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FINITE DIFFERENCE APPROXIMATION FOR PRICING THE AMERICAN LOOKBACK OPTION *

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Abstract

In this paper we are concerned with the pricing of lookback options with American type constrains. Based on the differential linear complementary formula associated with the pricing problem, an implicit difference scheme is constructed and analyzed. We show that there exists a unique difference solution which is unconditionally stable. Using the notion of viscosity solutions, we also prove that the finite difference solution converges uniformly to the viscosity solution of the continuous problem. Furthermore, by means of the variational inequality analysis method, the $\mathcal{O}(\Delta t+\Delta x^2)$-order error estimate is derived in the discrete L_2-norm provided that the continuous problem is sufficiently regular. In addition, a numerical example is provided to illustrate the theoretical results.

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1. Introduction

The pricing and hedging of derivative securities is a subject of considerable practical importance in finance. One basic type of derivative is an option which is a financial contract entered into by two parties, a buyer and a seller. The buyer of the contract obtains the right to trade an underlying asset, such as a stock, for a specified price, called the strike price, on or before a special time, called the expiration date. Options which provide the right to buy (sell) the underlying asset are known as call (put) options. When the option contract is entered into, the option buyer pays a price to the seller. In return, for the price the seller agrees to meet any obligations arising from the contract. Options which can be exercised only on the expiration date are called European, whereas options which can be exercised any time up to and including the expiration date are classified as American.

Generally speaking, the problems for pricing European options can be mathematically modelled by the well-known Black-Scholes partial differential equation [2] associated with initial/final and boundary value conditions, and some analytic solutions can be obtained for the

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cases with simple payoffs. However, for most American options or European options with complex payoffs, no analytic solutions are available. Thus, the research of effective numerical methods is of considerable importance in the field of option pricing.

The objective of this article is to develop a finite difference method for pricing the lookback options with American type constrains. Lookback options, which may be European or American type, are a kind of path-dependent options whose payoffs depend on the history of the underlying asset values over some period of the options' lifetime. In recent years, many numerical methods, such as the binomial methods, finite difference and finite element methods, etc., have been proposed for pricing path-dependent options including lookback options. See, for instance, [1, 3, 7, 10, 11, 14, 15], and the references cited therein. However, for the lookback option with American style, it is still difficult to establish the convergence analysis and error estimates because of the complexity of the problems, and few results can be found in this aspect.

In this paper, we will analyze the pricing problem of lookback options on the basis of the differential linear complementary formula. First, we introduce a variable transformation to reduce the problem to an one-dimensional problem in space. Then, a fully implicit and unconditionally stable difference scheme is carefully constructed. In view of the lower regularity of the problem, we will use the notion of viscosity solution, and follow the idea and the method proposed by Barles et al. [4, 5] to give the convergence analysis. The basic principle is that any stable, consistent and monotonic discretization scheme converges. Provided that the continuous problem satisfies a strong comparison principle. Furthermore, we discuss the error estimate under the assumption that the exact solution is smooth enough. Again, we transform the discrete linear complementary problem into an equivalent variational inequality problem, which allows us to derive the error estimates more readily. The use of an implicit difference method results in a set of variational inequalities which have to be solved at each time step. We will briefly discuss the method of solving the discrete system of inequalities.

The plan of this paper is as follows. In Section 2, we review the mathematical model of pricing lookback options and establish the finite difference approximation. In Section 3, we analyze the stability and convergence for the discrete approximation problem. In Section 4, an $\mathcal{O}(\Delta t + \Delta x^2)$-order error estimate is derived in the discrete L^2-norm. Finally, in Section 5 the projected SOR method of solving the discrete system of inequalities is established, and a numerical example is provided to confirm the theoretical analysis and the efficiency of the algorithm.

2. The Finite Difference Approximation

Let $S = S(t)$ be the underlying asset price. As usual, assume that S follows the lognormal diffusion with constant volatility σ and expected return μ:

$$dS = \mu\, S\, dt + \sigma S\, dZ, \tag{2.1}$$

where $\{Z(t) : t \geq 0\}$ is a standard Brownian motion. A lookback option is a derivative product whose payoff depends on the maximum or the minimum of the realized asset price over the lifetime of the option. In this paper we will consider the lookback put option with floating strike. In this case, the payoff function can be expressed as:

$$G(t, S, J) = J - S, \quad J(t) = \max_{0 \leq \tau \leq t} S(\tau). \tag{2.2}$$

Then, the value V of a lookback put option is a function of t, S, and path-dependent variable J, namely, $V = V(t,S,J)$. We will study how to determine numerically the value V for the lookback put option with American style.

