) \lesssim \|\mathbf{v}_h\|_{H(\mathbf{div})}\|q_h\|_0, \quad \forall \mathbf{v}_h \in V_h^0, \forall q_h \in W_h^0. \tag{4.5}$$

In the lemma below, we shall show that any stable Stokes elements satisfying (**H1**) and (**H2**) are also stable for the reduced problem (4.1).

Lemma 4.1. *Suppose that the finite element spaces $V_h \subset V^0$ and $W_h \subset L_0^2$ satisfy the assumptions* (**H1**) *and* (**H2**)*. Then the stability conditions* (4.2)-(4.5) *hold for* $\{V_h, W_h\}$*.*

Proof. The condition (4.2) is trivial by the assumption (**H2**). Under the assumption (**H1**), (4.3) follows from the fact that $\|\mathbf{v}_h\|_{H(\mathbf{div})} \lesssim \|\mathbf{v}_h\|_{1,h}$, $\forall \mathbf{v}_h \in V_h$. From the Cauchy-Schwartz inequality, the conditions (4.4) and (4.5) also follow immediately. □

In consequence, we have

Theorem 4.1. *Under the conditions in Lemma* 4.1*, the problem* (4.1) *admits a unique solution* $\{\mathbf{u}_h, p_h\} \in V_h \times W_h$*, such that*

$$\|\mathbf{u}-\mathbf{u}_h\|_{H(\mathbf{div})} \lesssim \inf_{\mathbf{v}_h\in V_h} \|\mathbf{u}-\mathbf{v}_h\|_{H(\mathbf{div})}, \tag{4.6}$$

$$\|p-p_h\|_0 \lesssim \inf_{q_h\in W_h} \|p-q_h\|_0 + \inf_{\mathbf{v}_h\in V_h} \|\mathbf{u}-\mathbf{v}_h\|_{H(\mathbf{div})}. \tag{4.7}$$

In addition, if the assumption (**H2′**) *holds, we have*

$$\|\mathbf{div}(\mathbf{u}-\mathbf{u}_h)\|_0 = \|(I-Q_h)\mathbf{div}\mathbf{u}\|_0. \tag{4.8}$$

Now, we want to construct modified $H(\mathbf{div})$ elements to approximate H^1 space on the basis of $H(\mathbf{div})$ stable elements. In practice for common choices of V_h^0 and W_h^0, the stability conditions (4.2) and (4.3) amount to

(**S1**)
$$\begin{cases} \text{There exists } \Pi_h : V^0 \to V_h^0 \text{ such that } \forall \mathbf{v} \in V^0, \\ \mathbf{div}\Pi_h\mathbf{v} = Q_h\mathbf{div}\mathbf{v} \ \text{ and } \ \|\Pi_h\mathbf{v}\|_{H(\mathbf{div})} \lesssim \|\mathbf{v}\|_{H(\mathbf{div})}, \end{cases}$$

and

(**S2**)
$$\mathbf{div}V_h^0 = W_h^0.$$

In fact it is easy to see that, under the assumption (**S1**), the assumption (**S2**) is equivalent to a weaker one:

$$\mathbf{div}V_h^0 \subseteq W_h^0. \tag{4.9}$$

For example, the Raviart-Thomas [34,35] or Brezzi-Douglas-Marini elements [36,37] satisfy (**S1**) and (**S2**). In these elements, Π_h is the canonical interpolation operator defined element by element. In addition, Π_h satisfies H^1 bound property, namely $\forall \mathbf{v} \in V^0$,

$$\|\Pi_h\mathbf{v}\|_{1,h} \lesssim \|\mathbf{v}\|_{1,h}. \tag{4.10}$$

In fact, this condition is crucial to the uniform stability for the original problem (2.13).

Lemma 4.2. *Suppose that the H^1 bound condition* (4.10) *and the $H(\mathbf{div})$ stability conditions,* (**S1**) *and* (**S2**)*, hold. Then the assumptions* (**H1**) *and* (**H2**) *hold for V_h^0 and W_h^0.*

Proof. By Fortin's Lemma, (**S1**), (**S2**), and (4.10) imply the assumption (**H1**). The assumption (**H2**) is a direct consequence of (**S2**). □

However, in general, the spaces V_h^0 and W_h^0 do not usually work for the original problem, since these spaces lead to the nonconforming approximation and a function in V_h^0 has no continuity of tangential component. In order to have desired accuracy for the consistency error estimate, at least we need to impose some weak continuity of tangential component of the velocity approximation. Now our task is to construct new element spaces V_h and W_h to approximate the original problem while preserving the structure of V_h^0 and W_h^0, namely, by satisfying the conditions (**S1**) and (**S2**).

