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RECOVERY A POSTERIORI ERROR ESTIMATES FOR GENERAL CONVEX ELLIPTIC OPTIMAL CONTROL PROBLEMS SUBJECT TO POINTWISE CONTROL CONSTRAINTS*

Yanping Chen
School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China
Email: yanpingchen@scnu.edu.cn
Yao Fu
College of Mechanical Engineering, Jiangxi Yu Zhou Scientific and Technological Institute, Xinyu 338029, China
Email: fuyao0222@163.com
Huanwen Liu
School of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning 530006, China
Email: mengtian29@163.com
Yongquan Dai and Huayi Wei
Hunan Key Laboratory for Computation and Simulation in Science and Engineering, School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China
Email: daiyongquan@126.com; weihy1984@163.com

Abstract

Superconvergence and recovery a posteriori error estimates of the finite element approximation for general convex optimal control problems are investigated in this paper. We obtain the superconvergence properties of finite element solutions, and by using the superconvergence results we get recovery a posteriori error estimates which are asymptotically exact under some regularity conditions. Some numerical examples are provided to verify the theoretical results.

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1. Introduction

Efficient numerical methods are essential to successful applications of optimal control problems (see, e.g., [17, 27, 33]) in practical areas. It is well known that finite element methods are undoubtedly the most widely used numerical methods in solving optimal control problems. There have been extensive studies in convergence of the finite element approximation for various optimal control problems (see, e.g., [2, 14, 16, 21, 22, 34]). Recently, a priori error estimates of the finite element approximation for optimal control problems governed by linear state equations can be found in [3], and a posteriori error estimates in [4, 5, 19, 20, 25, 28–31]. Some primary works on sharp a posteriori error estimates and a priori error estimates of mixed

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finite element methods for optimal control problems were obtained in [11–13, 36, 37]. Adaptive finite element methods are among the most important classes of numerical methods to boost accuracy and efficiency of the finite element discretization. The literature in this area is huge, see, e.g., [1, 6, 35, 40, 42–44]. The superconvergence property of finite element solutions has also been an active research area in numerical analysis for optimal control problems (see, e.g., [10, 26, 32, 39]). Very recently, superconvergence of mixed finite element methods for optimal control problems has been studied in [7–9, 12, 41]. The main objective for investigating the superconvergence property is to improve the existing approximation accuracy by applying certain postprocessing techniques which are easy to implement. For the quadratic optimal control problems, some superconvergence results have been established (see [26,39]). In [18], Hinze presented a method that is not necessary to discretize the control variable for linear quadratic optimal control problems.

Finite element recovery techniques are post-processing methods that reconstruct numerical approximations from finite element solutions to achieve better results. To be practically useful, a good recovery method should have the following three features: (i) It is simple to implement and cost effective. In practice, a recovery procedure takes only very small portion of the whole computation cost; (ii) It is applicable to higher dimensions; and (iii) It is problem independent, i.e., a recovery process uses only numerical solution data.

This paper is concerned with the following general convex optimal control problem:

$$\min_{u\in K\subset L^2(\Omega_U)}\{g(y)+h(u)\} \tag{1.1}$$

$$-\operatorname{div}(A\nabla y)+a_0y=f+Bu, \qquad \text{in } \Omega\ , \tag{1.2}$$

$$y=0, \qquad \text{on } \partial\Omega, \tag{1.3}$$

where g and h are convex functionals, K is a closed convex set in $L^2(\Omega_U)$, Ω and Ω_U are two bounded open subsets in R^n $(n\le 3)$ with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$, respectively. Let f be a given function of the space $L^2(\Omega)$ and B be a continuous linear operator from $L^2(\Omega_U)$ to $L^2(\Omega)$. The coefficient matrix $A(\cdot)=(a_{ij}(\cdot))_{n\times n}$ is symmetric and positive definite. Moreover, we require $0\leqslant a_0\in L^\infty(\Omega)$.

Denote by $W^{m,p}(\Omega)$ the usual Sobolev space on Ω with norm and semi-norm defined by

$$\begin{aligned}\|\phi\|^p_{m,p,\Omega} &= \sum_{|\alpha|\le m}\int_\Omega |\ \partial^\alpha\phi\ |^p\ dx,\\ |\ \phi\ |^p_{m,p,\Omega} &= \sum_{|\alpha|=m}\int_\Omega |\ \partial^\alpha\phi\ |^p\ dx,\end{aligned}$$

where $\phi\in W^{m,p}(\Omega)$. We set $W^{m,p}_0(\Omega)=\{\phi\in W^{m,p}(\Omega):\phi\ |_{\partial\Omega}=0\}$. In particular, we write $H^m(\Omega)=W^{m,2}(\Omega)$ $(H^m_0(\Omega)=W^{m,2}_0(\Omega))$ and $\|\cdot\|_{m,\Omega}=\|\cdot\|_{m,p,\Omega}$ $(\|\cdot\|_{H^m_0(\Omega)}=\|\cdot\|_{W^{m,p}_0(\Omega)})$, $|\cdot|_{m,\Omega}=|\cdot|_{m,p,\Omega}$ $(|\cdot|_{H^m_0(\Omega)}=|\cdot|_{W^{m,p}_0(\Omega)})$ for $p=2$. Besides, c or C denotes a general positive constant independent of h.

In this paper, we adopt the same recovery operators mentioned in [26] to solve general convex optimal control problems. We get the superconvergence property of finite element solutions, by which recovery a posteriori error estimates are obtained. The control variable is approximated by piecewise constant functions, and both the state y and the co-state p by piecewise linear finite element functions. We prove the superconvergence error estimate in L^2-norm between the approximated solution and the L^2-projection of the control, and superconvergence error estimates in H^1-norm between the approximated solutions and the elliptic projections of the state and co-state. By using the superconvergence results, recovery a posteriori error estimators are obtained, which are the bases to judge whether further refinement of meshes is necessary in adaptive grid methods. Another important issue is that different adaptive meshes should be used for the control and the states, as generally they are of very different natures of singularities. In our experiments, because different meshes are used for the approximation of the state and the control, we must look for a suitable preconditioning for the projection algorithm. It is observed that the preconditioning used in [26] is inefficient for our new model. To overcome this difficulty, we adopt an interpolation function as a preconditioning, which is found efficient for our present model and quadratic convex optimal control problems. Numerical examples are also shown that the estimators are simple to implement and applicable to 3-dimension space. In other words, method we will use satisfies the three features above.

The paper is organized as follows. In Section 2, we formulate the finite element approximation for general convex optimal control problems. In Section 3, we concentrate on the superconvergence analysis for the control problem. Recovery a posteriori error estimators for the L^2-error in discrete solutions are considered in Section 4. Numerical examples are provided in Section 5 to verify the theoretical results. We conclude with some further comments in Section 6.