It is well known that by applying Ito's lemma and using arbitrage-free argument, the option pricing problem for an American lookback put option can be mathematically modelled by the following differential linear complementary problem [13]:

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV\right)(V - G) = 0, \tag{2.3}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \le 0, \tag{2.4}$$

$$V(t,S,J) \ge G(t,S,J), \ \ 0 \le t < T, \ \ 0 \le S \le J < \infty, \tag{2.5}$$

where r is the risk-free interest rate, T is the expiration date, and the final and the boundary value conditions read as:

$$V(T,S,J) = G(T,S,J), \ \ 0 \le S \le J < \infty, \tag{2.6}$$

$$\frac{\partial V}{\partial J}(t,J,J) = 0, \ 0 \le t \le T, \ 0 \le S \le J < \infty. \tag{2.7}$$

Notice that the independent variable J only appears as a parameter in the above equations, but it also features in the final and boundary conditions.

Problem (2.3)–(2.7) is two-dimensional in space and we know little information about the boundary value condition, which results in great difficulties in applying numerical methods. However, the problem has a particular mathematical structure that permits a reduction in dimensionality by using a variable transformation. Introduce the transformation as follows:

$$\tau = \frac{1}{2}\sigma^2(T-t), \ x = \ln(J/S), \ V(t,S,J) = Su(\tau,x). \tag{2.8}$$

Under this transformation, we can convert the problem (2.3)–(2.5) into the following form:

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} + \sigma_r\frac{\partial u}{\partial x}\right)(u - g) = 0, \tag{2.9}$$

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} + \sigma_r\frac{\partial u}{\partial x} \ge 0, \tag{2.10}$$

$$u(\tau,x) \ge g(x), \ g(x) = e^x - 1, \ 0 < \tau \le T_1, \ \ 0 \le x < \infty, \tag{2.11}$$

where $T_1 = \sigma^2 T/2, \sigma_r = 1 + 2r/\sigma^2$. In order to enhance the stability of the difference scheme, we will remove the first-order derivative term from (2.10) and (2.11). For this purpose, we further use the following formula:

$$-d^{-1}(x)\frac{\partial}{\partial x}\left(d(x)\frac{\partial u}{\partial x}\right) = -\frac{\partial^2 u}{\partial x^2} + \sigma_r\frac{\partial u}{\partial x}, \quad d(x) = e^{-\sigma_r x}.$$

Thus, from (2.9)–(2.11) we see that under transformation (2.8), problem (2.3)–(2.7) can be changed into

$$\left(d(x)\frac{\partial u}{\partial \tau} - \frac{\partial}{\partial x}\left(d(x)\frac{\partial u}{\partial x}\right)\right)(u - g) = 0, \tag{2.12}$$

$$d(x)\frac{\partial u}{\partial \tau} - \frac{\partial}{\partial x}\left(d(x)\frac{\partial u}{\partial x}\right) \ge 0, \tag{2.13}$$

$$u(\tau,x) \ge g(x), \ g(x) = e^x - 1, \ 0 < \tau \le T_1, \ \ 0 \le x < \infty, \tag{2.14}$$

with the associated initial and the boundary value conditions (see (2.6)–(2.8))

$$u(0,x) = g(x), \quad \frac{\partial u}{\partial x}(\tau, 0) = 0. \tag{2.15}$$

Let $0 = \tau_0 < \tau_1 < \cdots < \tau_M = T_1$ be a partition of the interval $[0, T_1]$ with the step size $\Delta\tau = \tau_{n+1} - \tau_n$. For the space domain $0 \le x < \infty$, we will restrict ourselves on a finite domain, and consider the equation only for $0 \le x \le X$ with X suitably large, and thus we need to assign a boundary value condition on $x = X$ reasonably. In fact, for an American put option, it is well known that (see, for example, [12]) there exists a free boundary,

$$S_f = S_f(t), \quad S_f(t) \ge \frac{J(t)}{1 + \sigma^2/(2r)}, \quad 0 \le t \le T,$$

which is the optimal exercise boundary, so that when $0 \le S \le S_f, V = J - S$ holds. Then, by transformation (2.8), we have

$$u(\tau, X) = e^X - 1, \quad X \ge \ln\left(1 + \frac{\sigma^2}{2r}\right). \tag{2.16}$$

In the space direction, let $0 = x_0 < x_1 < \cdots < x_N = X$ be a regular partition with the step size $h = x_j - x_{j-1}$. Denoting the semi-node point $x_{j+\frac{1}{2}} = (j + \frac{1}{2})h$, we will use the difference formula

$$\frac{\partial}{\partial x}\left(d\frac{\partial u}{\partial x}\right)(x_j) = \frac{1}{h}\left(d_{j+\frac{1}{2}}\frac{u_{j+1} - u_j}{h} - d_{j-\frac{1}{2}}\frac{u_j - u_{j-1}}{h}\right) + \mathcal{O}(h^2).$$

