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ON MAXWELL EQUATIONS WITH THE TRANSPARENT BOUNDARY CONDITION*

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Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

In this paper we show the well-posedness and stability of the Maxwell scattering problem with the transparent boundary condition. The proof depends on the well-posedness of the time-harmonic Maxwell scattering problem with complex wave numbers which is also established.

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1. Introduction

We consider the electromagnetic scattering problem with the perfect conducting boundary condition on the obstacle

$$\varepsilon\frac{\partial \mathbf{E}}{\partial t} - \nabla\times\mathbf{H} = \mathbf{J} \quad \text{in } [\mathbb{R}^3\backslash\bar{D}]\times(0,T), \tag{1.1}$$

$$\mu\frac{\partial \mathbf{H}}{\partial t} + \nabla\times\mathbf{E} = 0 \quad \text{in } [\mathbb{R}^3\backslash\bar{D}]\times(0,T), \tag{1.2}$$

$$\mathbf{n}\times\mathbf{E} = 0 \quad \text{on } \Gamma_D\times(0,T), \tag{1.3}$$

$$\mathbf{E}|_{t=0} = \mathbf{E}_0, \quad \mathbf{H}|_{t=0} = \mathbf{H}_0. \tag{1.4}$$

Here $D\subset\mathbb{R}^3$ is a bounded domain with Lipschitz boundary Γ_D, $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, and $\mathbf{n}$ is the unit outer normal to Γ_D. The applied current $\mathbf{J}$ and the initial conditions $\mathbf{E}_0, \mathbf{H}_0$ are assumed to be supported in the circle $B_R = \{x\in\mathbb{R}^2 : |x| < R\}$ for some $R > 0$. The electric permittivity ε and magnetic permeability μ are assumed to be positive constants. We remark that the results in this paper can be easily extended to solve scattering problems with other boundary conditions such as the impedance boundary condition on Γ_D.

One of the fundamental problems in the efficient simulation of the wave propagation is the reduction of the exterior problem which is defined in the unbounded domain to the problem in the bounded domain. The transparent boundary condition plays an important role in the construction of various approximate absorbing boundary conditions for the simulation of wave

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propagation, see the review papers Givoli [5], Tsynkov [11], Hagstrom [7] and the references therein. The purpose of this paper is to study the transparent boundary condition for Maxwell scattering problems.

For any $s \in \mathbb{C}$ such that $\mathrm{Re}\,(s) > 0$, we let $\mathbf{E}_{\mathrm{L}} = \mathscr{L}(\mathbf{E})$ and $\mathbf{H}_{\mathrm{L}} = \mathscr{L}(\mathbf{H})$ be respectively the Laplace transform of $\mathbf{E}$ and $\mathbf{H}$ in time

$$\mathbf{E}_{\mathrm{L}}(x,s) = \int_0^\infty e^{-st}\mathbf{E}(x,t)dt, \qquad \mathbf{H}_{\mathrm{L}}(x,s) = \int_0^\infty e^{-st}\mathbf{H}(x,t)dt.$$

Since $\mathscr{L}(\partial_t \mathbf{E}) = s\mathbf{E}_{\mathrm{L}} - \mathbf{E}_0$ and $\mathscr{L}(\partial_t \mathbf{H}) = s\mathbf{H}_{\mathrm{L}} - \mathbf{H}_0$, by taking the Laplace transform of (1.1) and (1.2) we get

$$\varepsilon(s\mathbf{E}_{\mathrm{L}} - \mathbf{E}_0) - \nabla\times\mathbf{H}_{\mathrm{L}} = \mathbf{J}_{\mathrm{L}} \quad \text{in } \mathbb{R}^3\backslash\bar{D}, \tag{1.5}$$

$$\mu(s\mathbf{H}_{\mathrm{L}} - \mathbf{H}_0) + \nabla\times\mathbf{E}_{\mathrm{L}} = 0 \quad \text{in } \mathbb{R}^3\backslash\bar{D}, \tag{1.6}$$

where $\mathbf{J}_{\mathrm{L}} = \mathscr{L}(\mathbf{J})$. Because $\mathbf{J}$, $\mathbf{E}_0$, $\mathbf{H}_0$ are supported inside $B_R = \{x\in\mathbb{R}^2 : |x| < R\}$, we know that $\mathbf{E}_{\mathrm{L}}$ satisfies the time-harmonic Maxwell equation outside B_R

$$\nabla\times\nabla\times\mathbf{E} - k^2\mathbf{E}_{\mathrm{L}} = \mathbf{0} \quad \text{in } \mathbb{R}^{\mathbf{3}}\backslash\bar{\mathbf{D}},$$

where the wave number $k = \mathbf{i}\sqrt{\varepsilon\mu}s$ so that $\mathrm{Im}\,(k) = \sqrt{\varepsilon\mu}s_1 > 0$. Let $G_e : \mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_{\mathrm{R}}) \to \mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_{\mathrm{R}})$ be the Dirichlet to Neumann operator

$$G_e(\hat{\mathbf{x}}\times\mathbf{E}_{\mathrm{L}}) = \frac{1}{\mathbf{i}k}\hat{\mathbf{x}}\times(\nabla\times\mathbf{E}_{\mathrm{L}}) = -\frac{1}{\sqrt{\varepsilon\mu}}\frac{1}{s}\hat{\mathbf{x}}\times(\nabla\times\mathbf{E}_{\mathrm{L}}).$$

By using (1.6) we have

$$G_e(\hat{\mathbf{x}}\times\mathbf{E}_{\mathrm{L}}) = \sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{x}}\times\mathbf{H}_{\mathrm{L}} \quad \text{on } \Gamma_R. \tag{1.7}$$

For $\hat{\mathbf{x}}\times\mathbf{E}_{\mathrm{L}}|_{\Gamma_R} = \sum_{n=1}^\infty\sum_{m=-n}^n a_{mn}\mathbf{U}_n^m(\hat{\mathbf{x}}) + b_{mn}\mathbf{V}_n^m(\hat{\mathbf{x}})$, we know that (cf., e.g., in Monk [9] and also the discussion in Section 2)

$$G_e(\hat{\mathbf{x}}\times\mathbf{E}_{\mathrm{L}}) = \sum_{n=1}^\infty\sum_{m=-n}^n \frac{-\mathbf{i}kRb_{mn}h_n^{(1)}(kR)}{z_n^{(1)}(kR)}\mathbf{U}_n^m + \frac{a_{mn}z_n^{(1)}(kR)}{\mathbf{i}kRh_n^{(1)}(kR)}\mathbf{V}_n^m,$$

where $\mathbf{U}_n^m, \mathbf{V}_n^m$ are the vector spherical harmonics, $h_n^{(1)}(z)$ is the spherical Hankel function of the first order of order n, and $z_n^{(1)}(z) = h_n^{(1)}(z) + zh_n^{(1)'}(z)$.