2. Finite Element Methods for Optimal Control Problems

Let the state space

$$V = H_0^1(\Omega), \tag{2.1}$$

and the control space

$$U = L^2(\Omega_U). \tag{2.2}$$

We denote

$$H = L^2(\Omega). \tag{2.3}$$

Let the observation space $Y = L^2(\Omega)$. We further assume that g and h are continuously differentiable and bounded below on the observation space Y. Let

$$K = \{v \in U : v \geqslant 0\}. \tag{2.4}$$

To consider the finite element approximation of the general convex optimal control problem, we need a weak formulation for the state equation. Let

$$a(y,v) = \int_\Omega (A\nabla y) \cdot \nabla v + a_0 yv, \qquad \forall y,\ v \in V.$$

By the assumptions on A, there are positive constants c and C such that $\forall\ y, v \in V$,

$$a(y,y) \geq c\|y\|_{1,\Omega}^2, \qquad |\ a(y,v)\ | \leq C\|y\|_{1,\Omega}\|v\|_{1,\Omega}. \tag{2.5}$$

We recast (1.1)-(1.3) in the following weak form, find $(y,u) \in V \times U$ such that: (CCP)

$$\min_{u\in K\subset U} \{g(y) + h(u)\} \tag{2.6}$$

$$a(y(u),v) = (f + Bu, v), \qquad \forall v \in V = H_0^1(\Omega), \tag{2.7}$$

where the inner product in $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. It is well known (see, e.g., [27]) that the convex control problem (CCP) (2.6)-(2.7) has a unique solution (y,u), and that a pair $(y,u)\in V\times U$ is the solution of (CCP) (2.6)-(2.7) if and only if there is a co-state $p\in V$ such that the triplet (y,p,u) satisfies the following optimal conditions: (CCP-OPT)

$$a(y,v)=(f+Bu,v), \tag{2.8}$$
$$a(q,p)=(g'(y),q), \tag{2.9}$$
$$(h'(u)+B^*p,\tilde{u}-u)_U\geq 0, \tag{2.10}$$

for all $v\in V,\ q\in V$, and $\tilde{u}\in K$, where B^* is the adjoint operator of B, and g', h' are the deviratives of g, h, respectively. The inner product in $L^2(\Omega_U)$ is denoted by $(\cdot,\cdot)_U$.

Also we note that for any $(y,u)\in V\times U$, $g'(y)$ and $h'(u)$ are in $Y=Y'=L^2(\Omega)$ and $U'=U=L^2(\Omega_U)$ respectively. Therefore, they can be viewed as functions in $Y=L^2(\Omega)$ and $U=L^2(\Omega_U)$ respectively, from the well-known representation theorem in a Hilbert space.

For ease of exposition we will assume that Ω and Ω_U are both polygons. Let Ω^h and Ω_U^h be two polygonal approximations to Ω and Ω_U, so that $\Omega^h=\Omega$ and $\Omega_U^h=\Omega_U$. Let T^h and T_U^h be two partitioning of Ω^h and Ω_U^h into disjoint regular n-simplices τ and τ_U. So that $\bar{\Omega}^h=\cup_{\tau\in T^h}\bar{\tau}$, $\bar{\Omega}_U^h=\cup_{\tau_U\in T_U^h}\bar{\tau}_U$. We assume that $\bar{\tau}$ $(\bar{\tau}_U)$ and $\bar{\tau}'$ $(\bar{\tau}'_U)$ have either only one common vertex or a whole edge or face or are disjoint if τ (τ_U) and τ' $(\tau'_U)\in T^h$ (T_U^h). Moreover, we set

$$U^h=\{u\in U: u\,|_{\tau_U}\ \text{ is constant on all } \tau_U\in T_U^h\}, \tag{2.11}$$
$$V^h=\{y_h\in V:\ y_h\in P_1,\forall\tau\in T_h\}. \tag{2.12}$$

Let h_τ (h_{τ_U}) denote the maximum diameter of the element τ (τ_U) in T^h (T_U^h). Let $h=\max_{\tau\in T^h}\{h_\tau\}$ and $h_U=\max_{\tau_U\in T_U^h}\{h_{\tau_U}\}$. In computations, the element sizes in T^h are required to be larger than those in T_U^h. Therefore, we assume that $(h_U/h)\leq C$ in this paper.

The finite element approximation of (CCP) (2.6)-(2.7) is to find $(y_h,u_h)\in V^h\times U^h$ such that: $(\text{CCP})^h$

$$\min_{u_h\in K^h\subset U^h}\{g(y_h)+h(u_h)\} \tag{2.13}$$
$$a(y_h,v_h)=(f+Bu_h,v_h),\qquad \forall v_h\in V^h, \tag{2.14}$$

where K^h is a closed convex set in U^h. The control problem $(\text{CCP})^h$ (2.13)-(2.14) has a unique solution (y_h,u_h), and a pair $(y_h,u_h)\in V^h\times U^h$ is the solution of $(\text{CCP})^h$ (2.13)-(2.14) if and only if there is a co-state $p_h\in V^h$, such that the triplet (y_h,p_h,u_h) satisfies the following discretized optimality conditions: $(\text{CCP-OPT})^h$

$$a(y_h,v_h)=(f+Bu_h,v_h), \tag{2.15}$$
$$a(q_h,p_h)=(g'(y_h),q_h), \tag{2.16}$$
$$(h'(u_h)+B^*p_h,\tilde{u}_h-u_h)_U\geq 0, \tag{2.17}$$

for all $v_h\in V^h,\ q_h\in V^h$, and $\tilde{u}_h\in K^h$.

It is well known that for the problem (2.8)-(2.10) and its finite element approximation (2.15)-(2.17), the following error estimate hold:

$$\|u-u_h\|_{0,\Omega}+\|y-y_h\|_{1,\Omega}+\|p-p_h\|_{1,\Omega}\leq C(h_U+h), \tag{2.18}$$

if $y,\ p\in H^2(\Omega)$, and $u\in H^1(\Omega_U)$.

Remark 2.1. Falk [16] noticed that $u \in H^1(\Omega_U)$ needs some further data regularity. In this paper, take $h(u) = \frac{1}{2}||u||^2_{0,\Omega}$, $B = I$ for example, we know that if the solution of the optimal control u for this problem is given by $u = \max(0, -p)$, then we have $u \in H^1(\Omega_U)$.