Now we define the fully implicit difference approximation for problem (2.12)–(2.15) by finding $\{U_j^n\}$ such that

$$\left(d_j\frac{U_j^n - U_j^{n-1}}{\triangle\tau} - \frac{1}{h}\left(d_{j+\frac{1}{2}}\frac{U_{j+1}^n - U_j^n}{h} - d_{j-\frac{1}{2}}\frac{U_j^n - U_{j-1}^n}{h}\right)\right)\left(U_j^n - g_j\right) = 0, \tag{2.17}$$

$$d_j\frac{U_j^n - U_j^{n-1}}{\triangle\tau} - \frac{1}{h}\left(d_{j+\frac{1}{2}}\frac{U_{j+1}^n - U_j^n}{h} - d_{j-\frac{1}{2}}\frac{U_j^n - U_{j-1}^n}{h}\right) \ge 0, \quad U_j^n \ge g_j, \tag{2.18}$$

$$U_j^0 = g_j, \quad U_1^n = U_0^n, \quad U_N^n = g_N, \quad j = 1, \cdots, N-1, \quad n = 1, \ldots, M, \tag{2.19}$$

where for the boundary value approximation, we have used $(U_1 - U_0)/h = 0$ and (2.16). The above difference scheme can be rewritten as follows:

$$\left((d_j + \alpha(d_{j+\frac{1}{2}} + d_{j-\frac{1}{2}}))U_j^n - \alpha(d_{j-\frac{1}{2}}U_{j-1}^n + d_{j+\frac{1}{2}}U_{j+1}^n) - d_jU_j^{n-1}\right)(U_j^n - g_j) = 0, \tag{2.20}$$

$$\left(d_j + \alpha(d_{j+\frac{1}{2}} + d_{j-\frac{1}{2}})\right)U_j^n - \alpha\left(d_{j-\frac{1}{2}}U_{j-1}^n + d_{j+\frac{1}{2}}U_{j+1}^n\right) - d_jU_j^{n-1} \ge 0, \quad U_j^n \ge g_j, \tag{2.21}$$

$$U_j^0 = g_j, \quad U_1^n = U_0^n, \; U_N^n = g_N, \quad j = 1, \cdots, N-1, \; n = 1, \cdots, M, \tag{2.22}$$

where $\alpha = \Delta\tau/h^2$ is the mesh ratio.

3. The Stability and Convergence

3.1. The unique existence and stability

Let $\mathbf{U}^n = (U_1^n, \cdots, U_{N-1}^n)^T$ and $\mathbf{g} = (g_1, \ldots, g_{N-1})^T$ be the $N-1$ dimensional vectors arising from the problem (2.20)–(2.22). Introduce the closed convex subset in R^{N-1} by setting

$\Omega^{N-1} = \{\, \mathbf{U} \in R^{N-1} :\ \mathbf{U} \geq \mathbf{g} \,\}$. Then, the discrete problem (2.20)–(2.22) now can be expressed by matrices as the form: Find $\mathbf{U}^n \in \Omega^{N-1}$ and $\mathbf{U}^0 = \mathbf{g}$ such that

$$\big((\mathbf{D} + \alpha\, \mathbf{A})\mathbf{U}^n - F(\mathbf{U}^{n-1}),\ \mathbf{U}^n - \mathbf{g} \big) = 0, \tag{3.1}$$

$$(\mathbf{D} + \alpha\, \mathbf{A})\mathbf{U}^n - F(\mathbf{U}^{n-1}) \geq \mathbf{0},\ n = 1, \ldots, M, \tag{3.2}$$

where $(\cdot, \cdot)$ denotes the vector inner product in R^{N-1} with the associated norm $\|\mathbf{x}\|^2 = (\mathbf{x}, \mathbf{x})$, $F(\mathbf{U}^{n-1}) = \mathbf{D}\mathbf{U}^{n-1} + (0, \ldots, 0,\ \alpha\, d_{N-\frac{1}{2}} g_N)^T$, $\mathbf{D}$ =diag$(d_1, d_2, \ldots, d_{N-1})$ is a diagonal matrix, and $\mathbf{A} = (a_{ij})$ is a tridiagonal matrix whose elements are as follows:

$$\begin{aligned} &a_{11} = d_{\frac{3}{2}},\ a_{12} = -d_{\frac{3}{2}}, \\ &a_{i,i} = d_{i+\frac{1}{2}} + d_{i-\frac{1}{2}},\ \ a_{i,i-1} = -d_{i-\frac{1}{2}},\ \ a_{i,i+1} = -d_{i+\frac{1}{2}},\ \ i \geq 2. \end{aligned} \tag{3.3}$$

Lemma 1. *For any given* $\mathbf{U}^{n-1}$, *the linear complementary problem* (3.1)–(3.2) *admits a unique solution* $\mathbf{U}^n \in \Omega^{N-1}$.

