Journal of Computational Mathematics, Vol.26, No.3, 2008, 404–409.

AN EXPLICIT MULTI-CONSERVATION FINITE-DIFFERENCE SCHEME FOR SHALLOW-WATER-WAVE EQUATION*

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Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

An explicit multi-conservation finite-difference scheme for solving the spherical shallow-water-wave equation set of barotropic atmosphere has been proposed. The numerical scheme is based on a special semi-discrete form of the equations that conserves four basic physical integrals including the total energy, total mass, total potential vorticity and total enstrophy. Numerical tests show that the new scheme performs closely like but is much more time-saving than the implicit multi-conservation scheme.

Mathematics subject classification: 65N06, 65L12.
Key words: Explicit finite difference scheme, Multi-conservation, Shallow-water-wave, Physical integral.

1. Introduction

The spherical shallow-water-wave equation set of barotropic atmosphere, a representative atmospheric equation set, conserves five basic physical integrals including the total energy, total mass, total vorticity, total enstrophy and total angular momentum. These constant integrals imply important physical characteristic and mathematical significance of atmospheric motions [1,2]. To conserve these integrals as many as possible in a discrete scheme of the equation set is very necessary, which is one of the essential criterions to evaluate the scheme. For this reason, many efforts have been made on designing multi-conservation schemes for atmospheric equations [3–6]. The available multi-conservation schemes, however, are implicit and time-consuming due to a number of iterations for getting their solutions. Whether and how can an explicit multi-conservation scheme be constructed? This is an interesting question. A significant attempt to design an explicit multi-conservation finite-difference scheme is made in this paper, based on a special semi-discrete form of the spherical shallow-water-wave equation set of barotropic atmosphere that was applied to construct an implicit scheme with 4 conservation properties by Wang and Ji [6].

2. Equations and Conservations

The shallow-water-wave equation set of barotropic atmosphere in spherical coordinate system is originally formulated as

* Received March 2, 2008 / accepted March 14, 2008 /

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{1}{a\cos\theta}\left[\dfrac{\partial \varphi}{\partial \lambda} + u\dfrac{\partial u}{\partial \lambda} + v^*\dfrac{\partial u}{\partial \theta}\right] + f^* v, \\ \dfrac{\partial v}{\partial t} = -\dfrac{1}{a\cos\theta}\left[\cos\theta\dfrac{\partial \varphi}{\partial \theta} + u\dfrac{\partial v}{\partial \lambda} + v^*\dfrac{\partial v}{\partial \theta}\right] - f^* u\ , \\ \dfrac{\partial \varphi}{\partial t} = -\dfrac{1}{a\cos\theta}\left[\dfrac{\partial}{\partial \lambda}(u\varphi) + \dfrac{\partial}{\partial \theta}(v^*\varphi)\right]\ , \end{cases} \tag{2.1}$$

where θ, λ are the latitude and longitude respectively; a denotes the radius of earth, u,v and φ represent the zonal wind, meridional wind and geopotential height; $v^* = v\cos\theta$; $f^* = 2\omega_0 \sin\theta + ua^{-1}\tan\theta$, ω_0 is the angular velocity of the earth. It can be expressed into a concise form:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{a\cos\theta}\dfrac{\partial E}{\partial \lambda} - \eta v = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{1}{a}\dfrac{\partial E}{\partial \theta} + \eta u = 0, \\ \dfrac{\partial \varphi}{\partial t} + \dfrac{1}{a\cos\theta}\left[\dfrac{\partial}{\partial \lambda}(u\varphi) + \dfrac{\partial}{\partial \theta}(v^*\varphi)\right] = 0, \end{cases} \tag{2.2}$$

where

$$\begin{cases} E = \dfrac{1}{2}\left(u^2+v^2\right) + \varphi, \\ \eta = \dfrac{1}{a\cos\theta}\left[\dfrac{\partial v}{\partial \lambda} - \dfrac{\partial \Omega}{\partial \theta}\right] = q + 2\omega\sin\theta, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \Omega = u\cos\theta + a\omega\cos^2\theta, \\ q = \dfrac{1}{a\cos\theta}\left[\dfrac{\partial v}{\partial \lambda} - \dfrac{\partial}{\partial \theta}(u\cos\theta)\right]. \end{cases} \tag{2.4}$$

