Journal of Computational Mathematics, Vol.26, No.1, 2008, 76–97.

A MONOTONE DOMAIN DECOMPOSITION ALGORITHM FOR SOLVING WEIGHTED AVERAGE APPROXIMATIONS TO NONLINEAR SINGULARLY PERTURBED PARABOLIC PROBLEMS*

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Abstract

This paper presents and analyzes a monotone domain decomposition algorithm for solving nonlinear singularly perturbed reaction-diffusion problems of parabolic type. To solve the nonlinear weighted average finite difference scheme for the partial differential equation, we construct a monotone domain decomposition algorithm based on a Schwarz alternating method and a box-domain decomposition. This algorithm needs only to solve linear discrete systems at each iterative step and converges monotonically to the exact solution of the nonlinear discrete problem. The rate of convergence of the monotone domain decomposition algorithm is estimated. Numerical experiments are presented.

Mathematics subject classification: 65M06, 65M12, 65M55.
Key words: Parabolic reaction-diffusion problem, Boundary layers, θ-method, Monotone domain decomposition algorithm, Uniform convergence.

1. Introduction

We are interested in monotone iterative methods for solving nonlinear singularly perturbed problems which correspond to a reaction-diffusion problem of parabolic type

$$
\begin{aligned}
&-\mu^2\left(u_{xx}+u_{yy}\right)+u_t=-f(x,y,t,u), && (1.1)\\
&(x,y,t)\in Q=\omega\times(0,T], \quad \omega=\{0<x<1\}\times\{0<y<1\},\\
&0\le f_u\le c^*=\text{const}, \quad (x,y,t,u)\in\overline{Q}\times(-\infty,\infty), \quad (f_u\equiv\partial f/\partial u), && (1.2)
\end{aligned}
$$

where μ is a small positive parameter. The initial-boundary conditions are

$$
\begin{aligned}
&u(x,y,0)=u^0(x,y), \quad (x,y)\in\overline{\omega},\\
&u=g, \quad (x,y,t)\in\partial\omega\times(0,T],
\end{aligned}
$$

where $\partial\omega$ is the boundary of $\overline{\omega}$. If f, g and u^0 are sufficiently smooth, then under suitable continuity and compatibility conditions on the data, a unique solution u of (1.1) exists [6]. We mention that the assumption $f_u\ge 0$ in (1.1) can always be obtained via a change of variables.

For $\mu\ll 1$, the reaction-diffusion problem (1.1) is singularly perturbed and characterized by the boundary layers of width $\mathcal{O}(\mu|\ln\mu|)$ near $\partial\omega$, see, e.g., [1].

* Received February 28, 2007 / accepted July 19, 2007 /

We shall employ the weighted average scheme for solving problem (1.1). This nonlinear ten-point difference scheme can be regarded as taking a weighted average of the explicit and implicit schemes. It was proved in [2] that, for certain piecewise equidistant spatial meshes, the weighted average scheme converges μ-uniformly to the exact solution of problem (1.1). The truncation error analysis of [4] proved μ-uniform convergence on special fitted piecewise uniform and log-meshes.

In order to practically compute the nonlinear weighted average scheme, one requires a robust and efficient algorithm. A fruitful method is the method of upper and lower solutions and its associated monotone iterations. Since the initial data in the monotone iterative method is either an upper or lower solution, constructed directly from the difference equations without any knowledge of the exact solution, this method simplifies the search for the initial data and thus gives a practical advantage over Newton's method in the computation of numerical solutions. Based on the method of upper and lower solutions, the monotone iterative method of [2] converges μ-uniformly to the solution of problem (1.1) and only requires the solution of linear systems at each iterative step. The numerical experiments of [2] and [4] confirmed the theoretical rates of convergence on piecewise equidistant and log-meshes.

Iterative domain decomposition algorithms based on Schwarz-type alternating procedures have received much attention for their potential as efficient algorithms for parallel computing. In [3], we proposed an iterative algorithm for solving the nonlinear implicit finite difference scheme approximation of the partial differential equation (1.1). This algorithm combines the monotone approach and an iterative domain decomposition method based on the Schwarz alternating procedure. The spatial computational domain is partitioned into nonoverlapping box-subdomains. At each horizontal and vertical boundary, a small interfacial subdomain is introduced and an associated linear problem generates boundary values for the nonoverlapping box-subdomains. Thus, this approach may be considered as a variant of a block Gauss-Seidel iteration (or in the parallel context as a multicoloured algorithm) for the nonoverlapping box-subdomains with a Dirichlet-Dirichlet coupling through the interface variables. In this paper, we generalize the monotone box-domain decomposition algorithm of [3] from the nonlinear implicit scheme to the nonlinear weighted average scheme.

The structure of the paper is as follows. In Section 2, we present the nonlinear weighted average scheme and discuss the stability of two different weightings. Section 3 proposes a monotone domain decomposition algorithm based on the box-domain decomposition from [3]. We develop estimates of the rate of convergence and prove that on the piecewise uniform meshes the monotone domain decomposition algorithm converges μ-uniformly to the solution of (1.1). The numerical experiments of Section 4 correlate the convergence behaviour of the algorithm with the theoretical convergence parameter derived in Section 3. The experiments demonstrate the surprising result that, for sufficiently small perturbation parameter μ, this paper's domain decomposition generalization of the algorithm from [2] executes more quickly than the original undecomposed algorithm.

2. A Weighted Average Scheme

On $\overline{Q}$ introduce a rectangular mesh $\overline{\omega}^h \times \overline{\omega}^\tau$, $\overline{\omega}^h = \overline{\omega}^{hx} \times \overline{\omega}^{hy}$:

$$\overline{\omega}^{hx} = \{x_i,\ 0 \le i \le N_x;\ x_0 = 0,\ x_{N_x} = 1;\ h_{xi} = x_{i+1} - x_i\}, \tag{2.1a}$$

$$\overline{\omega}^{hy} = \left\{y_j,\ 0 \le j \le N_y;\ y_0 = 0,\ y_{N_y} = 1;\ h_{yj} = y_{j+1} - y_j\right\}, \tag{2.1b}$$

$$\overline{\omega}^{\tau} = \{t_k = k\tau,\ 0 \le k \le N_\tau,\ N_\tau \tau = T\}. \tag{2.1c}$$

For a mesh function $U(P,t)$, $P=(x,y)\in\overline{\omega}^h,\ t\in\overline{\omega}^\tau$, we use the weighted average or θ- method

$$\theta\mathcal{L}_h U(P,t)+(1-\theta)\mathcal{L}_h U(P,t-\tau)+\frac{1}{\tau}\left[U(P,t)-U(P,t-\tau)\right]=-\mathcal{F}, \tag{2.2a}$$

$$\mathcal{F}\equiv\theta f(P,t,U)+(1-\theta)f(P,t-\tau,U),\quad (P,t)\in\omega^h\times\omega^\tau, \tag{2.2b}$$

$$U(P,0)=u^0(P),\quad P\in\overline{\omega}^h,\quad U(P,t)=g(P,t),\quad (P,t)\in\partial\omega^h\times\omega^\tau, \tag{2.2c}$$

where $\theta=\text{const}$ and, when no confusion arises, we write $f(P,t,U(P,t))=f(P,t,U)$. $\mathcal{L}_h U$ is defined by

$$\mathcal{L}_h U=-\mu^2\left(\mathcal{D}_x^2 U+\mathcal{D}_y^2 U\right),$$

where $\mathcal{D}_x^2 U$ and $\mathcal{D}_y^2 U$ are the central difference approximations to the second derivatives

$$\mathcal{D}_x^2 U_{ij}^k=(\hbar_{xi})^{-1}\left[\left(U_{i+1,j}^k-U_{ij}^k\right)(h_{xi})^{-1}-\left(U_{ij}^k-U_{i-1,j}^k\right)(h_{x,i-1})^{-1}\right],$$

$$\mathcal{D}_y^2 U_{ij}^k=(\hbar_{yj})^{-1}\left[\left(U_{i,j+1}^k-U_{ij}^k\right)(h_{yj})^{-1}-\left(U_{ij}^k-U_{i,j-1}^k\right)(h_{y,j-1})^{-1}\right],$$

$$\hbar_{xi}=\frac{1}{2}\left(h_{x,i-1}+h_{xi}\right),\quad \hbar_{yj}=\frac{1}{2}\left(h_{y,j-1}+h_{yj}\right),$$

where $U_{ij}^k\equiv U\left(x_i,y_j,t_k\right)$.

This 10-point difference scheme can be regarded as taking a weighted average of the explicit scheme ($\theta=0$) and the fully implicit scheme ($\theta=1$). We assume that we are using an average with nonnegative weights, so that $0\le\theta\le1$.

Introduce the notation

$$\begin{aligned}
&v_i=v_{xi}^l+v_{xi}^r,\quad v_{xi}^l=(\hbar_{xi})^{-1}(h_{x,i-1})^{-1},\quad v_{xi}^r=(\hbar_{xi})^{-1}(h_{xi})^{-1},\\
&w_j=w_{yj}^l+w_{yj}^r,\quad w_{yj}^l=(\hbar_{yj})^{-1}(h_{y,j-1})^{-1},\quad w_{yj}^r=(\hbar_{yj})^{-1}(h_{yj})^{-1},\\
&\bar{v}=\max_{1\le i\le N_x-1} v_i,\quad \bar{w}=\max_{1\le j\le N_y-1} w_j.
\end{aligned} \tag{2.3}$$

We suppose that the time mesh spacing τ satisfies the constraint

$$\tau(1-\theta)\le\frac{1}{\mu^2\left(\bar{v}+\bar{w}\right)+c^*}. \tag{2.4}$$

The condition (2.4), known as the CFL condition, guarantees the discrete maximum principle on the computational domain $\overline{\omega}^h\times\overline{\omega}^\tau$. This imposes no time step restriction on the implicit scheme, for which $\theta=1$. A more interesting question is the stability of the Crank-Nicolson scheme [5], for which $\theta=0.5$. For a linear problem (1.1) with constant coefficients, Fourier analysis places no stability restriction on the Crank-Nicolson scheme, in contrast to condition (2.4). One can see that the CFL condition (2.4) is sharp by considering the one-dimensional linear problem $-\mu^2 u_{xx}+u_t=0$ with initial data

$$u^0(x)=\{2x,0\le x\le0.5;2(1-x),0.5\le x\le1\},$$

boundary conditions $g(0,t)=g(1,t)=0$ and $N_x=2$ mesh intervals [4].

Thus the maximum principle analysis can be viewed as an alternative means of obtaining stability conditions. It has the advantage over Fourier analysis that it is easily extended to problems with variable coefficients and to nonlinear problems. We mention that, in general, the maximum principle analysis gives only sufficient conditions for stability of difference schemes.

3. Monotone Domain Decomposition Algorithm

As in [3], we consider a rectangular decomposition of the spatial domain $\overline{\omega}$ into $(M \times L)$ nonoverlapping subdomains $\overline{\omega}_{ml}$, $m = 1, \cdots, M$, $l = 1, \cdots, L$:

$$\omega_{ml} = (x_{m-1}, x_m) \times (y_{l-1}, y_l), \quad x_0 = 0, \quad x_M = 1, \quad y_0 = 0, \quad y_L = 1.$$

We also introduce vertical interfacial subdomains η_m, $m = 1, \cdots, M-1$ (vertical strips):

$$\begin{aligned} &\eta_m = \eta_m^x \times \omega^y = \{x_m^b < x < x_m^e\} \times \{0 < y < 1\}, \quad \eta_{m-1} \cap \eta_m = \emptyset, \\ &\gamma_m^b = \{x = x_m^b, 0 \le y \le 1\}, \quad \gamma_m^e = \{x = x_m^e, 0 \le y \le 1\}, \\ &x_m^b < x_m < x_m^e, \quad \gamma_m^0 = \partial\omega \cap \partial\eta_m, \end{aligned}$$

and horizontal interfacial subdomains ϑ_l, $l = 1, \cdots, L-1$ (horizontal strips):

$$\begin{aligned} &\vartheta_l = \omega^x \times \vartheta_l^y = \{0 < x < 1\} \times \{y_l^b < y < y_l^e\}, \quad \vartheta_{l-1} \cap \vartheta_l = \emptyset, \\ &\rho_l^b = \{0 \le x \le 1, y = y_l^b\}, \quad \rho_l^e = \{0 \le x \le 1, y = y_l^e\}, \\ &y_l^b < y_l < y_l^e, \quad \rho_l^0 = \partial\omega \cap \partial\vartheta_l. \end{aligned}$$

On $\overline{\omega}_{ml}$, $m = 1, \cdots, M$, $l = 1, \cdots, L$; $\overline{\eta}_m$, $m = 1, \cdots, M-1$ and $\overline{\vartheta}_l$, $l = 1, \cdots, L-1$, introduce meshes:

$$\begin{aligned} &\overline{\omega}_{ml}^h = \overline{\omega}_{ml} \cap \overline{\omega}^h, \quad \overline{\eta}_m^h = \overline{\eta}_m \cap \overline{\omega}^h, \quad \overline{\vartheta}_l^h = \overline{\vartheta}_l \cap \overline{\omega}^h, \\ &\left\{x_m^b, x_m, x_m^e\right\}_{m=1}^{M-1} \in \omega^{hx}, \quad \left\{y_l^b, y_l, y_l^e\right\}_{l=1}^{L-1} \in \omega^{hy}, \\ &\gamma_m^{h0,b,e} = \gamma_m^{0,b,e} \cap \overline{\omega}^h, \quad \rho_l^{h0,b,e} = \rho_l^{0,b,e} \cap \overline{\omega}^h, \end{aligned}$$

where ω^{hx}, ω^{hy} are defined by (2.1).

