Journal of Computational Mathematics
Vol.27, No.5, 2009, 563–572.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2009.27.5.011

OPTIMAL ERROR ESTIMATES FOR NEDELEC EDGE ELEMENTS FOR TIME-HARMONIC MAXWELL'S EQUATIONS*

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Abstract

In this paper, we obtain optimal error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-norm for the Nédélec edge finite element approximation of the time-harmonic Maxwell's equations on a general Lipschitz domain discretized on quasi-uniform meshes. One key to our proof is to transform the L^2 error estimates into the L^2 estimate of a discrete divergence-free function which belongs to the edge finite element spaces, and then use the approximation of the discrete divergence-free function by the continuous divergence-free function and a duality argument for the continuous divergence-free function. For Nédélec's second type elements, we present an optimal convergence estimate which improves the best results available in the literature.

Mathematics subject classification: 65N30, 35Q60.
Key words: Edge finite element, Time-harmonic Maxwell's equations.

1. Introduction

Convergence analysis for edge element discretizations of the time-harmonic Maxwell's equations have been much studied in the literatures, see [4, 5, 7, 10, 12, 13]. Monk [12] first proved error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-norm under the assumption that Ω is convex, but both the exponent and the constant of convergence rate in L^2 error estimate involve an arbitrarily small constant $\varepsilon > 0$. Afterward, Hiptmair [10] and Monk [13] obtained asymptotic quasi-optimality of error estimates in $\boldsymbol{H}(\mathbf{curl})$-norm for a general Lipschitz polyhedron. Recently, Buffa [5] presented an abstract convergence theory for a class of noncoercive problems and then applied it to this model.

In this paper, we obtain optimal error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-norm for the Nédélec edge finite element approximation of the time-harmonic Maxwell's equations on a general Lipschitz domain and quasi-uniform meshes. First of all, we use the discrete Helmholtz decomposition for the difference between the Nédélec finite element solution and a finite element function, then obtain the discrete divergence-free function $\boldsymbol{w}_h$ which belongs to the edge finite element spaces. Secondly, we transform error estimate in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-seminorm into the L^2 estimate of $\boldsymbol{w}_h$ by proving that the error function is discrete divergence-free. Thirdly,

* Received November 3, 2008 / Revised version received December 12, 2008 / Accepted February 5, 2009 /

we obtain the L^2 estimates of $\boldsymbol{w}_h$ by using its special approximation $\boldsymbol{w}$ which is a continuous divergence-free function and a duality argument for $\boldsymbol{w}$. Finally, we obtain optimal error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-norm. Compare with the results in [12], the exponent and the constant of convergence rate in our error estimates are independent of the constant ε, thereby we improve the L^2 error estimate in [12].

Combining optimal L^2 error estimates with the corresponding interpolation error estimates for Nédélec's second type elements, we obtain the convergence order of the error function, and the order only depends on the Lipschitz domain and the smoothness of the solution. Especially, for the convex domain, we obtain an optimal convergence order. It should be noted that the L^2 error estimates are one order higher than the $\boldsymbol{H}(\mathbf{curl})$-norm estimates for Nédélec's second type elements, however, it is not correct for Nédélec's first type elements, because when restricted to the elements of the triangulation they fails to provide a complete space of polynomial (see [12]).

To avoid the repeated use of generic but unspecified constants, following [18], we use the notation $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$, the above generic constants C are independent of the function under consideration, but they may depend on Ω and the shape-regularity of the meshes.

The rest of the paper is organized as follows. In Section 2, we introduce the time-harmonic Maxwell's equations, then present its corresponding equivalent variational problem and the well-posedness. In Section 3, we present the discrete variational problem and some preliminaries. In Section 4, we obtain optimal error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-norm, and present an optimal convergence order for Nédélec's second type elements.