It is easy to prove that the equation set has five basic constant integrals [6]:

$$\begin{cases} \dfrac{\partial}{\partial t}\iint\limits_D \left(e + \dfrac{1}{2}\varphi\right)\varphi ds = 0, & \dfrac{\partial}{\partial t}\iint\limits_D \varphi ds = 0, \\ \dfrac{\partial}{\partial t}\iint\limits_D \xi\varphi ds = 0, & \dfrac{\partial}{\partial t}\iint\limits_D \xi^2\varphi ds = 0, \\ \dfrac{\partial}{\partial t}\iint\limits_D \Omega\varphi ds = 0, & \end{cases} \tag{2.5}$$

where the area unit ds is defined as: $ds = a^2\cos\theta d\lambda d\theta$, D is the integration region (here it is the whole spherical surface), e and ξ are respectively the kinetic energy and potential vorticity, which are defined as follows:

$$e = \frac{1}{2}\left(u^2+v^2\right), \quad \xi = \eta/\varphi. \tag{2.6}$$

3. Semi-discrete Equation Set

After discretizing the spatial differential terms of Eq. (2.2), a semi-discrete spherical shallow-water-wave equation set of barotropic atmosphere on an Arakawa A-grid system has been obtained:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{a\cos\theta}\bar{E}_\lambda^\lambda - (\eta_d + \varepsilon\ A\eta_d)v = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{1}{a}\bar{E}_\theta^\theta + (\eta_d + \varepsilon\ A\eta_d)u = 0, \\ \dfrac{\partial \varphi}{\partial t} + A\varphi = 0, \end{cases} \tag{3.1}$$

where ε is an adjustable real number, A is a discrete advection operator

$$AF = -\frac{1}{a\cos\theta}\left[\overline{(uF)}_\lambda^\lambda + \overline{(v^*F)}_\theta^\theta\right]. \tag{3.2}$$

In Eqs. (3.1) and (3.2), some marks respectively expressing the mean operations and the difference quotient operations are introduced

$$\begin{cases} \bar{F}^\lambda = \dfrac{F_{i+\frac{1}{2},j} + F_{i-\frac{1}{2},j}}{2}, \ \bar{F}^\theta = \dfrac{F_{i,j+\frac{1}{2}} + F_{i,j-\frac{1}{2}}}{2}, \\ F_\lambda = \dfrac{F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j}}{\Delta\lambda}, \ F_y = \dfrac{F_{i,j+\frac{1}{2}} - F_{i,j-\frac{1}{2}}}{\Delta\theta}, \end{cases} \tag{3.3}$$

where $\Delta\lambda$ and $\Delta\theta$ are respectively the grid intervals along zonal and meridional directions. The semi-discrete equation set is able to keep four of the five constant integrals in the discrete space

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\displaystyle\sum_{j=1}^{M}\sum_{i=1}^{N}\left(e_{i,j} + \frac{1}{2}\varphi_{i,j}\right)\varphi_{i,j}a^2\Delta\lambda\Delta\theta\cos\theta_j\right) = 0, \\ \dfrac{\partial}{\partial t}\left(\displaystyle\sum_{j=1}^{M}\sum_{i=1}^{N}\varphi_{i,j}a^2\Delta\lambda\Delta\theta\cos\theta_j\right) = 0, \\ \dfrac{\partial}{\partial t}\left(\displaystyle\sum_{j=1}^{M}\sum_{i=1}^{N}(\xi_d)_{i,j}\,\varphi_{i,j}, a^2\Delta\lambda\Delta\theta\cos\theta_j\right) = 0 \\ \dfrac{\partial}{\partial t}\left(\displaystyle\sum_{j=1}^{M}\sum_{i=1}^{N}\xi_{i,j}^2\varphi_{i,j}a^2\Delta\lambda\Delta\theta\cos\theta_j\right) = 0, \end{cases} \tag{3.4}$$