A natural choice for pressure space is $W_h = W_h^0$. This implies that, in order to have (**S2**) for V_h and W_h, our new space may take the form:

$$V_h = V_h^0 + \mathbf{curl} S. \tag{4.11}$$

Here S is some piecewise polynomial space on $\mathcal{T}_h$. Recall that the **curl** operator on a scalar function q in 2D is defined by

$$\mathbf{curl} q = \left(-\frac{\partial q}{\partial x_2}, \frac{\partial q}{\partial x_1}\right)^T,$$

and on a vector function $\mathbf{q}$ in 3D is defined by

$$\mathbf{curl}\mathbf{q} = \left(\frac{\partial q_2}{\partial x_3} - \frac{\partial q_3}{\partial x_2}, \frac{\partial q_3}{\partial x_1} - \frac{\partial q_1}{\partial x_3}, \frac{\partial q_1}{\partial x_2} - \frac{\partial q_2}{\partial x_1}\right)^T,$$

where $\mathbf{q} = (q_1,\ q_2,\ q_3)^T$. Now we look at the degrees of freedom of $\mathbf{v} \in V_h$. Let $\Pi_h : V \to V_h$ be the canonical interpolation operator defined element by element. We first note the identity by Green's formula,

$$\begin{aligned}\int_T (\mathbf{div}\Pi_h\mathbf{v} - Q_h\mathbf{div}\mathbf{v})q_h dx &= \int_T \mathbf{div}(\Pi_h\mathbf{v} - \mathbf{v})q_h \\ = -\int_T (\Pi_h\mathbf{v} - \mathbf{v})\nabla q_h dx &+ \int_{\partial T} (\Pi_h\mathbf{v} - \mathbf{v}) \cdot \mathbf{n} q_h ds.\end{aligned}$$

In order to have the commutativity property in (**S1**), we can take the degrees of freedom used in the standard Raviart-Thomas [34,35] or Brezzi-Douglas-Marini elements [36,37]. Further we need additional degrees of freedom to ensure certain weak continuity of the velocity approximation.

Examples. We define the velocity finite element space on the element T as following:

$$V_T := V_T^0 + \mathbf{curl}(bY), \tag{4.12}$$

here b is the bubble function, namely $b = \Pi_{i=1}^{d+1}\lambda_i$, and λ_i, $i = 1, \cdots, d+1$, is the barycentric coordinate of T. For the space Y, choose the following polynomial spaces:

$$Y = \begin{cases} Y_1 := P_1(T) & \text{2D}, \\ Y_2 := P_1^3(T) & \text{3D}, \\ Y_3 := P_1^3(T)/\text{span}\{(\lambda_i - \frac{1}{3})\nabla\lambda_i\}_{i=1}^4 & \text{3D}. \end{cases} \tag{4.13}$$

By construction, it is easy to see that $\forall \mathbf{q} \in \mathbf{curl}(bY)$ satisfies

$$\mathbf{div}\mathbf{q} = 0 \quad \text{and} \quad \mathbf{q}\cdot\mathbf{n}|_{\partial T} = 0. \tag{4.14}$$

For the space V_T^0, we choose the following well-known $H(\mathbf{div})$-conforming finite element spaces.

$$V_T^0 = \begin{cases} RT_1(T) := P_1^2(T) + \tilde{P}_1(T)\mathbf{x} & \text{2D},\\ BDM_1(T) := P_1^2(T) & \text{2D},\\ RT_1(T) := P_1^3(T) + \tilde{P}_1(T)\mathbf{x} & \text{3D},\\ BDM_1(T) := P_1^3(T) & \text{3D}, \end{cases} \tag{4.15}$$

where $\tilde{P}_1(T) := P_1(T)/P_0(T)$ (i.e., the homogeneous polynomial space of degree 1). $RT_1(T)$ denotes both the first order 2D Raviart-Thomas [34] and 3D Nedelec [35] finite element spaces. $BDM_1(T)$ denotes both the first order 2D Brezzi-Douglas-Marini [36] and 3D Brezzi-Douglas-Duran-Fortin [37] finite element spaces.

For different choices of V_T^0 and Y in (4.15) and (4.13), we give the corresponding degrees of freedom in the Table 4.1. In this table, $RT_0(f)$ is the zeroth-order Raviart-Thomas element space [34], i.e.,

$$RT_0(f) := P_0^2(f) + P_0\mathbf{x},$$

$\mathbf{n}$ is the unit normal vector to an edge $e \in \mathcal{E}(T)$ or a face $f \in \mathcal{F}(T)$, and $\mathbf{t}$ is the unit tangent vector along the edge e. For the element diagrams, see the Figs. 4.1-4.3.