By taking the inverse Laplace transform of (1.7) we obtain the following transparent boundary condition for the electromagnetic scattering problems

$$\sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{x}}\times\mathbf{H} = (\mathscr{L}^{-1}\circ G_e\circ\mathscr{L})(\hat{\mathbf{x}}\times\mathbf{E}|_{\Gamma_R}) \quad \text{on } \Gamma_R, \tag{1.8}$$

where

$$\begin{aligned}
&(\mathscr{L}^{-1}\circ G_e\circ\mathscr{L})(\hat{\mathbf{x}}\times\mathbf{E}|_{\Gamma_R}) \\
&= \sum_{n=1}^\infty\sum_{m=-n}^n \left[\mathscr{L}^{-1}\left(\frac{\sqrt{\varepsilon\mu}sRh_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}{z_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}\right) * b_{mn}(R,t)\right]\mathbf{U}_n^m \\
&\qquad - \left[\mathscr{L}^{-1}\left(\frac{z_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}{\sqrt{\varepsilon\mu}sRh_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}\right) * a_{mn}(R,t)\right]\mathbf{V}_n^m,
\end{aligned} \tag{1.9}$$

with

$$a_{mn}(R,t) = \int_{\Gamma_R} (\hat{\mathbf{x}} \times \mathbf{E}) \cdot \mathbf{U}_n^m d\hat{\mathbf{x}}, \quad b_{mn}(R,t) = \int_{\Gamma_R} (\hat{\mathbf{x}} \times \mathbf{E}) \cdot \mathbf{V}_n^m d\hat{\mathbf{x}}.$$

The objective of this paper is to prove the well-posedness and stability of the system (1.1)-(1.4) with the boundary condition (1.8). The proof depends on the abstract inversion theorem of the Laplace transform and the a priori estimate for the time-harmonic Maxwell scattering problem with complex wave number which seems to be new and is of independent interest. In Lax and Phillips [8], the scattering problem of the wave equation is studied by using the semigroup theory of operators in the absence of the source function. We remark that the well-posedness of scattering problems in the frequency domain is well-known for real wave numbers (cf., e.g., Colton and Kress [2], Nédélec [10], and Monk [9]).

2. The Time-Harmonic Maxwell Equation with Complex Wave Numbers

In this section we consider the following time-harmonic Maxwell scattering problem with complex wave numbers

$$\nabla \times \nabla \times \mathbf{E} - k^2 \mathbf{E} = \mathbf{J}_s \quad \text{in } \mathbb{R}^3 \backslash \bar{D}, \tag{2.1}$$

$$\mathbf{n} \times \mathbf{E} = 0 \quad \text{on } \Gamma_D, \tag{2.2}$$

$$|\mathbf{x}| \left[(\nabla \times \mathbf{E}) \times \hat{\mathbf{x}} - \mathbf{i} k \mathbf{E} \right] \to 0, \quad \text{as } |\mathbf{x}| \to \infty. \tag{2.3}$$

We assume the wave number k is complex such that $\mathrm{Im}\,(k) > 0$. The applied current $\mathbf{J}_s$ is assumed to be support inside some ball B_R.

We first recall the series solution of the scattering problem (2.1)-(2.3) outside the ball B_R by following the development in Monk [9]. Let $Y_n^m(\hat{\mathbf{x}})$, $m = -n, \cdots, n$, $n = 1, 2, \cdots$, be the *spherical harmonics* which satisfies

$$\Delta_{\partial B_1} Y_n^m(\hat{\mathbf{x}}) + n(n+1) Y_n^m(\hat{\mathbf{x}}) = 0 \quad \text{on } \partial B_1, \tag{2.4}$$

where

$$\Delta_{\partial B_1} = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2}$$

is the Laplace-Beltrami operator for the surface of the unit sphere ∂B_1. The set of all spherical harmonics $\{Y_n^m(\hat{\mathbf{x}}) : m = -n, \cdots, n, n = 1, 2, \cdots\}$ forms a complete orthonormal basis of $L^2(\partial B_1)$.

Denote the *vector spherical harmonics*

$$\mathbf{U}_n^m = \frac{1}{\sqrt{n(n+1)}} \nabla_{\partial B_1} Y_n^m, \qquad \mathbf{V}_n^m = \hat{\mathbf{x}} \times \mathbf{U}_n^m, \tag{2.5}$$

where

$$\nabla_{\partial B_1} Y_n^m = \frac{\partial Y_n^m}{\partial\theta} \mathbf{e}_\theta + \frac{1}{\sin\theta} \frac{\partial Y_n^m}{\partial\phi} \mathbf{e}_\phi,$$

and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ are the unit vectors of the spherical coordinates. The set of all vector spherical harmonics $\{\mathbf{U}_n^m, \mathbf{V}_n^m : m = -n, \cdots, n, n = 1, 2, \cdots\}$ forms a complete orthonormal basis of $\mathbf{L}_t^2(\partial B_1) = \{\mathbf{u} \in L^2(\partial B_1)^3 : \mathbf{u} \cdot \hat{\mathbf{x}} = 0 \text{ on } \partial B_1\}$.

For any $\mathbf{\Phi} \in H(\mathbf{curl}, B_R)$, $\hat{\mathbf{x}} \times \mathbf{\Phi}|_{\Gamma_R}$ is in the trace space $\mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})$, whose norm, for any $\lambda = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_{nm}\mathbf{U}_n^m + b_{nm}\mathbf{V}_n^m$, is defined by

$$\| \lambda \|^2_{\mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \sqrt{n(n+1)}|a_{nm}|^2 + \frac{1}{\sqrt{n(n+1)}}|b_{nm}|^2. \tag{2.6}$$

It is also known that for $\mathbf{\Phi} \in H(\mathbf{curl}; B_R)$, the tangential component $(\hat{\mathbf{x}} \times \mathbf{\Phi}) \times \hat{\mathbf{x}}|_{\Gamma_R}$ belongs to $H^{-1/2}(\mathrm{Curl};\Gamma_\mathrm{R})$ which is the dual space of $\mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})$ with respect to the scalar product in $\mathbf{L}_t^2(\Gamma_R)$ [10, Theorem 5.4.2, Lemma 5.3.1]. In the following we will always denote by $\langle\cdot,\cdot\rangle_{\Gamma_R}$ the duality pairing between $\mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})$ and $H^{-1/2}(\mathrm{Curl};\Gamma_\mathrm{R})$.

Let $h_n^{(1)}(z)$ be the spherical Hankel function of the first kind of order n. We introduce the *vector wave functions*

$$\mathbf{M}_n^m(r,\hat{\mathbf{x}}) = \nabla \times \{\mathbf{x}h_n^{(1)}(kr)Y_n^m(\hat{\mathbf{x}})\}, \quad \mathbf{N}_n^m(r,\hat{\mathbf{x}}) = \frac{1}{\mathbf{i}k}\nabla \times \mathbf{M}_n^m(r,\hat{\mathbf{x}}),$$

which are the radiation solutions of the Maxwell equation (2.1) in $\mathbb{R}^3\backslash\{0\}$.

Given the tangential vector $\lambda = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_{nm}\mathbf{U}_n^m + b_{nm}\mathbf{V}_n^m$ on Γ_R, the solution $\mathbf{E}$ of (2.1)-(2.3) in the domain $\mathbb{R}^3\backslash\bar{B}_R$ can be written as

$$\mathbf{E}(r,\hat{\mathbf{x}}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \frac{a_{nm}\mathbf{M}_n^m(r,\hat{\mathbf{x}})}{h_n^{(1)}(kR)\sqrt{n(n+1)}} + \frac{\mathbf{i}kRb_{nm}\mathbf{N}_n^m(r,\hat{\mathbf{x}})}{z_n^{(1)}(kR)\sqrt{n(n+1)}}. \tag{2.7}$$

The series in (2.7) converges uniformly for $r > R$ if $\lambda \in \mathbf{L}_t^2(\Gamma_R) = \{\mathbf{u} \in L^2(\Gamma_R)^3 : \mathbf{u}\cdot\hat{\mathbf{x}} = 0 \text{ on } \Gamma_R\}$ (cf., e.g., [9, Theorem 9.17]).