2. Formulation of the Problem

For simplicity, we assume that Ω is a bounded Lipschitz polyhedron in $\mathbb{R}^3$ with connected boundary Γ and unit outward normal $\boldsymbol{\nu}$. For any $m \geq 1$ and $p \geq 1$, we denote the standard Sobolev space by $W^{m,p}(\Omega)$. Especially, when $p = 2$, we denote the space by $H^m(\Omega) = W^{m,2}(\Omega)$, and $H_0^1(\Omega) = \{u \in H^1(\Omega) :\ u|_\Gamma = 0\}$. Furthermore, we also need some other Sobolev functional spaces (see [9, 14]):

$$\boldsymbol{H}_0(\mathbf{curl};\Omega) = \left\{\boldsymbol{u} \in (L^2(\Omega))^3 \;\middle|\; \nabla \times \boldsymbol{u} \in (L^2(\Omega))^3,\ \boldsymbol{\nu} \times \boldsymbol{u} = \mathbf{0} \text{ on } \Gamma\right\},$$
$$\boldsymbol{H}^s(\mathbf{curl};\Omega) = \left\{\boldsymbol{u} \in (H^s(\Omega))^3 \;\mid\; \nabla \times \boldsymbol{u} \in (H^s(\Omega))^3\right\},$$

where $s > 0$, and the above spaces are equipped with the norms, respectively,

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = \left(\|\boldsymbol{v}\|_0^2 + \|\nabla \times \boldsymbol{v}\|_0^2\right)^{1/2} \quad \forall\ \boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega),$$
$$\|\boldsymbol{v}\|_{\boldsymbol{H}^s(\mathbf{curl};\Omega)} = \left(\|\boldsymbol{v}\|_{H^s(\Omega)}^2 + \|\nabla \times \boldsymbol{v}\|_{H^s(\Omega)}^2\right)^{1/2} \quad \forall\ \boldsymbol{v} \in \boldsymbol{H}^s(\mathbf{curl};\Omega).$$

Here, $\|\cdot\|_0$ denotes the norm in $(L^2(\Omega))^3$.

We consider the following classical time-harmonic Maxwell's equations (c.f. [10, 12, 14])

$$\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E}) - \omega^2\eta\boldsymbol{E} = \boldsymbol{F} \quad \text{in } \Omega, \tag{2.1}$$
$$\boldsymbol{\nu} \times \boldsymbol{E} = \mathbf{0} \quad \text{on } \Gamma, \tag{2.2}$$

where μ is called the magnetic permeability, $\omega > 0$ is called the angular frequency, $\eta = \epsilon + i\sigma/\omega$, where $i = \sqrt{-1}$, ϵ and σ are called, respectively, the electric permittivity, and conductivity of the homogeneous isotropic body occupying Ω, $\boldsymbol{F} = i\omega\boldsymbol{J}$ with the applied current density $\boldsymbol{J}$.

In general, μ and ϵ are positive definite functions, σ is positive definite in a conductor and vanishes in an insulator. For simplicity, we assume that $\mu = 1, \alpha := \omega^2\epsilon \in \mathbb{R}, \beta := \omega\sigma \in \mathbb{R}$, where both ϵ and σ are constants. We also suppose that $\boldsymbol{F} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)'$ and $\boldsymbol{H}_0(\mathbf{curl};\Omega)'$ is the dual space of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ with respect to the $(L^2(\Omega))^3$ inner product.

Remark 2.1. 1. When $\epsilon = 0$, Eqs. (2.1) and (2.2) describe the eddy current model (c.f. [1,17]):

$$\nabla \times (\nabla \times \boldsymbol{E}) - i\beta\boldsymbol{E} = \boldsymbol{F} \quad \text{in } \Omega,$$
$$\boldsymbol{\nu} \times \boldsymbol{E} = \boldsymbol{0} \quad \text{on } \Gamma.$$

2. When $\sigma = 0$, Eqs. (2.1) and (2.2) describe the lossless case of the time-harmonic Maxwell's equations

$$\nabla \times (\nabla \times \boldsymbol{E}) - \alpha\boldsymbol{E} = \boldsymbol{F} \quad \text{in } \Omega, \tag{2.3}$$
$$\boldsymbol{\nu} \times \boldsymbol{E} = \boldsymbol{0} \quad \text{on } \Gamma. \tag{2.4}$$

In this case, α is real, and both $\boldsymbol{E}$ and $\boldsymbol{F}$ are also real.

The variational formulation of problems (2.1)-(2.2) is to find $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ such that

$$\hat{a}(\boldsymbol{E},\boldsymbol{\psi}) = (\boldsymbol{F},\boldsymbol{\psi}) \qquad \forall\, \boldsymbol{\psi} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \tag{2.5}$$

where

$$\hat{a}(\boldsymbol{E},\boldsymbol{\psi}) = (\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{\psi}) - ((\alpha + i\beta)\boldsymbol{E},\boldsymbol{\psi}). \tag{2.6}$$

Here, $(\cdot,\cdot)$ denotes the inner product in $(L^2(\Omega))^3$.