Table 4.1: The Six Modified $H(\mathbf{div})$ Elements.

Elements	V_T^0	Y	Degrees of Freedom (DOF)	# of DOF
First 2D Element	$RT_1(T)$	Y_1	$\int_e \mathbf{v}\cdot\mathbf{n}q ds, \quad \forall q \in P_1(e)$, $\int_T \mathbf{v}\cdot\mathbf{q} dx, \quad \forall \mathbf{q} \in P_0^2(T)$ $\int_e \mathbf{v}\cdot\mathbf{t} ds$	11
Second 2D Element	$BDM_1(T)$	Y_1	$\int_e \mathbf{v}\cdot\mathbf{n}q ds, \quad \forall q \in P_1(e)$, $\int_e \mathbf{v}\cdot\mathbf{t} ds$	9
First 3D Element	$RT_1(T)$	Y_2	$\int_f \mathbf{v}\cdot\mathbf{n}q ds, \quad \forall q \in P_1(f)$ $\int_T \mathbf{v}\cdot\mathbf{q} ds, \quad \forall \mathbf{q} \in P_0^3(T)$ $\int_f (\mathbf{v}\times\mathbf{n})\cdot\mathbf{r} ds, \quad \forall \mathbf{r} \in RT_0(f)$	27
Second 3D Element	$BDM_1(T)$	Y_2	$\int_f \mathbf{v}\cdot\mathbf{n}q ds, \quad \forall q \in P_1(f)$ $\int_f (\mathbf{v}\times\mathbf{n})\cdot\mathbf{r} ds, \quad \forall \mathbf{r} \in RT_0(f)$	24
Third 3D Element	$RT_1(T)$	Y_3	$\int_f \mathbf{v}\cdot\mathbf{n}q ds, \quad \forall q \in P_1(f)$ $\int_T \mathbf{v}\cdot\mathbf{q} ds, \quad \forall \mathbf{q} \in P_0^3(T)$ $\int_f (\mathbf{v}\times\mathbf{n})\cdot\mathbf{r} ds, \quad \forall \mathbf{r} \in P_0^2(f)$	23
Fourth 3D Element	$BDM_1(T)$	Y_3	$\int_f \mathbf{v}\cdot\mathbf{n}q ds, \quad \forall q \in P_1(f)$ $\int_f (\mathbf{v}\times\mathbf{n})\cdot\mathbf{r} ds, \quad \forall \mathbf{r} \in P_0^2(f)$	20

Among these six elements, the second 2D element and the second 3D element have been also proposed by Mardal-Tai-Winther [38] and Tai-Winther [39] respectively for the problem (1.1) with $\nu(x) \equiv \epsilon^2$ and $\alpha(x) \equiv 1$.

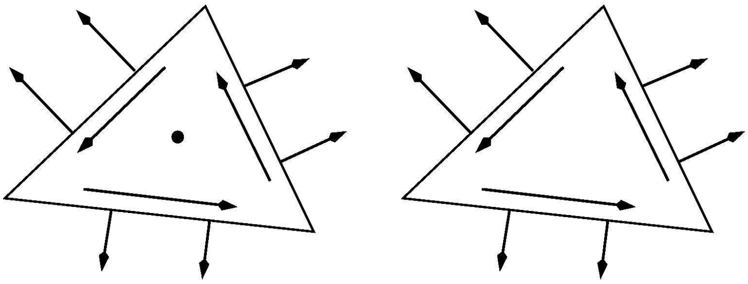

Fig. 4.1. DOF for the first 2D modified RT element (left) and for the second 2D modified BDM element (right).

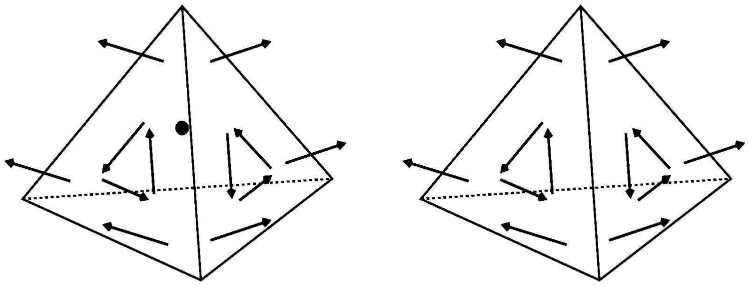

Fig. 4.2. DOF for the first 3D modified RT element (left) and for the second 3D modified BDM element (right).

