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A FEM-BEM FORMULATION FOR AN EXTERIOR QUASILINEAR ELLIPTIC PROBLEM IN THE PLANE*

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Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

In this paper, the finite element method and the boundary element method are combined to solve numerically an exterior quasilinear elliptic problem. Based on an appropriate transformation and the Fourier series expansion, the exact quasilinear artificial boundary conditions and a series of the corresponding approximations for the given problem are presented. Then the original problem is reduced into an equivalent problem defined in a bounded computational domain. We provide error estimate for the Galerkin method. Numerical results are presented to illustrate the theoretical results.

Mathematics subject classification: 65N38.

Key words: Boundary element, Coupling, Finite element, Quasilinear problems.

1. Introduction

In this paper, we consider a discretization procedure for an exterior quasilinear problem which combines the finite element method (FEM) and the boundary element method (BEM). This technique has been used to solve many linear problems, see, e.g., [6, 10, 15–17]. It has also been successfully generalized to nonlinear boundary value problems [4, 8, 9, 12]. In these extensions, the error analysis is often given when the coefficients satisfy conditions that make the nonlinear operator strongly monotone and Lipschitz continuous, see, e.g., [4, 9]. The advantage in this case is that Céa's lemma is satisfied. When these conditions do not hold, Xu [13] provides a useful tool by linearizing the nonlinear partial differential equation at a given isolated solution and considering its finite element discretization. Meddahi [11] extends this approach and gives the error analysis. However, all the problems considered are subject to the assumptions that they are homogeneous and linear with constant coefficients outside a bounded domain. In this paper, we shall consider more general quasilinear problems on the exterior region and give the error estimates.

Let Ω_0 is a bounded and simple connected domain in $\mathbb{R}^2$ with sufficiently smooth boundary Γ_0. $\Omega := \mathbb{R}^2/\overline{\Omega_0}$. We consider continuous nonlinear functions α_{kl} and β_i: $\Omega \times \mathbb{R} \to \mathbb{R}$ $(i = 0, 1, 2; k, l = 1, 2)$ such that the derivatives $(\partial\beta_i/\partial s)$, $(\partial\alpha_{kl}/\partial s)$, $(\partial^2\beta_i/\partial s^2)$, $(\partial^2\alpha_{kl}/\partial s^2)$, $(\partial\beta_i/\partial x_j), (\partial\alpha_{kl}/\partial x_j)(j = 1, 2)$ are continuous in $\Omega \times \mathbb{R}$. We need to approximate a function u that satisfies

$$\begin{cases} -div(\alpha(x,u)\nabla u + \beta(x,u)) + \beta_0(x,u) = f(x) & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \Gamma_0, \\ u(x) = \mathcal{O}(1), & \text{when} \quad |x| \to +\infty, \end{cases} \tag{1.1}$$

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where $\beta(x,u)=(\beta_1(x,u),\beta_2(x,u))^T$, $\alpha(x,u)=(\alpha_{kl})_{k,l=1}^2$.

Some existence and uniqueness results for this type of problem are given in [5] under some conditions on the coefficients α,β_i. We will not consider such issues, but instead, we assume that (1.1) has at least one solution. Our main purpose is to provide artificial boundary conditions for general quasilinear problem and error estimates for an approximate solution obtained from a FEM-BEM discretization scheme.

Assume that the given function $\beta_0,\beta\in L^2(\Omega)$ and $f(x)\in L^2(\Omega)$ has compact support, i.e., there is a constant $R_0>0$, such that

$$supp\ \beta_0, supp\ \beta\subset\Omega_{R_0}:=\{x\in\mathbb{R}^2||x|\le R_0\},\qquad supp\ f(x)\subset\Omega_{R_0}.$$

Moreover, we assume that there exists constant $C_0>0$, such that

$$\xi^T\alpha(x,u)\xi\ge C_0|\xi|^2,\qquad\forall u\in\mathbb{R},\quad\forall\xi\in\mathbb{R}^2,\quad x\in\overline{\Omega}_{R_0}\tag{1.2}$$

$$\alpha(x,u)=\tilde{\alpha}(u),\qquad\text{when}\quad|x|\ge R_0\tag{1.3}$$

We introduce an artificial boundary

$$\Gamma_R=\{x\in\mathbb{R}^2||x|=R\}\quad\text{with}\ \ R\ge R_0.$$

Γ_R divides Ω into two regions, a bounded domain $\Omega_i=\{x\in\Omega||x|\le R\}$, and Ω_e which is the unbounded region exterior to Γ_R. Then the problem (1.1) can be rewritten in the coupled form:

$$\begin{cases}-div(\alpha(x,u)\nabla u+\beta(x,u))+\beta_0(x,u)=f(x) & \text{in}\quad\Omega_i,\\ u=0, & \text{on}\quad\Gamma_0,\end{cases}\tag{1.4}$$

$$\begin{cases}-div(\tilde{\alpha}(u)\nabla u)=0 & \text{in}\quad\Omega_e,\\ u(x)=\mathcal{O}(1), & \text{when}\ \ |x|\to+\infty,\end{cases}\tag{1.5}$$

$$u(x)\ \text{and}\ \tilde{\alpha}(u)\partial u/\partial n\ \text{are continuous on}\ \Gamma_R.\tag{1.6}$$

Obviously, if $\alpha(x,u)\equiv a$ when $|x|\ge R_0$, the problem (1.5) is simplified to a linear exterior elliptic problem [11].