The Calderon operator $G_e : \mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R}) \to \mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})$ is the Dirichlet to Neumann operator defined by

$$G_e(\lambda) = \frac{1}{\mathbf{i}k}\hat{\mathbf{x}} \times (\nabla \times \mathbf{E}),$$

where $\mathbf{E}$ satisfies (2.1)-(2.3). Since

$$\frac{1}{\mathbf{i}k}\nabla \times \mathbf{M}_n^m = \mathbf{N}_m^n, \quad -\frac{1}{\mathbf{i}k}\nabla \times \mathbf{N}_n^m = \mathbf{M}_n^m,$$

we have

$$\frac{1}{\mathbf{i}k}\nabla \times \mathbf{E} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \frac{a_{mn}\mathbf{N}_n^m}{h_n^{(1)}(kR)\sqrt{n(n+1)}} - \frac{\mathbf{i}kRb_{mn}\mathbf{M}_n^m}{z_n^{(1)}(kR)\sqrt{n(n+1)}}.$$

On the other hand, it is easy to check that the vector wave functions satisfy

$$\begin{aligned}\mathbf{M}_n^m(r,\hat{\mathbf{x}}) &= h_n^{(1)}(kr)\nabla_{\partial B_1}Y_n^m(\hat{\mathbf{x}}) \times \hat{\mathbf{x}},\\ \mathbf{N}_n^m(r,\hat{\mathbf{x}}) &= \frac{\sqrt{n(n+1)}}{\mathbf{i}kr}z_n^{(1)}(kr)\mathbf{U}_n^m(\hat{\mathbf{x}}) + \frac{n(n+1)}{\mathbf{i}kr}h_n^{(1)}(kr)Y_n^m(\hat{\mathbf{x}})\hat{\mathbf{x}}.\end{aligned}$$

Thus

$$\hat{\mathbf{x}} \times \mathbf{M}_n^m = \sqrt{n(n+1)}h_n^{(1)}(kr)\mathbf{U}_n^m(\hat{\mathbf{x}}), \tag{2.8}$$

$$\hat{\mathbf{x}} \times \mathbf{N}_n^m = \frac{\sqrt{n(n+1)}}{\mathbf{i}kr}z_n^{(1)}(kr)\mathbf{V}_n^m(\hat{\mathbf{x}}), \tag{2.9}$$

which implies

$$G_e(\lambda) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \frac{-\mathbf{i}kRb_{nm}h_n^{(1)}(kR)}{z_n^{(1)}(kR)}\mathbf{U}_n^m(\hat{\mathbf{x}}) + \frac{a_{nm}z_n^{(1)}(kR)}{\mathbf{i}kRh_n^{(1)}(kR)}\mathbf{V}_n^m(\hat{\mathbf{x}}). \tag{2.10}$$

Let $a : H(\mathbf{curl}, \Omega_R) \times H(\mathbf{curl}, \Omega_R) \to \mathbb{C}$ be the sesquilinear form

$$a(\mathbf{E}, \mathbf{\Phi}) = \frac{1}{\mathbf{i}k}\int_{\Omega_R} (\nabla \times \mathbf{E} \cdot \nabla \times \bar{\mathbf{\Phi}} - k^2\mathbf{E}\cdot\bar{\mathbf{\Phi}})d\mathbf{x} + \langle G_e(\hat{\mathbf{x}} \times \mathbf{E}), (\hat{\mathbf{x}} \times \mathbf{\Phi}) \times \hat{\mathbf{x}}\rangle_{\Gamma_R}.$$

The scattering problem (2.1)-(2.3) is equivalent to the following weak formulation: Given $\mathbf{J}_s \in L^2(\Omega_R)^3$, find $\mathbf{E} \in H_D(\mathbf{curl}; \Omega_R)$ such that

$$a(\mathbf{E}, \mathbf{\Phi}) = \frac{1}{\mathbf{i}k}(\mathbf{J}_s, \mathbf{\Phi}), \quad \forall \mathbf{\Phi} \in H_D(\mathbf{curl}; \Omega_R), \tag{2.11}$$

where $H_D(\mathbf{curl}; \Omega_R) = \{\mathbf{v} \in H(\mathbf{curl}; \Omega_R) : \mathbf{n} \times \mathbf{v} = 0 \text{ on } \Gamma_D\}$.

We need the following lemma on the modified Bessel function which is a direct consequence of the Macdonald formula in Watson [14, P.439]. The proof of this lemma can be found in Chen and Liu [1].

Lemma 2.1. *For any $\nu \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $\mathrm{Im}\,(z) > 0$, we have*

$$|H_\nu^{(1)}(z)|^2 = \frac{2}{\pi^2}\int_0^\infty e^{-\frac{|z|^2}{2w} + \frac{z^2+\bar{z}^2}{2|z|^2}w} K_\nu(w)\frac{dw}{w}.$$

The following lemma can be proved by using Lemma 2.1, see [1, Lemma 2.2].

Lemma 2.2. *For any $\nu \in \mathbb{R}, z \in \mathbb{C}$ such that $\mathrm{Im}\,(z) > 0$, and $\Theta \in \mathbb{R}$ such that $0 < \Theta < |z|$, we have*

$$|H_\nu^{(1)}(z)| \le e^{-\mathrm{Im}\,(z)(1-\Theta^2/|z|^2)^{1/2}}|H_\nu^{(1)}(\Theta)|.$$

The following two lemmas on the spherical Bessel functions for the complex wave number k extend the corresponding results for the positive wave number in Nédélec [10, Theorem 2.6.1].

Lemma 2.3. *Let $R > 0, n \in \mathbb{Z}$, and $k \in \mathbb{C}$ such that $\mathrm{Im}\,(k) > 0$, we have*

$$\mathrm{Re}\left(\frac{z_n^{(1)}(kR)}{h_n^{(1)}(kR)}\right) < 0.$$

Proof. First we note that

$$\mathrm{Re}\left(\frac{z_n^{(1)}(kR)}{h_n^{(1)}(kR)}\right) = \mathrm{Re}\left(\frac{|h_n^{(1)}(kR)|^2 + kRh_n^{(1)'}(kR)\overline{h_n^{(1)}(kR)}}{|h_n^{(1)}(kR)|^2}\right).$$

For $z \in \mathbb{C}$, since $h_n^{(1)}(z) = \sqrt{\frac{\pi}{2z}}H_{n+1/2}^{(1)}(z)$, by Lemma 2.1 we have

$$|z||h_n^{(1)}(z)|^2 = \frac{\pi}{2}|H_{n+1/2}^{(1)}(z)|^2 = \frac{1}{\pi}\int_0^\infty e^{-\frac{|z|^2}{2w} + \frac{z^2+\bar{z}^2}{|z|^2}w} K_{n+1/2}(w)\frac{dw}{w}.$$

Thus, for any $r > 0$,

$$|kr||h_n^{(1)}(kr)|^2 = \frac{1}{\pi}\int_0^\infty e^{-\frac{|k|^2r^2}{2w} + \frac{k^2+\bar{k}^2}{2|k|^2}w} K_{n+1/2}(w)\frac{dw}{w}.$$

This implies $r|h_n^{(1)}(kr)|^2$ is a strictly decreasing function for $r \in (0,\infty)$. Consequently,

$$\frac{d}{dr}\left[r|h_n^{(1)}(kr)|^2\right]\Big|_{r=R} = |h_n^{(1)}(kR)|^2 + \mathrm{Re}\left[kRh_n^{(1)'}(kR)\overline{h_n^{(1)}(kR)}\right] < 0.$$

This completes the proof. □

Lemma 2.4. *Let $R>0, n\in\mathbb{Z}$, $k=k_1+\mathbf{i}k_2, k_1,k_2\in\mathbb{R}$ such that $k_2>0$, we have*

$$\mathrm{Im}\left(k_1\frac{z_n^{(1)}(kR)}{h_n^{(1)}(kR)}\right)\geq 0.$$

Proof. By the definition of the vector wave function $\nabla\times\mathbf{M}_n^m = \mathbf{i}k\mathbf{N}_n^m$ and (2.8)-(2.9) we have

$$\begin{aligned}&\langle\nabla\times\mathbf{M}_n^m\times\hat{\mathbf{x}},\hat{\mathbf{x}}\times\mathbf{M}_n^m\times\hat{\mathbf{x}}\rangle_{\Gamma_R}\\ &= \mathbf{i}k\langle\mathbf{N}_n^m\times\hat{\mathbf{x}},\hat{\mathbf{x}}\times\mathbf{M}_n^m\times\hat{\mathbf{x}}\rangle_{\Gamma_R} = n(n+1)Rz_n^{(1)}(kR)\overline{h_n^{(1)}(kR)}.\end{aligned}\tag{2.12}$$