4.1. Unisolvence

Denote the dimension of a polynomial space by $\mathbf{dim}(\cdot)$. It is easy to see that the following relation holds:

$$\mathbf{dim}(V_T^0 + \mathbf{curl}(bY)) = \mathbf{dim}(V_T^0) + \mathbf{dim}(\mathbf{curl}(bY)) = \mathbf{dim}(V_T^0) + \mathbf{dim}(Y).$$

From this relation, we can see that the dimension of polynomial space V_T is the same as the number of degrees of freedom for each element. In addition, the following lemma is useful for the unisolvence of the 3D elements.

Lemma 4.3. ([39]) *Assume that* $\mathbf{v} \in P_1^2(f)$ *is of the form*

$$\mathbf{v} = \sum_{i=1}^{3} c_i(\lambda_i - \frac{1}{3})\nabla\lambda_i,$$

and satisfies $\int_f b_f \mathbf{v} \cdot \mathbf{r} dx = 0$. *Here* $\mathbf{r}$ *is the position vector:* $\mathbf{r} = (x, y)^T$, *and* b_f *is the cubic bubble function associated with face* f. *Then* $c_1 + c_2 + c_3 = 0$.

Now, we give a unified unisolvence proof for all of the six elements.

Lemma 4.4. *For all the six elements,* $\forall \mathbf{v} \in V_T$ *is uniquely determined by the corresponding degrees of freedom.*

Proof. Assume that all the degrees of freedom are zeros. Let $\mathbf{v} = \mathbf{v}^0 + \mathbf{curl}(b\mathbf{q})$, with $\mathbf{v}^0 \in V_T^0$ and $\mathbf{q} \in Y$ (in 2D, $\mathbf{q}$ is scalar). In 3D,

$$\mathbf{curl}(b\mathbf{q})\cdot\mathbf{n} = \mathbf{curl}_f(b\mathbf{q})_f = 0,$$

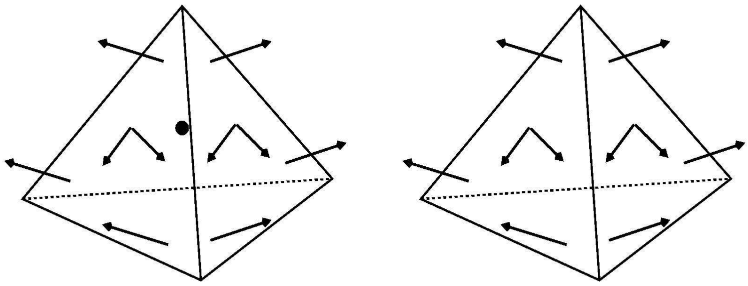

Fig. 4.3. DOF for the third 3D modified RT element (left) and for the fourth 3D modified BDM element (right).

with $(b\mathbf{q})_f$ the tangential component of $b\mathbf{q}$ on f. In 2D,

$$\mathbf{curl}(bq) \cdot \mathbf{n} = \frac{\partial(bq)}{\partial \mathbf{t}} = 0.$$

Thus, $\mathbf{v} \cdot \mathbf{n} = \mathbf{v}^0 \cdot \mathbf{n} \in P_1(\partial T)$. Furthermore,

$$\begin{cases} \int_e \mathbf{v^0} \cdot \mathbf{n} r ds = 0, \quad \forall r \in P_1(e) \quad \forall e \in \mathcal{E}(T) \qquad \text{in 2D,} \\ \int_f \mathbf{v^0} \cdot \mathbf{n} w ds = 0, \quad \forall w \in P_1(f) \quad \forall f \in \mathcal{F}(T) \qquad \text{in 3D.} \end{cases} \tag{4.16}$$

Also, by Stoke's theorem,

$$\int_T \mathbf{curl}(b\mathbf{q}) \cdot \mathbf{r} dx = 0, \quad \forall \mathbf{r} \in P_0^d(T), \quad d = 2, 3.$$

Hence

$$\int_T \mathbf{v^0} \cdot \mathbf{r} dx = \int_T \mathbf{v} \cdot \mathbf{r} dx = 0, \qquad \forall \mathbf{r} \in P_0^d(T). \tag{4.17}$$

It is well known that 2D Raviart-Thomas [34], 3D Nedelec [35], 2D Brezzi-Douglas-Marini [36], and 3D Brezzi-Douglas-Duran-Fortin [37] elements are all unisolvent. Therefore, $\mathbf{v^0} = \mathbf{0}$.