In order to ensure the well-posedness of variational problem (2.5), we will always make the following two assumptions:

$$\beta > 0 \ \text{ in } \overline{\Omega}, \tag{2.7}$$

or

$$\beta = 0, \text{ and } \alpha \text{ is not an eigenvalue of Eqs. (2.3) and (2.4).} \tag{2.8}$$

The following lemma presents the well-posedness of variational problem (2.5) and its proof can be found in [10, 12].

Lemma 2.1. *Assume that Ω is a bounded Lipschitz polyhedron with connected boundary, and that* (2.7) *or* (2.8) *holds, then there exists a unique solution $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ of the variational problem* (2.5)*, and we have*

$$\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \|\boldsymbol{F}\|_0.$$

3. Edge Element Discretization

Assume that Ω is covered by a regular mesh of tetrahedron $\mathcal{T}_h$, where h is the maximum diameter of the tetrahedron in $\mathcal{T}_h$. We introduce the following Nédélec's first type elements space $\boldsymbol{V}_h^{k,1}$ and second type elements space $\boldsymbol{V}_h^{k,2}$ (see [15, 16])

$$\boldsymbol{V}_h^{k,1} = \left\{ \boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \;\middle|\; \boldsymbol{v}_h|_K \in \boldsymbol{\mathcal{R}}_k \ \text{ for all } K \in \mathcal{T}_h \right\},$$
$$\boldsymbol{V}_h^{k,2} = \left\{ \boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \;\middle|\; \boldsymbol{v}_h|_K \in (\mathcal{P}_k)^3 \ \text{ for all } K \in \mathcal{T}_h \right\},$$

where

$$\boldsymbol{\mathcal{R}}_k = (\mathcal{P}_{k-1})^3 \oplus \left\{ \boldsymbol{p} \in (\tilde{\mathcal{P}}_k)^3 \mid \boldsymbol{x} \cdot \boldsymbol{p} = 0 \right\},$$

$\mathcal{P}_k$ is the set of polynomials of total degree at most k and $\tilde{\mathcal{P}}_k$ denotes the space of homogeneous polynomials of order k.

Let $\boldsymbol{V}_h$ denotes $\boldsymbol{V}_h^{k,1}$ (or $\boldsymbol{V}_h^{k,2}$). Now, we introduce the discrete variational problem of (2.5): Find $\boldsymbol{E}_h \in \boldsymbol{V}_h$ such that

$$\hat{a}(\boldsymbol{E}_h, \boldsymbol{\psi}_h) = (\boldsymbol{F}, \boldsymbol{\psi}_h) \quad \forall\ \boldsymbol{\psi}_h \in \boldsymbol{V}_h. \tag{3.1}$$

In the following, we introduce some preliminaries which will be used in the error estimates.

Using the degrees of freedom of Nédélec edge element space $\boldsymbol{V}_h^{k,l}(l=1,2)$, we can define the corresponding edge interpolations $\Pi_h^{\mathbf{curl},l}\boldsymbol{u} \in \boldsymbol{V}_h^{k,l}$, for any $\boldsymbol{u} \in \boldsymbol{H}^{1/2+\bar{\delta}}(\mathbf{curl}; K)$ with constant $\bar{\delta} > 0$ and $K \in \mathcal{T}_h$ (c.f. [15,16]).

The next lemma states the interpolation error estimate, see Theorem 8.15 in [14], Lemma 3.2 and Lemma 3.3 in [6].

Lemma 3.1. *Let $\boldsymbol{V}_h^{k,2}$ and $\Pi_h^{\mathbf{curl},2}$ be constructed as above. Then*

1. *If $\boldsymbol{u} \in (H^{s+1}(\Omega))^3$ for $1 \le s \le k$, we have*

$$\|\boldsymbol{u} - \Pi_h^{\mathbf{curl},2}\boldsymbol{u}\|_0 + h\|\nabla \times (\boldsymbol{u} - \Pi_h^{\mathbf{curl},2}\boldsymbol{u})\|_0 \lesssim h^{s+1}\|\boldsymbol{u}\|_{(H^{s+1}(\Omega))^3}. \tag{3.2}$$

2. *If $\boldsymbol{u} \in \boldsymbol{H}^{\delta}(\mathbf{curl};\Omega)$ for $1/2 < \delta \le 1$, we have*

$$\|\boldsymbol{u} - \Pi_h^{\mathbf{curl},2}\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h^{\delta}\|\boldsymbol{u}\|_{\boldsymbol{H}^{\delta}(\mathbf{curl};\Omega)}. \tag{3.3}$$

The estimate of (3.3) also holds for the interpolation $\Pi_h^{\mathbf{curl},1}$.