We further make the following assumptions:

- g' is Lipshitz continuous;
- There is a constant $c > 0$ such that

$$(h'(u) - h'(\tilde{u}), u - \tilde{u})_U \geq c\|u - \tilde{u}\|^2_{0,\Omega_U}, \quad \forall\ u,\ \tilde{u} \in U, \tag{2.19}$$

  the convex functional g also has such a property as h;

- Let $h(u) = \int_{\Omega_U} j(u)$. Then $(h'(u), v) = (j'(u), v)$, where $j(\cdot)$ is a smooth and convex function such that $j''(u) \in W^{1,\infty}(\Omega_U)$ and $j'''(\cdot) \in L^\infty(R)$;

- Let

$$\Omega_U^+ = \{\cup\tau_U : \tau_U \subset \Omega_U,\ u|_{\tau_U} > 0\},$$
$$\Omega_U^0 = \{\cup\tau_U : \tau_U \subset \Omega_U,\ u|_{\tau_U} = 0\}, \quad \Omega_U^b = \Omega_U \setminus (\Omega_U^+ \cup \Omega_U^0).$$

  In this paper, we assume that u and τ_h are regular such that meas $(\Omega_U^b) \leq Ch_U$.

3. Superconvergence Analysis

In this section, we provide the superconvergence results for general convex optimal control problems (1.1). Firstly, let us prove the following superconvergence property between $\pi^c u$ and u_h which are the L^2-projection and the approximated solution of u, respectively.

Definition 3.1. *Let $\pi^c u \in K^h$ be the L^2-projection of u, such that*

$$\pi^c u|_{\tau_U} = \frac{\int_{\tau_U} u}{\int_{\tau_U} 1}. \tag{3.1}$$

By the definition of $\pi^c u$, we have the following orthogonal property (see, e.g., [26, 39]):

$$(u - \pi^c u, u_h)_U = 0, \qquad \forall u_h \in U^h. \tag{3.2}$$

Theorem 3.1. *Let u and u_h be the solution of* (2.10) *and* (2.17)*, respectively. Assume that $h'(u) + B^*p \in W^{1,\infty}(\Omega_U)$, and Ω is convex. Then,*

$$\|u_h - \pi^c u\|_{0,\Omega_U} \leq C\left(h_U^{\frac{3}{2}} + h^2\right). \tag{3.3}$$

Proof. Note that u_h, $\pi^c u \in K^h \subset K$. From (2.10) and (2.17), by inserting $\tilde{u} = u_h$ and $\tilde{u}_h = \pi^c u$ respectively, we have

$$(h'(u) + B^*p,\ u_h - u)_U \geq 0, \tag{3.4}$$
$$(h'(u_h) + B^*p_h,\ \pi^c u - u_h)_U \geq 0. \tag{3.5}$$

By means of (2.19), (3.4), and (3.5), we have

$$\begin{aligned}c\|u_h-\pi^c u\|_{0,\Omega_U}^2 &\le (h'(u_h)-h'(\pi^c u),u_h-\pi^c u)_U\\ &\le (-B^*p_h,u_h-\pi^c u)_U-(h'(\pi^c u),u_h-\pi^c u)_U\\ &=(B^*p,u-u_h)_U+(B^*p,\pi^c u-u)_U\\ &\quad+(B^*(p-p_h),u_h-\pi^c u)_U+(h'(\pi^c u),\pi^c u-u_h)_U\\ &\le (h'(\pi^c u)-h'(u),\pi^c u-u_h)_U+(h'(u)+B^*p,\pi^c u-u)_U\\ &\quad+(B^*(p-p(u_h)),u-\pi^c u)_U+(B^*(p-p(u_h)),u_h-u)_U\\ &\quad+(B^*(p(u_h)-p_h),u_h-\pi^c u)_U, \end{aligned}\tag{3.6}$$

where $p(u_h)$ is the solution of the auxiliary equations:

$$a(y(u_h),v)=(f+Bu_h,v),\qquad \forall v\in V=H_0^1(\Omega),\tag{3.7}$$

$$a(q,p(u_h))=(g'(y(u_h)),q),\qquad \forall q\in V=H_0^1(\Omega).\tag{3.8}$$

Note that Ω is convex. We have that $p\in H^2(\Omega)\cap W^{1,\infty}(\Omega)$, and $u\in W^{1,\infty}(\Omega_U)$.

From (3.2) and Taylor's expansion of the function $j(\cdot)$, there exists a positive constant $0\le\theta\le 1$ such that

$$\begin{aligned}&(h'(\pi^c u)-h'(u),\pi^c u-u_h)_U\\ =&(j''(u)(\pi^c u-u)+\frac{1}{2}j'''(u+\theta(\pi^c u-u))(\pi^c u-u)^2,\pi^c u-u_h)_U\\ =&((j''(u)-\pi^c(j''(u)))(\pi^c u-u),\pi^c u-u_h)_U+\frac{1}{2}(j'''(u+\theta(\pi^c u-u))(\pi^c u-u)^2,\pi^c u-u_h)_U\\ \le&Ch_U\|j''(u)\|_{1,\infty,\Omega_U}\cdot\|\pi^c u-u\|_{0,\Omega_U}\cdot\|\pi^c u-u_h\|_{0,\Omega_U}\\ &+C\frac{1}{2}\|j'''(\cdot)\|_{0,\infty}\cdot\|\pi^c u-u\|_{0,4,\Omega_U}^2\cdot\|\pi^c u-u_h\|_{0,\Omega_U}\\ \le&Ch_U^2\|\pi^c u-u_h\|_{0,\Omega_U}.\end{aligned}\tag{3.9}$$

Similar to [26], we can get that

$$(B^*(p-p(u_h)),u-\pi^c u)_U\le Ch_U^2(h_U+\|u_h-\pi^c u\|_{0,\Omega_U}),\tag{3.10}$$

$$(B^*(p-p(u_h)),u_h-u)_U=(g'(y)-g'(y(u_h)),y(u_h)-y)\le 0,\tag{3.11}$$

where we used the fact that $g(\cdot)$ is a convex functional. Using Schwarz inequality and Young's inequality with ϵ, we have

$$\begin{aligned}(B^*(p(u_h)-p_h),u_h-\pi^c u)_U&\le C\|B^*(p(u_h)-p_h)\|_{0,\Omega_U}\|u_h-\pi^c u\|_{0,\Omega_U}\\ &\le C(\epsilon)\|p(u_h)-p_h\|_{0,\Omega}^2+C\epsilon\|u_h-\pi^c u\|_{0,\Omega_U}^2,\end{aligned}\tag{3.12}$$

where ϵ is an arbitrary small positive constant and $C(\epsilon)$ is a constant dependent on ϵ. In [26], it has been proved that

$$\|p(u_h)-p_h\|_{0,\Omega}\le Ch^2.\tag{3.13}$$

Then, it follows from (3.6)-(3.13) that

$$\begin{aligned}&c\|u_h-\pi^c u\|_{0,\Omega_U}^2\\ \le&Ch_U^2\|u_h-\pi^c u\|_{0,\Omega_U}+(h'(u)+B^*p,\ \pi^c u-u)_U\\ &+Ch_U^2(h_U+\|u_h-\pi^c u\|_{0,\Omega_U})+Ch^4+\epsilon\|u_h-\pi^c u\|_{0,\Omega_U}^2\\ \le&(h'(u)+B^*p,\pi^c u-u)_U+Ch_U^3+Ch^4+C\epsilon\|u_h-\pi^c u\|_{0,\Omega_U}^2,\end{aligned}\tag{3.14}$$

where we used $Ch_U^2||u_h-\pi^c u||_{0,\Omega_U}\le C(\epsilon)h_U^4+C\epsilon||u_h-\pi^c u||_{0,\Omega_U}^2$. Thus,

$$\|u_h-\pi^c u\|_{0,\Omega_U}^2\le C(h'(u)+B^*p,\ \pi^c u-u)_U+Ch_U^3+Ch^4. \tag{3.15}$$