3.1. Statement of the algorithm

We represent the difference equation from (2.2) in the equivalent form

$$\mathcal{G}_1(P,t,U) + \mathcal{G}_2(P,t-\tau,U) = 0, \quad (P,t) \in \omega^h \times \omega^\tau, \tag{3.1a}$$

$$\mathcal{G}_1(P,t,U) \equiv \left(\theta\mathcal{L}_h + \tau^{-1}\right) U(P,t) + \theta f(P,t,U), \tag{3.1b}$$

$$\mathcal{G}_2(P,t-\tau,U) \equiv \left[(1-\theta)\mathcal{L}_h - \tau^{-1}\right] U(P,t-\tau) + (1-\theta) f(P,t-\tau,U). \tag{3.1c}$$

We say that on a time level $t \in \omega^\tau$, $\overline{V}(P,t)$ is an upper solution with respect to a given function $V(P,t-\tau)$ if it satisfies

$$\begin{aligned} &\mathcal{G}_1\left(P,t,\overline{V}\right) + \mathcal{G}_2(P,t-\tau,V) \ge 0, \quad (P,t) \in \omega^h \times \omega^\tau, \\ &\overline{V}(P,t) = g(P,t), \quad P \in \partial\omega^h. \end{aligned}$$

Similarly, $\underline{V}(P,t)$ is called a lower solution with respect to a given function $V(P,t-\tau)$ if it satisfies the reversed inequality and the boundary condition.

Introduce the following notation:

$$\mathcal{L} \equiv \theta\mathcal{L}_h + \tau^{-1} + \theta c^*, \tag{3.2a}$$

$$\mathcal{G}\left(V(P,t), W(P,t-\tau)\right) \equiv \mathcal{G}_1(P,t,V) + \mathcal{G}_2(P,t-\tau,W), \tag{3.2b}$$

where c^* is defined in (1.2).

On each time level $t \in \omega^\tau$, we calculate n_* iterates $V^{(n)}(P,t)$, $P \in \overline{\omega}^h$, $n = 1, \cdots, n_*$ as follows.

Step 0. On the whole mesh $\overline{\omega}^h$ choose an upper or lower solution $V^{(0)}(P,t)$ satisfying the boundary condition $V^{(0)}(P,t) = g(P,t),\ P \in \partial\omega^h$.
For $n = 0$ to $n_* - 1$ do Steps 1-4
Step 1. For each subdomain $\overline{\omega}^h_{ml}$, solve the linear problem

$$\mathcal{L}Z^{(n+1)}_{ml}(P,t) = -\mathcal{G}\left(V^{(n)}(P,t), V(P,t-\tau)\right), \quad P \in \omega^h_{ml}, \tag{3.3}$$

with $Z^{(n+1)}_{ml}(\partial\omega^h_{ml}, t) = 0$, where $\mathcal{L}$ and $\mathcal{G}$ are defined in (3.2).
Step 2. For each vertical interfacial subdomain $\overline{\eta}^h_m$, solve the linear problem

$$\mathcal{L}Z^{(n+1)}_{m}(P,t) = -\mathcal{G}\left(V^{(n)}(P,t), V(P,t-\tau)\right), \quad P \in \eta^h_{m}, \tag{3.4}$$

with $Z^{(n+1)}_m(\partial\eta^h_m, t)$ defined by the mesh functions computed in Step 1.
Step 3. For each horizontal interfacial subdomain solve the linear problem

$$\mathcal{L}\widetilde{Z}^{(n+1)}_{l}(P,t) = -\mathcal{G}\left(V^{(n)}(P,t), V(P,t-\tau)\right), \quad P \in \vartheta^h_{l}, \tag{3.5}$$

with $\widetilde{Z}^{(n+1)}_l(\partial\vartheta^h_l, t)$ defined by the mesh functions computed in Steps 1 and 2.
Step 4. Piece together the mesh functions from Steps 1 through 3:

$$V^{(n+1)}(P,t) = \begin{cases} V^{(n)}(P,t) + \widetilde{Z}^{(n+1)}_l(P,t), & P \in \overline{\vartheta}^h_l; \\ V^{(n)}(P,t) + Z^{(n+1)}_m(P,t), & P \in \overline{\eta}^h_m \setminus \overline{\vartheta}^h; \\ V^{(n)}(P,t) + Z^{(n+1)}_{ml}(P,t), & P \in \overline{\omega}^h_{ml} \setminus \left(\overline{\eta}^h \cup \overline{\vartheta}^h\right), \end{cases} \tag{3.6}$$

where we use the notation $\overline{\eta}^h = \bigcup_{m=1}^{M-1} \overline{\eta}^h_m$, $\overline{\vartheta}^h = \bigcup_{l=1}^{L-1} \overline{\vartheta}^h_l$.
Step 5. Set up

$$V(P,t) = V^{(n_*)}(P,t), \quad P \in \overline{\omega}^h. \tag{3.7}$$

Algorithm (3.3)-(3.7) can be carried out by parallel processing. Steps 1, 2 and 3 must be performed sequentially, but on each step, the independent subproblems may be assigned to different computational nodes.

3.2. Monotone convergence of algorithm (3.3)-(3.7)

We have the following convergence property of algorithm (3.3)-(3.7).

Theorem 3.1. *Let $V(P,t-\tau)$ be given and $\overline{V}^{(0)}(P,t)$, $\underline{V}^{(0)}(P,t)$ be upper and lower solutions corresponding to $V(P,t-\tau)$. Suppose that f satisfies* (1.2). *Then the upper sequence $\{\overline{V}^{(n)}(P,t)\}$ generated by* (3.3)-(3.7) *converges monotonically from above to the unique solution $V^*(P,t)$ of the problem*

$$\begin{aligned} &\mathcal{G}\left(V(P,t), V(P,t-\tau)\right) = 0, \quad P \in \omega^h, \\ &V(P,t) = g(P,t), \quad P \in \partial\omega^h, \end{aligned}$$

and the lower sequence $\{\underline{V}^{(n)}(P,t)\}$ generated by (3.3)-(3.7) *converges monotonically from below* *to* $V^*(P,t)$*:*

$$\begin{aligned}&V^*(P,t)\le \overline{V}^{(n+1)}(P,t)\le \overline{V}^{(n)}(P,t)\le \overline{V}^{(0)}(P,t),\quad P\in\overline{\omega}^h,\\&\underline{V}^{(0)}(P,t)\le \underline{V}^{(n)}(P,t)\le \underline{V}^{(n+1)}(P,t)\le V^*(P,t),\quad P\in\overline{\omega}^h.\end{aligned}$$

The proof of the theorem is similar to the proof of Theorem 1 from [3].

Remark 3.1. Consider the following approach for constructing initial upper and lower solutions $\overline{V}^{(0)}(P,t)$ and $\underline{V}^{(0)}(P,t)$. Suppose that, for t fixed, a mesh function $R(P,t)$ is defined on $\overline{\omega}^h$ and satisfies the boundary condition $R(P,t)=g(P,t)$ on $\partial\omega^h$. Introduce the following difference problems:

$$\begin{aligned}&\left(\theta\mathcal{L}_h+\tau^{-1}\right)Z_q^{(0)}(P,t)=q\left|\mathcal{G}\left(R(P,t),V(P,t-\tau)\right)\right|,\quad P\in\omega^h,\\&Z_q^{(0)}(P,t)=0,\quad P\in\partial\omega^h,\quad q=1,-1.\end{aligned}\tag{3.8}$$

Then the functions

$$\overline{V}^{(0)}(P,t)=R(P,t)+Z_1^{(0)}(P,t)$$

and

$$\underline{V}^{(0)}(P,t)=R(P,t)+Z_{-1}^{(0)}(P,t)$$

are upper and lower solutions, respectively. The proof of this result can be found in [2].

Remark 3.2. Since the initial iteration in the monotone domain decomposition algorithm (3.3)-(3.7) is either an upper or lower solution which can be constructed directly from the difference equation without any knowledge of the exact solution, as we have suggested in the previous remark, this algorithm eliminates the search for the initial iteration as is often needed in Newton's method. This elimination offers a practical advantage in the computation of numerical solutions.

3.3. Convergence analysis of algorithm (3.3)-(3.7)

On each time level, we consider the linear difference problem

$$\mathcal{L}W(P,t)=F(P,t),\quad P\in\omega^h,\quad W(P,t)=W^0(P,t),\quad P\in\partial\omega^h,\tag{3.9}$$

where $\mathcal{L}$ is defined in (3.2). We now formulate a discrete maximum principle and give an estimate on the solution to (3.9).

Lemma 3.1. *(i) If W(P,t) satisfies the conditions*

$$\mathcal{L}W(P,t)-F(P,t)\ge 0\ (\le 0),\quad P\in\omega^h;\quad W(P,t)\ge 0\ (\le 0),\quad P\in\partial\omega^h,$$

then $W(P,t)\ge 0$ (≤ 0), $P\in\overline{\omega}^h$.

(ii) The following estimate of the solution to (3.9) *holds true*

$$\begin{aligned}&\|W(t)\|_{\overline{\omega}^h}\le\max\left[\left\|W^0(t)\right\|_{\partial\omega^h},\quad \|F(t)\|_{\omega^h}/\left(\tau^{-1}+\theta c^*\right)\right],\\&\|W^0(t)\|_{\partial\omega^h}\equiv\max_{P\in\partial\omega^h}|W^0(P,t)|,\quad \|F(t)\|_{\omega^h}\equiv\max_{P\in\omega^h}|F(P,t)|.\end{aligned}\tag{3.10}$$

The proof of the lemma can be found in [8].

We now establish convergence properties of algorithm (3.3)-(3.7). If we denote

$$Z^{(n+1)}(P,t)=V^{(n+1)}(P,t)-V^{(n)}(P,t),\quad P\in\overline{\omega}^h,$$

then from (3.3)-(3.6), $Z^{(n+1)}$ can be written in the form

$$Z^{(n+1)}(P,t)=\begin{cases} Z_{ml}^{(n+1)}(P,t), & P\in\overline{\omega}_{ml}^h\setminus\left(\overline{\eta}^h\cup\overline{\vartheta}^h\right),\\ Z_m^{(n+1)}(P,t), & P\in\overline{\eta}_m^h\setminus\overline{\vartheta}^h,\\ \widetilde{Z}_l^{(n+1)}(P,t), & P\in\overline{\vartheta}_l^h.\end{cases}\tag{3.11}$$

Introduce the notation

$$\hbar_{xm}^{b,e}=\frac{1}{2}\left(h_{xm}^{b-,e-}+h_{xm}^{b+,e+}\right),\quad \hbar_{yl}^{b,e}=\frac{1}{2}\left(h_{yl}^{b-,e-}+h_{yl}^{b+,e+}\right),$$

where $h_{xm}^{b\pm,e\pm}$ are the mesh step sizes on the right and left from points $x_m^{b,e}$ and $h_{yl}^{b\pm,e\pm}$ are the mesh step sizes on the top and bottom from points $y_l^{b,e}$, and

$$\begin{aligned}
&\kappa_{xm}^b\equiv\frac{\theta\mu^2}{\left(\theta c^*+\tau^{-1}\right)\hbar_{xm}^b h_{xm}^{b+}},\quad \kappa_{xm}^e\equiv\frac{\theta\mu^2}{\left(\theta c^*+\tau^{-1}\right)\hbar_{xm}^e h_{xm}^{e-}},\\
&\kappa_{yl}^b\equiv\frac{\theta\mu^2}{\left(\theta c^*+\tau^{-1}\right)\hbar_{yl}^b h_{yl}^{b+}},\quad \kappa_{yl}^e\equiv\frac{\theta\mu^2}{\left(\theta c^*+\tau^{-1}\right)\hbar_{yl}^e h_{yl}^{e-}},\\
&r^I=\max_{1\le m\le M-1}\left\{\kappa_{xm}^b;\kappa_{xm}^e\right\},\quad r^{II}=\max_{1\le l\le L-1}\left\{\kappa_{yl}^b;\kappa_{yl}^e\right\}.
\end{aligned}$$

On each time level $t\in\omega^\tau$, we have the following convergence property of algorithm (3.3)-(3.7).