We introduce the so-called Kirchhoff transformation

$$w(x)=\int_0^{u(x)}\tilde{\alpha}(\xi)d\xi,\qquad x\in\Omega_e\tag{1.7}$$

which gives

$$\nabla w=\tilde{\alpha}(u)\nabla u.\tag{1.8}$$

From (1.5) we have that w satisfies the following problem

$$\begin{cases}-\triangle w=0 & \text{in}\quad\Omega_e,\\ w(x)=\mathcal{O}(1), & \text{when}\ \ |x|\to+\infty.\end{cases}\tag{1.9}$$

Let W_p^m be the standard Sobolev spaces with norm $\|\cdot\|_{m,p,\Omega_i}$ and semi-norms $|\cdot|_{m,p,\Omega_i}$. For $p=2$, we denote $H^m(\Omega_i)=W_2^m$, $\|\cdot\|_{m,\Omega_i}=\|\cdot\|_{m,2,\Omega_i}$ and $|\cdot|_{m,\Omega_i}=|\cdot|_{m,2,\Omega_i}$.

The rest of this paper is organized as follows. In Section 2, we give the exact quasilinear artificial boundary condition on the artificial boundary, and present a new version of FEM-BEM formulation. In Section 3, the error analysis of the coupling method is given. Finally, Section 4 is devoted to numerical experiments to illustrate our theoretical results.

2. Exact Quasilinear Artificial Boundary Condition

Suppose that $w(x)$ is the solution of the problem (1.9). We have the Fourier expansion [16]:

$$w(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(\frac{R}{r}\Big)^n (a_n \cos n\theta + b_n \sin n\theta)$$

with

$$a_n = \frac{1}{\pi}\int_0^{2\pi} w(R,\theta)\cos n\theta d\theta, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} w(R,\theta)\sin n\theta d\theta, \quad n = 1,2,\cdots.$$

It is easy to show that

$$\frac{\partial w}{\partial r}(r,\theta)\Big|_{r=R} = -\frac{1}{R\pi}\sum_{n=1}^{\infty} n\int_0^{2\pi} w(R,\varphi)\cos n(\varphi-\theta)d\varphi. \tag{2.1}$$

It follows from (1.8) that

$$\frac{\partial w}{\partial r} = \tilde{\alpha}(u)\frac{\partial u}{\partial r}, \tag{2.2}$$

We then get the exact artificial boundary condition of u on the Γ_R:

$$\begin{aligned}&(\tilde{\alpha}(u)\frac{\partial u}{\partial r})\Big|_{\Gamma_R}\\ =& -\frac{1}{R\pi}\sum_{n=1}^{\infty}\int_0^{2\pi}\Big(\int_0^{u(R,\varphi)}\tilde{\alpha}(\xi)d\xi\Big) n\cos n(\varphi-\theta)d\varphi \triangleq -K_\infty(u(R,\theta)),\end{aligned} \tag{2.3}$$

where K_∞ is the natural integer operator. Then (1.1) is equivalent to the following problem:

$$\begin{cases} -div(\alpha(x,u)\nabla u + \beta(x,u)) + \beta_0(x,u) = f(x) & \text{in} \quad \Omega_i \\ u = 0, & \text{on} \quad \Gamma_0, \\ \tilde{\alpha}(u)\dfrac{\partial u}{\partial r} = -K_\infty(u(R,\theta)), & \text{on} \quad \Gamma_R. \end{cases} \tag{2.4}$$

Let us introduce the space

$$X := \{v \in H^1(\Omega_i); v|_{\Gamma_0} = 0\}.$$

We assume that the solution of problem (1.1) u satisfies

$$u|_{\Omega_i} \in X \cap W^2_{2+\epsilon}(\Omega_i), \qquad (0 < \epsilon).$$

Then boundary value problem (2.4) is equivalent to the following variational problem

$$Find \;\; u \in X, \; such \; that \;\; A(u,v) + B(u,v) = F(v), \qquad \forall v \in X, \tag{2.5}$$

where

$$\begin{aligned} A(u,v) &= \int_{\Omega_i}\alpha(x,u)\nabla u\cdot\nabla v dx + \int_{\Omega_i}\beta(x,u)\cdot\nabla v dx + \int_{\Omega_i}\beta_0(x,u)v dx,\\ B(u,v) &= \int_{\Gamma_R} K_\infty(u)v ds, \quad F(v) = \int_{\Omega_i} f(x)v(x)dx. \end{aligned}$$

In practice, we need to truncate the series in (2.3)

$$\begin{aligned}&(\tilde{\alpha}(u)\frac{\partial u}{\partial r})\Big|_{\Gamma_R}\\ &=-\frac{1}{R\pi}\sum_{n=1}^{N}\int_0^{2\pi}\Big(\int_0^{u(R,\varphi)}\tilde{\alpha}(\xi)d\xi\Big)n\cos n(\varphi-\theta)d\varphi\triangleq -K_N(u(R,\theta)),\end{aligned}\tag{2.6}$$

Then the approximate problem of (2.4) is

$$\begin{cases}-div(\alpha(x,u^N)\nabla u^N+\beta(x,u^N))+\beta_0(x,u^N)=f(x) & \text{in}\quad \Omega_i\\ u^N=0, & \text{on}\quad \Gamma_0,\\ \tilde{\alpha}(u^N)\dfrac{\partial u^N}{\partial r}=-K_N(u(R,\theta)), & \text{on}\quad \Gamma_R.\end{cases}\tag{2.7}$$