Note that

$$\begin{aligned}&(h'(u)+B^*p,\pi^c u-u)_U\\ =&\int_{\Omega_U^+}(h'(u)+B^*p)(\pi^c u-u)+\int_{\Omega_U^0}(h'(u)+B^*p)(\pi^c u-u)+\int_{\Omega_U^b}(h'(u)+B^*p)(\pi^c u-u),\end{aligned}$$

and $(\pi^c u-u)|_{\Omega_U^0}=0$. From (2.10), we have pointwise a.e. $h'(u)+B^*p\ge 0$. In (2.10), we choose $\tilde{u}|_{\Omega_U^+}=0$ and $u|_{\Omega_U\setminus\Omega_U^+}=u$, so that

$$(h'(u)+B^*p,u)_{\Omega_U^+}\le 0.$$

Therefore,

$$(h'(u)+B^*p)|_{\Omega_U^+}=0.$$

Then, by (3.2) we have

$$\begin{aligned}&(h'(u)+B^*p,\pi^c u-u)_U\\ =&\int_{\Omega_U^b}(h'(u)+B^*p)(\pi^c u-u)\\ =&\sum_{\tau_U\subset\Omega_U^b}\int_{\tau_U}(h'(u)+B^*p-\pi^c(h'(u)+B^*p))(\pi^c u-u)\\ \le& C\sum_{\tau_U\subset\Omega_U^b}h_{\tau_U}^2|h'(u)+B^*p|_{1,\tau_U}|u|_{1,\tau_U}\\ \le& Ch_U^2\|h'(u)+B^*p\|_{1,\Omega^b}\|u\|_{1,\Omega^b}\\ \le& Ch_U^2\|h'(u)+B^*p\|_{1,\infty,\Omega_U}\|u\|_{1,\infty,\Omega_U}\mathrm{meas}(\Omega^b)\le Ch_U^3.\end{aligned} \tag{3.16}$$

Finally, we combine (3.15) and (3.16) to derive (3.3). □

Similarly as [26], we have the following corollary.

Corollary 3.1. *Let u and u_h be the solutions of* (2.10) *and* (2.17)*, respectively. Assume that $h'(u)+B^*p\in W^{1,\infty}(\Omega_U)$, and Ω is convex. Then*

$$\|u-u_h\|_{-1,\Omega_U}\le C\left(h_U^{\frac{3}{2}}+h^2\right). \tag{3.17}$$

Then, we will establish the following superconvergence property for the state y and the co-state p by using the standard superconvergence results of Theorem 3.1 and Corollary 3.1.

Theorem 3.2. *Let y, p be the solutions of* (2.8) *and* (2.9)*, and y_h, p_h be the solutions of* (2.15) *and* (2.16)*. Let y_I and p_I be the piecewise linear Lagrange interpolations of y and p. Assume that all the conditions in Theorem* 3.1 *are valid. Moreover, assume that the mesh T^h is uniform, and y, $p\in H^3(\Omega)$. Then,*

$$|y_h-y_I|_{1,\Omega}+|p_h-p_I|_{1,\Omega}\le C\left(h_U^{\frac{3}{2}}+h^2\right). \tag{3.18}$$

The proof is similar to that of [26]. Furthermore, we can also compare the approximated solutions with the elliptic projection of the exact solutions. Given u, y the exact solutions, define the elliptic projections $\hat{y}_h$, $\hat{p}_h$ by the means of the following elliptic problems:

$$a(\hat{y}_h, v_h) = (f + Bu, v_h), \tag{3.19}$$
$$a(q_h, \hat{p}_h) = (g'(y), q_h), \tag{3.20}$$

for all $v_h \in V_h$, $q_h \in V_h$. Substracting (2.15)-(2.16) from (3.19)-(3.20), we have the following error equations of the approximated solutions and the elliptic projections:

$$a(\hat{y}_h - y_h, v_h) = (B(u - u_h), v_h), \tag{3.21}$$
$$a(q_h, \hat{p}_h - p_h) = (g'(y) - g'(y_h), q_h), \tag{3.22}$$

for all $v_h \in V_h$, $q_h \in V_h$.

Thus, we can prove another superconvergence result between the elliptic projections and the approximated solutions.

Theorem 3.3. *Suppose that all the conditions of Theorems* 3.1 *and* 3.2 *are valid. Then,*

$$\|\hat{y}_h - y_h\|_{1,\Omega} + \|\hat{p}_h - p_h\|_{1,\Omega} \leq C\left(h_U^{\frac{3}{2}} + h^2\right). \tag{3.23}$$

Proof. Let $v_h = \hat{y}_h - y_h$. Here we assume the continuous linear operator B can be expressed as $B = \alpha(x) \in W^{1,\infty}(\Omega)$. An application of Theorem 3.1 and (3.21) yields

$$\begin{aligned}
&c\|\hat{y}_h - y_h\|_{1,\Omega}^2 \\
\leq& a(\hat{y}_h - y_h, v_h) = (B(u - u_h), v_h) \\
=& (B(u - \pi^c u), v_h) + (B(\pi^c u - u_h), v_h) \\
=& ((\alpha(x) - \pi^c \alpha(x))(u - \pi^c u), v_h) + \|\pi^c u - u_h\|_{0,\Omega_U} \|B^* v_h\|_{0,\Omega_U} \\
\leq& C h_U^2 \|u\|_{1,\Omega_U} \|v_h\|_{0,\Omega} + C(h_U^{\frac{3}{2}} + h^2)\|v_h\|_{0,\Omega} \\
\leq& C(h_U^{\frac{3}{2}} + h^2)\|v_h\|_{1,\Omega},
\end{aligned} \tag{3.24}$$

which gives

$$\|\hat{y}_h - y_h\|_{1,\Omega} \leq C(h_U^{\frac{3}{2}} + h^2). \tag{3.25}$$

Similarly, let $q_h = \hat{p}_h - p_h$. By Schwarz inequality, Poincare inequality, Theorem 3.2, and the interpolation theory in Sobolev space (see, e.g., [15]), we have