In what follows, we shall show that $\mathbf{q} = \mathbf{0}$. For the 2D elements, $\forall q \in P_1(e)$,

$$0 = \int_e \mathbf{v} \cdot \mathbf{t} ds = \int_e \mathbf{curl}(bq) \cdot \mathbf{t} ds = \int_e \nabla(bq) \cdot \mathbf{n} ds = \int_e \frac{\partial b}{\partial \mathbf{n}} q ds.$$

Since $\partial b / \partial \mathbf{n}$ remains the same sign on each edge e, we know that q has zero point in the interior of e. Thus, $q = 0$. For the 3D elements, on each face f, it is easy to calculate that

$$\mathbf{v} \times \mathbf{n} = \mathbf{curl}(b\mathbf{q}) \times \mathbf{n} = \frac{\partial b}{\partial \mathbf{n}} (\mathbf{n} \times \mathbf{q}) \times \mathbf{n}.$$

Here $\partial b / \partial \mathbf{n}$ is proportional to b_f on the face f. Then we get

$$\int_f b_f (\mathbf{n} \times \mathbf{q}) \times \mathbf{n} \cdot \mathbf{r} ds = 0, \qquad \forall \mathbf{r} \in RT_0(f) \text{ or } \in P_0^2(f), \quad \forall f \in \mathcal{F}(T). \tag{4.18}$$

The remaining proof is essentially from Lemma 3 by Tai and Winther [39]. Note $\mathbf{r} \in P_0^2(f) \subset RT_0(f)$. Then it is easy to get

$$\mathbf{q_t}(x_f^b) = \mathbf{0}, \qquad f \in \mathcal{F}(T). \tag{4.19}$$

Here $\mathbf{q_t} := (\mathbf{n} \times \mathbf{q}) \times \mathbf{n}$ is the tangential component of $\mathbf{q}$, x_f^b is the barycenter of the face f.

For $\mathbf{q} \in P_1^3(T)$, from (4.19) we easily know that $\mathbf{q}$ can be formulated as

$$\mathbf{q} = \sum_{i=1}^{4} c_i(\lambda_i - \frac{1}{3})\nabla\lambda_i. \tag{4.20}$$

Here $c_1, c_2, \cdots, c_4$ are arbitrary constants. Then, the tangential component of $\mathbf{q}$ on the face f_1 has the form

$$\mathbf{q_t} = \sum_{i=2}^{4} c_i(\lambda_i - \frac{1}{3})(\nabla\lambda_i)_{\mathbf{t}}. \tag{4.21}$$

For the first and second 3D elements, from Lemma 4.3, we get $c_2 + c_3 + c_4 = 0$. By considering the other three faces we can also get $c_1 + c_3 + c_4 = 0$, $c_1 + c_2 + c_4 = 0$ and $c_1 + c_2 + c_3 = 0$. These four relations imply that $c_i = 0, i = 1, 2, 3, 4$. For the third and fourth elements, by the construction of the space Y_3, it follows immediately $c_i = 0, i = 1, 2, 3, 4$. Therefore, $\mathbf{q} = \mathbf{0}$. □

4.2. Verification of assumptions

Let $V_h^{(i)}$, $i = 1, 2, \cdots, 6$, be the finite dimensional velocity spaces corresponding to the six velocity finite elements listed in the Table 4.1 with all degrees of freedom of $\mathbf{v} \in V_h^{(i)}$ being zero on $\partial\Omega$. It is easy to see $V_h^{(i)} \subset H(\mathbf{div}, \Omega)$, but $\not\subset H_0^1(\Omega)$. These choices of spaces lead to nonconforming finite element methods of the problem (2.13).

For the pressure, let the finite dimensional space $W_h^{(i)}$, $i = 1, 2, \cdots, 6$, be as follows:

$$\begin{cases} W_h^{(1)} := \{q \in W : \quad q|_T \in P_1(T)\}, \\ W_h^{(2)} := \{q \in W : \quad q|_T \in P_0(T)\}, \\ W_h^{(3)} := \{q \in W : \quad q|_T \in P_1(T)\}, \\ W_h^{(4)} := \{q \in W : \quad q|_T \in P_0(T)\}, \\ W_h^{(5)} := \{q \in W : \quad q|_T \in P_1(T)\}, \\ W_h^{(6)} := \{q \in W : \quad q|_T \in P_0(T)\}. \end{cases}$$

Taking $V_h = V_h^{(i)}$ and $W_h = W_h^{(i)}$, the assumption (**S2**) is trivially satisfied by the constructions.