Theorem 3.2. *For algorithm* (3.3)-(3.7)*, the following estimate holds true*

$$\left\|Z^{(n+1)}(t)\right\|_{\overline{\omega}^h}\le\widetilde{r}\left\|Z^{(n)}(t)\right\|_{\overline{\omega}^h},\quad \widetilde{r}=r+r^I+r^{II},\quad t\in\omega^\tau,\tag{3.12}$$

where

$$Z^{(n)}(P,t)=V^{(n)}(P,t)-V^{(n-1)}(P,t),\quad r=\theta c^*/\left(\theta c^*+\tau^{-1}\right).$$

Proof. Suppose that the sequence $\left\{V^{(n)}\right\}$ generated by (3.3)-(3.6) is an upper sequence. From (3.3), we use (3.10) to get the estimate

$$\begin{aligned}
&\left\|Z_{ml}^{(n+1)}(t)\right\|_{\overline{\omega}_{ml}^h}\le\frac{1}{\theta c^*+\tau^{-1}}\left\|\mathcal{G}^{(n)}(t)\right\|_{\omega^h},\\
&\mathcal{G}^{(n)}(P,t)\equiv\mathcal{G}\left(V^{(n)}(P,t),V(P,t-\tau)\right).
\end{aligned}$$

From here and estimating (3.4) by (3.10), we conclude that

$$\begin{aligned}
&\left\|Z_m^{(n+1)}(t)\right\|_{\overline{\eta}_m^h}\le\max\left\{\frac{1}{\theta c^*+\tau^{-1}}\left\|\mathcal{G}^{(n)}(t)\right\|_{\omega^h};\max_{1\le l\le L}\left[\left\|Z_{ml}^{(n+1)}(t)\right\|_{\gamma_{ml}^{hb}};\right.\right.\\
&\qquad\left.\left.\left\|Z_{m+1,l}^{(n+1)}(t)\right\|_{\gamma_{ml}^{he}}\right]\right\}\le\frac{1}{\theta c^*+\tau^{-1}}\left\|\mathcal{G}^{(n)}(t)\right\|_{\omega^h},\\
&\gamma_{ml}^{hb}=\gamma_m^{hb}\cap\overline{\omega}_{ml}^h,\quad \gamma_{ml}^{he}=\gamma_m^{he}\cap\overline{\omega}_{m+1,l}^h.
\end{aligned}$$

Similarly, from here and (3.5), we can obtain the estimate

$$\left\|\widetilde{Z}_l^{(n+1)}(t)\right\|_{\overline{\vartheta}_l^h} \leq \frac{1}{\theta c^* + \tau^{-1}} \left\|\mathcal{G}^{(n)}(t)\right\|_{\omega^h}.$$

Thus, by the definition of $Z^{(n+1)}$ in (3.11), we have

$$\left\|Z^{(n+1)}(t)\right\|_{\overline{\omega}^h} \leq \frac{1}{\theta c^* + \tau^{-1}} \left\|\mathcal{G}^{(n)}(t)\right\|_{\omega^h}. \tag{3.13}$$

From Lemma 3.1, by the maximum principle for the difference operator $\mathcal{L}$, it follows that

$$Z_{ml}^{(n+1)}(P,t) \leq 0, \quad P \in \overline{\omega}_{ml}^h. \tag{3.14}$$

Using the mean-value theorem and the equation for $Z_{ml}^{(n+1)}$ from (3.3), we have

$$\begin{aligned} &\mathcal{G}\left(V_{ml}^{(n+1)}, V\right) = -\theta\left(c^* - f_{u,ml}^{(n)}\right) Z_{ml}^{(n+1)}(P,t) \geq 0, \quad P \in \omega_{ml}^h, \\ &f_{u,ml}^{(n)} \equiv f_u\left[P, t, V^{(n)}(P,t) + \Theta_{ml}^{(n)}(P,t) Z_{ml}^{(n+1)}(P,t)\right], \quad 0 < \Theta_{ml}^{(n)}(P,t) < 1, \end{aligned} \tag{3.15}$$

where

$$V_{ml}^{(n+1)} = V^{(n)} + Z_{ml}^{(n+1)}$$

and nonnegativeness of the right hand side follows from (1.2) and (3.14). If no confusion arises, we write

$$\mathcal{G}(S(P,t), W(P,t-\tau)) = \mathcal{G}(S,W).$$

Taking into account (3.14) and $V^{(n)}$ is an upper solution, by the maximum principle in Lemma 3.1, it follows from (3.4) and (3.5) that

$$\begin{aligned} &Z_m^{(n+1)}(P,t) \leq 0, \quad P \in \overline{\eta}_m^h, \quad m = 1, \cdots, M-1, \\ &\widetilde{Z}_l^{(n+1)}(P,t) \leq 0, \quad P \in \overline{\vartheta}_l^h, \quad l = 1, \cdots, L-1. \end{aligned} \tag{3.16}$$

Similar to (3.15), we obtain the difference problems for $V_m^{(n+1)} = V^{(n)} + Z_m^{(n+1)}$

$$\mathcal{G}\left(V_m^{(n+1)}, V\right) = -\theta\left(c^* - f_{u,m}^{(n)}\right) Z_m^{(n+1)}(P,t) \geq 0, \quad P \in \eta_m^h, \tag{3.17}$$

$$V_m^{(n+1)}(P,t) = \begin{cases} g(P,t), & P \in \gamma_m^{h0}; \\ V_{ml}^{(n+1)}(P,t), & P \in \gamma_m^{hb} \cap \overline{\omega}_{ml}^h; \\ V_{m+1,l}^{(n+1)}(P,t), & P \in \gamma_m^{he} \cap \overline{\omega}_{m+1,l}^h, \end{cases}$$

and for $\widetilde{V}_l^{(n+1)} = V^{(n)} + \widetilde{Z}_l^{(n+1)}$

$$\mathcal{G}\left(\widetilde{V}_l^{(n+1)}, V\right) = -\theta\left(c^* - f_{u,l}^{(n)}\right) \widetilde{Z}_l^{(n+1)}(P,t) \geq 0, \quad P \in \vartheta_l^h,$$

$$\widetilde{V}_l^{(n+1)}(P,t) = \begin{cases} g(P,t), & P \in \rho_l^{h0}; \\ V_{ml}^{(n+1)}(P,t), & P \in (\rho_l^{hb} \setminus \eta^h) \cap \overline{\omega}_{ml}^h; \\ V_{m,l+1}^{(n+1)}(P,t), & P \in (\rho_l^{he} \setminus \eta^h) \cap \overline{\omega}_{m,l+1}^h; \\ V_m^{(n+1)}(P,t), & P \in \partial\vartheta_l^h \cap \eta_m^h, \end{cases}$$

where nonnegativeness of the right hand sides of the difference equations follows from (1.2) and (3.16). From here at the iterative step n, (3.15), (3.17) and using the definition of $Z^{(n)}$ in (3.11), we represent $\mathcal{G}^{(n)}(P,t)$ in the form

$$\mathcal{G}^{(n)}(P,t) = -\theta\left(c^* - f_u^{(n-1)}\right) Z^{(n)}(P,t), \quad P \in \widetilde{\omega}^h, \quad \widetilde{\omega}^h = \omega^h \setminus \left(\widetilde{\gamma}^h \cup \rho^h\right),$$

where $\widetilde{\gamma}^h$ and ρ^h are defined as follows

$$\widetilde{\gamma}_{ml}^{hb,e} = \left\{x_i = x_m^{b,e}, y_{l-1}^e < y_j < y_l^b\right\}, \quad \widetilde{\gamma}_m^{hb,e} = \bigcup_{l=1}^{L} \widetilde{\gamma}_{ml}^{hb,e}, \quad y_0^e = 0,\ y_L^b = 1,$$
$$\widetilde{\gamma}^{hb} = \bigcup_{m=1}^{M-1} \widetilde{\gamma}_m^{hb}, \quad \widetilde{\gamma}^{he} = \bigcup_{m=1}^{M-1} \widetilde{\gamma}_m^{he}, \quad \widetilde{\gamma}^h = \widetilde{\gamma}^{hb} \cup \widetilde{\gamma}^{he},$$
$$\rho^{hb} = \bigcup_{l=1}^{L-1} \rho_l^{hb}, \quad \rho^{he} = \bigcup_{l=1}^{L-1} \rho_l^{he}, \quad \rho^h = \rho^{hb} \cup \rho^{he}.$$

By (1.2),

$$\frac{1}{\theta c^* + \tau^{-1}} \left\| \mathcal{G}^{(n)}(t) \right\|_{\widetilde{\omega}^h} \leq r \left\| Z^{(n)}(t) \right\|_{\overline{\omega}^h}. \tag{3.18}$$

Now we estimate $\mathcal{G}^{(n)}(P,t)$ on $\widetilde{\gamma}^h$. On $\widetilde{\gamma}_{ml}^{hb} = \left\{x_i = x_m^b, y_{l-1}^e < y_j < y_l^b\right\}$, we represent $\mathcal{G}^{(n)}(P,t)$ in the form

$$\begin{aligned}\mathcal{G}^{(n)}\left(\widetilde{P}_m^b, t\right) &= \mathcal{G}\left(V_{ml}^{(n)}\left(\widetilde{P}_m^b, t\right), V\left(\widetilde{P}_m^b, t-\tau\right)\right) \\ &\quad - \frac{\theta\mu^2}{\hbar_{xm}^b h_{xm}^{b+}} \left(V_m^{(n)}\left(\widetilde{P}_m^{b+}, t\right) - V_{ml}^{(n)}\left(\widetilde{P}_m^{b+}, t\right)\right), \\ \widetilde{P}_m^b &= \left(x_m^b, y_j\right) \in \widetilde{\gamma}_{ml}^{hb}, \quad \widetilde{P}_m^{b+} = \left(x_m^b + h_{xm}^{b+}, y_j\right).\end{aligned}$$

From (3.14) at the iterative step n and the definition of $V^{(n)}$ in (3.6), we have

$$\begin{aligned}&V_{ml}^{(n)}\left(\widetilde{P}_m^{b+}, t\right) - V_m^{(n)}\left(\widetilde{P}_m^{b+}, t\right) \\ &\leq V^{(n-1)}\left(\widetilde{P}_m^{b+}, t\right) - V^{(n)}\left(\widetilde{P}_m^{b+}, t\right) = -Z^{(n)}\left(\widetilde{P}_m^{b+}, t\right).\end{aligned}$$

From here, (3.15) and taking into account that $Z_{ml}^{(n)}(P,t) = Z^{(n)}(P,t)$, $P \in \widetilde{\gamma}_{ml}^{hb}$, it follows that

$$\frac{1}{\theta c^* + \tau^{-1}} \left\| \mathcal{G}^{(n)}(t) \right\|_{\widetilde{\gamma}_{ml}^{hb}} \leq \left(r + \kappa_{xm}^b\right) \left\| Z^{(n)}(t) \right\|_{\overline{\omega}^h}.$$

Similarly, we can prove the estimate

$$\frac{1}{\theta c^* + \tau^{-1}} \left\| \mathcal{G}^{(n)}(t) \right\|_{\widetilde{\gamma}_{ml}^{he}} \leq \left(r + \kappa_{xm}^e\right) \left\| Z^{(n)}(t) \right\|_{\overline{\omega}^h}.$$

Thus, on $\widetilde{\gamma}^h$, we conclude the estimate

$$\frac{1}{\theta c^* + \tau^{-1}} \left\| \mathcal{G}^{(n)}(t) \right\|_{\widetilde{\gamma}^h} \leq \left(r + r^I\right) \left\| Z^{(n)}(t) \right\|_{\overline{\omega}^h}. \tag{3.19}$$