$$\begin{aligned}
&c\|\hat{p}_h - p_h\|_{1,\Omega}^2 \leq a(q_h, \hat{p}_h - p_h) \\
=& (g'(y) - g'(y_h), q_h) \leq C\|y - y_h\|_{0,\Omega}\|q_h\|_{0,\Omega} \\
\leq& C(\|y - y_I\|_{0,\Omega} + \mid y_I - y_h \mid_{1,\Omega})\|q_h\|_{1,\Omega} \\
\leq& C(h^2 \mid y \mid_{2,\Omega} + h_U^{\frac{3}{2}} + h^2))\|q_h\|_{1,\Omega} \\
\leq& C(h_U^{\frac{3}{2}} + h^2)\|q_h\|_{1,\Omega},
\end{aligned} \tag{3.26}$$

which yields

$$\|\hat{p}_h - p_h\|_{1,\Omega} \leq C(h_U^{\frac{3}{2}} + h^2). \tag{3.27}$$

Therefore, the desired inequality follows from (3.25) and (3.27). □

4. Global L^2 Superconvergence by Recovery

Theorem 3.1 shows that the error order of $\|u_h - \pi^c u\|_{0,\Omega_U}$ is one half order higher than the optimal error for the piecewise constant finite element space. To provide the global superconvergence for the control and state, we use the recovery techniques on uniform meshes.

In this section, we use the recovery operators R_h and G_h in [26] and [39]. Without zero boundary constraint, let $R_h v \in V^h$ be a continuous piecewise linear function. Similar to the *Z-Z* patch recovery (see, e.g., [43, 44]), the values of $R_h v$ on the nodes are defined by least-squares argument on an element patches surrounding the nodes. For example, let z be a node, $\omega_z = \cup_{z\in\bar{\tau}_U}\tau_U$, and V_z be the space of linear functions on ω_z. Set $R_h v_z = \sigma_z(z)$, where

$$E(\sigma_z) = \min_{w\in V_z} E(w), \quad \text{with} \quad E(w) = \sum_{\tau_U\subset\omega_z}\left(\int_{\tau_U} w - \int_{\tau_U} v\right)^2.$$

When $z \in \partial\Omega_U$, we should add a few extra neighbor elements to ω_z such that ω_z contains more than three elements. For the regular mesh and the suitable choice of ω_z, we can conclude that for any $v \in L^2(\Omega)$, $R_h v$ exists. Moreover, for any domain $D \subset \Omega$, $R_h v = v$ on D if v is a linear function on $\hat{D}$, where $\hat{D} = \{\cup\tau_U : \bar{\tau}_U \cap \bar{D} \neq \emptyset\}$.

We construct the gradient recovery operator $G_h v = (R_h v_x, R_h v_y)$ for the gradient of y and p. In the piecewise linear case, it is noted that G_h is the same as the *Z-Z* gradient recovery (see, e.g., [43, 44]).

As in [26], we can prove the following superconvergence results by recovery.

Lemma 4.1. *Suppose that all the conditions of Theorem* 3.1 *are valid. Then*

$$\|R_h u - u\|_{0,\Omega_U} \le Ch_U^{\frac{3}{2}}. \tag{4.1}$$

Theorem 4.1. *Suppose that all the conditions of Theorem* 3.1 *are valid. Then,*

$$\|R_h u_h - u\|_{0,\Omega_U} \le C\left(h_U^{\frac{3}{2}} + h^2\right). \tag{4.2}$$

Proof. By using Theorem 3.1 and Lemma 4.1, we obtain Theorem 4.1 (see, [26]). □

Theorem 4.2. *Suppose that all the conditions of Theorems* 3.1 *and* 3.2 *are valid. Then,*

$$\|G_h y_h - \nabla y\|_{0,\Omega} + \|G_h p_h - \nabla p\|_{0,\Omega} \le C(h_U^{\frac{3}{2}} + h^2). \tag{4.3}$$

Proof. Theorem 4.2 follows from Theorem 3.2 and the standard interpolation error estimate technique (see, e.g., [27]), and the details can be found in [26]. □

It is of great importance for a finite element method to have a computable a posteriori error estimator by which we can evaluate the accuracy of the finite element solutions in applications. One way to construct error estimators is to employ certain superconvergence properties of the finite element solutions. Thus, according to above global superconvergence, we can obtain the following recovery a posteriori error estimates for the general convex optimal control problems.

Theorem 4.3. *Suppose that all the conditions of Theorem* 3.1 *are valid. Then,*

$$\|R_h u_h - u_h\|_{0,\Omega_U} = \|u - u_h\|_{0,\Omega_U} + \mathcal{O}(h_U^{\frac{3}{2}} + h^2), \tag{4.4}$$

$$\|G_h y_h - \nabla y_h\|_{0,\Omega} = \|\nabla(y - y_h)\|_{0,\Omega} + \mathcal{O}(h_U^{\frac{3}{2}} + h^2), \tag{4.5}$$

$$\|G_h p_h - \nabla p_h\|_{0,\Omega} = \|\nabla(p - p_h)\|_{0,\Omega} + \mathcal{O}(h_U^{\frac{3}{2}} + h^2). \tag{4.6}$$

Furthermore, there hold

$$\lim_{h\to 0}\frac{\|R_h u_h - u_h\|_{0,\Omega_U}}{\|u-u_h\|_{0,\Omega_U}} = 1, \tag{4.7}$$

$$\lim_{h\to 0}\frac{\|G_h y_h - \nabla y_h\|_{0,\Omega}}{\|\nabla(y-y_h)\|_{0,\Omega}} = 1, \tag{4.8}$$

$$\lim_{h\to 0}\frac{\|G_h p_h - \nabla p_h\|_{0,\Omega}}{\|\nabla(p-p_h)\|_{0,\Omega}} = 1. \tag{4.9}$$

Proof. The above results can be obtained by combining Lemma 4.1, and Theorems 4.1 and 4.2. □

Therefore, the recovery type a posteriori error estimators defined above are asymptotically exact under some regularity conditions.

5. Numerical Examples

In this section, we present three numerical experiments to illustrate the error estimators shown in Theorem 4.3, which is crucial to refinement. To implement adaptive multi-mesh schemes, it is known that the best choice is h-method. The general idea of the h-method is to refine the meshes such that the error estimators are equally distributed over the computational mesh. Assume that an a posteriori error estimator η has the form $\eta^2 = \sum_{e_i}\eta_{e_i}^2$, where e_i is a finite element. At first, we can calculate the average quantity of $\eta_{e_i}^2$, which is then compared by each of $\eta_{e_i}^2$. The element e_i is to be refined or coarsened if $\eta_{e_i}^2$ is larger or smaller than the quantity. So $\eta_{e_i}^2$ reflects the distribution of the total approximation error over e_i, which guarantees that a higher density of nodes is distributed over the area where the error is larger.