For each finite element space V_h, the canonical interpolation operator Π_h: $V \to V_h$ is defined by the corresponding degrees of freedom in V_h. Thus, $\forall q \in W_h^{(i)}$ and $\forall \mathbf{v} \in V$,

$$\begin{aligned} \int_T \mathbf{div}\Pi_h \mathbf{v} q dx &= -\int_T \Pi_h \mathbf{v} \cdot \nabla q dx + \int_{\partial T} \Pi_h \mathbf{v} \cdot \mathbf{n} q ds, \\ = -\int_T \mathbf{v} \cdot \nabla q dx + \int_{\partial T} \mathbf{v} \cdot \mathbf{n} q ds &= \int_T \mathbf{div}\mathbf{v} q dx = \int_T Q_h \mathbf{div}\mathbf{v} q dx. \end{aligned}$$

Therefore, by the assumption (**S2**), we get the commutativity property

$$\mathbf{div}\Pi_h \mathbf{v} = Q_h \mathbf{div}\mathbf{v}. \tag{4.22}$$

Furthermore, since the operator Π_h preserves linear polynomials locally, we can prove that there hold the interpolation error estimates.

$$\|\mathbf{v} - \Pi_h \mathbf{v}\|_{j,h} \lesssim h^{k+1-j} |\mathbf{v}|_{k+1,h}, \quad 0 \le j \le k \le 1, \tag{4.23}$$

and the H^1 bound property

$$\|\Pi_h \mathbf{v}\|_{1,h} \lesssim \|\mathbf{v}\|_{1,h}. \tag{4.24}$$

Notice that these elements are not invariant under the Piola transformation. Consequently, a different argument is required to prove the interpolation error estimate. The analysis can be done by scaling to a similar element of unit diameter using translation, rotation, and dilation and using the compactness argument [40, 41]. Thus, the assumption (**S1**) and (4.10) hold.

Now we shall derive consistency error estimates for all the six elements.

We consider the detailed discussion in 3D case (2D case is similar and easier). By the interface condition $[\sigma(\mathbf{u},p)\mathbf{n}]\,|_{\Gamma_{ij}} = 0$, we can rewrite the consistency error term (3.7) as

$$E_h(\mathbf{u},p,\mathbf{v}_h) = \sum_{f\in\mathcal{F}(T)} \int_f \sigma(\mathbf{u},p)\mathbf{n}\cdot[\mathbf{v}_h]ds. \tag{4.25}$$

On the face f, decompose the vector $\sigma(\mathbf{u},p)\mathbf{n}$ and $\mathbf{v}$ along the normal direction $\mathbf{n}$ and along the tangential direction to the face f, i.e., $\sigma(\mathbf{u},p)\mathbf{n} = (\sigma(\mathbf{u},p)\mathbf{n}\cdot\mathbf{n})\mathbf{n} + \mathbf{n}\times(\sigma(\mathbf{u},p)\mathbf{n}\times\mathbf{n})$, and $\mathbf{v}_h = (\mathbf{v}_h\cdot\mathbf{n})\mathbf{n} + \mathbf{n}\times(\mathbf{v}_h\times\mathbf{n})$. Then we get

$$E_h(\mathbf{u},p,\mathbf{v}_h) = \sum_{f\in\mathcal{F}(T)} \int_f (\sigma(\mathbf{u},p)\mathbf{n}\times\mathbf{n})\cdot[\mathbf{v}_h\times\mathbf{n}]ds. \tag{4.26}$$

Let T_f^- and T_f^+ denote the two tetrahedrons sharing the same face f. Denote

$$w^+ := w|_{\partial T_f^+\cap f}, \quad w^- := w|_{\partial T_f^-\cap f},$$

here w can be either a scalar or a vector. In addition, denote $\sigma^+(\mathbf{u},p) := \sigma(\mathbf{u}^+,p^+)$ and $\sigma^-(\mathbf{u},p) := \sigma(\mathbf{u}^-,p^-)$. For all the four 3D elements in Section 4, the following uniform consistency error estimate holds.

Lemma 4.5. *For* $\mathbf{u}\in H_0^1$, $\forall \mathbf{v}_h \in V_h^{(i)}, i = 3,4,5,6,$

$$|E_h(\mathbf{u},p,\mathbf{v}_h)| \lesssim h|\nu^{1/2}\nabla\mathbf{u}|_{1,h}|\!|\!|\mathbf{v}_h|\!|\!|_h. \tag{4.27}$$