Proof. We only need to show that $\mathbf{D} + \alpha\, \mathbf{A}$ is a positive definite matrix (see, e.g., [9]), namely

$$((\mathbf{D} + \alpha\, \mathbf{A})\mathbf{x},\, \mathbf{x}) > 0\ \ \forall\, \mathbf{x} \neq \mathbf{0} \in R^{N-1}. \tag{3.4}$$

Obviously, $\mathbf{A}$ is a real symmetric matrix, which implies that all the eigenvalues $\{\lambda\}$ of $\mathbf{A}$ are real. By means of the Gerschgorin's disk theory and (3.3), we have

$$|\lambda - a_{ii}| \leq |a_{ii-1}| + |a_{ii+1}|, \ \text{ or }\ \lambda \geq a_{ii} - |a_{ii-1}| - |a_{ii+1}| \geq 0,$$

which implies that $\mathbf{A}$ is a semi-positive definite matrix. In fact, $\mathbf{A}$ is positive, since $\mathbf{D}$ is diagonal with positive diagonals. Thus, we complete the proof. $\square$

The use of an implicit difference scheme results in a set of variational inequalities (3.1)–(3.2) which have to be solved at each time step. We will give the method to solve the system of inequalities (see Lemma 5).

Lemma 2. *The discrete problem* (2.20)–(2.22) *is unconditionally stable in the sense that*

$$\|\mathbf{U}^n\|_\infty \leq \|\mathbf{U}^0\|_\infty, \ \ n = 1, 2, \ldots, M,$$

which is independent of the time step and the spatial mesh.

Proof. Let $\{\mathbf{U}^n\}$ be the solution to the problem (2.20)–(2.22). Denote $U_j^n = \|\mathbf{U}^n\|_\infty = \max\limits_{1 \leq i \leq N-1} \{\, U_i^n \,\}$ (note that $U_i^k \geq g_i \geq 0$). If $U_j^n = g_j$, we have

$$\|\mathbf{U}^n\|_\infty = g_j \leq \mathbf{U}_j^{n-1} \leq \|\mathbf{U}^{n-1}\|_\infty,\ n = 1, \ldots, M.$$

Otherwise, by (2.20) we have

$$\Big(d_j + \alpha(d_{j+\frac{1}{2}} + d_{j-\frac{1}{2}})\Big)\, U_j^n - \alpha\left(d_{j-\frac{1}{2}} U_{j-1}^n + d_{j+\frac{1}{2}} U_{j+1}^n\right) - d_j U_j^{n-1} = 0,$$

and hence

$$\Big(d_j + \alpha(d_{j+\frac{1}{2}} + d_{j-\frac{1}{2}})\Big)\, \|\mathbf{U}^n\|_\infty \leq \alpha\left(d_{j-\frac{1}{2}} + d_{j+\frac{1}{2}}\right) \|\mathbf{U}^n\|_\infty + d_j \|\mathbf{U}^{n-1}\|_\infty.$$

Consequently,

$$\|\mathbf{U}^n\|_\infty \leq \|\mathbf{U}^{n-1}\|_\infty,\ n = 1, \cdots, M.$$

The proof is complete by recurring on n. $\square$

3.2. The convergence

In this subsection, we will employ the notion of viscosity solutions to show the convergence of difference scheme (2.20)–(2.22). First, let us briefly recall the notion of viscosity solutions introduced by Crandall and Lions [8]. For convenience, we introduce the partial differential operator

$$\pounds u = d(x)\frac{\partial u}{\partial \tau} - \frac{\partial}{\partial x}\left(d(x)\frac{\partial u}{\partial x}\right).$$

Then, problem (2.12)–(2.15) can be rewritten as

$$Q(\tau, x, u, u_\tau, u_x, u_{xx}) = 0 \quad \text{in } [0, T_1] \times \overline{\Omega}, \tag{3.5}$$

where Q is given by

$$Q(\tau, x, u, u_\tau, u_x, u_{xx}) = \begin{cases} \min(\pounds u, u - g) & \text{in } (0, T_1] \times \Omega, \\ u(0, x) - g(x) & \text{in } \Omega, \\ u_x(\tau, 0) & \text{in } (0, T_1], \end{cases}$$

and $\Omega = (0, \infty)$. In the following, let z^* and z_* denote the upper semi-continuous and lower semi-continuous envelope of the function $z : [0, T_1] \times D \to R$, where D is a compact subset of R, defined by

$$z^*(\tau, x) = \limsup_{(t,y)\to(\tau,x)} z(t, y), \quad z_*(\tau, x) = \liminf_{(t,y)\to(\tau,x)} z(t, y).$$