In our numerical experiments, the following type of convex objective functionals governed by elliptic state equations is used:

$$\min_{u\in K\subset L^2(\Omega_U)} \quad \frac{1}{4}\int_\Omega (y-y_0)^4 + \frac{1}{4}\int_{\Omega_U}(u-u_0)^4$$

where $\Omega_U = \Omega = [0,1]^n,\ n=2,3$. Let Ω^h and Ω_U^h be partitioned into T^h and T_U^h as described in Sect. 2. The discretization was already described in previous sections: The control function u is discretized by piecewise constant functions, whereas the state y and the co-state p are approximated by linear finite element functions. Different meshes are used for the approximation of the state and the control. In our experiments, we shall use $\|R_h u_h - u_h\|_{0,\Omega_U}$ as the control mesh refinement indicator, and $\|G_h y_h - \nabla y_h\|_{0,\Omega} + \|G_h p_h - \nabla p_h\|_{0,\Omega}$ for the state's and co-state's (see [26]).

All of the optimization problems here are solved numerically with codes developed based on AFEPack, which provides a general tool of mesh adaptation for multi-meshes. The package is freely available and the details can be found in [24]. One of the key difficulties in implementing a multi-mesh scheme is to efficiently handle numerical integrals, which involves the base functions on different mesh spaces, and can easily consume much computational work. Another difficulty is to interpolate a finite element function into the other finite element space when their meshes are different. In fact, AFEPack has effectively overcome these difficulties by using the following structured multi-meshes: two elements τ, τ_U always have the relationship either $\bar\tau \subset \bar\tau_U$ or $\bar\tau_U \subset \bar\tau$ provided $\tau\cap\tau_U \neq \emptyset$, so that a tree structure for fast searching can be formed. When an element is to be refined, it is refined into 2^k smaller simplex elements as described in [23].

In [26], the bilinear form $b(\cdot,\cdot)$ was used as the preconditioning for the projection algorithm. However, we find that this kind of preconditioning is inefficient for our model. In order to overcome this difficulty, we adopt the interpolation function $p_I^{(k)}$ as a preconditioning for the projection algorithm P_K, see below:

$$\begin{cases} a(y^{(k)},w)=(f+Bu^{(k)},w), & y^{(k)}\in V^h, \quad \forall w\in V^h,\\ a(q,p^{(k)})=((y^{(k)}-y_0)^3,q), & p^{(k)}\in V^h, \quad \forall q\in V^h,\\ p_I^{(k)}|_{\tau_U}=(B^*p^{(k)})(S), & p_I^{(k)}\in U^h,\\ u^{(k+1)}=P_K(u_0-\sqrt[3]{p_I^{(k)}}), & \end{cases} \tag{5.1}$$

where the subscript h have been omitted, k corresponds to the iterations $(k=0,1,2,\cdots)$, S is the center point of the finite element τ_U, $p_I^{(k)}$ is an interpolation function of $p^{(k)}$ in U^h, which is a suitable preconditioning for the projection algorithm P_K. In the experiments, we test this kind of interpolate preconditioning and find that it is efficient for our model as well as the quadratic convex optimal control problems. The main computational effort of this system is

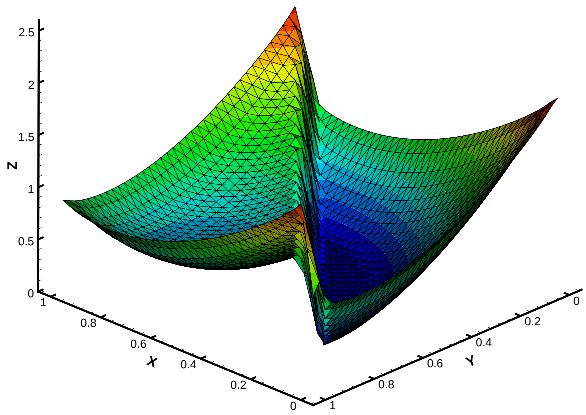

Fig. 5.1. Example 1: the profile of u.

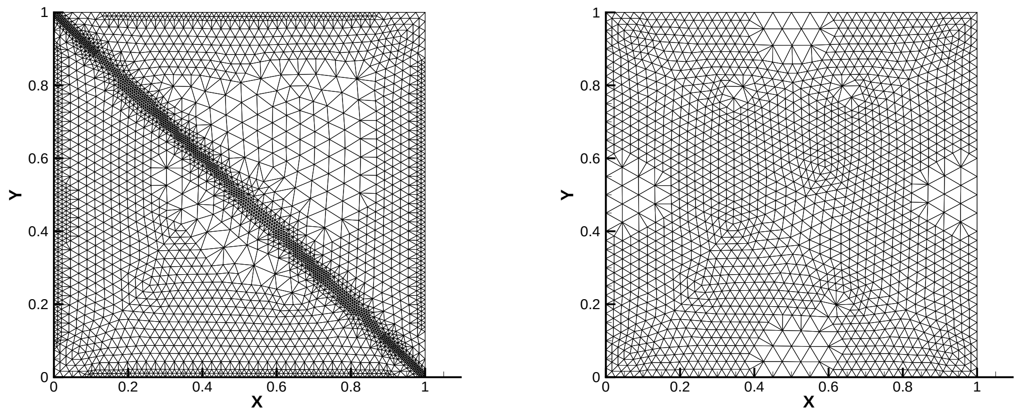

Fig. 5.2. Example 1: the mesh of u (left) and the mesh of y and p (right).

to solve the two state equations, and to compute the projection $P_K(u_0 - \sqrt[3]{p_I^{(k)}})$. As in [26], we also use a fast algebraic multigrid solver to solve the state equations in this paper. The projection operator $P_K: U^h \to K^h$ is defined: For given $w \in U^h$, find $P_K w \in K^h$, such that

$$P_K w|_{\tau_U} = \max(0, \pi^c w)|_{\tau_U}, \tag{5.2}$$

where $\pi^c w$ is the L^2-projection of w.

Example 1. The first example is to solve the following convex control problem

$$\begin{aligned} &\min_{u\geq 0} \ \frac{1}{4}\int_\Omega (y-y_0)^4 dx + \frac{1}{4}\int_\Omega (u-u_0)^4 dx \\ &-\Delta y + y = f + u, \end{aligned} \tag{5.3}$$

where $\Omega = [0,1]\times[0,1]$, and

$$\begin{aligned} &z = \begin{cases} 1.0, & x_1 + x_2 > 1, \\ 0.0, & x_1 + x_2 \leq 1, \end{cases} \\ &y = \sin(\pi x_1)\sin(\pi x_2), \quad p = \sin(\pi x_1)\sin(\pi x_2), \\ &u_0 = 2.0 - \sin\frac{\pi x_1}{2} - \sin\frac{\pi x_2}{2} + z, \\ &u = \max(u_0 - \sqrt[3]{p}, 0), \quad f = -\Delta y + y - u, \\ &y_0 = y + \sqrt[3]{\Delta p - p}\ . \end{aligned} \tag{5.4}$$

The dual equation of the state equation is

$$-\Delta p + p = (y-y_0)^3. \tag{5.5}$$

Table 5.1: Numerical results for Example 1 on uniform meshes.