Proof. By the continuity of the normal component of the stress tensor, on the face f we have that $\sigma(\mathbf{u},p)\mathbf{n} = \sigma^+(\mathbf{u},p)\mathbf{n} = \sigma^-(\mathbf{u},p)\mathbf{n}$. We first estimate

$$\begin{aligned}
&\left|\int_f \big(\sigma^+(\mathbf{u},p)\mathbf{n}\times\mathbf{n}\big)\cdot[\mathbf{v}_h\times\mathbf{n}]ds\right| \\
=&\left|\int_f (\nu^+\nabla\mathbf{u}^+\mathbf{n}\times\mathbf{n})\cdot[\mathbf{v}_h\times\mathbf{n}]ds - \int_f \big((p^+\mathbf{I})\mathbf{n}\times\mathbf{n}\big)\cdot[\mathbf{v}_h\times\mathbf{n}]ds\right| \\
=&\left|\int_f (\nu^+\nabla\mathbf{u}^+\mathbf{n}\times\mathbf{n})\cdot[\mathbf{v}_h\times\mathbf{n}]ds\right| = \left|\nu^+\int_f (\nabla\mathbf{u}^+\mathbf{n}\times\mathbf{n}-\lambda)[\mathbf{v}_h\times\mathbf{n}-\mu]ds\right| \\
\le&\, \nu^+ \inf_{\lambda\in\mathbf{R}^2}\|\nabla\mathbf{u}^+\mathbf{n}\times\mathbf{n}-\lambda\|_{0,f} \inf_{\mu\in\mathbf{R}^2}\|[\mathbf{v}_h\times\mathbf{n}-\mu]\|_{0,f} \\
\lesssim&\, \nu^+ h|\mathbf{u}|_{2,T_f^+}|\mathbf{v}_h|_{1,T_f^+}.
\end{aligned} \tag{4.28}$$

Here the third equality follows from the definition of degrees of freedom, the first inequality from Cauchy-Schwartz inequality, and the second inequality from the standard scaling argument and Bramble-Hilbert Lemma. Similarly, we can get

$$\left|\int_f \big(\sigma^-(\mathbf{u},p)\mathbf{n}\times\mathbf{n}\big)\cdot[\mathbf{v}_h\times\mathbf{n}]ds\right| \lesssim \nu^- h|\mathbf{u}|_{2,T_f^-}|\mathbf{v}_h|_{1,T_f^-}. \tag{4.29}$$

From the above two estimates, (4.28) and (4.29), it follows that

$$\begin{aligned}&\left|\int_f (\sigma(\mathbf{u},p)\mathbf{n}\times\mathbf{n})\cdot[\mathbf{v}_h\times\mathbf{n}]ds\right|\\ &\lesssim h\left(\nu^+|\mathbf{u}|^2_{2,T_f^+}+\nu^-|\mathbf{u}|^2_{2,T_f^-}\right)^{1/2}\left(\nu^+|\mathbf{v}_h|^2_{1,T_f^+}+\nu^-|\mathbf{v}_h|^2_{1,T_f^-}\right)^{1/2}.\end{aligned}$$

Finally, applying Cauchy-Schwartz inequality, we have the consistency error estimate

$$\begin{aligned}&|E_h(\mathbf{u},p,\mathbf{v}_h)|\\ \lesssim\;& h\left(\sum_{f\in\mathcal{F}(T)}\left(\nu^+|\mathbf{u}|^2_{2,T_f^+}+\nu^-|\mathbf{u}|^2_{2,T_f^-}\right)\right)^{1/2}\left(\sum_{f\in\mathcal{F}(T)}\left(\nu^+|\mathbf{v}_h|^2_{1,T_f^+}+\nu^-|\mathbf{v}_h|^2_{1,T_f^-}\right)\right)^{1/2}\\ \lesssim\;& h|\nu^{1/2}\nabla\mathbf{u}|_{1,h}|\!|\!|\mathbf{v}_h|\!|\!|_h.\end{aligned}$$

□

For the 2D case, on the edge e, decompose the vector $\sigma(\mathbf{u},p)\mathbf{n}$ and $\mathbf{v}_h$ along the normal direction $\mathbf{n}$ and along the tangential direction $\mathbf{t}$, i.e., $\sigma(\mathbf{u},p)\mathbf{n}=(\sigma(\mathbf{u},p)\mathbf{n}\cdot\mathbf{n})\mathbf{n}+(\sigma(\mathbf{u},p)\mathbf{n}\cdot\mathbf{t})\mathbf{t}$ and $\mathbf{v}_h=(\mathbf{v}_h\cdot\mathbf{n})\mathbf{n}+(\mathbf{v}_h\cdot\mathbf{t})\mathbf{t}$. Then, we get

$$E_h(\mathbf{u},p,\mathbf{v}_h)=\sum_{e\in\mathcal{E}(T)}\int_e(\sigma(\mathbf{u},p)\mathbf{n}\cdot\mathbf{t})[\mathbf{v}_h\cdot\mathbf{t}]ds. \tag{4.30}$$

Similar to the 3D case, the following uniform consistency error estimate holds for the two 2D elements in Section 4.