Definition 1. *A locally bounded function $u : [0, T_1] \times \overline{\Omega} \to R$ is a viscosity subsolution (respectively supersolution) of Eq.* (3.5) *if and only if for all $\varphi \in C^{1,2}([0, T_1] \times \overline{\Omega})$ and for all local maximum* (*respectively minimum*) *points (τ, x) of $u^* - \varphi$* (*respectively $u_* - \varphi$*), *we have $Q(\tau, x, u^*, \varphi_\tau, \varphi_x, \varphi_{xx}) \le 0$* (*respectively $Q(\tau, x, u_*, \varphi_\tau, \varphi_x, \varphi_{xx}) \ge 0$*). *A locally bounded function is a viscosity solution of Eq.* (3.5) *if it is both a viscosity subsolution and a viscosity supersolution.*

Next we will apply the results of Barles [4, 5] to show the convergence of the discrete scheme (2.20)–(2.22) to the viscosity solution of (3.5), which essentially says that any stable, consistent and monotone difference scheme converges to the viscosity solution of (3.5), provided (3.5) satisfies the strong comparison principle. Barles, Daher, and Romano have shown that the strong comparison principle is true for lookback options (see [5, 6] for the proof). Thus, we have the following lemma.

Lemma 3. *The strong comparison principle holds for Eq.* (3.5), *namely, if the locally bounded upper semi-continuous* (*lower semi-continuous*) *function u* (v) *is a viscosity subsolution* (*supersolution*) *of* (3.5), *then $u \le v$.*

Now introduce the following notation:

$$S_j(U_j^n, U_{j-1}^n, U_{j+1}^n, U_j^{n-1}) = a_j U_j^n - b_j U_{j-1}^n - c_j U_{j+1}^n - F_j(\mathbf{U}^{n-1}) \quad 1 \le j \le N-1,$$

with the following coefficients:

$$\begin{aligned} &a_1 = d_1 + \alpha\, d_{\frac{3}{2}},\ b_1 = 0,\ c_1 = \alpha\, d_{\frac{3}{2}}, \\ &a_j = d_j + \alpha\,(d_{j-\frac{1}{2}} + d_{j+\frac{1}{2}}),\ b_j = \alpha\, d_{j-\frac{1}{2}},\ c_j = \alpha\, d_{j+\frac{1}{2}},\ \ j = 2, \cdots, N-2, \\ &a_{N-1} = d_{N-1} + \alpha\,(d_{N-\frac{1}{2}} + d_{N-\frac{3}{2}}), \quad b_{N-1} = \alpha\, d_{N-\frac{3}{2}},\ c_{N-1} = 0. \end{aligned}$$

Then, the discrete scheme (2.20)–(2.22) can be rewritten in the following form:

$$\begin{aligned}&Q_j(U_j^n, U_{j-1}^n, U_{j+1}^n, U_j^{n-1})\\&= \min\big(S_j(U_j^n, U_{j-1}^n, U_{j+1}^n, U_j^{n-1})\,,\ U_j^n - g_j\big) = 0, \quad j = 1, \cdots, N-1.\end{aligned} \tag{3.6}$$

Definition 2. *A discretization of the form* (3.6) *is called to be monotonic if*

$$\begin{aligned}&Q_j(U_j^n, U_{j-1}^n + \varepsilon_1, U_{j+1}^n + \varepsilon_2, U_j^{n-1} + \varepsilon_3) \le Q_j(U_j^n, U_{j-1}^n, U_{j+1}^n, U_j^{n-1}),\\&\forall j, \forall\, \varepsilon_1 \ge 0,\ \varepsilon_2 \ge 0,\ \varepsilon_3 \ge 0;\\&Q_j(U_j^n + \varepsilon, U_{j-1}^n, U_{j+1}^n, U_j^{n-1}) \ge Q_j(U_j^n, U_{j-1}^n, U_{j+1}^n, U_j^{n-1}),\ \ \forall\, j, \forall\, \varepsilon \ge 0.\end{aligned}$$

Now we need to check the monotonicity of (3.6), such that the conditions, under which (3.6) converges to the viscosity solution of (3.5), are guaranteed.

Lemma 4. *The difference scheme* (3.6) *is monotonic which is independent of the choice of* $\Delta\tau$ *and* h.