uniform	1	2	3	4	5
nodes	121	441	1681	6561	25921
$\|u-u_h\|_{L^2}$	1.81758e-01	1.22801e-01	7.90325e-02	5.30981e-02	3.73144e-02
$\mid y-y_h \mid_{H^1}$	3.46711e-01	1.74193e-01	8.72019e-02	4.36148e-02	2.18091e-02
$\mid p-p_h \mid_{H^1}$	3.49164e-01	1.74511e-01	8.72524e-02	4.36231e-02	2.18105e-02
$\|R_h u_h - u_h\|_{L^2}$	1.40482e-01	1.01953e-01	6.38542e-02	4.18971e-02	3.00517e-02
$\|G_h y_h - \nabla y_h\|_{L^2}$	3.53011e-01	1.75876e-01	8.75720e-02	4.36976e-02	2.18283e-02
$\|G_h p_h - \nabla p_h\|_{L^2}$	3.45485e-01	1.74906e-01	8.74528e-02	4.36831e-02	2.18266e-02

Table 5.2: Numerical results for Example 1 on adaptive meshes.

adaptive	1	2	3	4	5
nodes(u)	139	513	1649	2634	3471
nodes(y,p)	139	513	1773	1773	1773
$\|u-u_h\|_{L^2}$	9.84953e-02	6.99288e-02	5.00637e-02	3.59728e-02	2.75903e-02
$\mid y-y_h \mid_{H^1}$	2.46085e-01	1.23659e-01	6.43379e-02	6.43375e-02	6.43376e-02
$\mid p-p_h \mid_{H^1}$	2.46936e-01	1.23793e-01	6.43583e-02	6.43526e-02	6.43522e-02
$\|R_h u_h - u_h\|_{L^2}$	1.22371e-01	8.34112e-02	5.59271e-02	4.15201e-02	3.21595e-02
$\|G_h y_h - \nabla y_h\|_{L^2}$	2.59330e-01	1.25796e-01	6.54861e-02	6.54935e-02	6.54939e-02
$\|G_h p_h - \nabla p_h\|_{L^2}$	2.56654e-01	1.25469e-01	6.54375e-02	6.54443e-02	6.54449e-02

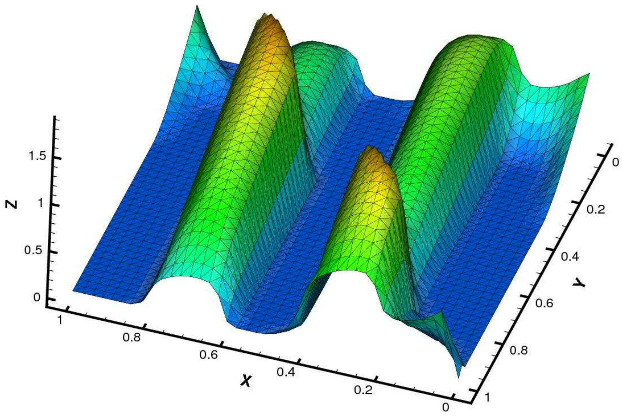

Fig. 5.3. Example 2: the profile of u.

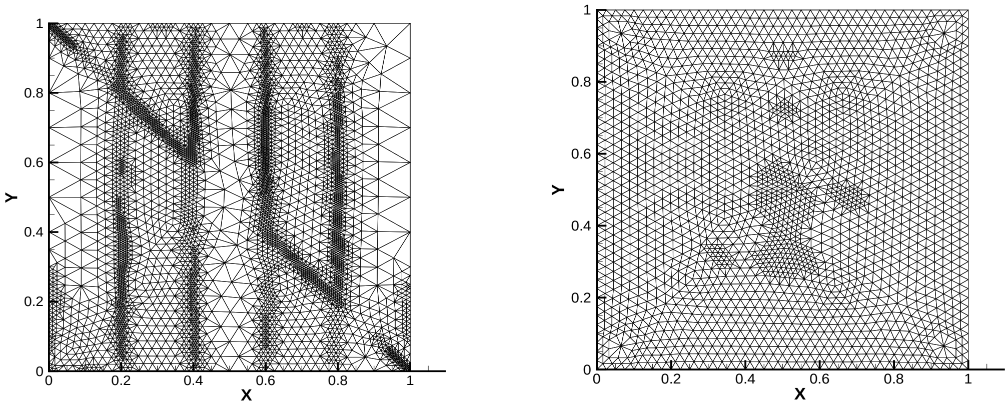

Fig. 5.4. Example 2: The mesh of u (left) and the mesh of y and p (right).

In Figure 5.1, the profile of the control u for Example 1 is plotted. We can clearly see that the control u is smooth everywhere except along the line $x_1 + x_2 = 1$, where u is discontinuous. It is observed from Tables 5.1 and 5.2 that the adaptive meshes generated via the error indicators can save substantial computational work compared with the uniform meshes, and verified that the error indicators are asymptotically exact. In Figure 5.2, it is seen that the u-mesh adapts very well to the neighborhood of the discontinuous line $x_1 + x_2 = 1$, and a higher density of node points are indeed distributed along the line. Furthermore, the optimal meshes for the control and the states are very different as seen in Figure 5.2.

Example 2. This is an example of nonlinear case

$$\begin{aligned} &\min_{u\geq 0} \quad \frac{1}{4}\int_\Omega (y-y_0)^4 dx + \frac{1}{4}\int_\Omega (u-u_0)^4 dx \\ &-\Delta y + y^3 = f + u, \end{aligned} \tag{5.6}$$

The dual equation of the state equation is

$$-\Delta p + 3y^2 p = (y-y_0)^3. \tag{5.7}$$

where $\Omega = [0,1] \times [0,1]$, and

$$
\begin{aligned}
& z = \begin{cases} 1.0, & x_1 + x_2 > 1.0, \\ 0.0, & x_1 + x_2 \le 1.0, \end{cases} \\
& y = \sin(\pi x_1)\sin(5\pi x_2), \quad p = \sin(5\pi x_1)\sin(\pi x_2), \\
& u_0 = 1.0 - \sin\frac{\pi x_1}{2} - \sin\frac{\pi x_2}{2} + z, \\
& u = \max(u_0 - \sqrt[3]{p}, 0), \quad f = -\Delta y + y^3 - u, \\
& y_0 = y + \sqrt[3]{\Delta p - 3y^2 p}\ .
\end{aligned} \tag{5.8}
$$

The numerical results for this test problem on both uniform and adaptive meshes are listed in Tables 5.3 and 5.4, respectively. It is clearly observed that the adaptive mesh results are better than those of the unform mesh results. The profile of the control u is plotted in Figure 5.3, and the meshes for u and for y and p are plotted in Figure 5.4.

With this example, it seems that we can use the error indicators in nonlinear cases. However, to reach this conclusion further theoretical analysis and numerical tests are required.