Thus we only need to prove

$$\mathrm{Im}\left(k_1\langle\nabla\times\mathbf{M}_n^m\times\hat{\mathbf{x}},\hat{\mathbf{x}}\times\mathbf{M}_n^m\times\hat{\mathbf{x}}\rangle_{\Gamma_R}\right)\geq 0.$$

Since $\mathbf{M}_n^m$ satisfies the Maxwell equation

$$\nabla\times\nabla\times\mathbf{M}_n^m - k^2\mathbf{M}_n^m = 0 \quad \text{in } \mathbb{R}^3\backslash\{0\}.$$

By multiplying the above equation by $\overline{\mathbf{M}}_n^m$ and integrating over $\Omega_{R,\rho} = B_\rho\backslash\bar{B}_R$, we obtain

$$\|\nabla\times\mathbf{M}_n^m\|^2_{L^2(\Omega_{R,\rho})} - k^2\|\mathbf{M}_n^m\|^2_{L^2(\Omega_{R,\rho})} - \langle\nabla\times\mathbf{M}_n^m\times\mathbf{n},\mathbf{n}\times\mathbf{M}_n^m\times\mathbf{n}\rangle_{\Gamma_R\cup\Gamma_\rho} = 0.$$

Notice that $\mathrm{Im}\,(-k_1k^2) = -2k_1^2k_2\leq 0$, we have

$$\begin{aligned}&-\mathrm{Im}\left(k_1\langle\nabla\times\mathbf{M}_n^m\times\mathbf{n},\mathbf{n}\times\mathbf{M}_n^m\times\mathbf{n}\rangle_{\Gamma_R}\right)\\ &\geq \mathrm{Im}\left(k_1\langle\nabla\times\mathbf{M}_n^m\times\mathbf{n},\mathbf{n}\times\mathbf{M}_n^m\times\mathbf{n}\rangle_{\Gamma_\rho}\right),\end{aligned}$$

which implies, since $\mathbf{n}=-\hat{\mathbf{x}}$ on Γ_R and $\mathbf{n}=\hat{\mathbf{x}}$ on Γ_ρ,

$$\begin{aligned}&\mathrm{Im}\left(k_1\langle\nabla\times\mathbf{M}_n^m\times\hat{\mathbf{x}},\hat{\mathbf{x}}\times\mathbf{M}_n^m\times\hat{\mathbf{x}}\rangle_{\Gamma_R}\right)\\ &\geq \mathrm{Im}\left(k_1\langle\nabla\times\mathbf{M}_n^m\times\hat{\mathbf{x}},\hat{\mathbf{x}}\times\mathbf{M}_n^m\times\hat{\mathbf{x}}\rangle_{\Gamma_\rho}\right).\end{aligned}\tag{2.13}$$

By (2.12)

$$\begin{aligned}&\mathrm{Im}\left(k_1\langle\nabla\times\mathbf{M}_n^m\times\hat{\mathbf{x}},\hat{\mathbf{x}}\times\mathbf{M}_n^m\times\hat{\mathbf{x}}\rangle_{\Gamma_\rho}\right)\\ &= n(n+1)\mathrm{Im}\left(k_1k\rho^2h_n^{(1)'}(k\rho)\overline{h_n^{(1)}(k\rho)}\right).\end{aligned}$$

We are now going to show that

$$|k\rho^2h_n^{(1)'}(k\rho)\overline{h_n^{(1)}(k\rho)}|\to 0, \quad \text{as } \rho\to\infty.\tag{2.14}$$

Since $h_n^{(1)'}(z) = -\frac{n+1}{z}h_n^{(1)}(z) - h_{n-1}^{(1)}(z)$, we have

$$|k\rho^2h_n^{(1)'}(k\rho)\overline{h_n^{(1)}(k\rho)}| \leq (n+1)\rho|h_n^{(1)}(k\rho)|^2 + |k\rho^2||h_n^{(1)}(k\rho)||h_{n-1}^{(1)}(k\rho)|.$$

On the other hand, for any $\Theta > 0$ such that $\Theta < |k\rho|$, by Lemma 2.2

$$|h_n^{(1)}(k\rho)| = \sqrt{\frac{\pi}{2|k\rho|}}|H_{n+1/2}^{(1)}(k\rho)| \le \sqrt{\frac{\pi}{2|k\rho|}}e^{-k_2\rho\left(1-\frac{\Theta^2}{|k\rho|^2}\right)^{1/2}}|H_{n+1/2}^{(1)}(\Theta)|,$$

that is, $|h_n^{(1)}(k\rho)$ decays exponentially as $\rho\to\infty$. This proves (2.14) and completes the proof. □

Lemma 2.5. *For any* $\mathbf{\Phi} \in H(\mathbf{curl};\Omega_R)$, *we have*

$$\mathrm{Re}\,\langle G_e(\hat{\mathbf{x}} \times \mathbf{\Phi}), \hat{\mathbf{x}} \times \mathbf{\Phi} \times \hat{\mathbf{x}}\rangle_{\Gamma_R} \le 0.$$

Proof. Let $\hat{\mathbf{x}} \times \mathbf{\Phi}|_{\Gamma_R} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_{mn}\mathbf{U}_n^m(\hat{\mathbf{x}}) + b_{mn}\mathbf{V}_n^m(\hat{\mathbf{x}})$. Then

$$\hat{\mathbf{x}} \times \mathbf{\Phi} \times \hat{\mathbf{x}}|_{\Gamma_R} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} -a_{mn}\mathbf{V}_n^m(\hat{\mathbf{x}}) + b_{mn}\mathbf{U}_n^m(\hat{\mathbf{x}}).$$

Thus, by (2.10),

$$\begin{aligned}&\langle G_e(\hat{\mathbf{x}} \times \mathbf{\Phi}), \hat{\mathbf{x}} \times \mathbf{\Phi} \times \hat{\mathbf{x}}\rangle_{\Gamma_R}\\ &= -\sum_{n=1}^{\infty}\sum_{m=-n}^{n} \frac{z_n^{(1)}(kR)}{\mathbf{i}kRh_n^{(1)}(kR)}|a_{mn}|^2 + \frac{\mathbf{i}kRh_n^{(1)}(kR)}{z_n^{(1)}(kR)}|b_{mn}|^2.\end{aligned}$$

Denote by $z_n^{(1)}(kR)/h_n^{(1)}(kR) = a_n(kR) + \mathbf{i}b_n(kR)$, where $a_n(kR), b_n(kR)$ are the real and imaginary part of $z_n^{(1)}(kR)/h_n^{(1)}(kR)$. By Lemmas 2.3 and 2.4 we know that $a_n(kR) < 0$ and $k_1b_n(kR) \ge 0$. Hence

$$\begin{aligned}\mathrm{Re}\left(\frac{z_n^{(1)}(kR)}{\mathbf{i}kRh^{(1)}(kR)}\right) &= \mathrm{Re}\left(\frac{-\mathbf{i}\bar{k}(a_n(kR) + \mathbf{i}b_n(kR))}{|k|^2R}\right)\\ &= \frac{-k_2a_n(kR) + k_1b_n(kR)}{|k|^2R} > 0.\end{aligned}$$

Since $\mathrm{Re}\,(z^{-1}) = |z|^{-2}\mathrm{Re}\,(z)$, we then have

$$\mathrm{Re}\left(\frac{\mathbf{i}kRh^{(1)}(kR)}{z_n^{(1)}(kR)}\right) > 0.$$

This completes the proof. □

The following theorem is the main result of this section.