Proof. For any $\varepsilon_1 \ge 0$, $\varepsilon_2 \ge 0$, $\varepsilon_3 \ge 0$, we have

$$\begin{aligned}&S_j(U_j^n, U_{j-1}^n + \varepsilon_1, U_{j+1}^n + \varepsilon_2, U_j^{n-1} + \varepsilon_3)\\&= S_j(U_j^n, U_{j-1}^n, U_{j+1}^n, U_j^{n-1}) - b_j\,\varepsilon_1 - c_j\varepsilon_2 - d_j\varepsilon_3\\&\le S_j(U_j^n, U_{j-1}^n, U_{j+1}^n, U_j^{n-1}),\end{aligned}$$

and hence, by the definition of Q_j,

$$Q_j(U_j^n, U_{j-1}^n + \varepsilon_1, U_{j+1}^n + \varepsilon_2, U_j^{n-1} + \varepsilon_3) \le Q_j(U_j^n, U_{j-1}^n, U_{j+1}^n, U_j^{n-1}).$$

Moreover, for any $\varepsilon \ge 0$,

$$\begin{aligned}&Q_j(U_j^n + \varepsilon, U_{j-1}^n, U_{j+1}^n, U_j^{n-1})\\&= \min\big(S_j(U_j^n, U_{j-1}^n, U_{j+1}^n, U_j^{n-1}) + a_j\,\varepsilon,\ U_j^n + \varepsilon - g_j\big)\\&\ge \min\big(S_j(U_j^n, U_{j-1}^n, U_{j+1}^n, U_j^{n-1}),\ U_j^n - g_j\big) = Q_j(U_j^n, U_{j-1}^n, U_{j+1}^n, U_j^{n-1}).\end{aligned}$$

Thus, we complete the proof. □

Combining the above results allows us to state the following theorem.

Theorem 1. *As* $\Delta\tau, h \to 0$, *the solution of fully implicit difference scheme* (2.20)–(2.22) *converges uniformly to the viscosity solution of* (3.5) *in any compact subset of* $(0, T_1) \times (0, X)$.

Proof. In Barles [4] it has been shown that a stable, consistent and monotone discretization converges to the viscosity solution, provided that the corresponding continuous problem satisfies the strong comparison principle. Obviously, the difference scheme (2.20)–(2.22) is consistent, and then Theorem 1 follows directly from the results of Barles and Lemmas 1-4. □

4. Error Estimates

In this section, we will assume that the solution $u(\tau, x)$ of the problem (2.12)–(2.15) is smooth enough and bounded for our purpose. Under this assumption we can derive better convergence result for the discrete approximation scheme (2.20)–(2.22).

First we transform the problem (3.1)–(3.2) into an equivalent variational inequality problem.

Lemma 5. *The problem* (3.1)–(3.2) *is equivalent to finding* $\mathbf{U}^n \in \Omega^{N-1}$ *and* $\mathbf{U}^0 = \mathbf{g}$, *such that*

$$((\mathbf{D} + \alpha\,\mathbf{A})\mathbf{U}^n - F(\mathbf{U}^{n-1}),\, \mathbf{V} - \mathbf{U}^n) \geq 0 \;\; \forall\, \mathbf{V} \in \Omega^{N-1}, \;\; n = 1, \ldots, M. \tag{4.1}$$

Proof. First of all, let $\mathbf{U}^n$ be the solution of the problem (3.1)–(3.2). Then, for $\mathbf{V} \in \Omega^{N-1}$ (note that $\mathbf{V} \geq \mathbf{g}$), it follows from (3.2) that

$$\big((\mathbf{D} + \alpha\,\mathbf{A})\mathbf{U}^n - F(\mathbf{U}^{n-1}),\, \mathbf{V} - \mathbf{g}\big) \geq 0,$$

from which subtracting (3.1) we see that (4.1) holds. On the other hand, let $\mathbf{U}^n \in \Omega^{N-1}$ be the solution of problem (4.1). Take $\mathbf{V} = \mathbf{U}^n + \mathbf{C}$ in (4.1) with $\mathbf{C} \geq \mathbf{0}$ to obtain

$$\big((\mathbf{D} + \alpha\,\mathbf{A})\mathbf{U}^n - F(\mathbf{U}^{n-1}),\, \mathbf{C}\big) \geq 0.$$

From the arbitrariness of the non-negative vector $\mathbf{C}$ we know that (3.2) is true. Now taking $\mathbf{C} = \mathbf{U}^n - \mathbf{g} \geq \mathbf{0}$, we have

$$\big((\mathbf{D} + \alpha\,\mathbf{A})\mathbf{U}^n - F(\mathbf{U}^{n-1}),\, \mathbf{U}^n - \mathbf{g}\big) \geq 0,$$

which, together with setting $\mathbf{V} = \mathbf{g}$ in (4.1), leads to (3.1). □

Lemma 5 provides an approach of solving the difference problem (2.20)–(2.22) or (4.1). When $\mathbf{D} + \alpha\mathbf{A}$ is a symmetric and positive definite matrix (see Lemma 1), the projected SOR method can be used for solving problem (4.1) (see, e.g., [9]).