On $\widetilde{\rho}_{ml}^{hb} = \left\{x_{m-1}^e < x_i < x_m^b, y_j = y_l^b\right\}$, we represent $\mathcal{G}^{(n)}(P,t)$ in the form

$$\begin{aligned}\mathcal{G}^{(n)}\left(P_l^b, t\right) &= \mathcal{G}\left(V_{ml}^{(n)}\left(P_l^b, t\right), V\left(P_l^b, t-\tau\right)\right) \\ &\quad - \frac{\theta\mu^2}{\hbar_{yl}^b h_{yl}^{b+}} \left(\widetilde{V}_l^{(n)}\left(P_l^{b+}, t\right) - V_{ml}^{(n)}\left(P_l^{b+}, t\right)\right), \\ P_l^b &= \left(x_i, y_l^b\right) \in \widetilde{\rho}_{ml}^{hb}, \quad P_l^{b+} = \left(x_i, y_l^b + h_{yl}^{b+}\right).\end{aligned}$$

From (3.14) at the iterative step n and the definition of $V^{(n)}$ in (3.6), we have

$$\begin{aligned}&V_{ml}^{(n)}\left(P_l^{b+},t\right)-\widetilde{V}_l^{(n)}\left(P_l^{b+},t\right)\\ &\le V^{(n-1)}\left(P_l^{b+},t\right)-V^{(n)}\left(P_l^{b+},t\right)=-Z^{(n)}\left(P_l^{b+},t\right).\end{aligned}$$

From here and (3.15), and taking into account that $Z_{ml}^{(n)}(P,t)=Z^{(n)}(P,t)$, $P\in\widetilde{\rho}_{ml}^{hb}$, we get the estimate

$$\frac{1}{\theta c^*+\tau^{-1}}\left\|\mathcal{G}^{(n)}(t)\right\|_{\widetilde{\rho}_{ml}^{hb}}\le\left(r+\kappa_{yl}^b\right)\left\|Z^{(n)}(t)\right\|_{\overline{\omega}^h}.$$

Similarly, we can prove the estimate

$$\frac{1}{\theta c^*+\tau^{-1}}\left\|\mathcal{G}^{(n)}(t)\right\|_{\widetilde{\rho}_{ml}^{he}}\le\left(r+\kappa_{yl}^e\right)\left\|Z^{(n)}(t)\right\|_{\overline{\omega}^h}.$$

On $\hat{\rho}_{ml}^{hb}=\left\{x_m^b<x_i<x_m^e,y_j=y_l^b\right\}$, we represent $\mathcal{G}^{(n)}(P,t)$ in the form

$$\begin{aligned}\mathcal{G}^{(n)}\left(P_l^b,t\right)&=\mathcal{G}\left(V_m^{(n)}\left(P_l^b,t\right),V\left(P_l^b,t-\tau\right)\right)\\ &\quad-\frac{\theta\mu^2}{\hbar_{yl}^b h_{yl}^{b+}}\left(\widetilde{V}_l^{(n)}\left(P_l^{b+},t\right)-V_m^{(n)}\left(P_l^{b+},t\right)\right),\\ P_l^b&=\left(x_i,y_l^b\right)\in\hat{\rho}_{ml}^{hb},\quad P_l^{b+}=\left(x_i,y_l^b+h_{yl}^{b+}\right).\end{aligned}$$

From (3.16) at the iterative step n and the definition of $V^{(n)}$ in (3.6), we have

$$\begin{aligned}&V_m^{(n)}\left(P_l^{b+},t\right)-\widetilde{V}_l^{(n)}\left(P_l^{b+},t\right)\\ &\le V^{(n-1)}\left(P_l^{b+},t\right)-V^{(n)}\left(P_l^{b+},t\right)=-Z^{(n)}\left(P_l^{b+},t\right).\end{aligned}$$

From here and (3.17), and taking into account that $Z_m^{(n)}(P,t)=Z^{(n)}(P,t)$, $P\in\hat{\rho}_{ml}^{hb}$, we get the estimate

$$\frac{1}{\theta c^*+\tau^{-1}}\left\|\mathcal{G}^{(n)}(t)\right\|_{\hat{\rho}_{ml}^{hb}}\le\left(r+\kappa_{yl}^b\right)\left\|Z^{(n)}(t)\right\|_{\overline{\omega}^h}.$$

Similarly, we can prove the estimate

$$\frac{1}{\theta c^*+\tau^{-1}}\left\|\mathcal{G}^{(n)}(t)\right\|_{\hat{\rho}_{ml}^{he}}\le\left(r+\kappa_{yl}^e\right)\left\|Z^{(n)}(t)\right\|_{\overline{\omega}^h}.$$

At $P_{ml}^b=\left(x_m^b,y_l^b\right)$, we represent $\mathcal{G}^{(n)}(P,t)$ in the form

$$\begin{aligned}\mathcal{G}^{(n)}\left(P_{ml}^b,t\right)&=\mathcal{G}\left(V_{ml}^{(n)}\left(P_{ml}^b,t\right),V\left(P_{ml}^b,t-\tau\right)\right)-\frac{\theta\mu^2}{\hbar_{xm}^b h_{xm}^{b+}}\left(V_m^{(n)}\left(P_{ml}^{bx+},t\right)\right.\\ &\quad\left.-V_{ml}^{(n)}\left(P_{ml}^{bx+},t\right)\right)-\frac{\theta\mu^2}{\hbar_{yl}^b h_{yl}^{b+}}\left(\widetilde{V}_l^{(n)}\left(P_{ml}^{by+},t\right)-V_{ml}^{(n)}\left(P_{ml}^{by+},t\right)\right),\\ P_{ml}^{bx+}&=\left(x_m^b+h_{xm}^{b+},y_l^b\right),\quad P_{ml}^{by+}=\left(x_m^b,y_l^b+h_{yl}^{b+}\right).\end{aligned}$$

From (3.14) at the iterative step n and the definition of $V^{(n)}$ in (3.6), we have

$$\begin{aligned}&V_{ml}^{(n)}\left(P_{ml}^{bx+},t\right)-V_m^{(n)}\left(P_{ml}^{bx+},t\right)\\ &\le V^{(n-1)}\left(P_{ml}^{bx+},t\right)-V^{(n)}\left(P_{ml}^{bx+},t\right)=-Z^{(n)}\left(P_{ml}^{bx+},t\right),\\ &V_{ml}^{(n)}\left(P_{ml}^{by+},t\right)-\widetilde{V}_l^{(n)}\left(P_{ml}^{by+},t\right)\\ &\le V^{(n-1)}\left(P_{ml}^{by+},t\right)-V^{(n)}\left(P_{ml}^{by+},t\right)=-Z^{(n)}\left(P_{ml}^{by+},t\right).\end{aligned}$$

From here and (3.15), and taking into account that $Z^{(n)}_{ml}\left(P^b_{ml},t\right)=Z^{(n)}\left(P^b_{ml},t\right)$, we get the estimate

$$\frac{1}{\theta c^*+\tau^{-1}}\left|\mathcal{G}^{(n)}(t)\left(P^b_{ml},t\right)\right|\leq\left(r+\kappa^b_{xm}+\kappa^b_{yl}\right)\left\|Z^{(n)}(t)\right\|_{\overline{\omega}^h}.$$

By the same reasonings, the following estimate holds true

$$\frac{1}{\theta c^*+\tau^{-1}}\left|\mathcal{G}^{(n)}(t)\left(P^e_{m-1,l}\right)\right|\leq\left(r+\kappa^e_{x,m-1}+\kappa^b_{yl}\right)\left\|Z^{(n)}(t)\right\|_{\overline{\omega}^h},$$
$$P^e_{m-1,l}=\left(x^e_{m-1},y^b_l\right).$$

On ρ^{hb}_l, we conclude the estimate

$$\frac{1}{\theta c^*+\tau^{-1}}\left\|\mathcal{G}^{(n)}(t)\right\|_{\rho^{hb}_l}\leq\left(r+r^I+r^{II}\right)\left\|Z^{(n)}(t)\right\|_{\overline{\omega}^h}.$$

The same estimate holds true on ρ^{he}_l, and on ρ^h we get the estimate

$$\frac{1}{\theta c^*+\tau^{-1}}\left\|\mathcal{G}^{(n)}(t)\right\|_{\rho^h}\leq\left(r+r^I+r^{II}\right)\left\|Z^{(n)}(t)\right\|_{\overline{\omega}^h}.$$

From here, (3.18) and (3.19), we conclude the estimate

$$\frac{1}{\theta c^*+\tau^{-1}}\left\|\mathcal{G}^{(n)}(t)\right\|_{\omega^h}\leq\left(r+r^I+r^{II}\right)\left\|Z^{(n)}(t)\right\|_{\overline{\omega}^h}, \tag{3.20}$$

and by (3.13), prove the theorem. □

Theorem 3.3. *Let $V^{(0)}(P,t)$ be an upper or lower solution in the domain decomposition algorithm* (3.3)-(3.7)*, and let f satisfy* (1.2)*. If the CFL condition* (2.4) *holds true, then the following estimate on convergence rate holds*

$$\max_{t_k\in\omega^\tau}\left\|V\left(t_k\right)-U\left(t_k\right)\right\|\leq C\left(\theta c^*+\tau^{-1}\right)\widetilde{r}^{n_*}, \tag{3.21}$$

where $\widetilde{r}$ is defined in (3.12)*, $U(P,t)$ is the solution to* (2.2) *and the constant C is independent of τ. Furthermore, the sequence $\left\{V^{(n)}(P,t)\right\}$ converges monotonically on each time level.*

Proof. The difference problem (2.2) can be represented in the form

$$\mathcal{G}\left(U\left(P,t\right),U\left(P,t-\tau\right)\right)=0,\quad P\in\omega^h,$$
$$U\left(P,t\right)=g\left(P,t\right),\quad P\in\partial\omega^h.$$

If we add and subtract the term $\mathcal{G}\left(V\left(P,t\right),V\left(P,t-\tau\right)\right)$ on the left hand side of the equation then, by the mean-value theorem, we get the difference problem for $W\left(P,t\right)=U\left(P,t\right)-V\left(P,t\right)$

$$\begin{aligned}&\left(\theta\widetilde{\mathcal{L}}_h(P,t)+\tau^{-1}\right)W(P,t)\\ &=-\left[(1-\theta)\widetilde{\mathcal{L}}_h(P,t-\tau)-\tau^{-1}\right]W(P,t-\tau)-\mathcal{G}\left(V^{(n_*)}(P,t),V(P,t-\tau)\right),\quad P\in\omega^h,\\ &W(P,t)=0,\ P\in\partial\omega^h,\quad \widetilde{\mathcal{L}}_h(P,t)\equiv\mathcal{L}_h+f_u(P,t),\end{aligned}$$

where

$$f_u(P,t)\equiv f_u\left[P,t,V(P,t)+\Theta(P,t)W(P,t)\right],\quad 0<\Theta(P,t)<1$$

and $V(P,t) = V^{(n_*)}(P,t)$. At the mesh point (x_i, y_j, t_k), represent the above difference equation in the following equivalent form

$$\begin{aligned}&\left(1+\theta s_{ij}^k\right) W_{ij}^k = \theta\left[\mathcal{M}_x\left(W_{ij}^k\right)+\mathcal{M}_y\left(W_{ij}^k\right)\right]+(1-\theta)\left[\mathcal{M}_x\left(W_{ij}^{k-1}\right)+\mathcal{M}_y\left(W_{ij}^{k-1}\right)\right]\\&\qquad\qquad\quad+\left[1-(1-\theta)s_{ij}^{k-1}\right]W_{ij}^{k-1}-\tau\mathcal{G}_{ij}^{(n_*),k},\\&s_{ij}^k=\tau\mu^2\left(v_i+w_j\right)+\tau f_{u,ij}^k,\quad \mathcal{M}_x\left(W_{ij}^k\right)\equiv\tau\mu^2\left(v_{xi}^r W_{i+1,j}^k+v_{xi}^l W_{i-1,j}^k\right),\\&\mathcal{M}_y\left(W_{ij}^k\right)\equiv\tau\mu^2\left(w_{yj}^r W_{i,j+1}^k+w_{yj}^l W_{i,j-1}^k\right),\end{aligned}$$

where we use the notation from (2.3). Under the hypotheses of the theorem, all the coefficients on the right hand side of the difference equation are nonnegative, and we conclude the estimate

$$\|W(t)\|_{\overline{\omega}^h} \le \|W(t-\tau)\|_{\overline{\omega}^h} + \tau\left\|\mathcal{G}^{(n_*)}(t)\right\|_{\omega^h}.$$

Taking into account $W(P,0)=0$, from (3.12) and (3.20), it follows that

$$\|W(t_k)\| \le \left(\sum_{l=1}^k \zeta_l\right)\tau\left(\theta c^*+\tau^{-1}\right)\widetilde{r}^{n_*},\quad k=1,\cdots,N_\tau;\quad \zeta_l=\left\|Z^{(1)}(t_l)\right\|_{\overline{\omega}^h}.$$

Applying (3.10) to estimate consecutively $Z_{ml}^{(1)}(P,t)$, $Z_m^{(1)}(P,t)$ and $\widetilde{Z}_l^{(1)}(P,t)$ from (3.3)-(3.5), respectively, and using the definition of $Z^{(1)}(P,t)$ in (3.11), we get

$$\left\|Z^{(1)}(t_l)\right\|_{\overline{\omega}^h} \le \tau\left\|\mathcal{G}\left(V^{(0)}(t_l), V(t_l-\tau)\right)\right\|_{\omega^h} \le C_l, \tag{3.22}$$

where C_l is independent of τ. Denoting

$$C_0 = \max_{1\le l\le N_\tau} C_l,$$

and taking into account that $N_\tau\tau = T$, we prove the estimate (3.21) with $C = TC_0$. $\square$

Remark 3.3. For the undecomposed algorithm, with $M=1$ and $L=1$, one has $\widetilde{\omega}^h = \omega^h$ in (3.18) which together with (3.13) gives estimate (3.21) with $\widetilde{r} = r < 1$, where (see, e.g., [2])

$$r = \theta c^*/(\theta c^* + \tau^{-1}) \le \theta c^* \tau.$$

3.4. Estimates on the convergence rate of algorithm (3.3)-(3.7)

We now analyze the convergence rate of algorithm (3.3)-(3.7) defined on piecewise uniform meshes of Shishkin-type [7].