Theorem 2.1. *The variational problem* (2.11) *has a unique weak solution* $\mathbf{E} \in H_D(\mathbf{curl};\Omega_R)$ *which satisfies*

$$\|\,\nabla \times \mathbf{E}\,\|_{L^2(\Omega_R)} + \|\,k\mathbf{E}\,\|_{L^2(\Omega_R)} \le Ck_2^{-1}\|\,\mathbf{J}_s\,\|_{L^2(\Omega_R)}. \tag{2.15}$$

Proof. Since $\mathrm{Re}\,(1/\mathbf{i}k) = -k_2/|k|^2$, $\mathrm{Re}\,(\mathbf{i}k) = -k_2$, by Lemma 2.5 we know that, for any $\mathbf{\Phi} \in H(\mathbf{curl};\Omega_R)$,

$$|a(\mathbf{\Phi},\mathbf{\Phi})| \ge \mathrm{Re}\,(-a(\mathbf{\Phi},\mathbf{\Phi})) \ge \frac{k_2}{|k|^2}\left(\|\,\nabla \times \mathbf{\Phi}\,\|_{L^2(\Omega_R)}^2 + \|\,k\mathbf{\Phi}\,\|_{L^2(\Omega_R)}^2\right). \tag{2.16}$$

By the Lax-Milgram lemma we know that the problem (2.11) has a unique solution. To show the stability estimate (2.15), we know from (2.11) that

$$|a(\mathbf{E},\mathbf{E})| \le |k|^{-2} \| \mathbf{J}_s \|_{L^2(\Omega_R)} \| k\mathbf{E} \|_{L^2(\Omega_R)}.$$

Therefore, by (2.16),

$$\| \nabla\times\mathbf{E} \|_{L^2(\Omega_R)} + \| k\mathbf{E} \|_{L^2(\Omega_R)} \le C k_2^{-1} \| \mathbf{J}_s \|_{L^2(\Omega_R)}.$$

This completes the proof. □

3. The Maxwell Scattering Problem

We first give the assumptions required on the boundary and initial data:

(H1) $\mathbf{E}_0, \mathbf{H}_0, \nabla\times\mathbf{E}_0, \nabla\times\mathbf{H}_0 \in H(\mathbf{curl};\Omega_R)$ and $\mathrm{supp}(\mathbf{E}_0), \mathrm{supp}(\mathbf{H}_0) \subset B_R$;
(H2) $\mathbf{J} \in H^1(0,T;L^2(\Omega_R))^3$, $\mathbf{J}|_{t=0} = 0$, and $\mathrm{supp}(\mathbf{J}) \subset B_R \times (0,T)$.

In the rest of this paper, we will always assume that $\mathbf{J}$ is extended so that

$$\mathbf{J} \in H^1(0,+\infty;L^2(\Omega_R))^3, \quad \|\mathbf{J}\|_{H^1(0,+\infty;L^2(\Omega_R))} \le C\|\mathbf{J}\|_{H^1(0,T;L^2(\Omega_R))}.$$

The following lemma can be proved by the standard energy argument.

Lemma 3.1. *Let $\hat{\mathbf{E}}, \hat{\mathbf{H}}$ be the solution of the following problem*

$$\begin{aligned}
&\varepsilon\frac{\partial\hat{\mathbf{E}}}{\partial t} - \nabla\times\hat{\mathbf{H}} = 0 \quad \text{in } \Omega_R\times(0,T),\\
&\mu\frac{\partial\hat{\mathbf{H}}}{\partial t} + \nabla\times\hat{\mathbf{E}} = 0 \quad \text{in } \Omega_R\times(0,T),\\
&\mathbf{n}\times\hat{\mathbf{E}} = 0 \quad \text{on } \Gamma_D\cup\Gamma_R,\\
&\hat{\mathbf{E}}|_{t=0} = \mathbf{E}_0, \quad \hat{\mathbf{H}}|_{t=0} = \mathbf{H}_0.
\end{aligned}$$

Then

$$\begin{aligned}
&\| \hat{\mathbf{E}} \|_{L^2(\Omega_R)} + \| \hat{\mathbf{H}} \|_{L^2(\Omega_R)} \le C\| \mathbf{E}_0 \|_{L^2(\Omega_R)} + C\| \mathbf{H}_0 \|_{L^2(\Omega_R)},\\
&\| \partial_t\hat{\mathbf{E}} \|_{L^2(\Omega_R)} + \| \partial_t\hat{\mathbf{H}} \|_{L^2(\Omega_R)} \le C\| \nabla\times\mathbf{E}_0 \|_{L^2(\Omega_R)} + C\| \nabla\times\mathbf{H}_0 \|_{L^2(\Omega_R)},\\
&\| \partial_t^2\hat{\mathbf{E}} \|_{L^2(\Omega_R)} + \| \partial_t^2\hat{\mathbf{H}} \|_{L^2(\Omega_R)} \le C\| \nabla\times\nabla\times\mathbf{E}_0 \|_{L^2(\Omega_R)} + C\| \nabla\times\nabla\times\mathbf{H}_0 \|_{L^2(\Omega_R)}.
\end{aligned}$$

Let $\mathbf{E}' = \mathbf{E} - \hat{\mathbf{E}}, \mathbf{H}' = \mathbf{H} - \hat{\mathbf{H}}$. Then by (1.1)-(1.2) we know that

$$\varepsilon\frac{\partial\mathbf{E}'}{\partial t} - \nabla\times\mathbf{H}' = \mathbf{J} \quad \text{in } [\mathbb{R}^3\backslash\bar{D}]\times(0,T), \tag{3.1}$$

$$\mu\frac{\partial\mathbf{H}'}{\partial t} + \nabla\times\mathbf{E}' = 0 \quad \text{in } [\mathbb{R}^3\backslash\bar{D}]\times(0,T). \tag{3.2}$$

The boundary condition (1.3) becomes

$$\mathbf{n}\times\mathbf{E}' = 0 \quad \text{on } \Gamma_D\times(0,T). \tag{3.3}$$

By (1.8) we have

$$\sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{x}}\times\mathbf{H}' = (\mathscr{L}^{-1}\circ G_e\circ\mathscr{L})(\hat{\mathbf{x}}\times\mathbf{E}'|_{\Gamma_R}) - \sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{x}}\times\hat{\mathbf{H}} \quad \text{on } \Gamma_R. \tag{3.4}$$

It is obvious that

$$\mathbf{E}'|_{t=0} = 0, \qquad \mathbf{H}'|_{t=0} = 0. \tag{3.5}$$

Let $\mathbf{E}'_{\text{L}} = \mathscr{L}(\mathbf{E}'), \mathbf{H}'_{\text{L}} = \mathscr{L}(\mathbf{H}')$. Then by taking the Laplace transform of (3.1)-(3.4) we obtain

$$\nabla\times\nabla\times\mathbf{E}'_{\text{L}} - k^2\mathbf{E}'_{\text{L}} = -\sqrt{\frac{\mu}{\varepsilon}}\mathbf{i}k\mathbf{J}_{\text{L}} \quad \text{in } \Omega_R, \tag{3.6}$$

$$\mathbf{n}\times\mathbf{E}'_{\text{L}} = 0 \quad \text{on } \Gamma_D, \tag{3.7}$$

$$\frac{1}{\mathbf{i}k}\hat{\mathbf{x}}\times(\nabla\times\mathbf{E}'_{\text{L}}) = G_e(\hat{\mathbf{x}}\times\mathbf{E}'_{\text{L}}) - \sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{x}}\times\hat{\mathbf{H}}_{\text{L}} \quad \text{on } \Gamma_R. \tag{3.8}$$

By Theorem 2.1 we know that the problem (3.6)-(3.8) has a unique solution. Our strategy to show the well-posedness of (3.1)-(3.5) and thus (1.1)-(1.4), (1.8) is to show the inverse Laplace transform of the solution $\mathbf{E}'_{\text{L}}$ of (3.6)-(3.8) is existent.