Now, by means of Lemma 5, we can give the following error estimate result. Denote $\mathbf{D}^{\frac{1}{2}} = \mathrm{diag}(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_{N-1}}\,)$, and $\|\mathbf{x}\|_D = \|\mathbf{D}^{\frac{1}{2}}\mathbf{x}\| = (\mathbf{D}\mathbf{x}, \mathbf{x})$.

Theorem 2. *Let* $u(\tau, x)$ and $\mathbf{U}^n$ *be the solutions of the problems* (2.12)–(2.15) *and* (2.20)–(2.22), *respectively. Then, we have the following error estimate in the discrete* L_2*-norm*:

$$\|\mathbf{u}^n - \mathbf{U}^n\|_h = \left(\sum_{j=1}^{N-1} h|\, u_j^n - U_j^n\,|^2\right)^{\frac{1}{2}} \leq C(\Delta\tau + h^2),$$

where $\mathbf{u}^n = (u_1^n, u_2^n, \ldots, u_{N-1}^n)^T$ *and* C *is a constant independent of* $\Delta\tau$ *and* h.

Proof. At the mesh point (τ_n, x_j), we discretize the problem (2.12)–(2.15) in the same way as that in (2.17)–(2.19). Then, by a similar argument to that for Lemma 5, it is easy to see that $\mathbf{u}^n \in \Omega^{N-1}$ satisfies the following variational inequality equation:

$$\big((\mathbf{D} + \alpha\,\mathbf{A})\mathbf{u}^n - F(\mathbf{u}^{n-1}) + \Delta\tau\,\varepsilon^n,\, \mathbf{v} - \mathbf{u}^n\big) \geq 0 \;\; \forall\, \mathbf{v} \in \Omega^{N-1}, \;\; n = 1, \ldots, M, \tag{4.2}$$

where $\varepsilon^n = (\,\varepsilon_1^n, \ldots, \varepsilon_{N-1}^n)^T$ is the truncation error vector, and $\varepsilon_j^n = \mathcal{O}(\Delta\tau + h^2)$ when $u(t,x)$ is smooth enough. Let $\mathbf{e}^n = \mathbf{u}^n - \mathbf{U}^n$. Setting $\mathbf{v} = \mathbf{U}^n$ and $\mathbf{V} = \mathbf{u}^n$, we obtain from (4.1) and (4.2) that

$$\begin{aligned}
((\mathbf{D} + \alpha\,\mathbf{A})\mathbf{e}^n, \mathbf{e}^n) &= ((\mathbf{D} + \alpha\,\mathbf{A})\mathbf{u}^n, \mathbf{u}^n - \mathbf{U}^n) - ((\mathbf{D} + \alpha\,\mathbf{A})\mathbf{U}^n, \mathbf{u}^n - \mathbf{U}^n) \\
&\leq \big(F(\mathbf{u}^{n-1}) - \Delta\tau\,\varepsilon^n, \mathbf{u}^n - \mathbf{U}^n\big) - \big(F(\mathbf{U}^{n-1}), \mathbf{u}^n - \mathbf{U}^n\big) \\
&= (\mathbf{D}\mathbf{e}^{n-1} - \Delta\tau\,\varepsilon^n, \mathbf{e}^n) \leq \Big(\|\mathbf{e}^{n-1}\|_D + \Delta\tau\,\|\,\mathbf{D}^{-\frac{1}{2}}\varepsilon^n\|\Big)\,\|\mathbf{e}^n\|_D,
\end{aligned}$$

and hence

$$\|\mathbf{e}^n\|_D^2 + \alpha\,(\mathbf{A}\mathbf{e}^n, \mathbf{e}^n) \le \left(\|\mathbf{e}^{n-1}\|_D + \Delta\tau\,\|\mathbf{D}^{-\frac{1}{2}}\varepsilon^n\|\right)\|\mathbf{e}^n\|_D, \quad n = 1, \ldots, M. \tag{4.3}$$

From the argument of Lemma 1, we know that $\mathbf{A}$ is a semi-positive definite matrix. Therefore, it follows from (4.3) that

$$\|\mathbf{e}^n\|_D \le \|\mathbf{e}^{n-1}\|_D + \Delta\tau\,\|\mathbf{D}^{-\frac{1}{2}}\varepsilon^n\|, \quad n = 1, \ldots, M.$$