Next, we define the following Lagrange finite element space corresponding to $H_0^1(\Omega)$:

$$S_h^k = \left\{ p_h \in H_0^1(\Omega) \cap C(\bar{\Omega}) \ \middle|\ p_h|_K \in \mathcal{P}_k,\ \forall\ K \in \mathcal{T}_h \right\}.$$

Definition 3.1. *A function $\boldsymbol{v} \in (L^2(\Omega))^3$ is called discrete divergence-free for $\boldsymbol{V}_h^{k,l}(l=1,2)$ if there holds*

$$(\boldsymbol{v}, \nabla q_h) = 0 \quad \forall\ q_h \in S_h^{k+l-1}. \tag{3.4}$$

In view of the above definition, we know that a discrete divergence-free function does not possess normal continuity.

From the exact sequence property of discrete finite element space, we know that $\mathbf{grad}S_h^{k+l-1}$ is the kernel of operator $\mathbf{curl}$ in $\boldsymbol{V}_h^{k,l}, l=1,2$. Hence, we have the following discrete Helmholtz decomposition for $\boldsymbol{V}_h^{k,l}$ (see [10,11,14])

$$\boldsymbol{V}_h^{k,l} = \boldsymbol{V}_{0,h}^{k,l} + \mathbf{grad}S_h^{k+l-1}, \tag{3.5}$$

where

$$\boldsymbol{V}_{0,h}^{k,l} := \{\boldsymbol{u}_h \in \boldsymbol{V}_h^{k,l} \mid (\boldsymbol{u}_h, \nabla p_h) = 0 \ \text{ for } \forall\ p_h \in S_h^{k+l-1}\}.$$

The following lemma shows that the discrete divergence-free function can be well approximated by a continuous divergence-free function. The construction for continuous divergence-free function was used for example by Girault [8] and Monk [12]. However a clearer analysis is from Amrouche et al. [2], Arnold , Falk and Winther [3], Hiptmair [10] or Monk [13,14].

Lemma 3.2. *For any given* $\boldsymbol{u}_h \in \boldsymbol{V}_{0,h}^{k,l}$, *there exists a* $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ *that satisfies*

$$\nabla \times \boldsymbol{u} = \nabla \times \boldsymbol{u}_h, \ \nabla \cdot \boldsymbol{u} = 0, \tag{3.6}$$

and

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \lesssim h^\delta \|\nabla \times \boldsymbol{u}_h\|_0 \tag{3.7}$$

with a constant $\delta \in (0.5, 1]$ *and* $\delta = 1$ *for the case that* Ω *is smooth or convex.*

4. Error Estimates

In this section, we will discuss the finite element error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-norm.

Let $\boldsymbol{V}_{0,h}$ denotes $\boldsymbol{V}_{0,h}^{k,1}$ (or $\boldsymbol{V}_{0,h}^{k,2}$) and S_h denotes S_h^k (or S_h^{k+1}). For any $\boldsymbol{v}_h \in \boldsymbol{V}_h$, using the discrete Helmholtz decompositions (3.5) for $\boldsymbol{v}_h - \boldsymbol{E}_h$, we have

$$\boldsymbol{v}_h - \boldsymbol{E}_h = \boldsymbol{w}_h + \nabla q_h, \tag{4.1}$$

where $\boldsymbol{w}_h \in \boldsymbol{V}_{0,h}$ and $\boldsymbol{q}_h \in S_h$. In view of (4.1), we obtain

$$\nabla \times \boldsymbol{w}_h = \nabla \times (\boldsymbol{v}_h - \boldsymbol{E}_h). \tag{4.2}$$

At this stage we do not know that $\boldsymbol{E}_h$ exists, but if it does exist we define $\boldsymbol{e}_h := \boldsymbol{E} - \boldsymbol{E}_h$. In the following, we present error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-seminorm, respectively.