We first recall the following theorem in Treves [12, Theorem 43.1] which is the analog of the Paley-Wiener-Schwarz theorem for the Fourier transform of the distributions with compact support in the case of Laplace transform .

Lemma 3.2. *Let* $\mathbf{h}(s)$ *denote a holomorphic function in the half-plane* $\operatorname{Re}(s) > \sigma_0$*, valued in the Banach space* E*. The following conditions are equivalent:*
(i) *there is a distribution* $T\in\mathcal{D}'_+(E)$ *whose Laplace transform is equal to* $\mathbf{h}(s)$*;*
(ii) *there is a* σ_1 *real,* $\sigma_0\le\sigma_1<\infty$*, a constant* $C>0$*, and an integer* $k\ge 0$ *such that, for all complex numbers* s*,* $\operatorname{Re}(s)>\sigma_1$*,*

$$\|\,\mathbf{h}(s)\,\|_E \le C(1+|s|)^k.$$

Here $\mathcal{D}'_+$ *is the space of distributions on the real line which vanish identically in the open negative half-line.*

Lemma 3.3. *There exists a constant* C *independent of* s *such that*

$$\begin{aligned}&\|\,\nabla\times\mathbf{E}'_{\text{L}}\,\|_{L^2(\Omega_R)} + \|\,k\mathbf{E}'_{\text{L}}\,\|_{L^2(\Omega_R)}\\ &\le \frac{C}{k_2}\left(\|\,k\mathbf{J}_{\text{L}}\,\|_{L^2(\Omega_R)} + \|\,k\hat{\mathbf{x}}\times\hat{\mathbf{H}}_{\text{L}}\,\|_{H^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})} + \|\,|k|^2\hat{\mathbf{x}}\times\hat{\mathbf{H}}_{\text{L}}\,\|_{H^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})}\right).\end{aligned}$$

Proof. By testing (3.6) with $\bar{\mathbf{E}}'_{\text{L}}\in H_D(\mathbf{curl};\Omega_R)$ we know that

$$a(\mathbf{E}'_{\text{L}},\mathbf{E}'_{\text{L}}) = -\sqrt{\frac{\mu}{\varepsilon}}(\mathbf{J}_{\text{L}},\mathbf{E}'_{\text{L}}) + \sqrt{\frac{\mu}{\varepsilon}}\langle\hat{\mathbf{x}}\times\hat{\mathbf{H}}_{\text{L}},\hat{\mathbf{x}}\times\mathbf{E}'_{\text{L}}\times\hat{\mathbf{x}}\rangle_{\Gamma_R}.$$

By (2.16)

$$\begin{aligned}&\frac{k_2}{|k|^2}\left(\|\,\nabla\times\mathbf{E}'_{\text{L}}\,\|^2_{L^2(\Omega_R)} + \|\,k\mathbf{E}'_{\text{L}}\,\|^2_{L^2(\Omega_R)}\right)\\ &\le C\|\,\bar{k}^{-1}\mathbf{J}_{\text{L}}\,\|_{L^2(\Omega_R)}\|\,k\mathbf{E}'_{\text{L}}\,\|_{L^2(\Omega_R)} + C\|\,\hat{\mathbf{x}}\times\hat{\mathbf{H}}_{\text{L}}\,\|_{H^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})}\|\,\mathbf{E}'_{\text{L}}\,\|_{H(\mathbf{curl};\Omega_R)}\\ &\le C\|\,\bar{k}^{-1}\mathbf{J}_{\text{L}}\,\|_{L^2(\Omega_R)}\|\,k\mathbf{E}'_{\text{L}}\,\|_{L^2(\Omega_R)} + C\|\,\hat{\mathbf{x}}\times\hat{\mathbf{H}}_{\text{L}}\,\|_{H^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})}\|\,\nabla\times\mathbf{E}'_{\text{L}}\,\|_{L^2(\Omega_R)}\\ &\quad + C\|\,\bar{k}^{-1}\hat{\mathbf{x}}\times\hat{\mathbf{H}}_{\text{L}}\,\|_{H^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})}\|\,k\mathbf{E}'_{\text{L}}\,\|_{L^2(\Omega_R)}.\end{aligned}$$

This completes the proof. $\square$

Theorem 3.1. *Let the assumptions (H1)-(H2) be satisfied. Then the problem* (1.1)-(1.4), (1.8) *has a unique solution* $(\mathbf{E},\mathbf{H})$ *such that*

$$\mathbf{E}\in L^2(0,T;H_D(\mathbf{curl};\Omega_R))\cap H^1(0,T;L^2(\Omega_R)^3),$$
$$\mathbf{H}\in L^2(0,T;H(\mathbf{curl},\Omega_R))\cap H^1(0,T;L^2(\Omega_R)^3),$$

$\mathbf{E}|_{t=0}=\mathbf{E}_0$, $\mathbf{H}|_{t=0}=\mathbf{H}_0$, *and*

$$\int_0^T\left[\varepsilon\Big(\frac{\partial\mathbf{E}}{\partial t},\mathbf{\Phi}\Big)-(\mathbf{H},\nabla\times\mathbf{\Phi})-\sqrt{\frac{\varepsilon}{\mu}}\langle(\mathscr{L}^{-1}\circ G_e\circ\mathscr{L})(\hat{\mathbf{x}}\times\mathbf{E}),\mathbf{\Phi}\rangle_{\Gamma_R}\right]dt$$
$$=\int_0^T(\mathbf{J},\mathbf{\Phi})dt,\qquad\forall\mathbf{\Phi}\in L^2(0,T;H_D(\mathbf{curl};\Omega_R)),\tag{3.9}$$

$$\int_0^T\left[\mu\Big(\frac{\partial\mathbf{H}}{\partial t},\mathbf{\Psi}\Big)+(\nabla\times\mathbf{E},\mathbf{\Psi})\right]dt=0,\qquad\forall\mathbf{\Psi}\in L^2(0,T;L^2(\Omega_R)^3).\tag{3.10}$$

Here $(\mathscr{L}^{-1}\circ G\circ\mathscr{L})(\hat{\mathbf{x}}\times\mathbf{E})\in L^2(0,T;H^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R}))$*. Moreover,* $(\mathbf{E},\mathbf{H})$ *satisfies the following stability estimate*

$$\max_{0\le t\le T}\left(\|\,\partial_t\mathbf{E}\,\|_{L^2(\Omega_R)}+\|\,\nabla\times\mathbf{E}\,\|_{L^2(\Omega_R)}+\|\,\partial_t\mathbf{H}\,\|_{L^2(\Omega_R)}+\|\,\nabla\times\mathbf{H}\,\|_{L^2(\Omega_R)}\right)$$
$$\le C\|(\mathbf{E}_0,\mathbf{H}_0)\|_{\Omega_R}+C\|\partial_t\mathbf{J}\|_{L^1(0,T;L^2(\Omega_R))},\tag{3.11}$$

where

$$\|(\mathbf{E}_0,\mathbf{H}_0)\|_{\Omega_R}=\|\,\mathbf{E}_0\,\|_{H(\mathbf{curl};\Omega_R)}+\|\,\mathbf{H}_0\,\|_{H(\mathbf{curl};\Omega_R)}.$$

Proof. Our starting point is the solution $\mathbf{E}'_\mathrm{L},\mathbf{H}'_\mathrm{L}$ of the following scattering problem

$$\varepsilon s\mathbf{E}'_\mathrm{L}-\nabla\times\mathbf{H}'_\mathrm{L}=\mathbf{J}_\mathrm{L}\quad\text{in }\mathbb{R}^3\backslash\bar{D},\tag{3.12}$$
$$\mu s\mathbf{H}'_\mathrm{L}+\nabla\times\mathbf{E}'_\mathrm{L}=0\quad\text{in }\mathbb{R}^3\backslash\bar{D},\tag{3.13}$$
$$\mathbf{n}\times\mathbf{E}'_\mathrm{L}=0\quad\text{on }\Gamma_D,\tag{3.14}$$
$$\sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{x}}\times\mathbf{H}'_\mathrm{L}=G_e(\hat{\mathbf{x}}\times\mathbf{E}'_\mathrm{L})-\sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{x}}\times\hat{\mathbf{H}}_\mathrm{L}\qquad\text{on }\Gamma_R.\tag{3.15}$$