if ε is determined according to the following formula

$$\varepsilon = \left[\frac{1}{2}\left(\xi_d^2,\ A\varphi\right) - (\xi_d,\ A\eta_d)\right]\Big/\left(\xi_d,\ A^2\eta_d\right), \tag{3.5}$$

where the discrete inner product operation $(\cdot,\cdot)$ is defined as

$$(F,\ G) = \sum_{j=1}^{M}\sum_{i=1}^{N}F_{i,j}G_{i,j}a^2\Delta\lambda\Delta\theta\cos\theta_j. \tag{3.6}$$

4. Explicit Multi-conservation Scheme

Wang and Ji [6] proposed the following implicit scheme with 4 conservation properties:

$$\begin{cases} u^{n+1} = u^n - \tau\ L_1u^{n+\frac{1}{2}}, \ u^{n+\frac{1}{2}} = \dfrac{1}{2}\left(u^{n+1} + u^n\right), \\ v^{n+1} = v^n - \tau\ L_2v^{n+\frac{1}{2}}, \ v^{n+\frac{1}{2}} = \dfrac{1}{2}\left(v^{n+1} + v^n\right), \\ \varphi^{n+1} = \varphi^n - \tau\ A\varphi^{n+\frac{1}{2}}, \ \varphi^{n+\frac{1}{2}} = \dfrac{1}{2}\left(\varphi^{n+1} + \varphi^n\right), \end{cases} \tag{4.1}$$

where τ is the time step, and the operators L_1 and L_2 are defined as follows

$$\begin{cases} L_1 u = \dfrac{1}{a\cos\theta}\bar{E}_\lambda^\lambda - (\eta_d + \varepsilon\ A\eta_d)v, \\ L_2 v = \dfrac{1}{a}\bar{E}_\theta^\theta + (\eta_d + \varepsilon\ A\eta_d)u. \end{cases} \tag{4.2}$$

It is easy to prove that the operators L_1, $L_2$and A satisfy the following equality

$$(L_1 u,\ \varphi\ u) + (L_2 v,\ \varphi\ v) + (A\varphi,\ E) = 0. \tag{4.3}$$

The implicit solution to Eq. (4.1) was obtained iteratively:

$$\begin{cases} u^{n+1,k} = u^n - \tau\ L_1 u^{n+\frac{1}{2},k-1},\ u^{n+\frac{1}{2},k-1} = \dfrac{1}{2}\left(u^{n+1,k-1} + u^n\right), \\ v^{n+1,k} = v^n - \tau\ L_2 v^{n+\frac{1}{2},k-1},\ v^{n+\frac{1}{2},k-1} = \dfrac{1}{2}\left(v^{n+1,k-1} + v^n\right), \\ \varphi^{n+1,k} = \varphi^n - \tau\ A\varphi^{n+\frac{1}{2},k-1},\ \varphi^{n+\frac{1}{2},k-1} = \dfrac{1}{2}\left(\varphi^{n+1,k-1} + \varphi^n\right), \end{cases} \tag{4.4}$$
$$(k = 1,\ 2,\ \cdots;\ u^{n+1,0} = u^n,\ v^{n+1,0} = v^n,\ \varphi^{n+1,0} = \varphi^n).$$