Problem (2.7) is equivalent to the following variational problem

$$\textit{Find}\ \ u^N\in X,\ \textit{such that}\ \ A(u^N,v)+B_N(u^N,v)=F(v),\qquad \forall v\in X.\tag{2.8}$$

Lemma 2.1. *The bilinear forms $B(u,v)$ and $B_N(u^N,v)$ are bounded, i.e., there exists a constant $C>0$, such that*

$$\begin{aligned}&|B(u,v)|\le C\|u\|_{1,\Omega_i}\|v\|_{1,\Omega_i},\qquad \forall u,v\in X,\\ &|B_N(u^N,v)|\le C\|u\|_{1,\Omega_i}\|v\|_{1,\Omega_i},\qquad \forall u,v\in X.\end{aligned}$$

Furthermore, $B(u,u)\ge C_0|v|_1^2$.

Proof. The proof may be found in [7]. □

We introduce the nonlinear form

$$\begin{aligned}&\bar{A}(u,v):=A(u,v)+B(u,v),\\ &\bar{A}_N(u^N,v):=A(u^N,v)+B_N(u^N,v).\end{aligned}$$

Then problems (2.5) and (2.8) are reduced to

$$\textit{Find}\ \ u\in X,\ \textit{such that}\ \ \bar{A}(u,v)=F(v),\qquad \forall v\in X,\tag{2.9}$$

and

$$\textit{Find}\ \ u^N\in X,\ \textit{such that}\ \ \bar{A}_N(u^N,v)=F(v),\qquad \forall v\in X.\tag{2.10}$$

Let us introduce the bilinear form $A'(u;\cdot,\cdot)$ and $A'_N(u;\cdot,\cdot)$ defined by

$$\begin{aligned}&A'(u;v,z)\\ &=\int_{\Omega_i}\frac{\partial\alpha}{\partial s}(x,u)v\nabla u\cdot\nabla z dx+\int_{\Omega_i}\alpha(x,u)\nabla v\cdot\nabla z dx+\int_{\Omega_i}\frac{\partial\beta}{\partial s}(x,u)\cdot\nabla z v dx\\ &\quad+\int_{\Omega_i}\frac{\partial\beta_0}{\partial s}(x,u)vz dx+\int_0^{2\pi}\int_0^{2\pi}\frac{\partial\tilde{\alpha}}{\partial s}(u)v\frac{\partial u}{\partial\varphi}(R,\varphi)\frac{\partial z}{\partial\theta}(R,\theta)\sum_{n=1}^{\infty}\frac{\cos n(\varphi-\theta)}{n\pi}d\theta d\varphi\\ &\quad+\int_0^{2\pi}\int_0^{2\pi}\tilde{\alpha}(u)\frac{\partial v}{\partial\varphi}(R,\varphi)\frac{\partial z}{\partial\theta}(R,\theta)\sum_{n=1}^{\infty}\frac{\cos n(\varphi-\theta)}{n\pi}d\theta d\varphi;\end{aligned}$$

$$
\begin{aligned}
&A'_N(u^N; v, z) \\
&= \int_{\Omega_i} \frac{\partial \alpha}{\partial s}(x, u^N) v \nabla u^N \cdot \nabla z dx + \int_{\Omega_i} \alpha(x, u^N) \nabla v \cdot \nabla z dx + \int_{\Omega_i} \frac{\partial \beta}{\partial s}(x, u^N) \cdot \nabla z v dx \\
&\quad + \int_{\Omega_i} \frac{\partial \beta_0}{\partial s}(x, u^N) v z dx + \int_0^{2\pi} \int_0^{2\pi} \frac{\partial \tilde{\alpha}}{\partial s}(u^N) v \frac{\partial u^N}{\partial \varphi}(R, \varphi) \frac{\partial z}{\partial \theta}(R, \theta) \sum_{n=1}^{N} \frac{\cos n(\varphi - \theta)}{n\pi} d\theta d\varphi \\
&\quad + \int_0^{2\pi} \int_0^{2\pi} \tilde{\alpha}(u^N) \frac{\partial v}{\partial \varphi}(R, \varphi) \frac{\partial z}{\partial \theta}(R, \theta) \sum_{n=1}^{N} \frac{\cos n(\varphi - \theta)}{n\pi} d\theta d\varphi,
\end{aligned}
$$

where

$$
\Big(\frac{\partial \beta}{\partial s}\Big)(x, u) = \Big(\Big(\frac{\partial \beta_1}{\partial s}\Big)(x, u), \Big(\frac{\partial \beta_2}{\partial s}\Big)(x, u)\Big)^T.
$$

Let X' be the dual of X. Notice that $A'(u; \cdot, \cdot)$ is bounded on $X \times X$ since the functions $(\partial \beta_i / \partial s)(\cdot, u(\cdot)), (i = 0, 1, 2)$ are continuous in Ω_i. Then there exists an operator $T : X \to X'$ such that

$$
(Tv, z) = A'(u; v, z), \qquad \forall v, z \in X. \tag{2.11}
$$

Lemma 2.2. *The bilinear form (Tv, v) defined by $A'(u; v, v)$ satisfies the following inequality:*

$$
(Tv, v) + K(\|v\|^2_{0,\Omega_i} + \|v\|^2_{1/2,\Gamma_R}) \geq \alpha_1 \|v\|^2_{1,\Omega_i}, \qquad \forall v \in X, \tag{2.12}
$$

where $K \geq 0$ is a sufficiently large constant and $\alpha_1 > 0$ is a constant.