By a recursive operation and noticing $\mathbf{e}^0 = \mathbf{0}$, we have

$$\|\mathbf{e}^n\|_D \le \Delta\tau \sum_{k=1}^{n} \|\mathbf{D}^{-\frac{1}{2}}\varepsilon^k\| \le \tau_n \max_k \|\mathbf{D}^{-\frac{1}{2}}\varepsilon^k\| \le T_1 \frac{1}{\min\sqrt{d(x)}} \max_k \|\varepsilon^k\|.$$

Note that $\|\mathbf{e}^n\|_D \ge \min\sqrt{d(x)}\ \|\mathbf{e}\|$. Thus, we can obtain

$$\|\mathbf{e}^n\|_h \le T_1 \frac{1}{\min\, d(x)} \max_k \|\varepsilon^k\|_h \le T_1 e^{\sigma_r X} \max_{k,\,j} |\varepsilon_j^k| \le C\,T_1 e^{\sigma_r X}(\,\Delta\tau + h^2\,).$$

The proof is complete. □

5. The Numerical Example

In this section, we present a numerical example to demonstrate the theoretical result and the efficiency of the algorithm.

Consider the system of inequalities (2.20)–(2.22) (or (4.1) in matrix), in which coefficient matrix $\mathbf{D} + \alpha\mathbf{A}$ with element $m_{ii} = d_i + \alpha\, a_{ii}$, $m_{ij} = \alpha\, a_{ij}$, is tridiagonal, and

$$F(\mathbf{U}^{n-1}) = \mathbf{D}\mathbf{U}^{n-1} + \mathbf{b}$$

with $\mathbf{b} = (0, \ldots, 0, \alpha\, d_{N-\frac{1}{2}} g_N)^T$. We use the projected SOR method [9] to solve (4.1). Let $1 < \omega < 2$ be the super-relaxation factor, and ε the iterative stopping criterion. Then, the procedure to solve (4.1) is as follows:

Step 1. Compute $\mathbf{U}^0 = \mathbf{g} = (\,g(x_0), g(x_1), \cdots, g(x_{N-1})\,)^T$, $\mathbf{F}^0 = \mathbf{D}\mathbf{U}^0 + \mathbf{b}$;

Step 2. For $k = 0, 1, \cdots, M-1$ perform Steps 3–6 recursively;

Step 3. $\mathbf{V}^0 = \max(\,\mathbf{g}, \mathbf{U}^k\,)$;

Step 4. For $i = 1, \cdots, N-1$ compute
$V_i = (\,F_i^k - m_{i,i-1}V_{i-1} - m_{i,i+1}V_{i+1}^0\,)/m_{ii}$,
$V_i = \max(\,g_i, V_i^0 + \omega(V_i - V_i^0)\,)$;

Step 5. Perform Step 6 when $||\mathbf{V} - \mathbf{V}^0||_0 \le \varepsilon$; otherwise, set $\mathbf{V}^0 = \mathbf{V}$ and return to Step 4;

Step 6. Set $\mathbf{U}^{k+1} = \mathbf{V}$, $\mathbf{F}^{k+1} = \mathbf{D}\mathbf{U}^{k+1} + \mathbf{b}$.

By virtue of the above computation procedure we can first obtain the intermediate variable $\mathbf{U}^M$, and then the option present value $V = SU_0^M$ corresponding to the stock price S by the transform (2.8).

Now we consider an American lookback put option on nondividend paying stocks during its lifetime. Assume that the given data are as follows: $r = 0.06$, $\sigma = 0.40$, the expiration dates $T = 3, 6, 9, 12$ months (or $1/4, 2/4, 3/4, 4/4$ years), and the stock price $S = 50$. Next we compute the option values for the above different expiration dates.

For the computation domain $(x, t) \in [0, X] \times [0, T_1]$, we take $X = 10$ and $T_1 = \frac{1}{2}\sigma^2 T$. In computation, we take the relaxation factor $\omega = 1.5$, the meshes $\Delta\tau = h^2$ and $h = X/N$. Table 1 gives the option present values with different expiration dates. The computation time costs only several seconds. Because no exact solution can be obtained for the differential linear complementary problem (2.3)–(2.7), here we only check the convergence of our method. The numerical example demonstrates that the finite difference method provided here is convergent and efficient for the valuation of American lookback put options.

Table 1: Numerical results for American lookback put options pricing

T	N=400	N=800	N=1000	N=1500	N=2000
3	8.1586	8.1685	8.1683	8.1686	8.1688
6	11.6473	11.6647	11.6653	11.6683	11.6684
9	14.2920	14.3693	14.3018	14.3956	14.3967
12	16.6943	16.7152	16.7185	16.7186	16.7188

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