Generally, it would converge after 6-8 steps or more of iteration. Obviously, it is time-consuming. In order to save the computing time, the iteration can be broken down after the 3^{rd} step, and an approximate solution to Eq. (4.1) is obtained:

$$\begin{cases} u^{n+1} = u^n - \tau\ L_1 u^{n+\frac{1}{2},3}, \\ v^{n+1} = v^n - \tau\ L_2 v^{n+\frac{1}{2},3}, \\ \varphi^{n+1} = \varphi^n - \tau\ A\varphi^{n+\frac{1}{2},3}, \end{cases} \tag{4.5}$$

which conserves the total mass and total potential vorticity naturally. This approximate solution, however, is unable to conserve the total energy and total enstrophy exactly due to the broken-down iteration. To make the total energy conserved, which is essential to ensure the computational stability, a flexible coefficient β_n is introduced to correct the approximate solution

$$\begin{cases} u^{n+1} = u^n - \beta_n\tau\ L_1 u^{n+\frac{1}{2},3}, \\ v^{n+1} = v^n - \beta_n\tau\ L_2 v^{n+\frac{1}{2},3}, \\ \varphi^{n+1} = \varphi^n - \beta_n\tau\ A\varphi^{n+\frac{1}{2},3}, \end{cases} \tag{4.6}$$

where β_n is determined by the following formula

$$a_n\tau^2\beta_n^2 - b_n\tau\beta_n + c_n = 0 \tag{4.7}$$

and

$$\begin{cases} a_n = \left(\left(L_1 u^{n+\frac{1}{2},3}\right)^2 + \left(L_2 v^{n+\frac{1}{2},3}\right)^2,\ A\varphi^{n+\frac{1}{2},3}\right), \\ b_n = \left(\left(L_1 u^{n+\frac{1}{2},3}\right)^2 + \left(L_2 v^{n+\frac{1}{2},3}\right)^2, \varphi^n\right) \\ \qquad + \left(2u^n L_1 u^{n+\frac{1}{2},3} + 2v^n L_2 v^{n+\frac{1}{2},3} + A\varphi^{n+\frac{1}{2},3}, A\varphi^{n+\frac{1}{2},3}\right), \\ c_n = 2\left[\left(L_1 u^{n+\frac{1}{2},3}, \varphi^n u^n\right) + \left(L_2 v^{n+\frac{1}{2},3}, \varphi^n v^n\right) + \left(A\varphi^{n+\frac{1}{2},3}, E^n\right)\right]. \end{cases} \tag{4.8}$$

Table 5.1: Temporal evolution of the five basic physical integrals simulated by the explicit multi-conservation finite-difference scheme (4.6)-(4.12).

Integration time (day)	Total energy ($\times 10^{12}$)	Total mass ($\times 10^{8}$)	Total enstrophy ($\times 10^{-10}$)	Total potential vorticity	Total angular momentum ($\times 10^{10}$)
0	7.61845750632341	1.75248645432350	2.00662133399936	0.0	5.82066986554327
10	7.61845750632917	1.75248645432350	2.00662133505423	-5.2×10^{-17}	5.82047589834651
20	7.61845750633071	1.75248645432347	2.00662133551195	1.2×10^{-17}	5.82054539378767
30	7.61845750633259	1.75248645432347	2.00662133589375	4.0×10^{-17}	5.82040192957590
40	7.61845750632594	1.75248645432348	2.00662133624930	-1.5×10^{-17}	5.82040766484019
50	7.61845750632493	1.75248645432348	2.00662133660218	-9.1×10^{-17}	5.82031332165226
60	7.61845750632882	1.75248645432348	2.00662133695058	-1.4×10^{-16}	5.82019225852469
70	7.61845750633128	1.75248645432348	2.00662133729320	-1.5×10-^{16}	5.82014362178185
80	7.61845750632727	1.75248645432348	2.00662133762288	-6.4×10^{-17}	5.82003761366708
90	7.61845750632524	1.75248645432348	2.00662133793366	-2.1×10^{-18}	5.82005239603197
100	7.61845750632337	1.75248645432349	2.00662133825054	-1.4×10^{-16}	5.81981469815627