Proof. We first observe that

$$
\begin{aligned}
&(Tv, v) + K\|v\|^2_{0,\Omega_i} \\
&= \int_{\Omega_i} \frac{\partial \alpha}{\partial s}(x, u) v \nabla u \cdot \nabla v dx + \int_{\Omega_i} \alpha(x, u) \nabla v \cdot \nabla v dx \\
&\quad + \int_{\Omega_i} \frac{\partial \beta}{\partial s}(x, u) \cdot \nabla v v dx + \int_{\Omega_i} (\frac{\partial \beta_0}{\partial s}(x, u) + K) v^2 dx \\
&\quad + \int_0^{2\pi} \int_0^{2\pi} \frac{\partial \tilde{\alpha}}{\partial s}(u) v \frac{\partial u}{\partial \varphi}(R, \varphi) \frac{\partial v}{\partial \theta}(R, \theta) \sum_{n=1}^{\infty} \frac{\cos n(\varphi - \theta)}{n\pi} d\theta d\varphi \\
&\quad + \int_0^{2\pi} \int_0^{2\pi} \tilde{\alpha}(u) \frac{\partial v}{\partial \varphi}(R, \varphi) \frac{\partial v}{\partial \theta}(R, \theta) \sum_{n=1}^{\infty} \frac{\cos n(\varphi - \theta)}{n\pi} d\theta d\varphi.
\end{aligned}
$$

By *Hölder* inequality and the continuous property of $\partial \alpha / \partial s, \partial \beta_i / \partial s (i = 0, 1, 2)$, it is easy to show that

$$
\begin{aligned}
&\Big| \int_{\Omega_i} \frac{\partial \alpha}{\partial s}(x, u) v \nabla u \cdot \nabla v dx \Big| \leq M_1 |v|_{1,\Omega_i} \|v\|_{0,\Omega_i}, \\
&\Big| \int_{\Omega_i} \frac{\partial \beta}{\partial s}(x, u) \cdot \nabla v v dx \Big| \leq M_2 |v|_{1,\Omega_i} \|v\|_{0,\Omega_i}, \\
&\Big| \int_0^{2\pi} \int_0^{2\pi} \frac{\partial \tilde{\alpha}}{\partial s}(u) v \frac{\partial u}{\partial \varphi}(R, \varphi) \frac{\partial v}{\partial \theta}(R, \theta) \cdot \sum_{n=1}^{\infty} \frac{\cos n(\varphi - \theta)}{n\pi} d\theta d\varphi \Big| \leq M_3 \|v\|^2_{1/2}.
\end{aligned}
$$

In virtue of Lemma 2.1, we obtain

$$\int_0^{2\pi}\int_0^{2\pi}\tilde{\alpha}(u)\frac{\partial v}{\partial\varphi}(R,\varphi)\frac{\partial v}{\partial\theta}(R,\theta)\sum_{n=1}^{\infty}\frac{\cos n(\varphi-\theta)}{n\pi}d\theta d\varphi\geq 0.$$

By the arithmetic-geometric mean inequality, we can obtain

$$\begin{aligned}&(Tv,v)+K\|v\|^2_{0,\Omega_i}\\ &\geq c_0|v|^2_{1,\Omega_i}-M_1|v|_{1,\Omega_i}\|v\|_{0,\Omega_i}-M_2|v|_{1,\Omega_i}\|v\|_{0,\Omega_i}\\ &\quad+\int_{\Omega_i}(\frac{\partial\beta_0}{\partial s}(x,u)+K)v^2dx-M_3\|v\|^2_{1/2,\Gamma_R}\\ &\geq\frac{c_0}{2}|v|^2_{1,\Omega_i}+\left(C_3+K-\frac{M_1^2+M_2^2}{C_0}\right)\|v\|^2_{0,\Omega_i}-M_3\|v\|^2_{1,\Omega_i},\end{aligned}$$

where

$$C_3:=essinf\left\{\frac{\partial\beta_0}{\partial s}(x,u):x\in\Omega_i\right\}.\tag{2.13}$$

Consequently, we conclude that

$$(Tv,v)+K(\|v\|^2_{0,\Omega_i}+\|v\|^2_{1/2,\Gamma_R})\geq\alpha_1\|v\|^2_{1,\Omega_i},\qquad\forall v\in X$$

provided that

$$K\geq\max\left\{\frac{C_0}{2}+\frac{M_1^2+M_2^2}{C_0}-C_3,M_3\right\}.$$

It is noted that K does not need not be positive if $C_3>0$. □

Let I: $X\to X'$ be the canonical injection. As X is compactly embedded in $L^2(\Omega_i)$, we deduce that operator J: $X\to X'$ defined by $J(v)=(I(v),0)$ is also compact. Thus the Fredholm alternative applies for T. We assume here that

$$A'(u;v,z)=0,\qquad\forall z\in X\Rightarrow v=0.\tag{2.14}$$

This implies that $T:X\to X'$ is an isomorphism.