Lemma 4.1. *The following estimate holds*

$$\|\boldsymbol{e}_h\|_0 \leq \sqrt{2} \left(\|\boldsymbol{E} - \boldsymbol{v}_h\|_0 + \|\boldsymbol{w}_h\|_0\right). \tag{4.3}$$

Proof. Subtracting (3.1) from (2.5), we obtain the Galerkin orthogonality

$$\hat{a}(\boldsymbol{e}_h, \boldsymbol{\psi}_h) = 0 \qquad \forall\ \boldsymbol{\psi}_h \in \boldsymbol{V}_h, \tag{4.4}$$

which leads to

$$((\alpha + i\beta)\boldsymbol{e}_h, \nabla \boldsymbol{p}_h) = 0 \qquad \forall\ \boldsymbol{p}_h \in S_h. \tag{4.5}$$

For arbitrary real-valued function $r_h \in S_h$, taking ∇r_h or $i\nabla r_h$ instead of $\nabla \boldsymbol{p}_h$ in (4.5), respectively, and using the assumptions (2.7) and (2.8), we conclude that $\boldsymbol{e}_h$ is discrete divergence-free, namely

$$(\boldsymbol{e}_h, \nabla \boldsymbol{p}_h) = 0 \quad \forall\ \boldsymbol{p}_h \in S_h. \tag{4.6}$$

Using (4.1) and (4.6), we have

$$\begin{aligned} \|\boldsymbol{e}_h\|_0^2 &= (\boldsymbol{e}_h, \boldsymbol{E} - \boldsymbol{v}_h) + (\boldsymbol{e}_h, \boldsymbol{v}_h - \boldsymbol{E}_h) \\ &= (\boldsymbol{e}_h, \boldsymbol{E} - \boldsymbol{v}_h) + (\boldsymbol{e}_h, \boldsymbol{w}_h + \nabla \boldsymbol{q}_h) \\ &= (\boldsymbol{e}_h, \boldsymbol{E} - \boldsymbol{v}_h) + (\boldsymbol{e}_h, \boldsymbol{w}_h). \end{aligned} \tag{4.7}$$

Combining (4.7) with Cauchy-Schwartz inequality yields

$$\|\boldsymbol{e}_h\|_0 \leq \sqrt{2} \left(\|\boldsymbol{E} - \boldsymbol{v}_h\|_0 + \|\boldsymbol{w}_h\|_0\right),$$

which completes the proof. □

Lemma 4.2. *The following estimate holds*

$$\|\nabla \times \boldsymbol{e}_h\|_0 \lesssim \|\boldsymbol{E} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{w}_h\|_0, \tag{4.8}$$

where the constant only depends on the parameters α and β.

Proof. By (2.6), (4.4), (4.1) and (4.5), we have

$$\begin{aligned}
\|\nabla \times \boldsymbol{e}_h\|_0^2 &= \hat{a}(\boldsymbol{e}_h, \boldsymbol{e}_h) + ((\alpha + i\beta)\boldsymbol{e}_h, \boldsymbol{e}_h) \\
&= \hat{a}(\boldsymbol{e}_h, \boldsymbol{E} - \boldsymbol{v}_h) + ((\alpha + i\beta)\boldsymbol{e}_h, \boldsymbol{e}_h) \\
&= (\nabla \times \boldsymbol{e}_h, \nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)) - ((\alpha + i\beta)\boldsymbol{e}_h, \boldsymbol{E} - \boldsymbol{v}_h - \boldsymbol{e}_h) \\
&= (\nabla \times \boldsymbol{e}_h, \nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)) + ((\alpha + i\beta)\boldsymbol{e}_h, \boldsymbol{v}_h - \boldsymbol{E}_h) \\
&= (\nabla \times \boldsymbol{e}_h, \nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)) + ((\alpha + i\beta)\boldsymbol{e}_h, \boldsymbol{w}_h).
\end{aligned}$$

Then using the Cauchy-Schwartz inequality, we have

$$\|\nabla \times \boldsymbol{e}_h\|_0^2 \lesssim \|\nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)\|_0^2 + \|\boldsymbol{e}_h\|_0^2 + \|\boldsymbol{w}_h\|_0^2,$$

where the constant only depends on the parameters α and β. Then we obtain

$$\|\nabla \times \boldsymbol{e}_h\|_0 \lesssim \|\nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)\|_0 + \|\boldsymbol{e}_h\|_0 + \|\boldsymbol{w}_h\|_0. \tag{4.9}$$

Substituting (4.3) into (4.9), we obtain

$$\|\nabla \times \boldsymbol{e}_h\|_0 \lesssim \|\boldsymbol{E} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{w}_h\|_0,$$

which completes the proof. □

It is clear that it suffices to estimate $\|\boldsymbol{w}_h\|_0$ for completing error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-seminorm. To this end, we use the approximation of the discrete divergence-free function by the continuous divergence-free function and a duality argument for the continuous divergence-free function.