The piecewise equidistant mesh of Shishkin-type is formed by the following manner. We divide each of the intervals $\overline{\omega}^x = [0,1]$ and $\overline{\omega}^y = [0,1]$ into three parts each $[0,\sigma_x]$, $[\sigma_x, 1-\sigma_x]$, $[1-\sigma_x, 1]$, and $[0,\sigma_y]$, $[\sigma_y, 1-\sigma_y]$, $[1-\sigma_y, 1]$, respectively. Assuming that N_x, N_y are divisible by 4, in the parts $[0,\sigma_x]$, $[1-\sigma_x,1]$ and $[0,\sigma_y]$, $[1-\sigma_y,1]$ we use uniform meshes with $N_x/4+1$ and $N_y/4+1$ mesh points, respectively, and in the parts $[\sigma_x, 1-\sigma_x]$, $[\sigma_y, 1-\sigma_y]$ we use uniform meshes with $N_x/2+1$ and $N_y/2+1$ mesh points, respectively. This defines the piecewise equidistant meshes in the x- and y-directions condensed in the boundary layers at

$x = 0, 1$ and $y = 0, 1$:

$$x_i = \begin{cases} ih_{x\mu}, & i = 0, 1, \cdots, N_x/4, \\ \sigma_x + (i - N_x/4)h_x, & i = N_x/4 + 1, \cdots, 3N_x/4, \\ 1 - \sigma_x + (i - 3N_x/4)h_{x\mu}, & i = 3N_x/4 + 1, \cdots, N_x, \end{cases}$$

$$y_j = \begin{cases} jh_{y\mu}, & j = 0, 1, \cdots, N_y/4, \\ \sigma_y + (j - N_y/4)h_y, & j = N_y/4 + 1, \cdots, 3N_y/4, \\ 1 - \sigma_y + (j - 3N_y/4)h_{y\mu}, & j = 3N_y/4 + 1, \cdots, N_y, \end{cases}$$

$$h_x = 2(1 - 2\sigma_x)N_x^{-1}, \quad h_{x\mu} = 4\sigma_x N_x^{-1}, \quad h_y = 2(1 - 2\sigma_y)N_y^{-1}, \quad h_{y\mu} = 4\sigma_y N_y^{-1},$$

where $h_{x\mu}, h_{y\mu}$ and h_x, h_y are the step sizes inside and outside the boundary layers, respectively. We choose the transition points σ_x, $(1 - \sigma_x)$ and σ_y, $(1 - \sigma_y)$ in Shishkin's sense (see [7] for details), i.e.,

$$\sigma_x = \min\left\{4^{-1}, \upsilon_1 \mu \ln N_x\right\}, \quad \sigma_y = \min\left\{4^{-1}, \upsilon_2 \mu \ln N_y\right\},$$

where υ_1 and υ_2 are positive constants. If $\sigma_{x,y} = 1/4$, then $N_{x,y}^{-1}$ are very small relative to μ, and in this case the difference scheme (2.2) can be analyzed using standard techniques. We therefore assume that

$$\sigma_x = \upsilon_1 \mu \ln N_x, \quad \sigma_y = \upsilon_2 \mu \ln N_y.$$

In this case the meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ are piecewise equidistant with the step sizes

$$\begin{aligned} N_x^{-1} < h_x < 2N_x^{-1}, \quad h_{x\mu} = 4\upsilon_1 \mu N_x^{-1} \ln N_x, \\ N_y^{-1} < h_y < 2N_y^{-1}, \quad h_{y\mu} = 4\upsilon_2 \mu N_y^{-1} \ln N_y. \end{aligned} \tag{3.23}$$

In [4], we proved that if the time mesh spacing τ satisfies the CFL condition (2.4), then the difference scheme (2.2) on the piecewise uniform mesh (3.23) converges μ-uniformly to the solution of (1.1):

$$\max_{t \in \overline{\omega}^\tau} \|U(t) - u(t)\|_{\overline{\omega}^h} \le D\left(N^{-1} \ln N + |\theta - 0.5|\tau + \tau^2\right), \tag{3.24}$$

where $N = \min\{N_x, N_y\}$ and constant D is independent of μ, N and τ.

Theorem 3.4. *Let the interfacial subdomains $\overline{\eta}_m^h$, $m = 1, \cdots, M - 1$ and $\overline{\vartheta}_l^h$, $l = 1, \cdots, L - 1$ be located in the x- and y-directions, respectively, outside the boundary layers (unbalanced decomposition). Suppose that $\mu \le \mu_0 \ll 1$, and that the following conditions are satisfied*

$$\overline{N} \le \frac{1}{\mu_0}, \quad \overline{N} = \max\{N_x, N_y\}, \quad \theta\tau \le \frac{1}{2 + c^*}.$$

If the number of iterates $n_* \ge 2$, then for the monotone domain decomposition algorithm (3.3)-(3.7) *on the piecewise uniform mesh* (3.23)*, the estimate* (3.21) *becomes*

$$\max_{t \in \omega^\tau} \|V(t) - U(t)\| \le C\theta(2 + c^*)\,\widetilde{r}^{n_* - 1}, \quad \widetilde{r} < \theta\tau(2 + c^*) \le 1,$$

where the constant C is independent of τ.

Proof. Since the interfacial subdomains are located outside the boundary layers, where the step sizes h_x and h_y are in use, then under the above assumption on $\overline{N}$, with the notation from (3.12), we have

$$\left(\theta c^* + \tau^{-1}\right)\left(r + r^I + r^{II}\right) < \theta(2 + c^*), \quad \widetilde{r} < \theta\tau(2 + c^*).$$

Thus, if

$$\theta\tau \le (2+c^*)^{-1},$$

as assumed in the theorem, then $\widetilde{r} < 1$, and we prove the theorem. □

Remark 3.4. From (3.24), we conclude that the Crank-Nicolson difference scheme (2.2), $\theta = 0.5$, is of second order with respect to τ. Thus, to guarantee the consistency of the global errors in the Crank-Nicolson difference scheme and in the monotone domain decomposition algorithm (3.3)-(3.7) with $\theta = 0.5$, we can choose $n_* = 3$. Similarly, for the fully implicit scheme (2.2), $\theta = 1$, we can choose $n_* = 2$.

Remark 3.5. Such domain decompositions, in which the interfacial subdomains are outside the boundary layers, are said to be *unbalanced*, since the distribution of mesh points among the nonoverlapping main subdomains is uneven. By contrast, a *balanced* domain decomposition is one in which the mesh points are equally distributed among the main subdomains. For balanced decompositions, the first and last interfacial subdomains each overlap the boundary layer.

3.5. Uniform convergence of algorithm (3.3)-(3.7)

Without loss of generality, we assume that the boundary condition $g(P,t) = 0$. This assumption can always be obtained by a change of variables. On each time level let an initial function $V^{(0)}(P,t)$ be chosen in the form of (3.8), i.e., $V^{(0)}(P,t)$ is the solution of the difference problem

$$\begin{aligned} &\left(\theta\mathcal{L}_h + \tau^{-1}\right) V^{(0)}(P,t) = q\left|\mathcal{G}\left(0, V(P,t-\tau)\right)\right|, \quad P \in \omega^h, \\ &V^{(0)}(P,t) = 0, \quad P \in \partial\omega^h, \quad q = 1, -1, \end{aligned} \tag{3.25}$$

where $R(P,t) = 0$. Then $\overline{V}^{(0)}(P,t)$, $\underline{V}^{(0)}(P,t)$ corresponding to $q = 1$ and $q = -1$, respectively, are upper and lower solutions.

Theorem 3.5. *Let the assumptions of Theorem* 3.4 *hold true. Suppose that* $V^{(0)}$ *is chosen in the form of* (3.25)*. Then the monotone domain decomposition algorithm* (3.3)-(3.7) *converges* μ*-uniformly to the solution of the continuous problem* (1.1)*:*

$$\max_{t\in\omega^\tau} \left\|V(t) - u(t)\right\| \le K\left(N^{-1}\ln N + |\theta - 0.5|\tau + \tau^2 + \theta\widetilde{r}^{n_*-1}\right),$$

where here and throughout K *denotes a generic constant which is independent of* μ*,* N_x*,* N_y *and* τ*.*

Proof. From (3.25), by (3.10),

$$\left\|V^{(0)}(t)\right\|_{\overline{\omega}^h} \le \tau\left\|\mathcal{G}\left(0, V(t-\tau)\right)\right\|_{\omega^h}. \tag{3.26}$$

Using the mean-value theorem, (3.25) and (1.2), it follows from (3.22) that

$$\begin{aligned} &\left\|Z^{(1)}(t_l)\right\|_{\overline{\omega}^h} \\ &\le \tau\Big[\left\|\left(\theta\mathcal{L}_h + \tau^{-1}\right)V^{(0)}(t_l)\right\|_{\omega^h} + \theta c^*\left\|V^{(0)}(t_l)\right\|_{\overline{\omega}^h} + \left\|\mathcal{G}\left(0, V(t_l-\tau)\right)\right\|_{\omega^h}\Big] \\ &\le \left(2\tau + \theta c^*\tau^2\right)\left[\theta\left\|f\left(P, t_l, 0\right)\right\|_{\overline{\omega}^h} + \left\|\mathcal{G}_2\left(V\left(t_{l-1}\right)\right)\right\|_{\omega^h}\right] \le C_l, \end{aligned}$$

where we use (3.1) and (3.2) with $\mathcal{G}_1(P,t_l,0)=\theta f\left(P,t_l,0\right)$.

If $\tau\left\|\mathcal{G}_2\left(V\left(t_l-\tau\right)\right)\right\|_{\omega^h}$ is independent of μ, N_x, N_y and τ, then all constants C_l are independent of μ, N_x, N_y and τ, where we assume $\tau\leq\tau_0$. We prove this result by induction.