3. Finite Element Approximation of the Coupling Method

Suppose ξ_h is a regular and quasi-uniform triangulation on Ω_i, $k\in\xi_h$ is a (curved) triangle. Denote h the maximum side of the triangles. Let

$$X_h=\{x_h\in C^0(\Omega_i),x_h|_k\ \ is\ \ a\ \ linear\ \ polynomial,\forall k\in\xi_h\}\subset X.\tag{3.1}$$

We consider the approximation problem of (2.9) and (2.10)

$$\begin{cases}Find\ \ u_h\in X_h,\ such\ that\\ \bar{A}(u_h,v_h)=F(v_h),\qquad\qquad\forall v_h\in X_h,\end{cases}\tag{3.2}$$

and

$$\begin{cases}Find\ \ u_h^N\in X_h,\ such\ that\\ \bar{A}_N(u_h^N,v_h)=F(v_h),\qquad\qquad\forall v_h\in X_h.\end{cases}\tag{3.3}$$

Theorem 10.1.2 of [2] assures that under the conditions (2.9), (2.12) and (2.14), there exists an $h_0 \in (0,1]$, such that the following inf-sup condition is satisfied:

$$\sup_{z\in X_h} \frac{A'(u;v,z)}{\|z\|_{1,\Omega_i}} \geq \alpha_1 \|v\|_{1,\Omega_i}, \qquad \forall v \in X_h \tag{3.4}$$

for some constant $\alpha_1 > 0$ independent of h $(h < h_0)$.

We define the Galerkin projection with respect to $A'(u;\cdot,\cdot)$, $P_h : X \to X_h$

$$A'(u, P_h v, z) = A'(u, v, z), \qquad \forall z \in X_h.$$

It is easy to deduce that the operator P_h satisfies

$$\|v - P_h v\|_{1,p,\Omega_i} \leq C \inf_{v_h \in X_h} \|v - v_h\|_{1,p,\Omega_i} \leq Ch^\sigma, \qquad 2 \leq p \leq \infty. \tag{3.5}$$

Lemma 3.1. *$u_h^N \in X_h$ is a solution of* (3.3) *if and only if the following equation is satisfied*

$$A'_N(u^N; u^N - u_h^N, v) = R(u^N; u_h^N, v), \qquad \forall v \in X_h,$$

where

$$\begin{aligned}
&R(u^N; u_h^N, v) \\
:= &\int_{\Omega_i} \Big(\int_0^1 \Big[(\frac{\partial^2 \alpha}{\partial s^2})(x, w_h^N) \nabla w_h^N \cdot \nabla v \Big] (1-t) dt \Big) \cdot (d_h^N)^2 dx \\
&+ 2 \int_{\Omega_i} \Big(\int_0^1 \Big[(\frac{\partial \alpha}{\partial s})(x, w_h^N) \nabla (d_h^N) \cdot \nabla v \Big] (1-t) dt \Big)\, d_h^N dx \\
&+ \int_{\Omega_i} \Big(\int_0^1 \Big[(\frac{\partial^2 \beta}{\partial s^2})(x, w_h^N) \cdot \nabla v + (\frac{\partial^2 \beta_0}{\partial s^2})(x, w_h^N) v \Big] (1-t) dt \Big) (d_h^N)^2 dx \\
&+ \int_0^{2\pi} \int_0^{2\pi} \Big(\int_0^1 \Big[(\frac{\partial^2 \tilde{\alpha}}{\partial s^2}) w_h^N \frac{\partial w_h^N}{\partial \varphi} \frac{\partial v}{\partial \theta} \cdot \sum_{n=1}^N \frac{\cos n(\varphi - \theta)}{n\pi} \Big] (1-t) dt \Big) (d_h^N)^2 d\theta d\varphi \\
&+ 2 \int_0^{2\pi} \int_0^{2\pi} \Big(\int_0^1 \Big[(\frac{\partial \tilde{\alpha}}{\partial s}) w_h^N \frac{\partial d_h^N}{\partial \varphi} \frac{\partial v}{\partial \theta} \cdot \sum_{n=1}^N \frac{\cos n(\varphi - \theta)}{n\pi} \Big] (1-t) dt \Big)\, d_h^N d\theta d\varphi,
\end{aligned}$$

and $w_h^N = u^N + t(u_h^N - u^N)$, $d_h^N = u_h^N - u^N$.

Proof. Let $\eta(t) := \bar{A}_N(u^N + t(u_h^N - u^N), v)$. The desired result follows from the identity

$$\eta(1) = \eta(0) + \eta'(0) + \int_0^1 \eta''(t)(1-t) dt$$

and the fact that $\bar{A}_N(u^N, v) = \bar{A}_N(u_h^N, v) = F(v)$, $\forall v \in X_h$. □

Lemma 3.2. *Let $M_h := \{v \in X_h; \|v\|_{1,\infty,\Omega_i} \leq 1 + \|u^N\|_{1,\infty,\Omega_i}\}$. Then there exists a constant $C > 0$ independent of h, such that*

$$|R(u^N; v, z)| \leq C\big(\|u^N - v\|_{1,\Omega_i}^2 + \|u^N - v\|_{1,\Omega_i}\big)\|z\|_{1,\Omega_i}, \qquad \forall v \in M_h, \qquad \forall z \in X_h.$$