Lemma 4.3. *There exists a constant $h_0 > 0$ independents of h, $\boldsymbol{E}$ and $\boldsymbol{E}_h$, such that for all $h < h_0$, we have*

$$\|\boldsymbol{w}_h\|_0 \lesssim \|\boldsymbol{E} - \boldsymbol{v}_h\|_0 + h^\delta \|\nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)\|_0, \tag{4.10}$$

where the constant δ is the exponent in Lemma 3.2.

Proof. For given $\boldsymbol{w}_h \in \boldsymbol{V}_{0,h}$ in (4.1), using Lemma 3.2, there exists a $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ satisfies

$$\nabla \times \boldsymbol{w} = \nabla \times \boldsymbol{w}_h, \quad \nabla \cdot \boldsymbol{w} = 0, \tag{4.11}$$

and

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_0 \lesssim h^\delta \|\nabla \times \boldsymbol{w}_h\|_0. \tag{4.12}$$

Using (4.12), (4.2) and the triangle inequality, we have

$$\begin{aligned}
\|\boldsymbol{w} - \boldsymbol{w}_h\|_0 &\lesssim h^\delta \|\nabla \times (\boldsymbol{v}_h - \boldsymbol{E}_h)\|_0 \\
&\leq h^\delta \left(\|\nabla \times \boldsymbol{e}_h\|_0 + \|\nabla \times (\boldsymbol{E} - \boldsymbol{v}_h)\|_0\right).
\end{aligned} \tag{4.13}$$

Using the triangle inequality, (4.13) and (4.8), we obtain

$$\begin{aligned}\|\boldsymbol{w}_h\|_0 &\le C_1\left(h^\delta\left(\|\nabla\times\boldsymbol{e}_h\|_0+\|\nabla\times(\boldsymbol{E}-\boldsymbol{v}_h)\|_0\right)+\|\boldsymbol{w}\|_0\right)\\ &\le C_2\left(h^\delta\|\boldsymbol{E}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}+h^\delta\|\boldsymbol{w}_h\|_0+\|\boldsymbol{w}\|_0\right).\end{aligned}$$

where the constants C_i $(i=1,2)$ independent of h, $\boldsymbol{E}$ and $\boldsymbol{E}_h$. Choosing $h_1>0$ satisfies $1-C_2h_1^\delta>0$, then for all $h<h_1$, we have

$$\|\boldsymbol{w}_h\|_0\lesssim h^\delta\|\boldsymbol{E}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}+\|\boldsymbol{w}\|_0. \tag{4.14}$$

Next, we will use a duality argument to obtain the L^2 estimate of $\boldsymbol{w}$.

Let $\boldsymbol{\phi}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$ solves the following auxiliary problem

$$\hat{a}(\boldsymbol{\psi},\boldsymbol{\phi})=(\boldsymbol{w},\boldsymbol{\psi})\quad\forall\,\boldsymbol{\psi}\in\ \boldsymbol{H}_0(\mathbf{curl};\Omega). \tag{4.15}$$

Taking $\boldsymbol{\psi}=\nabla q$ with some $q\in H_0^1(\Omega)$ in (4.15), and using the Green formula with the fact $\nabla\cdot\boldsymbol{w}=0$, we have

$$((\alpha+i\beta)\nabla q,\boldsymbol{\phi})=0. \tag{4.16}$$

Since $\nabla\cdot\boldsymbol{w}=0$, we have the following regularity result for auxiliary problem (4.15) (see [13])

$$\|\boldsymbol{\phi}\|_{H^\delta(\mathbf{curl};\Omega)}\lesssim\|\boldsymbol{w}\|_0. \tag{4.17}$$

Combining (4.2) and (4.11), we have

$$\nabla\times(\boldsymbol{w}-(\boldsymbol{v}_h-\boldsymbol{E}_h))=0. \tag{4.18}$$