For $l=1$, $V(P,0)=u^0(P)$, and

$$\mathcal{G}_2\left(P,0,u^0(P)\right)=\left[(1-\theta)\mathcal{L}_h-\tau^{-1}\right]u^0(P)+(1-\theta)f(P,0,u^0).$$

In [4], we proved the following estimates:

$$\left\|\mu^2\mathcal{D}_x^2\left(u^0\right)\right\|_{\omega^h}\leq\left\|\mu^2\frac{\partial^2u^0}{\partial x^2}\right\|_{\omega},\quad\left\|\mu^2\mathcal{D}_y^2\left(u^0\right)\right\|_{\omega^h}\leq\left\|\mu^2\frac{\partial^2u^0}{\partial y^2}\right\|_{\omega}.$$

It means that $\left\|\mathcal{L}_hu^0(P)\right\|_{\omega^h}$ is μ-uniformly bounded. Thus,

$$\tau\left\|\mathcal{G}_2\left(u^0\right)\right\|_{\omega^h}\leq K. \tag{3.27}$$

Hence, C_1 is bounded independently of μ, N_x, N_y and τ. From (3.3)-(3.6), we have

$$\begin{aligned}&\left(\theta\mathcal{L}_h+\tau^{-1}+\theta c^*\right)V^{(n+1)}(P,t)\\&=\theta c^*V^{(n)}(P,t)-\theta f\left(P,t,V^{(n)}\right)-\mathcal{G}_2\left(P,t-\tau,V\right),\\&P\in\widetilde{\omega}^h,\quad\widetilde{\omega}^h=\omega^h\setminus\left(\widetilde{\gamma}^h\cup\rho^h\right).\end{aligned} \tag{3.28}$$

Using the same reasonings as in Theorem 3.2, we can get the difference problem on $\widetilde{\gamma}^h\cup\rho^h$ in the following form

$$\begin{aligned}&\left(\theta\mathcal{L}_h+\tau^{-1}+\theta c^*\right)V^{(n+1)}(P,t)\\&=\theta c^*V^{(n)}(P,t)-\theta f\left(P,t,V^{(n)}\right)-\mathcal{G}_2\left(P,t-\tau,V\right)+\Delta^{(n)}(P,t),\\&P\in\widetilde{\gamma}^h\cup\rho^h,\quad\left\|\Delta^{(n)}(t)\right\|_{\omega^h}\leq\left(r^I+r^{II}\right)\left\|\mathcal{G}^{(n)}(t)\right\|_{\omega^h},\end{aligned} \tag{3.29}$$

where r^I and r^{II} are defined in (3.12), and

$$\mathcal{G}^{(n)}(P,t)\equiv\mathcal{G}\left(V^{(n)}(P,t),V(P,t-\tau)\right).$$

From (3.28) and (3.29), by (3.10),

$$\begin{aligned}\left\|V^{(n+1)}(t)\right\|_{\overline{\omega}^h}\leq\tau\theta\left(c^*\left\|V^{(n)}(t)\right\|_{\overline{\omega}^h}+\left\|f\left(P,t,V^{(n)}\right)\right\|_{\overline{\omega}^h}\right)+\\\tau\left(\left\|\mathcal{G}_2(V(t-\tau))\right\|_{\omega^h}+\left(r^I+r^{II}\right)\left\|\mathcal{G}^{(n)}(t)\right\|_{\omega^h}\right).\end{aligned} \tag{3.30}$$

At $t=t_1$, from (3.25), (3.26) and (3.27), by (3.10), we have

$$\left\|V^{(0)}(t_1)\right\|_{\overline{\omega}^h}\leq K,\quad\tau\left\|\mathcal{L}_hV^{(0)}(t_1)\right\|_{\omega^h}\leq K. \tag{3.31}$$

From here, it follows that

$$\tau\mathcal{G}_1\left(P,t_1,V^{(0)}\right)=\tau\left(\theta\mathcal{L}_h+\tau^{-1}\right)V^{(0)}(P,t_1)+\tau\theta f\left(P,t_1,V^{(0)}\right)$$

is bounded independently of μ, N_x, N_y and τ. Taking into account

$$\tau\mathcal{G}^{(0)}\left(P,t_1\right)=\tau\mathcal{G}_1\left(P,t_1,V^{(0)}\right)+\tau\mathcal{G}_2\left(P,0,u^0\right),$$

and (3.27), we conclude that

$$\tau \left\| \mathcal{G}^{(0)}(t_1) \right\|_{\omega^h} \leq K.$$

Since the interfacial subdomains are located outside the boundary layers, where the step sizes h_x and h_y are in use, then under the above assumption on $\overline{N}$ in Theorem 3.4, we have

$$r^I + r^{II} < 2\tau\theta \leq 2\tau_0.$$

From (3.30) at $t = t_1$ and $n = 0$, (3.31) and the last estimate, by (3.10), we conclude

$$\left\| V^{(1)}(t_1) \right\|_{\overline{\omega}^h} \leq K.$$

From here and (3.28), (3.29) at $t = t_1$ and $n = 0$, by (3.10), we have

$$\tau \left\| \mathcal{L}_h V^{(1)}(t_1) \right\|_{\omega^h} \leq K.$$

Now, by induction on n, we prove that $V(P, t_1) = V^{(n_*)}(P, t_1)$ satisfies the following estimates

$$\| V(t_1) \|_{\overline{\omega}^h} \leq K, \quad \tau \| \mathcal{L}_h V(t_1) \|_{\omega^h} \leq K.$$

Using these estimates, we conclude that

$$\begin{aligned} &\tau \mathcal{G}_2 (P, t_1, V(P, t_1)) \\ &= \tau \left[(1-\theta)\mathcal{L}_h - \tau^{-1} \right] V(P, t_1) + \tau(1-\theta) f(P, t_1, V) \end{aligned}$$

is bounded independently of μ, N_x, N_y and τ. Now, by induction on l, we prove that $\tau \| \mathcal{G}_2 (V(t_l - \tau)) \|_{\omega^h}$ is bounded independently of μ, N_x, N_y and τ. Hence, the constant C in Theorems 3.3 and 3.4 is independent of μ, N_x, N_y and τ, and we prove the theorem. □

4. Numerical Experiments

In [4], for the weighted average method we investigated μ-uniform numerical order of convergence with respect to N^{-1} and τ on the piecewise uniform mesh (3.23). For each of the implicit and Crank-Nicolson schemes, we found that the numerical order of convergence with respect to N^{-1} is between one and two. The numerical order of convergence with respect to τ is one for the implicit scheme and two for the Crank-Nicolson scheme. It was also found that the CFL condition (2.4) could be violated by an order of magnitude without loss of stability. For the experiments of [4], the nonlinear difference scheme was solved with the undecomposed monotone iterative algorithm ($M = 1$, $L = 1$). In this section, we are interested in the convergence and execution time of the monotone domain decomposition algorithm (3.3)-(3.7). For our numerical experiments, we take $N_x = N_y = N$. Because the mesh is only piecewise continuous, the linear systems can be nonsymmetric. Therefore, we employ the restarted GMRES algorithm from [9], suitable for nonsymmetric systems.

We consider the model problem

$$\begin{aligned} &-\mu^2(u_{xx} + u_{yy}) + u_t = -\frac{u-4}{5-u}, \\ &(x, y, t) \in \omega \times (0, 1], \quad \omega = \{0 < x < 1\} \times \{0 < y < 1\}, \end{aligned}$$

with the initial and boundary conditions:

$$u(\omega, 0) = 0, \quad u(\partial\omega, 0) = 1, \quad u(\partial\omega, t) = 1, \quad t \in (0, 1].$$

The steady state solution to the reduced problem ($\mu = 0$) is $u_r = 4$. For $\mu \ll 1$ the problem is singularly perturbed and the steady state solution increases sharply from $u = 1$ on $\partial\omega$ to $u = 4$ on the interior. The solution to the parabolic problem approaches this steady state with time.

Consider first the implicit scheme, for which $\theta = 1$. The numerical solution at $t = 0$ is given by the initial condition

$$V(\omega^h, 0) = 0, \quad V(\partial\omega^h, 0) = 1.$$

The mesh function $\underline{V}^{(0)}(P, t_1)$ defined by $\underline{V}^{(0)}(P, t_1) = V(P, 0)$ is clearly a lower solution with respect to $V(P, 0)$. We initiate the algorithm with $\underline{V}^{(0)}(P, t_1)$ and thus generate a sequence of lower solutions. At each time level t_k, we define a converged solution $V(P, t_k) = \underline{V}^{(n_*)}(P, t_k)$ with $n_* = n_*(t_k)$ minimal subject to

$$\|\underline{V}^{(n_*)}(t_k) - \underline{V}^{(n_*-1)}(t_k)\|_{\overline{\omega}^h} < \delta,$$

where δ is a specified tolerance. At the next time level, t_{k+1}, we require an initial iterate that is a lower solution with respect to $V(P, t_k)$. Since the boundary condition and function $f(u) = (u - 4)/(5 - u)$ are independent of time, and because of Theorem 3.1, we may choose $\underline{V}^{(0)}(P, t_{k+1}) = V(P, t_k)$. Now, again from Theorem 3.1, it follows by induction on k that the mesh function $\overline{V}(P, t_{k+1})$ defined by

$$\overline{V}(\omega^h, t_{k+1}) = 4, \quad \overline{V}(\partial\omega^h, t_{k+1}) = 1$$

is an upper solution with respect to $V(P, t_k)$ and thus our computed mesh functions satisfy

$$0 \leq \underline{V}^{(n)}(P, t_k) \leq 4, \quad P \in \overline{\omega}^h, \quad 0 \leq n \leq n_*, \quad 0 \leq k \leq N_\tau. \tag{4.1}$$

Hence we may suppose that $f_u = 1/(5 - u)^2$ is bounded below and above by $c_* = 1/25$ and $c^* = 1$, respectively.

For the Crank-Nicolson scheme with $\theta = 0.5$, the mesh function $V^{(0)}(P, t_{k+1}) = V(P, t_k)$ does not provide a lower solution with respect to $V(P, t_k)$. To generate an initial lower solution on time level t_{k+1} we solve (3.8) with $R(P, t_{k+1}) = V(P, t_k)$. We then define the initial lower solution by

$$\underline{V}^{(0)}(P, t_{k+1}) = V(P, t_k) + Z_{-1}^{(0)}(P, t_{k+1}).$$

Although the initial iterate $\underline{V}^{(0)}$ can be negative, violating (4.1), our numerical experiments indicate that the choice $c^* = 1$ is an upper bound on f_u for all computed iterates and that (4.1) is satisfied for all $n \geq 1$.

In all experiments, we take as our convergence tolerance $\delta = 10^{-5}$. This choice necessitated at least four iterations on each time step, thus guaranteeing the consistency of algorithm (3.3)-(3.7) and the corresponding nonlinear weighted average scheme (2.2) (see Remark 3.4).

We present results from balanced and unbalanced domain decompositions on the piecewise uniform mesh (3.23) with $\upsilon_{1,2} = 1/\sqrt{c_*} = 5$. We consider the implicit ($\theta = 1$) and Crank-Nicolson ($\theta = 0.5$) schemes over the parameter ranges $\mu = 10^{-2}$, 10^{-3}, 10^{-4}, $N = 128$, 256, 512 and $\{M, L\} \subset \{1, 4, 8, 16, 32\}$. For balanced domain decompositions, where there is some choice for the interfacial subdomain widths, we choose them to be all minimal or all maximal. For unbalanced decompositions, we choose the interfacial subdomains to be minimal. Each simulation comprises ten time steps of size $\tau = 0.1$.

Shown in Table 4.1 is the convergence parameter $\tilde{r}$ for all experiments of this paper. The value of $\tilde{r}$ is independent of the widths of the interfacial subdomains and depends only on

Table 4.1: The convergence parameter $\tilde{r}$ (balanced) and $\tilde{r}$ (unbalanced) for the implicit scheme and Crank-Nicolson scheme under balanced and unbalanced domain decomposition. The domain can be decomposed in none, one or both of the x- and y- directions.

μ	$M>1$ / $L>1$	Implicit scheme $N=128$	Implicit scheme $N=256$	Implicit scheme $N=512$	Crank-Nicolson scheme $N=128$	Crank-Nicolson scheme $N=256$	Crank-Nicolson scheme $N=512$
10^{-2}	0	0.09;0.09	0.09;0.09	0.09;0.09	0.05;0.05	0.05;0.05	0.05;0.05
	1	0.25;0.23	0.69;0.69	2.47;2.47	0.13;0.12	0.36;0.36	1.30;1.30
	2	0.41;0.37	1.28;1.28	4.86;4.86	0.21;0.19	0.67;0.67	2.54;2.54
$\leq 10^{-3}$	0	0.09;0.09	0.09;0.09	0.09;0.09	0.05;0.05	0.05;0.05	0.05;0.05
	1	0.25;0.09	0.58;0.09	1.62;0.09	0.13;0.05	0.36;0.05	1.30;0.05
	2	0.41;0.09	1.06;0.09	**3.15**;0.09	0.21;0.05	0.56;0.05	**1.65**;0.05

Table 4.2: Average convergence iteration count of the implicit ($\theta = 1.0$) and Crank-Nicolson ($\theta = 0.5$) schemes under balanced domain decomposition.