Theorem 3.1. *Let $u \in X \cap W^2_{2+\epsilon}(\Omega_i)$ be a solution of* (1.1) *with* $0 < \epsilon$, *and assume that $u|_{\Gamma_R} \in H^{3/2}(\Gamma_R)$ and* (2.14) *is satisfied. If h is sufficiently small, then the finite element equation* (3.3) *has a solution $u_h^N \in X_h$ satisfying*

$$\|u - u_h^N\|_{1,\Omega_i} \leq C\Big(h^\sigma + \frac{1}{(N+1)}(\frac{R_0}{R})^{N+1}\|u\|_{3/2,\Gamma_{R_0}}\Big), \tag{3.6}$$

where C is a constant independent of h and N. Furthermore, there exists a constant $\eta > 0$ such that u_h^N is the only solution satisfying

$$\|u - u_h^N\|_{1,\infty,\Omega_i} \leq \eta. \tag{3.7}$$

Proof. We shall divide the proof of Theorem 3.1 into 6 steps.
Step 1. For $\forall\, u^N \in X$, assume that

$$w^N(r,\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(\frac{R_0}{r})^n(a_n \cos n\varphi + b_n \sin n\varphi), \qquad \forall r \geq R_0,$$
$$v(R,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n \cos n\theta + d_n \sin n\theta).$$

Using (1.2), we obtain

$$\begin{aligned}
&|B(u^N,v) - B_N(u^N,v)| \\
=&\Big|\int_0^{2\pi}\int_0^{2\pi}\frac{\partial w^N}{\partial\varphi}(R,\varphi)\frac{\partial v}{\partial\theta}(R,\theta)\sum_{n=N+1}^{\infty}\frac{\cos n(\varphi-\theta)}{n\pi}d\theta d\varphi\Big| \\
=&\Big|\sum_{n=N+1}^{\infty}(\frac{R_0}{R})^n n\pi(a_n c_n + b_n d_n)\Big| \\
\leq&\pi(\frac{R_0}{R})^{N+1}\Big[\sum_{n=N+1}^{\infty}n(a_n^2+b_n^2)\Big]^{1/2}\Big[\sum_{n=N+1}^{\infty}n(c_n^2+d_n^2)\Big]^{1/2} \\
\leq&\pi\frac{1}{(N+1)}(\frac{R_0}{R})^{N+1}\Big[\sum_{n=N+1}^{\infty}n^3(a_n^2+b_n^2)\Big]^{1/2}\Big[\sum_{n=N+1}^{\infty}n(c_n^2+d_n^2)\Big]^{1/2} \\
\leq&\frac{C}{(N+1)}(\frac{R_0}{R})^{N+1}\|u\|_{3/2,\Gamma_R}\|v\|_{1,\Omega_i},
\end{aligned}$$

where the constant C is independent of N. It follows from (2.8) that

$$\bar{A}(u^N,v) = a(u^N,v) + B(u^N,v) = F(v) + B(u^N,v) - B_N(u^N,v).$$

Let $\eta(t) = \bar{A}(u + t(u^N - u), v)$. We obtain

$$\int_0^1 A'(u + t(u^N - u); u^N - u, v)dt = \bar{A}(u^N,v) - \bar{A}(u,v).$$

Using (2.9), (2.12), (2.14) and [2] gives

$$\begin{aligned}
\|u - u^N\|_{1,\Omega_i} &\leq C\sup_{v\in X}\frac{\int_0^1 A'(u+t(u^N-u);u^N-u,v)dt}{\|v\|_{1,\Omega_i}} \\
&\leq C\frac{|B(u^N,v) - B_N(u^N,v)|}{\|v\|_{1,\Omega_i}} \leq \frac{C}{(N+1)}(\frac{R_0}{R})^{N+1}\|u\|_{3/2,\Gamma_R}.
\end{aligned}$$

Step 2. We define the nonlinear mapping $\phi : X_h \to X_h$ as follows. Given $v \in X_h$, $\phi(v)$ is the unique solution of

$$A'(u, \phi(v), z) = A'(u; u, z) - R(u, v, z), \qquad \forall z \in X_h. \tag{3.8}$$

Therefore,

$$A'(u; \phi(v) - \phi(v_n), z) = R(u; v_n, z) - R(u; v, z). \tag{3.9}$$

Combining this with (3.4), we obtain the continuity of this operator. i.e.,

$$\lim_{v_n \to v} \phi(v_n) = \phi(v).$$

Step 3. We define the set

$$B_h := \{v \in X_h : \|v - P_h u^N\|_{1,\infty,\Omega_i} \le h^\sigma\}.$$

For any $v \in B_h$,

$$\|v\|_{1,\infty,\Omega_i} \le \|u^N - v\|_{1,\infty,\Omega_i} + \|u^N\|_{1,\infty,\Omega_i}, \tag{3.10}$$
$$\|u^N - v\|_{1,\infty,\Omega_i} \le \|u^N - P_h u^N\|_{1,\infty,\Omega_i} + \|P_h u^N - v\|_{1,\infty,\Omega_i}, \tag{3.11}$$
$$\|u^N - P_h u^N\|_{1,\infty,\Omega_i} \le \|u^N - \pi_h u^N\|_{1,\infty,\Omega_i} + \|\pi_h u^N - P_h u^N\|_{1,\infty,\Omega_i}. \tag{3.12}$$

Now we use the fact that ξ_h is quasi-uniform to obtain the following inverse inequality [14]

$$\|w\|_{1,\infty,\Omega_i} \le C(\log \frac{1}{h})^{1/2} \|w\|_{1,\Omega_i}, \qquad \forall w \in X_h. \tag{3.13}$$

Combining this with the definition of B_h, Lemma 2.2, and (3,5), we obtain

$$\|u^N - v\|_{1,\infty,\Omega_i} \le 1.$$

This implies that $v \in M_h$.