Based on the equality (4.3), the coefficient c_n can be rewritten into

$$\begin{cases} c_n = \tau\ \tilde{c}_n, \\ \tilde{c}_n = (L_1 u^{n+\frac{1}{2},3},\ r^n_{\varphi u}) + (L_2 v^{n+\frac{1}{2},3},\ r^n_{\varphi v}) + (A\varphi^{n+\frac{1}{2},3},\ r^n_E), \end{cases} \tag{4.9}$$

where

$$\begin{cases} r^n_{\varphi u} = \varphi^n L_1 u^{n+\frac{1}{2},2} + u^{n+\frac{1}{2},3} A\varphi^{n+\frac{1}{2},2}, \\ r^n_{\varphi v} = \varphi^n L_2 v^{n+\frac{1}{2},2} + v^{n+\frac{1}{2},3} A\varphi^{n+\frac{1}{2},2}, \\ r^n_E = A\varphi^{n+\frac{1}{2},2} + u^{n+\frac{1}{2},3} L_1 u^{n+\frac{1}{2},2} + v^{n+\frac{1}{2},3} L_2 v^{n+\frac{1}{2},2}. \end{cases} \tag{4.10}$$

In this way, Eq. (4.7) becomes the following form

$$a_n \tau \beta_n^2 - b_n \beta_n + \tilde{c}_n = 0. \tag{4.11}$$

Considering the finiteness of β_n when $\tau \to 0$, the solution to (4.11) can only be determined by

$$\beta_n = \begin{cases} 2\tilde{c}_n / \left(b_n + \sqrt{b_n - 4a_n \tilde{c}_n \tau} \right) & b_n \geq 0, \\ 2\tilde{c}_n / \left(b_n - \sqrt{b_n - 4a_n \tilde{c}_n \tau} \right) & b_n < 0. \end{cases} \tag{4.12}$$

Now, an explicit scheme (4.6)-(4.12) with 3 conservation properties is constructed.

5. Numerical Tests and Discussion

To examine the multi-conservation properties, the scheme (4.6)-(4.12) is implemented for a 100-day integration using the four-wave Rossby-Haurwitz waves as the initial conditions. The integration region is the whole spherical surface, and the horizontal resolutions is set to be $4.5^0 \times 4.5^0$. Table 5.1 shows the temporal evolution of the five simulated physical integrals by the scheme. It is observed that the proposed scheme can conserve the total energy, total mass and total potential vorticity well. The results verify the three conservation properties of the scheme obtained theoretically. Among them, the total potential vorticity is best conserved, which is zero in the initial time and keeps being a small number near to zero with not less than 16-digit precision in the integration. The total mass on each day keeps 14 digits invariable, and the relative errors to the initial value are limited in a range of 10^{-15}. The values of total

Table 5.2: CPU time of the two schemes for 100-day integration on the IBM ThinkPad T41 Laptop.

Scheme	Explicit multi-conservation scheme (4.6)-(4.12)	Implicit multi-conservation scheme (4.1)
CPU time	261s	814s

energy on different days have 11 same digits. To our surprise, the total enstrophy is conserved in 9-digit precision, although this can not be proved theoretically. The total angular momentum also keeps 4 digits constant in the whole 100-day integration. Comparing with the performance of the implicit multi-conservation scheme (4.1) described by Wang and Ji [6], the explicit multi-conservation scheme (4.6)-(4.12) behaves very close to the implicit scheme, but requires much less computational time (see Table 5.2). These convince us that the explicit multi-conservation scheme is more practicable. It is expected that the proposed scheme can be generalized to design multi-conservation dynamical cores for atmospheric general circulation model (AGCM).

Acknowledgements. This work was jointly supported by the National Key Development and Planning Project for the Basic Research (973) (Grant No. 2005CB321703) and the Science Funds for Creative Research Groups (Grant No. 40221503). The author thanks Dr. Hongwei Yang for preparing the paper in Latex.

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