Since $k=\mathbf{i}\sqrt{\varepsilon\mu}s$, by Lemma 3.3, there exists a constant C independent of s such that

$$\|\,\nabla\times\mathbf{E}'_\mathrm{L}\,\|_{L^2(\Omega_R)}+\|\,s\mathbf{E}'_\mathrm{L}\,\|_{L^2(\Omega_R)}$$
$$\le\frac{C}{s_1}\left(\|\,s\mathbf{J}_\mathrm{L}\,\|_{L^2(\Omega_R)}+\|\,s\hat{\mathbf{x}}\times\hat{\mathbf{H}}_\mathrm{L}\,\|_{H^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})}+\|\,|s|^2\hat{\mathbf{x}}\times\hat{\mathbf{H}}_\mathrm{L}\,\|_{H^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})}\right).\tag{3.16}$$

By (3.12)-(3.13),

$$\|\,\nabla\times\mathbf{H}'_\mathrm{L}\,\|_{L^2(\Omega_R)}+\|\,s\mathbf{H}'_\mathrm{L}\,\|_{L^2(\Omega_R)}$$
$$\le\frac{C}{s_1}\left(\|\,\mathbf{J}_\mathrm{L}\,\|_{L^2(\Omega_R)}+\|\,s\mathbf{J}_\mathrm{L}\,\|_{L^2(\Omega_R)}\right)$$
$$+\frac{C}{s_1}\left(\|\,s\hat{\mathbf{x}}\times\hat{\mathbf{H}}_\mathrm{L}\,\|_{H^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})}+\|\,|s|^2\hat{\mathbf{x}}\times\hat{\mathbf{H}}_\mathrm{L}\,\|_{H^{-1/2}(\mathrm{Div};\Gamma_\mathrm{R})}\right).\tag{3.17}$$

By [12, Lemma 44.1], $\mathbf{E}'_\mathrm{L},\mathbf{H}'_\mathrm{L}$ are holomorphic functions of s on the half plane $\mathrm{Re}\,(s)>\gamma>0$, where γ is any positive constant. By Lemma 3.2 the inverse Laplace transform of $\mathbf{E}'_\mathrm{L},\mathbf{H}'_\mathrm{L}$ are existent and supported in $[0,\infty]$. Denote by $\mathbf{E}'=\mathscr{L}^{-1}(\mathbf{E}'_{\rm L})$, $\mathbf{H}'=\mathscr{L}^{-1}(\mathbf{H}'_{\rm L})$. Then, since $\mathbf{E}'_{\rm L}=\mathscr{L}(\mathbf{E}')=\mathscr{F}(e^{-s_1t}\mathbf{E}')$, where $\mathscr{F}$ is the Fourier transform in s_2, by the Parseval identity and (3.16), we have

$$
\begin{aligned}
&\int_0^\infty e^{-2s_1t}\left(\|\nabla\times\mathbf{E}'\|^2_{L^2(\Omega_R)}+\|\partial_t\mathbf{E}'\|^2_{L^2(\Omega_R)}\right)dt\\
&=2\pi\int_{-\infty}^\infty\left(\|\nabla\times\mathbf{E}'_{\rm L}\|^2_{L^2(\Omega_R)}+\|s\mathbf{E}'_{\rm L}\|^2_{L^2(\Omega_R)}\right)ds_2\\
&\le\frac{C}{s_1^2}\int_{-\infty}^\infty\|s\mathbf{J}_{\rm L}\|^2_{L^2(\Omega_R)}ds_2\\
&\quad+\frac{C}{s_1^2}\int_{-\infty}^\infty\left(\|s\hat{\mathbf{x}}\times\hat{\mathbf{H}}_{\rm L}\|^2_{H^{-1/2}(\mathrm{Div};\Gamma_{\rm R})}+\||s|^2\hat{\mathbf{x}}\times\hat{\mathbf{H}}_{\rm L}\|^2_{H^{-1/2}(\mathrm{Div};\Gamma_{\rm R})}\right)ds_2.
\end{aligned}
$$

Since $\mathbf{J}|_{t=0}=0$ in Ω_R, $\hat{\mathbf{x}}\times\hat{\mathbf{H}}|_{t=0}=\partial_t(\hat{\mathbf{x}}\times\hat{\mathbf{H}})|_{t=0}=0$ on Γ_R, we have $\mathscr{L}(\partial_t\mathbf{J})=s\mathbf{J}_{\rm L}$ in Ω_R and $\mathscr{L}(\partial_t(\hat{\mathbf{x}}\times\mathbf{H}))=s\hat{\mathbf{x}}\times\mathbf{H}_{\rm L}$ on Γ_R. Moreover, notice that

$$
|s|^2\hat{\mathbf{x}}\times\hat{\mathbf{H}}_{\rm L}=(2s_1-s)s\hat{\mathbf{x}}\times\hat{\mathbf{H}}_{\rm L}=2s_1\mathscr{L}(\partial_t(\hat{\mathbf{x}}\times\hat{\mathbf{H}}))-\mathscr{L}(\partial_t^2(\hat{\mathbf{x}}\times\hat{\mathbf{H}}))\quad\text{on }\Gamma_R,
$$

we have

$$
\begin{aligned}
&\int_0^\infty e^{-2s_1t}\left(\|\nabla\times\mathbf{E}'\|^2_{L^2(\Omega_R)}+\|\partial_t\mathbf{E}'\|^2_{L^2(\Omega_R)}\right)dt\\
&\le\frac{C}{s_1^2}\int_{-\infty}^\infty\left(\|\mathscr{L}(\partial_t\mathbf{J})\|^2_{L^2(\Omega_R)}+\|\mathscr{L}(\hat{\mathbf{x}}\times\partial_t^2\hat{\mathbf{H}})\|^2_{H^{-1/2}(\mathrm{Div};\Gamma_{\rm R})}\right)ds_2\\
&\quad+C\left(1+\frac{1}{s_1^2}\right)\int_{-\infty}^\infty\|\mathscr{L}(\hat{\mathbf{x}}\times\partial_t\hat{\mathbf{H}})\|^2_{H^{-1/2}(\mathrm{Div};\Gamma_{\rm R})}ds_2.
\end{aligned}
$$

Again by the Parseval identity

$$
\begin{aligned}
&\int_0^\infty e^{-2s_1t}\left(\|\nabla\times\mathbf{E}'\|^2_{L^2(\Omega_R)}+\|\partial_t\mathbf{E}'\|^2_{L^2(\Omega_R)}\right)dt\\
&\le\frac{C}{s_1^2}\int_0^\infty e^{-2s_1t}\left(\|\partial_t\mathbf{J}\|^2_{L^2(\Omega_R)}+\|\hat{\mathbf{x}}\times\partial_t^2\hat{\mathbf{H}}\|^2_{H^{-1/2}(\mathrm{Div};\Gamma_{\rm R})}\right)dt\\
&\quad+C\left(1+\frac{1}{s_1^2}\right)\int_0^\infty e^{-2s_1t}\|\hat{\mathbf{x}}\times\partial_t\hat{\mathbf{H}}\|^2_{H^{-1/2}(\mathrm{Div};\Gamma_{\rm R})}dt.
\end{aligned}
$$

This proves $\mathbf{E}'\in L^2(0,T;H(\mathbf{curl};\Omega_R))\cap H^1(0,T;L^2(\Omega_R)^3)$. Similarly, by (3.17), we have $\mathbf{H}'\in L^2(0,T;H(\mathbf{curl};\Omega_R))\cap H^1(0,T;L^2(\Omega_R)^3)$. Moreover, by (3.15), we deduce that $(\mathscr{L}^{-1}\circ G_e\circ\mathscr{L})(\hat{\mathbf{x}}\times\mathbf{E}')\in L^2(0,T;H^{-1/2}(\mathrm{Div};\Gamma_{\rm R}))$. By taking the inverse Laplace transform in (3.12)-(3.13) and using the definition of $\mathbf{E}'=\mathbf{E}-\hat{\mathbf{E}}$, $\mathbf{H}'=\mathbf{H}-\hat{\mathbf{H}}$, one can easily show that $(\mathbf{E},\mathbf{H})$ satisfies (3.9)-(3.10).