Step 4. By the definition of P_h, (3.8) can be write as follows:

$$A'(u^N, \phi(v) - P_h u^N, z) = -R(u^N, v, z), \qquad \forall z \in X_h,$$

and

$$\begin{aligned}
&\|\phi(v) - P_h u^N\|_{1,\Omega_i} \\
\le& C \sup_{z \in X_h} \frac{A'(u, \phi(v) - P_h u^N, z)}{\|z\|_{1,\Omega_i}} \le C(\|u^N - v\|^2_{1,\Omega_i} + \|u^N - v\|_{1,\Omega_i}) \\
\le& C\{\|u^N - P_h u^N\|^2_{1,\Omega_i} + \|P_h u^N - v\|^2_{1,\Omega_i} + \|u^N - P_h u^N\|_{1,\Omega_i} + \|P_h u^N - v\|_{1,\Omega_i}\} \\
\le& h^\sigma.
\end{aligned}$$

This implies that $\phi : B_h \to B_h$.

Step 5. It follows from Brouwer's fixed point theorem that there exists $u_h^N \in X_h$, such that $\phi(u_h^N) = u_h^N$. Due to the Lemma 3.2, we deduce that u_h^N is a solution of (3.3). Furthermore,

$$\|u^N - u_h^N\|_{1,\Omega_i} \le \|u^N - P_h u^N\|_{1,\Omega_i} + \|P_h u^N - u_h^N\|_{1,\Omega_i} \le Ch^\sigma.$$

Table 4.1: The errors with N=10 for Example 1.

$M1$	M	$e_0(h,N)$	$ratio$	$e_1(h,N)$	$ratio$	$e_\infty(h,N)$
2	8	2.4160E-01	–	1.8383	–	2.2109E-01
4	16	6.5019E-02	3.7159	8.9936E-01	2.0440	6.0875E-02
8	32	1.6909E-02	3.8450	4.4990E-01	1.9989	1.7717E-02
16	64	4.3481E-03	3.8888	2.2534E-01	1.9965	5.4231E-03
32	128	1.2043E-03	3.6102	1.1444E-01	1.9690	2.6581E-03

Table 4.2: The errors with N=5 for Example 2.

$M1$	M	$e_0(h,N)$	$ratio$	$e_1(h,N)$	$ratio$	$e_\infty(h,N)$
2	8	1.6273E-01	–	6.1288E-01	–	9.8528E-02
4	16	4.6705E-02	3.4843	3.0000E-01	2.0429	2.9869E-02
8	32	1.2267E-02	3.8072	1.4831E-01	2.0227	8.4149E-03
16	64	3.1498E-03	3.8946	7.3899E-02	2.0069	2.4804E-03

Here we use (3.5) and the fact that $u_h^N \in B_h$.

$$\begin{aligned}\|u-u_h^N\|_{1,\Omega_i} &\le \|u-u^N\|_{1,\Omega_i} + \|u^N-u_h^N\|_{1,\Omega_i}\\ &\le C\Big(h^\sigma + \frac{1}{(N+1)}(\frac{R_0}{R})^{N+1}\|u\|_{3/2,\Gamma_{R_0}}\Big).\end{aligned}$$

Step 6. Let u_h^N and $\tilde{u}_h^N$ be two solutions of (3.3) satisfying (3.7). The same technique given in Step 1 can be easily reproduced here to prove a local uniqueness result if we let $\eta(t) = \bar{A}_N(u_h^N + t(\tilde{u}_h^N - u_h^N), v)$. The proof of this theorem is complete by combining the above steps. □

4. Numerical Examples

In this section, we present some numerical experiments to confirm our theoretical results.

Example 1. We take $\Omega_0 = \{(x,y)\in\mathbb{R}^2 : r=\sqrt{x^2+y^2}\le 1\}$ and the artificial boundary Γ_R is the circle centered at the origin of radius 2. We present results of numerical experiments for problems (1.1) when $\beta_0\equiv 0, \beta\equiv 0$ and

$$\alpha(x,u) = \begin{cases} 4-r^2+\dfrac{1}{1+u^2}, & 1\le r\le 2,\\ \dfrac{1}{1+u^2}, & r>2,\end{cases} \tag{4.1}$$

$$f(x) = \begin{cases} -\Big(1+\tan^2(\dfrac{y}{r^2})\Big)\Big(\dfrac{2y}{r^2}+\dfrac{2(4-r^2)}{r^4}\tan(\dfrac{y}{r^2})\Big), & 1\le r\le 2,\\ 0, & r>2.\end{cases} \tag{4.2}$$

The exact solution of Example 1 is $u(x)=\tan(y/r^2)$. Furthermore, we let

$$\begin{aligned}&\Delta r = \frac{1}{M1},\quad \Delta\theta = \frac{2\pi}{M},\quad e_0(h,N) = \|u-u_h^N\|_{L^2(\Omega_i)};\\ &e_1(h,N) = \|u-u_h^N\|_{H^1(\Omega_i)};\ \ e_\infty(h,N) = \|u-u_h^N\|_{L^\infty(\Omega_i)}.\end{aligned}$$

The numerical results are given in Fig. 4.1(a), Fig. 4.2(a) and Table 4.1.