Noting that $\boldsymbol{w}-(\boldsymbol{v}_h-\boldsymbol{E}_h)\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$ and (4.18), thus using the exact sequence property, there exists a $p\in H_0^1(\Omega)$, such that

$$\boldsymbol{w}-(\boldsymbol{v}_h-\boldsymbol{E}_h)=\nabla p. \tag{4.19}$$

Using (4.18), (4.19) and (4.16), we have

$$\begin{aligned}&\hat{a}(\boldsymbol{w}-(\boldsymbol{v}_h-\boldsymbol{E}_h),\boldsymbol{\phi})\\ &=(\nabla\times(\boldsymbol{w}-(\boldsymbol{v}_h-\boldsymbol{E}_h)),\nabla\times\boldsymbol{\phi})-((\alpha+i\beta)(\boldsymbol{w}-(\boldsymbol{v}_h-\boldsymbol{E}_h)),\boldsymbol{\phi})\\ &=((\alpha+i\beta)\nabla p,\boldsymbol{\phi})=0.\end{aligned} \tag{4.20}$$

Setting $\boldsymbol{\psi}=\boldsymbol{w}$ in (4.15), and from (4.20), (4.16), (4.4), (4.15), (3.3) and (4.17), we have

$$\begin{aligned}\|\boldsymbol{w}\|_0^2 &=\hat{a}(\boldsymbol{w},\boldsymbol{\phi})=\hat{a}(\boldsymbol{w}-(\boldsymbol{v}_h-\boldsymbol{E}_h),\boldsymbol{\phi})+\hat{a}(\boldsymbol{v}_h-\boldsymbol{E}_h,\boldsymbol{\phi})\\ &=\hat{a}(\boldsymbol{e}_h,\boldsymbol{\phi})-\hat{a}(\boldsymbol{E}-\boldsymbol{v}_h,\boldsymbol{\phi})\\ &=\hat{a}(\boldsymbol{e}_h,\boldsymbol{\phi}-\boldsymbol{\Pi}_h^{curl}\boldsymbol{\phi})-(\boldsymbol{w},\boldsymbol{E}-\boldsymbol{v}_h)\\ &\lesssim\|\boldsymbol{e}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\|\boldsymbol{\phi}-\boldsymbol{\Pi}_h^{\mathbf{curl}}\boldsymbol{\phi}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}+\|\boldsymbol{w}\|_0\|\boldsymbol{E}-\boldsymbol{v}_h\|_0\\ &\lesssim\|\boldsymbol{w}\|_0\left(h^\delta\|\boldsymbol{e}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}+\|\boldsymbol{E}-\boldsymbol{v}_h\|_0\right),\end{aligned}$$

where $\Pi_h^{\mathbf{curl}}$ denotes $\Pi_h^{\mathbf{curl},1}$ (or $\Pi_h^{\mathbf{curl},2}$). Hence we obtain

$$\|\boldsymbol{w}\|_0\lesssim\|\boldsymbol{E}-\boldsymbol{v}_h\|_0+h^\delta\|\boldsymbol{e}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}. \tag{4.21}$$

At last, substituting (4.21) into (4.14) and using Lemmas 4.1 and 4.2, we obtain

$$\begin{aligned}\|\boldsymbol{w}_h\|_0 &\le C_3\left(\|\boldsymbol{E}-\boldsymbol{v}_h\|_0 + h^\delta\|\nabla\times(\boldsymbol{E}-\boldsymbol{v}_h)\|_0 + h^\delta\|\boldsymbol{e}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\right)\\ &\le C_4\left(\|\boldsymbol{E}-\boldsymbol{v}_h\|_0 + h^\delta\|\nabla\times(\boldsymbol{E}-\boldsymbol{v}_h)\|_0 + h^\delta\|\boldsymbol{w}_h\|_0\right),\end{aligned}$$

where the constants C_i $(i=3,4)$ independent of h, $\boldsymbol{E}$ and $\boldsymbol{E}_h$. Choosing $h_2>0$ satisfies $1-C_4h_2^\delta>0$, then for all $h<h_0:=\min\{h_1,h_2\}$, we obtain the estimate (4.10). □

Now, we present our main result as follows.