Example 3. This is a 3D example on $\Omega = [0,1]^3$, and the model is

$$
\begin{aligned}
& \min_{u \ge 0} \ \frac{1}{4}\int_\Omega (y - y_0)^4 dx + \frac{1}{4}\int_\Omega (u - u_0)^4 dx \\
& -\Delta y = f + u,
\end{aligned} \tag{5.9}
$$

Table 5.3: Numerical results for Example 2 on uniform meshes.

uniform	1	2	3	4	5
nodes	121	441	1681	6561	25921
$\|u-u_h\|_{L^2}$	1.75544e-01	1.16451e-01	7.28518e-02	4.78305e-02	3.24052e-02
$\mid y-y_h \mid_{H^1}$	3.65326e+00	1.89533e+00	9.56588e-01	4.79421e-01	2.39851e-01
$\mid p-p_h \mid_{H^1}$	3.67290e+00	1.89904e+00	9.57102e-01	4.79486e-01	2.39860e-01
$\|R_h u_h - u_h\|_{L^2}$	2.27792e-01	1.55201e-01	9.43446e-02	5.78430e-02	3.59519e-02
$\|G_h y_h - \nabla y_h\|_{L^2}$	4.19991e+00	2.04813e+00	9.80511e-01	4.82679e-01	2.40288e-01
$\|G_h p_h - \nabla p_h\|_{L^2}$	4.21216e+00	2.04872e+00	9.80542e-01	4.82682e-01	2.40289e-01

Table 5.4: Numerical results for Example 2 on adaptive meshes.

adaptive	1	2	3	4	5
nodes(u)	139	473	1547	3178	4975
nodes(y,p)	139	513	1949	2199	2204
$\|u-u_h\|_{L^2}$	2.07087e-01	1.44693e-01	7.66059e-02	4.98255e-02	3.45941e-02
$\mid y-y_h \mid_{H^1}$	3.14318e+00	1.60448e+00	8.08421e-01	7.56503e-01	7.55581e-01
$\mid p-p_h \mid_{H^1}$	3.21706e+00	1.64044e+00	8.26328e-01	7.77704e-01	7.77350e-01
$\|R_h u_h - u_h\|_{L^2}$	2.06393e-01	1.36739e-01	8.21328e-02	5.25178e-02	3.72725e-02
$\|G_h y_h - \nabla y_h\|_{L^2}$	3.64520e+00	1.71285e+00	8.24303e-01	7.75038e-01	7.73692e-01
$\|G_h p_h - \nabla p_h\|_{L^2}$	3.75807e+00	1.75673e+00	8.43286e-01	7.96661e-01	7.96391e-01

where

$$
\begin{aligned}
&z = \begin{cases} 2.0, & x_1 + x_2 + x_3 > 1.0, \\ 0.0, & x_1 + x_2 + x_3 \leq 1.0, \end{cases} \\
&y = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3), \\
&p = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3), \\
&u_0 = 1.0 - \sin\frac{\pi x_1}{2} - \sin\frac{\pi x_2}{2} - \sin\frac{\pi x_3}{2} + z, \\
&u = \max(u_0 - \sqrt[3]{p}, 0), \\
&f = -\Delta y - u, \quad y_0 = y + \sqrt[3]{\Delta p}\ .
\end{aligned} \tag{5.10}
$$

The dual equation of the state equation is

$$
-\Delta p = (y - y_0)^3. \tag{5.11}
$$

From the numerical results summarized in Tables 5.5 and 5.6, we can easily find that the error estimators are efficient and the adaptive multi-meshes can save computational work substantially in 3D space.

6. Conclusions and Discussions

The paper discussed the superconvergence analysis and recovery a posteriori error estimates of the finite element approximation for general convex optimal control problems governed by linear state equation. We obtained the superconvergence properties of finite element solutions. Finally, numerical examples were shown to verify the theoretical results.

Our future work is to investigate the superconvergence for the general convex control problems subject to the nonlinear state equation:

$$
\begin{aligned}
&-\text{div}(A\nabla y) + \phi(y) = f + Bu, \quad \text{in } \Omega, \\
&y = 0, \quad \text{on } \partial\Omega,
\end{aligned}
$$

where ϕ is a nonlinear function.

Table 5.5: Numerical results for Example 3 on uniform meshes.

	uniform	1	2	3
u, y, p mesh info	nodes	270	1813	13145
	edges	1543	11332	86616
	faces	2380	18368	144256
	elements	1106	8848	70784
	Dofs of u	1106	8848	70784
	Dofs of y, p	270	1813	13145
error	$\|u - u_h\|_{L^2}$	1.94e-01	1.17e-01	7.40e-02
	$\mid y - y_h \mid_{H^1}$	1.11e+00	7.21e-01	4.87e-01
	$\mid p - p_h \mid_{H^1}$	1.15e+00	7.39e-01	4.93e-01
	$\|R_h u_h - u_h\|_{L^2}$	1.96e-01	1.26e-01	8.88e-02
	$\|G_h y_h - \nabla y_h\|_{L^2}$	8.33e-01	5.64e-01	3.74e-01
	$\|G_h p_h - \nabla p_h\|_{L^2}$	8.33e-01	5.02e-01	3.52e-01

Table 5.6: Numerical results for Example 3 on adaptive meshes.

	adaptive	1	2	3	4	4	6
u mesh info	nodes	270	1799	4109	5687	6290	6488
	edges	1543	11234	24511	32968	36575	37840
	faces	2380	18200	38841	51537	57257	59325
	elements	1106	8764	18438	24255	26971	27972
	Dofs of u	1106	8764	18438	24255	26971	27972
y, p mesh info	nodes	270	1777	2389	3061	3407	3517
	edges	1543	11098	14335	17971	19660	20165
	faces	2380	17981	22811	28295	30702	31405
	elements	1106	8659	10864	13384	14448	14756
	Dofs of y, p	270	1777	2389	3061	3407	3517
error	$\|u - u_h\|_{L^2}$	1.94e-01	1.17e-01	8.47e-02	8.24e-02	8.04e-02	7.87e-02
	$\| y - y_h \|_{H^1}$	1.11e+00	7.21e-01	5.56e-01	4.53e-01	3.97e-01	3.70e-01
	$\| p - p_h \|_{H^1}$	1.15e+00	7.39e-01	5.66e-01	4.59e-01	4.01e-01	3.74e-01
	$\|R_h u_h - u_h\|_{L^2}$	1.96e-01	1.26e-01	9.76e-02	9.02e-02	8.81e-02	8.64e-02
	$\|G_h y_h - \nabla y_h\|_{L^2}$	8.33e-01	5.64e-01	4.62e-01	3.99e-01	3.64e-01	3.48e-01
	$\|G_h p_h - \nabla p_h\|_{L^2}$	6.62e-01	5.02e-01	4.32e-01	3.83e-01	3.53e-01	3.39e-01

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