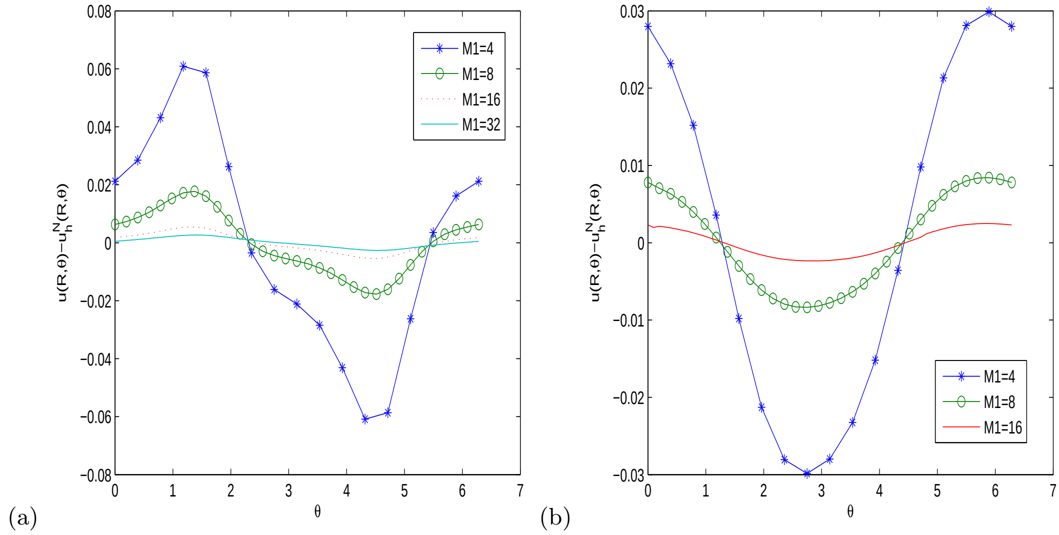

Fig. 4.1. The errors on artificial boundary for with different mesh sizes. (a): Example 1 with $N = 10$; (b): Example 2 with $N = 5$.

Example 2. We take $\Omega_0 = \{(x, y) \in \mathbb{R}^2 : r = \sqrt{x^2 + y^2} \le 1.5\}$ and the artificial boundary Γ_R is the circle centered at the origin of radius 3. We present results of numerical experiments for problems (1.1) when $\beta_0 \equiv 0, \beta \equiv 0$ and

$$\alpha(x, u) = \begin{cases} 9 - r^2 + \dfrac{1}{\sqrt{1-u^2}}, & 1.5 \le r \le 3, \\ \dfrac{1}{\sqrt{1-u^2}}, & r > 3, \end{cases} \tag{4.3}$$

$$f(x) = \begin{cases} \dfrac{9 - x^2 - y^2}{(x^2+y^2)^2} \sin(\dfrac{x}{x^2+y^2}) - \dfrac{2x}{x^2+y^2} \cos(\dfrac{x}{x^2+y^2}), & 1.5 \le r \le 3, \\ 0, & r > 3. \end{cases} \tag{4.4}$$

The exact solution of Example 2 is $u(x) = \sin(x/r^2)$. The numerical results are given in Fig. 4.1(b), Fig. 4.2 (b) and Table 4.2.

It is observed from the numerical results that increasing the order of the artificial boundary condition or refining the mesh can reduce the numerical errors. When a finer mesh cannot produce a much more accurate numerical solution, the error originated from the series truncating is dominating. These observations are in agreement with the error analysis we obtain. The numerical results above show that the coupling BEM and FEM technique can be used to deal with the quasilinear problems effectively.

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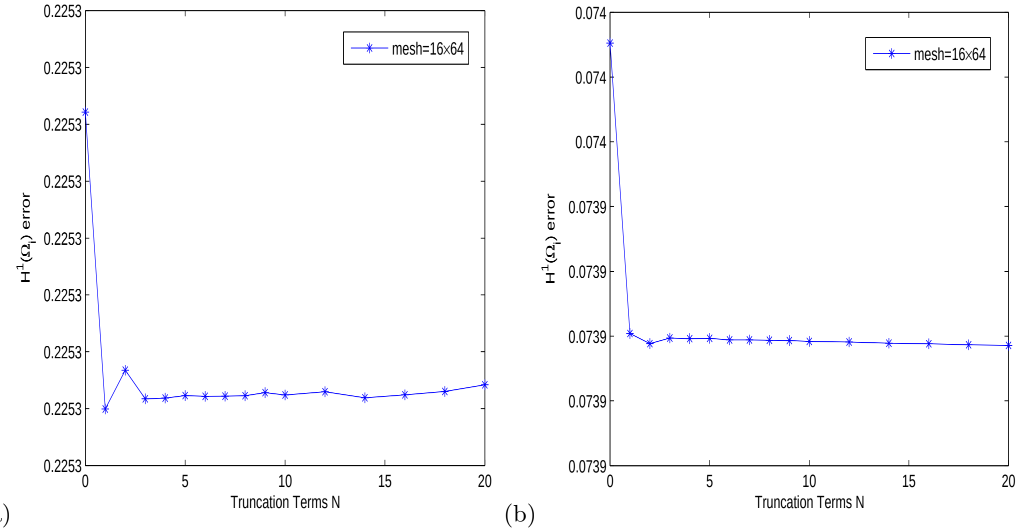

(a) (b)

Fig. 4.2. $H^1(\Omega_i)$ errors against N. (a): Example 1; (b): Example 2.

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