Theorem 4.1. *Let Ω be a bounded Lipschitz polyhedron with connected boundary, $\boldsymbol{E}$ and $\boldsymbol{E}_h$ satisfy* (2.5) *and* (3.1), *respectively, and* (2.7) *or* (2.8) *holds. Then there exists a constant $\delta\in(0.5,1]$ with $\delta=1$ for a convex domain, and a constant $h_0>0$ independents of h, $\boldsymbol{E}$ and $\boldsymbol{E}_h$, such that for all $h<h_0$, we have*

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0 \lesssim \inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\left(\|\boldsymbol{E}-\boldsymbol{v}_h\|_0 + h^\delta\|\nabla\times(\boldsymbol{E}-\boldsymbol{v}_h)\|_0\right), \tag{4.22}$$

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim \inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\|\boldsymbol{E}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}, \tag{4.23}$$

where the constants only depend on Ω, the shape-regularity of the meshes, the parameters α and β.

Remark 4.1. When Ω is convex, and (2.8) holds, the following error estimate can be found in [12]

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0 \le C(\varepsilon)\left(\|\boldsymbol{E}-\Pi_h^{\mathbf{curl}}\boldsymbol{E}\|_0 + h^{1-\varepsilon}\|\nabla\times(\boldsymbol{E}-\Pi_h^{\mathbf{curl}}\boldsymbol{E})\|_0\right), \tag{4.24}$$

where the constant $C(\varepsilon)$ depends on the parameter $\varepsilon>0$.

Proof. Substituting (4.10) into (4.3) and (4.8), respectively, we have

$$\|\boldsymbol{e}_h\|_0 \lesssim \|\boldsymbol{E}-\boldsymbol{v}_h\|_0 + h^\delta\|\nabla\times(\boldsymbol{E}-\boldsymbol{v}_h)\|_0, \tag{4.25}$$

$$\|\nabla\times\boldsymbol{e}_h\|_0 \lesssim \|\boldsymbol{E}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}, \tag{4.26}$$

which concludes the proof of the desired estimate (4.22). The estimate (4.23) is a direct consequence of (4.25) and (4.26). □

Corollary 4.1. *Under the hypotheses of Theorem* 4.1, *then for $\boldsymbol{F}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)'$, there exists a constant h_0 such that for all $h<h_0$, variational problem* (3.1) *has a unique solution.*

Corollary 4.2. *Assume that Ω is a bounded Lipschitz polyhedron with connected boundary,* (2.7) *or* (2.8) *holds, for some s with $1\le s\le k$, $\boldsymbol{E}\in(H^{s+1}(\Omega))^3$ and $\boldsymbol{E}_h\in\boldsymbol{V}_h^{k,2}$ satisfy* (2.5) *and* (3.1), *respectively. Then there exists a constant $\delta\in(0.5,1]$, and a constant $h_0>0$ independent of h, $\boldsymbol{E}$ and $\boldsymbol{E}_h$, such that for all $h<h_0$, we have*

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0 \lesssim h^{s+\delta}\|\boldsymbol{E}\|_{(H^{s+1}(\Omega))^3}.$$

Especially for the convex domain, we obtain optimal convergence

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_0 \lesssim h^{s+1}\|\boldsymbol{E}\|_{(H^{s+1}(\Omega))^3}.$$

Proof. Replacing $\boldsymbol{v}_h$ by $\Pi_h^{\mathbf{curl},2}\boldsymbol{E}$ in (4.22) and using (3.2), the desired estimates follow. □

Remark 4.2. The results in Corollary 4.2 can not hold for $\boldsymbol{V}_h^{k,1}$, since the estimate of (3.2) does not hold for the interpolation $\Pi_h^{\mathbf{curl},1}$.

Acknowledgments. The authors would like to thank Dr. Long Chen, University of California, Irvine, for helpful discussions. The authors are also grateful to the referees for their constructive comments which improve the presentation of the paper. The first and second author were supported in part by National Natural Science Foundation of China (Grant Nos. 10771178 and 10676031), National Key Basic Research Program of China (973 Program) (Grant No. 2005CB321702), the Key Project of Chinese Ministry of Education and Scientific Research Fund of Hunan Provincial Education Department (Grant Nos. 208093 and 07A068). Especially, the first author was also supported in part by Hunan Provincial Innovation Foundation for Postgraduate. The last author was partially supported by Alexander von Humboldt Research Award for Senior US Scientists, NSF DMS-0609727, NSFC-10528102 and Furong Professor Scholar Program of Hunan Province of China through Xiangtan University.

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