θ	M	1	4	4	8	8	8	16	16	16	16	32	32	32	32	32
μ	$N\backslash L$	1	1	4	1	4	8	1	4	8	16	1	4	8	16	32
1.0 10^{-2}	128	5.0	$\frac{6.0}{5.0}$	$\frac{6.7}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.0}{5.2}$	$\frac{7.0}{5.2}$	$\frac{7.0}{5.2}$	$\frac{7.0}{5.2}$	$\frac{7.0}{6.0}$
	256	5.0	$\frac{8.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{8.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{8.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{8.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$
	512	5.0	$\frac{13.0}{5.0}$	$\frac{16.0}{5.0}$	$\frac{13.0}{5.0}$	$\frac{16.0}{5.0}$	$\frac{16.0}{5.0}$	$\frac{13.0}{5.0}$	$\frac{16.0}{5.0}$	$\frac{16.0}{5.0}$	$\frac{16.0}{5.0}$	$\frac{13.0}{5.0}$	$\frac{16.0}{5.0}$	$\frac{16.0}{5.0}$	$\frac{16.0}{5.0}$	$\frac{16.0}{5.0}$
1.0 10^{-3}	128	5.0	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.0}{5.2}$	$\frac{6.0}{5.2}$	$\frac{7.0}{5.2}$	$\frac{7.0}{5.2}$	$\frac{7.0}{6.0}$
	256	5.0	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.6}{5.0}$	$\frac{7.6}{5.0}$	$\frac{9.0}{5.0}$	$\frac{7.6}{5.0}$	$\frac{7.6}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{7.7}{5.0}$	$\frac{8.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$
	512	5.0	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{11.0}{5.0}$	$\frac{11.0}{5.0}$	$\frac{13.0}{5.0}$	$\frac{11.0}{5.0}$	$\frac{11.0}{5.0}$	$\frac{13.0}{5.0}$	$\frac{13.0}{5.0}$	$\frac{11.0}{5.0}$	$\frac{11.0}{5.0}$	$\frac{13.0}{5.0}$	$\frac{13.0}{5.0}$	$\frac{13.0}{5.0}$
1.0 10^{-4}	128	5.0	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.0}{5.2}$	$\frac{6.0}{5.2}$	$\frac{7.0}{5.2}$	$\frac{7.0}{5.2}$	$\frac{7.0}{6.0}$
	256	5.0	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.6}{5.0}$	$\frac{7.6}{5.0}$	$\frac{9.0}{5.0}$	$\frac{7.6}{5.0}$	$\frac{7.6}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{7.7}{5.0}$	$\frac{7.7}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$
	512	5.0	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{11.0}{5.0}$	$\frac{11.0}{5.0}$	$\frac{13.0}{5.0}$	$\frac{11.0}{5.0}$	$\frac{11.0}{5.0}$	$\frac{13.0}{5.0}$	$\frac{\mathbf{13.0}}{5.0}$	$\frac{11.0}{5.0}$	$\frac{11.0}{5.0}$	$\frac{13.0}{5.0}$	$\frac{13.0}{5.0}$	$\frac{13.0}{5.0}$
0.5 10^{-2}	128	5.0	$\frac{5.1}{5.0}$	$\frac{6.0}{5.0}$	$\frac{5.1}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{5.1}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{5.1}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$
	256	5.0	$\frac{6.7}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.7}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.7}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.9}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$
	512	5.0	$\frac{10.3}{5.0}$	$\frac{12.0}{5.0}$	$\frac{10.3}{5.0}$	$\frac{12.0}{5.0}$	$\frac{12.0}{5.0}$	$\frac{10.3}{5.0}$	$\frac{12.0}{5.0}$	$\frac{12.0}{5.0}$	$\frac{12.0}{5.0}$	$\frac{10.3}{5.0}$	$\frac{12.0}{5.0}$	$\frac{12.0}{5.0}$	$\frac{12.0}{5.0}$	$\frac{12.0}{5.0}$
0.5 10^{-3}	128	5.0	$\frac{5.0}{5.0}$	$\frac{5.0}{5.0}$	$\frac{5.1}{5.0}$	$\frac{5.1}{5.0}$	$\frac{6.0}{5.0}$	$\frac{5.1}{5.0}$	$\frac{5.1}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{5.1}{5.0}$	$\frac{5.9}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$
	256	5.0	$\frac{5.5}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$
	512	5.0	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{10.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{10.0}{5.0}$	$\frac{10.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{10.0}{5.0}$	$\frac{10.0}{5.0}$	$\frac{10.0}{5.0}$
0.5 10^{-4}	128	5.0	$\frac{5.0}{5.0}$	$\frac{5.0}{5.0}$	$\frac{5.1}{5.0}$	$\frac{5.1}{5.0}$	$\frac{6.0}{5.0}$	$\frac{5.1}{5.0}$	$\frac{5.1}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{5.1}{5.0}$	$\frac{5.7}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$
	256	5.0	$\frac{5.2}{5.0}$	$\frac{5.7}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$	$\frac{7.0}{5.0}$
	512	5.0	$\frac{6.0}{5.0}$	$\frac{6.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{10.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{10.0}{5.0}$	$\frac{\mathbf{10.0}}{5.0}$	$\frac{9.0}{5.0}$	$\frac{9.0}{5.0}$	$\frac{10.0}{5.0}$	$\frac{10.0}{5.0}$	$\frac{10.0}{5.0}$

μ, N, whether the domain is decomposed in none, one or two directions and whether the decomposition is balanced or unbalanced.

In Tables 4.2-4.5 we give results for all $M \times L$ decompositions in which $M \geq L$. We mention that interchanging M and L gives an identical convergence iteration count and very similar execution time. Table 4.2 shows the average convergence iteration count per time step for balanced decompositions. From Table 4.1 we see that the value of $\tilde{r}$ for the Crank-Nicolson scheme on a given decomposition is about half of the corresponding value for the implicit scheme. For example, with $\mu = 10^{-4}$, $N = 512$ and the balanced, 16×16, minimal interfacial subdomain decomposition, the implicit scheme has $\tilde{r} = 3.15$ while the Crank-Nicolson scheme has $\tilde{r} = 1.65$. These table entries are shown in bold. We know from Theorem 3.1 that the sequence $\underline{V}^{(n)}(P,t)$ converges at each time level $t \in \omega^{\tau}$. Although the convergence estimate (3.12) is of no formal use when $\tilde{r} \geq 1$, the value of $\tilde{r}$ is reflected in the convergence behaviour of the algorithm if the interfacial subdomains are minimal. Thus, in Table 4.2 the corresponding average convergence iteration counts for the implicit and Crank-Nicolson schemes (shown in bold) are 13.0 and 10.0, respectively. On the other hand, if the interfacial subdomains are maximal then the parameter $\tilde{r}$ is not reflected in the convergence behaviour; the convergence iteration count is very close to that of the undecomposed algorithm.

Although maximal interfacial subdomains entail fewer global iterations, the problems on Steps 2 and 3 of the algorithm are larger than the main subdomain problems of Step 1. Hence, in Table 4.3 we find that the algorithm executes more quickly when the interfacial subdomains are chosen minimally.

Table 4.3: Execution time of the implicit ($\theta = 1.0$) and Crank-Nicolson ($\theta = 0.5$) schemes under balanced domain decomposition.

θ	M	1	4	4	8	8	8	16	16	16	16	32	32	32	32	32
μ	$N\backslash L$	1	1	4	1	4	8	1	4	8	16	1	4	8	16	32
1.0, 10^{-2}	128	1.0	$\frac{1.2}{1.5}$	$\frac{1.2}{2.1}$	$\frac{1.0}{1.5}$	$\frac{1.2}{2.1}$	$\frac{1.2}{2.1}$	$\frac{1.0}{1.4}$	$\frac{1.2}{2.0}$	$\frac{1.3}{2.1}$	$\frac{1.3}{2.0}$	$\frac{0.9}{1.1}$	$\frac{1.2}{2.0}$	$\frac{1.3}{2.0}$	$\frac{1.2}{2.0}$	$\frac{1.1}{1.9}$
	256	7.4	$\frac{14.9}{15.2}$	$\frac{15.0}{21.7}$	$\frac{13.1}{14.6}$	$\frac{13.7}{20.9}$	$\frac{12.4}{20.4}$	$\frac{10.1}{12.4}$	$\frac{12.2}{19.4}$	$\frac{12.4}{19.6}$	$\frac{12.1}{18.1}$	$\frac{8.1}{9.4}$	$\frac{11.2}{18.1}$	$\frac{11.6}{18.6}$	$\frac{11.3}{17.0}$	$\frac{9.0}{14.7}$
	512	68.0	$\frac{250}{167}$	$\frac{315}{241}$	$\frac{251}{182}$	$\frac{251}{247}$	$\frac{227}{245}$	$\frac{228}{168}$	$\frac{224}{229}$	$\frac{209}{239}$	$\frac{192}{228}$	$\frac{147}{115}$	$\frac{198}{198}$	$\frac{184}{208}$	$\frac{174}{203}$	$\frac{151}{168}$
1.0, 10^{-3}	128	1.3	$\frac{1.1}{1.6}$	$\frac{0.8}{2.0}$	$\frac{1.0}{1.5}$	$\frac{0.8}{2.0}$	$\frac{1.0}{2.0}$	$\frac{0.9}{1.3}$	$\frac{0.8}{1.9}$	$\frac{1.0}{1.9}$	$\frac{1.0}{1.8}$	$\frac{0.9}{1.1}$	$\frac{0.8}{1.7}$	$\frac{1.0}{1.7}$	$\frac{1.0}{1.7}$	$\frac{1.0}{1.8}$
	256	9.4	$\frac{9.7}{13.5}$	$\frac{5.9}{16.1}$	$\frac{10.2}{12.1}$	$\frac{6.7}{15.4}$	$\frac{7.6}{14.7}$	$\frac{9.0}{10.8}$	$\frac{6.2}{14.8}$	$\frac{7.8}{14.6}$	$\frac{7.5}{13.8}$	$\frac{7.9}{8.9}$	$\frac{5.9}{13.1}$	$\frac{6.9}{12.9}$	$\frac{7.1}{12.8}$	$\frac{6.7}{11.4}$
	512	75.5	$\frac{94.7}{114}$	$\frac{71.1}{142}$	$\frac{140}{125}$	$\frac{89.2}{145}$	$\frac{95.7}{153}$	$\frac{129}{117}$	$\frac{79.8}{140}$	$\frac{89.0}{149}$	$\frac{82.0}{146}$	$\frac{98.1}{91.1}$	$\frac{61.1}{119}$	$\frac{73.8}{131}$	$\frac{72.8}{128}$	$\frac{65.3}{109}$
1.0, 10^{-4}	128	1.3	$\frac{1.1}{1.6}$	$\frac{0.8}{2.0}$	$\frac{1.0}{1.5}$	$\frac{0.8}{2.0}$	$\frac{1.0}{2.0}$	$\frac{0.9}{1.4}$	$\frac{0.8}{1.9}$	$\frac{1.0}{1.9}$	$\frac{1.0}{1.8}$	$\frac{0.9}{1.1}$	$\frac{0.8}{1.7}$	$\frac{0.9}{1.7}$	$\frac{1.0}{1.7}$	$\frac{0.9}{1.7}$
	256	9.4	$\frac{9.7}{13.6}$	$\frac{5.8}{16.0}$	$\frac{10.2}{11.9}$	$\frac{6.6}{15.3}$	$\frac{7.5}{14.7}$	$\frac{9.0}{10.8}$	$\frac{6.1}{14.6}$	$\frac{7.4}{14.4}$	$\frac{7.4}{13.7}$	$\frac{7.9}{8.9}$	$\frac{5.5}{13.0}$	$\frac{6.6}{12.8}$	$\frac{6.7}{12.4}$	$\frac{6.1}{11.2}$
	512	78.6	$\frac{82.3}{120}$	$\frac{61.0}{148}$	$\frac{140}{125}$	$\frac{81.4}{148}$	$\frac{94.8}{154}$	$\frac{129}{122}$	$\frac{74.6}{145}$	$\frac{87.3}{149}$	$\frac{\mathbf{80.7}}{145}$	$\frac{96.4}{91.0}$	$\frac{56.8}{122}$	$\frac{71.1}{128}$	$\frac{68.9}{128}$	$\frac{59.3}{107}$
0.5, 10^{-2}	128	1.2	$\frac{1.2}{1.7}$	$\frac{1.3}{2.3}$	$\frac{1.1}{1.7}$	$\frac{1.3}{2.4}$	$\frac{1.4}{2.4}$	$\frac{1.1}{1.7}$	$\frac{1.3}{2.4}$	$\frac{1.4}{2.4}$	$\frac{1.4}{2.3}$	$\frac{1.1}{1.4}$	$\frac{1.3}{2.2}$	$\frac{1.3}{2.2}$	$\frac{1.4}{2.2}$	$\frac{1.4}{2.0}$
	256	7.3	$\frac{11.0}{13.0}$	$\frac{10.9}{17.5}$	$\frac{9.7}{12.6}$	$\frac{10.3}{17.5}$	$\frac{9.9}{17.5}$	$\frac{9.2}{11.7}$	$\frac{10.0}{17.1}$	$\frac{9.9}{17.4}$	$\frac{9.9}{17.0}$	$\frac{8.1}{9.8}$	$\frac{9.4}{15.6}$	$\frac{10.0}{16.3}$	$\frac{9.9}{16.2}$	$\frac{8.9}{14.8}$
	512	71.6	$\frac{163}{136}$	$\frac{197}{192}$	$\frac{163}{150}$	$\frac{157}{190}$	$\frac{146}{200}$	$\frac{153}{148}$	$\frac{147}{187}$	$\frac{139}{196}$	$\frac{127}{194}$	$\frac{107}{103}$	$\frac{133}{160}$	$\frac{123}{168}$	$\frac{124}{169}$	$\frac{112}{143}$
0.5, 10^{-3}	128	1.3	$\frac{1.1}{1.8}$	$\frac{1.0}{2.2}$	$\frac{1.1}{1.7}$	$\frac{1.0}{2.2}$	$\frac{1.2}{2.2}$	$\frac{1.1}{1.6}$	$\frac{1.0}{2.2}$	$\frac{1.2}{2.2}$	$\frac{1.2}{2.2}$	$\frac{1.1}{1.5}$	$\frac{1.1}{2.0}$	$\frac{1.2}{2.1}$	$\frac{1.3}{2.0}$	$\frac{1.3}{1.9}$
	256	8.7	$\frac{8.6}{12.2}$	$\frac{6.9}{14.8}$	$\frac{8.4}{11.2}$	$\frac{6.6}{14.5}$	$\frac{7.3}{14.2}$	$\frac{7.9}{10.6}$	$\frac{6.4}{14.1}$	$\frac{7.3}{14.0}$	$\frac{7.4}{13.9}$	$\frac{7.1}{9.6}$	$\frac{6.1}{13.2}$	$\frac{7.3}{13.2}$	$\frac{7.3}{13.1}$	$\frac{7.1}{12.4}$
	512	63.5	$\frac{71.2}{94.2}$	$\frac{55.5}{119}$	$\frac{95.1}{100}$	$\frac{63.8}{120}$	$\frac{67.4}{122}$	$\frac{87.8}{96.3}$	$\frac{60.2}{120}$	$\frac{65.0}{122}$	$\frac{61.1}{119}$	$\frac{71.6}{78.7}$	$\frac{53.1}{104}$	$\frac{58.4}{109}$	$\frac{59.3}{108}$	$\frac{56.2}{97.2}$
0.5, 10^{-4}	128	1.3	$\frac{1.2}{1.8}$	$\frac{1.0}{2.2}$	$\frac{1.1}{1.7}$	$\frac{1.0}{2.2}$	$\frac{1.2}{2.2}$	$\frac{1.1}{1.6}$	$\frac{1.0}{2.2}$	$\frac{1.2}{2.2}$	$\frac{1.2}{2.2}$	$\frac{1.1}{1.5}$	$\frac{1.1}{2.0}$	$\frac{1.2}{2.0}$	$\frac{1.2}{2.0}$	$\frac{1.2}{1.9}$
	256	8.8	$\frac{8.3}{12.3}$	$\frac{6.6}{14.8}$	$\frac{8.2}{11.5}$	$\frac{6.5}{14.4}$	$\frac{7.2}{14.1}$	$\frac{7.8}{10.7}$	$\frac{6.4}{14.1}$	$\frac{7.2}{13.9}$	$\frac{7.3}{14.0}$	$\frac{7.2}{9.7}$	$\frac{6.1}{13.3}$	$\frac{7.2}{13.1}$	$\frac{7.2}{13.0}$	$\frac{7.0}{12.1}$
	512	65.8	$\frac{71.5}{95.3}$	$\frac{54.9}{120}$	$\frac{93.6}{102}$	$\frac{62.5}{119}$	$\frac{66.7}{122}$	$\frac{89.8}{96.8}$	$\frac{58.4}{120}$	$\frac{64.0}{120}$	$\frac{\mathbf{60.7}}{121}$	$\frac{71.7}{78.8}$	$\frac{52.0}{105}$	$\frac{57.8}{108}$	$\frac{58.7}{108}$	$\frac{55.3}{96.3}$