It remains to prove the stability estimate (3.11). By (1.9) we know that

$$
\begin{aligned}
&\langle(\mathscr{L}^{-1}\circ G_e\circ\mathscr{L})(x\times\mathbf{E}),\hat{\mathbf{x}}\times\mathbf{E}\times\hat{\mathbf{x}}\rangle_{\Gamma_R}\\
&=R^2\sum_{n=1}^\infty\sum_{m=-n}^n\left[\mathscr{L}^{-1}\left(\frac{\sqrt{\varepsilon\mu}sRh_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}{z_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}\right)*b_{mn}(R,t)\right]\bar{b}_{mn}(R,t)\\
&\qquad-\left[\mathscr{L}^{-1}\left(\frac{z_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}{\sqrt{\varepsilon\mu}sRh_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}\right)*a_{mn}(R,t)\right]\bar{a}_{mn}(R,t).
\end{aligned}
$$

Denote $\tilde{a}_{mn} = a_{mn}\chi_{[0,T]}, \tilde{b}_{mn} = b_{mn}\chi_{[0,T]}$, where $\chi_{[0,T]}$ is the characteristic function of the interval $(0,T)$. Therefore

$$\begin{aligned}&\int_0^T e^{-2s_1 t}\langle(\mathscr{L}^{-1}\circ G_e\circ\mathscr{L})(x\times\mathbf{E}),\hat{\mathbf{x}}\times\mathbf{E}\times\hat{\mathbf{x}}\rangle_{\Gamma_R}dt\\&=R^2\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\int_{-\infty}^{\infty}e^{-2s_1 t}\left[\mathscr{L}^{-1}\left(\frac{\sqrt{\varepsilon\mu}sRh_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}{z_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}\right)*\tilde{b}_{mn}\right]\bar{\tilde{b}}_{mn}\\&\qquad+\int_{-\infty}^{\infty}e^{-2s_1 t}\left[\mathscr{L}^{-1}\left(\frac{z_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}{\sqrt{\varepsilon\mu}sRh_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}\right)*\tilde{a}_{mn}\right]\bar{\tilde{a}}_{mn}.\end{aligned}$$

Note that by the formula for the inverse Laplace transform we have

$$g(t)=\mathscr{F}^{-1}(e^{s_1 t}\mathscr{L}(g)(s_1+\mathbf{i}s_2)),$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform with respect to s_2. By the Plancherel identity we then obtain

$$\begin{aligned}&\int_0^T e^{-2s_1 t}\langle(\mathscr{L}^{-1}\circ G_e\circ\mathscr{L})(x\times\mathbf{E}),\hat{\mathbf{x}}\times\mathbf{E}\times\hat{\mathbf{x}}\rangle_{\Gamma_R}dt\\&=2\pi R^2\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\int_{-\infty}^{\infty}\left(\frac{\sqrt{\varepsilon\mu}sRh_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}{z_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}\right)|\mathscr{L}(\tilde{b}_{mn})|^2\\&\qquad+\int_{-\infty}^{\infty}\left(\frac{z_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}{\sqrt{\varepsilon\mu}sRh_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}\right)|\mathscr{L}(\tilde{a}_{mn})|^2.\end{aligned}$$

Since $k=\mathbf{i}\sqrt{\varepsilon\mu}s$ satisfies $\mathrm{Im}\,(k)>0$, by using Lemmas 2.3 and 2.4 we obtain

$$-\mathrm{Re}\int_0^T e^{-2s_1 t}\langle(\mathscr{L}^{-1}\circ G_e\circ\mathscr{L})(x\times\mathbf{E}),\hat{\mathbf{x}}\times\mathbf{E}\times\hat{\mathbf{x}}\rangle_{\Gamma_R}dt\geq 0.$$

For any $0<t^*<T$, by taking $\mathbf{\Phi}=e^{-2s_1 t}\mathbf{E}\chi_{(0,t^*)}$ in (3.9), $\mathbf{\Psi}=e^{-2s_1 t}\mathbf{H}\chi_{(0,t^*)}$ in (3.10), and adding the two equations, we obtain

$$\frac{1}{2}\int_0^{t^*}e^{-2s_1 t}\frac{d}{dt}\left(\varepsilon\|\,\mathbf{E}\,\|^2_{L^2(\Omega_R)}+\mu\|\,\mathbf{H}\,\|^2_{L^2(\Omega_R)}\right)dt\leq\int_0^{t^*}e^{-2s_1 t}(\mathbf{J},\mathbf{E})dt.$$

By standard argument we can deduce

$$\begin{aligned}&\max_{0\leq t\leq T}\left[e^{-2s_1 t}\left(\varepsilon\|\,\mathbf{E}\,\|^2_{L^2(\Omega_R)}+\mu\|\,\mathbf{H}\,\|^2_{L^2(\Omega_R)}\right)\right]\\&\leq C\left(\varepsilon\|\,\mathbf{E}_0\,\|^2_{L^2(\Omega_R)}+\mu\|\,\mathbf{H}_0\,\|^2_{L^2(\Omega_R)}\right)+C\|\,e^{-s_1 t}\mathbf{J}\,\|_{L^1(0,T;L^2(\Omega_R))}.\end{aligned}$$

By letting $s_1\to 0$, we obtain

$$\begin{aligned}&\max_{0\leq t\leq T}\left(\varepsilon\|\,\mathbf{E}\,\|^2_{L^2(\Omega_R)}+\mu\|\,\mathbf{H}\,\|^2_{L^2(\Omega_R)}\right)\\&\leq C\left(\varepsilon\|\,\mathbf{E}_0\,\|^2_{L^2(\Omega_R)}+\mu\|\,\mathbf{H}_0\,\|^2_{L^2(\Omega_R)}\right)+C\|\,\mathbf{J}\,\|_{L^1(0,T;L^2(\Omega_R))}.\end{aligned}\tag{3.18}$$

Since $\mathbf{E}_0,\mathbf{H}_0$ has a compact support inside B_R, $a_{mn}(R,0)=b_{mn}(R,0)=0$ on Γ_R. By differentiating (1.8) in time we know that

$$\sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{x}}\times\partial_t\mathbf{H}=(\mathscr{L}^{-1}\circ G_e\circ\mathscr{L})(\hat{\mathbf{x}}\times\partial_t\mathbf{E}|_{\Gamma_R})\quad\text{ on }\Gamma_R.$$

Now by differentiating (1.1)-(1.2) in time, we know that $\partial_t\mathbf{E}, \partial_t\mathbf{H}$ satisfy the same set of equations with the source $\partial_t\mathbf{J}$ and the initial condition $\partial_t\mathbf{E}|_{t=0} = \varepsilon^{-1}\nabla\times\mathbf{E}_0, \partial_t\mathbf{H}|_{t=0} = -\mu^{-1}\nabla\times\mathbf{E}_0$. Thus we can use (3.18) for $\partial_t\mathbf{E}, \partial_t\mathbf{H}$ to conclude the proof. □

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