Lemma 4.6. *For* $\mathbf{u}\in H^1_0$, $\forall\mathbf{v}_h\in V_h^{(i)}, i=1,2,$

$$|E_h(\mathbf{u},p,\mathbf{v}_h)|\lesssim h|\nu^{1/2}\nabla\mathbf{u}|_{1,h}|\!|\!|\mathbf{v}_h|\!|\!|_h. \tag{4.31}$$

5. Application of Stable Stokes Elements for a Modified Formulation

Brezzi, Frotin and Marini [22] studied the mixed form of Poisson equation and modified the variational formulation such that the coercivity condition automatically held on the discrete level. We can apply the same technique by considering the following equivalent formulation of (2.1). Find $\{\mathbf{u},p\}\in V\times W$ such that

$$\left\{\begin{array}{ll}a(\mathbf{u},\mathbf{v})+M(\mathbf{div}\mathbf{u},\mathbf{div}\mathbf{v})-(p,\mathbf{div}\mathbf{v})=<\mathbf{f},\mathbf{v}>+M<g,\mathbf{div}\mathbf{v}> & \forall\mathbf{v}\in V,\\ (\mathbf{div}\mathbf{u},q)=<g,q> & \forall q\in W,\end{array}\right. \tag{5.1}$$

where M is given by (2.4). Correspondingly, we have the following discrete weak formulation. Find $\{\mathbf{u}_h,p_h\}\in V_h\times W_h$ such that $\forall\mathbf{v}_h\in V_h$ and $\forall q_h\in W_h$,

$$\left\{\begin{array}{l}a_h(\mathbf{u}_h,\mathbf{v}_h)+M(\mathbf{div}\mathbf{u}_h,\mathbf{div}\mathbf{v}_h)-(p_h,\mathbf{div}\mathbf{v}_h)=<\mathbf{f},\mathbf{v}_h>+M<g,\mathbf{div}\mathbf{v}_h>,\\ (\mathbf{div}\mathbf{u}_h,q_h)=<g,q_h>.\end{array}\right. \tag{5.2}$$

Under this modified formulation, any pair of stable Stokes elements that satisfy the inf-sup condition

$$\sup_{\mathbf{v}_h\in V_h}\frac{(\mathbf{div}\mathbf{v}_h,q_h)}{\|\mathbf{v}_h\|_{1,h}}\gtrsim\|q_h\|_0\qquad\forall q_h\in W_h, \tag{5.3}$$

is uniformly stable under the norms given in (2.15) and (2.3).

Furthermore, by the standard saddle point theory [25–27], we have

$$\|\mathbf{u}-\mathbf{u}_h\|_h + \|p-p_h\| \lesssim \inf_{\mathbf{v}_h\in V_h}\|\mathbf{u}-\mathbf{v}_h\|_h + \inf_{q_h\in W_h}\|p-q_h\| + \sup_{\mathbf{v}_h\in V_h}\frac{|E_h(\mathbf{u},p,\mathbf{v}_h)|}{\|\mathbf{v}_h\|_h}. \tag{5.4}$$

Here the consistency error is defined by

$$E_h(\mathbf{u},p,\mathbf{v}_h) = \sum_{T\in\mathcal{T}_h}\int_{\partial T}\left(\sigma(\mathbf{u},p)\mathbf{n}\cdot\mathbf{v}_h - \mathbf{div}\mathbf{u}(\mathbf{v}_h\cdot\mathbf{n})\right)ds. \tag{5.5}$$

It is easy to see that under (**H2′**) the new formulation (5.2) is equivalent to the original one (2.13).

6. Concluding Remarks and Future Work

We show that any traditional stable Stokes element is also uniformly stable for the Darcy-Stokes-Brinkman equations with respect to the viscosity and zeroth-order term coefficient and their jumps if and only if the discretely divergence-free velocity implies almost everywhere divergence-free one. We also discuss the construction of uniformly stable elements on the basis of $H(\mathbf{div})$-conforming elements. By keeping the structure of standard $H(\mathbf{div})$-conforming elements, we construct several new uniformly stable 2D and 3D elements in a unified way. On the other hand, the original weak formulation of Darcy-Stokes-Brinkman equation can be equivalently modified in such a way that any traditional stable Stokes element is also uniformly stable. Among these three methods, the modified $H(\mathbf{div})$ elements have the exact sequence property which is an important tool to design and analyze preconditioner and multigrid method for the resulting linear systems. In this sense, we regard these modified $H(\mathbf{div})$ elements as "solver friendly" ones, and we will discuss it in future work.

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