Consider again the problem with $\mu = 10^{-4}$, $N = 512$ on the balanced, 16×16, minimal interfacial subdomain decomposition. In Table 4.3, the corresponding execution times are shown

Table 4.4: Average convergence iteration count of the implicit ($\theta = 1.0$) and Crank-Nicolson ($\theta = 0.5$) schemes under unbalanced domain decomposition.

θ	M	1	4	4	8	8	8	16	16	16	16	32	32	32	32	32
μ	$N \backslash L$	1	1	4	1	4	8	1	4	8	16	1	4	8	16	32
1.0	128	5.0	6.0	6.0	6.0	6.0	6.0	6.0	6.0	6.0	6.0	7.0	7.0	7.0	7.0	7.0
10^{-2}	256	5.0	8.0	9.0	8.0	9.0	9.0	8.0	9.0	9.0	9.0	8.0	9.0	9.0	9.0	10.0
	512	5.0	13.0	16.0	13.0	16.0	16.0	13.0	16.0	16.0	16.0	13.3	17.0	17.0	17.0	17.0
1.0	128	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
10^{-3}	256	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
	512	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
1.0	128	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
10^{-4}	256	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
	512	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
0.5	128	5.0	5.0	6.0	5.0	6.0	6.0	5.0	6.0	6.0	6.0	6.0	6.0	6.0	6.0	6.0
10^{-2}	256	5.0	6.7	7.0	6.7	7.0	7.0	6.7	7.0	7.0	7.0	7.0	7.0	7.0	7.0	7.2
	512	5.0	10.3	12.0	10.3	12.0	12.0	10.3	12.0	12.0	12.0	10.3	12.0	12.0	12.0	13.0
0.5	128	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
10^{-3}	256	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
	512	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
0.5	128	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
10^{-4}	256	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0
	512	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0	5.0

in bold for both the implicit and Crank-Nicolson schemes. The implicit scheme requires 80.7 seconds to execute 13 global iterations per time step. The Crank-Nicolson scheme requires 60.7 seconds to execute 10 global iterations per time step. In addition to these iterations, an initial iterate must be found on each time step. At Step 1 of the algorithm, we solve the following linear problems on the main subdomains ω_{ml}^h,

$$\begin{aligned}\left[\theta\mathcal{L}_h + \frac{1}{\tau} + \theta c^*\right] & V_{ml}^{(n+1)}(P,t) \\ &= \theta c^* \underline{V}^{(n)}(P,t) - \theta f(P,t,\underline{V}^{(n)}) - \mathcal{G}_2(P,t-\tau,V).\end{aligned}$$

Let us classify the main subdomains as F-F, F-C or C-C according to whether the mesh spacing is, respectively, fine in both the x- and y- directions, fine in one direction and coarse in the other direction or coarse in both directions. For the implicit scheme, the condition number of the matrix $\left[\theta\mathcal{L}_h + \tau^{-1} + \theta c^*\right]$ is 12.84, 7.006 and 1.000 for the main subdomains of class F-F, F-C and C-C, respectively. For the Crank-Nicolson scheme, these condition numbers are 7.287, 4.168 and 1.000, respectively. Thus, the main subdomain problems are more easily solved and the algorithm requires less time per global iteration when applied to the Crank-Nicolson scheme. Notwithstanding the need to compute the initial iterate on each time step, the Crank-Nicolson scheme requires fewer subsequent iterations and these are more easily computed. Thus, for $\mu = 10^{-4}$, $N = 512$ and the balanced 16×16, minimal interfacial subdomain decomposition, the algorithm solves the Crank-Nicolson scheme significantly more quickly than it does the implicit scheme. This is generally true of all balanced decompositions on the piecewise uniform mesh when $\mu \leq 10^{-3}$ and $N = 512$.

Table 4.5: Execution time of the implicit ($\theta = 1.0$) and Crank-Nicolson ($\theta = 0.5$) schemes under unbalanced domain decomposition.

θ	M	1	4	4	8	8	8	16	16	16	16	32	32	32	32	32
μ	$N\backslash L$	1	1	4	1	4	8	1	4	8	16	1	4	8	16	32
1.0	128	1.0	1.2	1.0	1.1	1.0	1.1	1.0	1.0	1.0	1.0	1.1	1.1	1.2	1.1	1.1
10^{-2}	256	7.4	15.0	14.6	13.3	13.8	13.3	11.7	13.2	12.7	11.6	10.7	12.1	11.8	11.0	11.2
	512	68.1	249	322	245	271	241	201	256	227	219	176	249	228	227	189
1.0	128	1.2	0.9	0.7	0.9	0.7	0.7	0.9	0.7	0.8	0.8	0.9	0.8	0.8	0.8	0.9
10^{-3}	256	9.4	7.6	5.0	7.1	5.0	4.8	6.8	5.0	4.9	4.9	6.7	5.0	4.9	5.0	5.1
	512	75.5	68.0	54.2	67.7	47.3	43.6	64.5	46.2	43.2	42.8	61.2	45.8	42.8	46.5	42.2
1.0	128	1.3	0.9	0.7	0.9	0.7	0.7	0.9	0.7	0.8	0.8	0.9	0.8	0.8	0.8	0.9
10^{-4}	256	9.4	8.1	4.9	7.5	4.8	4.7	7.3	4.9	4.8	4.8	7.2	4.9	4.9	5.0	5.1
	512	75.9	68.2	55.6	67.8	48.2	44.4	64.0	47.2	44.1	43.4	61.4	46.8	44.0	47.0	42.2
0.5	128	1.2	1.1	1.3	1.1	1.3	1.3	1.1	1.3	1.3	1.3	1.2	1.3	1.3	1.3	1.3
10^{-2}	256	7.3	11.0	10.8	10.3	10.5	10.3	9.4	10.2	10.1	9.6	9.3	9.8	9.7	9.4	9.2
	512	71.7	163	194	161	172	157	135	164	151	144	123	153	143	137	135
0.5	128	1.3	1.1	1.0	1.1	1.0	1.0	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.2
10^{-3}	256	8.6	7.5	5.9	7.2	5.9	5.8	7.1	5.9	5.9	5.9	7.1	6.0	6.0	6.1	6.1
	512	63.1	58.2	47.7	58.2	45.4	43.4	55.4	45.0	43.0	42.5	54.1	44.9	43.2	42.5	42.9
0.5	128	1.3	1.1	1.0	1.1	1.0	1.0	1.1	1.0	1.0	1.1	1.1	1.1	1.1	1.1	1.2
10^{-4}	256	8.7	7.6	6.1	7.2	6.0	5.9	7.1	6.0	6.0	6.1	7.1	6.1	6.1	6.2	6.3
	512	65.2	59.1	47.9	58.9	45.8	44.3	56.1	45.9	43.8	43.3	54.4	45.6	44.2	43.5	43.5

Considering those balanced decompositions of Table 4.3 for which $N \geq 256$, the undecomposed algorithm solves each scheme fastest when $\mu = 10^{-2}$ while it the 32×4, minimal interfacial subdomain decomposition that is most efficient when $\mu \leq 10^{-3}$.

Consider now the results from unbalanced domain decomposition, in which the interfacial subdomains are located outside the boundary layers. For $\mu = 10^{-2}$ and $N \geq 256$, $\sigma_{x,y} = 0.25$ and thus the mesh is uniform and the values of $\tilde{r}$ for balanced and unbalanced decomposition in Table 4.1 are the same. For $\mu \leq 10^{-3}$ we find that $\tilde{r}$ is independent of decomposition, as are the corresponding convergence iteration counts of Table 4.4. From Table 4.5 we see that if $\mu \leq 10^{-3}$ then all unbalanced domain decompositions reduce the execution time below that of the undecomposed algorithm. By comparing Table 4.5 (unbalanced decomposition) and Table 4.3 (balanced decomposition) we see that, if $\mu \leq 10^{-3}$, the algorithm executes more quickly when the decomposition is unbalanced.

5. Conclusions

We make the following observations from the results of this work:

- Although the convergence estimate (3.12) is of no formal use when $\tilde{r} \geq 1$, we find that, for balanced decompositions in which the interfacial subdomains are minimal, the convergence parameter $\tilde{r}$ is reflected in the convergence behaviour of the algorithm.
- If $\mu \leq 10^{-3}$ and $N = 512$, the Crank-Nicolson scheme is computed significantly more quickly than the implicit scheme on all balanced domain decompositions, including the case $M = 1$, $L = 1$; the undecomposed algorithm from [2].

- For unbalanced domain decompositions, the convergence iteration count is independent of M and L and minimal interfacial subdomains are sufficient. If $\mu \leq 10^{-3}$ then all unbalanced decompositions reduce the execution time below that of the undecomposed algorithm. For $\mu \leq 10^{-3}$ and $N = 512$, the minimum execution time over all unbalanced decompositions is between 15 % and 30 % lower than the minimum execution time over all balanced decompositions.

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