Journal of Computational Mathematics
Vol.27, No.4, 2009, 474–483.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2009.27.4.014

CONVERGENCE OF AN IMMERSED INTERFACE UPWIND SCHEME FOR LINEAR ADVECTION EQUATIONS WITH PIECEWISE CONSTANT COEFFICIENTS II: SOME RELATED BINOMIAL COEFFICIENT INEQUALITIES *

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Abstract

In this paper we give proof of three binomial coefficient inequalities. These inequalities are key ingredients in [Wen and Jin, J. Comput. Math. 26, (2008), 1-22] to establish the L^1-error estimates for the upwind difference scheme to the linear advection equations with a piecewise constant wave speed and a general interface condition, which were further used to establish the L^1-error estimates for a Hamiltonian-preserving scheme developed in [Jin and Wen, Commun. Math. Sci. 3, (2005), 285-315] to the Liouville equation with piecewise constant potentials [Wen and Jin, SIAM J. Numer. Anal. 46, (2008), 2688-2714].

Mathematics subject classification: 05A20, 05A10, 65M15.
Key words: Binomial coefficient, Linear advection equations, Immersed interface upwind scheme, Piecewise constant coefficients, Error estimates.

1. Introduction

In this paper we give proof of the following three binomial coefficient inequalities.

Theorem 1.1.

$$\sum_{l=0}^{n} \Gamma_{n,l}(\lambda)|n-n\lambda-l| \le \sqrt{\frac{2}{e}}\sqrt{\lambda(1-\lambda)(n+1)}, \quad \forall 0<\lambda<1, n\in\mathbb{N}, \tag{1.1}$$

where

$$\Gamma_{n,l}(\lambda) = C_n^l \lambda^{n-l}(1-\lambda)^l, \tag{1.2}$$

and C_n^l denote binomial coefficients.

Theorem 1.2. *Let $0<\lambda^-,\lambda^+<1, n\in\mathbb{N}, J\in\mathbb{Z},$ with $-n\lambda^-<J<0, K=\frac{\lambda^+}{\lambda^-}(J+n\lambda^-)$. Define*

$$T_1 = \nu(n,n+J+1,\lambda^-)\lambda^-, \quad \text{if } [K]^-=0, \tag{1.3}$$

$$\begin{aligned} T_1 = \nu(n,n+J+1,\lambda^-)\lambda^- + \sum_{l=n+J+1-[K]^-}^{n-1} \sum_{j=\max(n+1-l-[K]^-,1)}^{\min(n-l,-J)} \sum_{k=0}^{l} \Lambda_{j,k,l}^n \\ + \nu(n,n+1-[K]^-,\lambda^+)\lambda^+, \quad \text{if } [K]^->0, \end{aligned} \tag{1.4}$$

* Received April 25, 2008 / Revised version received July 21, 2008 / Accepted August 6, 2008 /

where $[x]^-$ *denotes the largest integer no more than* x*, and*

$$\nu(n,p,z)=\sum_{l=p}^{n}\Gamma_{n,l}(z)z^{-1}(l-p+1),\quad 0\le p\le n, 0<z<1, \tag{1.5}$$

$$\Lambda^n_{i,j,k}=C^k_{j+k-1}C^{l-k}_{n-j-k}(\lambda^+)^{n-l-j+1}(1-\lambda^+)^{l-k}(\lambda^-)^{j-1}(1-\lambda^-)^k. \tag{1.6}$$

Then

$$T_1\le\nu(n,n-[n\lambda^m]^++1,\lambda^m)\lambda^M, \tag{1.7}$$

where $[x]^+$ *denotes the smallest integer no less than* x*,* $\lambda^m=\min\{\lambda^-,\lambda^+\}$*, and* $\lambda^M=\max\{\lambda^-,\lambda^+\}$.

Theorem 1.3. *Let* $0<\lambda^-,\lambda^+<1, n\in\mathbb{N}, J\in\mathbb{Z}$*, with* $-n\lambda^-<J<0, K=\frac{\lambda^+}{\lambda^-}(J+n\lambda^-)$*. Define*

$$T_2=\sum_{l=0}^{n+J-[K]^+-1}\sum_{j=1-J}^{n-l-[K]^+}\sum_{k=0}^{l}\Lambda^n_{i,j,k}. \tag{1.8}$$

Then

$$T_2\le\eta(n,n-[n\lambda^m]^--1,\lambda^m)\lambda^M, \tag{1.9}$$

where

$$\eta(n,p,z)=\sum_{l=0}^{p}\Gamma^l_n(z)z^{-1}(p+1-l),\quad 0\le p\le n-1, 0<z<1. \tag{1.10}$$

These binomial coefficient inequalities have been used in [6] to derive the L^1-error estimates for the upwind difference scheme to the linear advection equation

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0,\qquad t>0, x\in\mathbb{R}, \tag{1.11}$$

$$u|_{t=0}=u_0(x), \tag{1.12}$$

with a step function wave speed

$$c(x)=\begin{cases}c^- & x<0,\\ c^+ & x>0,\end{cases} \tag{1.13}$$

where we consider $c(x)$ has definite sign.

Eqs. (1.11)-(1.13) is the simplest case of a hyperbolic equation with singular (discontinuous or measure-valued) coefficients. In [6], we proved that given a general interface condition

$$u(0^+,t)=\rho u(0^-,t),\qquad \rho>0, \tag{1.14}$$

the upwind difference scheme with the immersed interface condition converges in L^1-norm to Eqs. (1.11)-(1.13) with the corresponding interface condition, and derived the half-order L^1-error bounds with explicit coefficients for the numerical solutions. Due to the linearities of both Eq. (1.11) and the upwind difference scheme, the error estimates for general BV initial data can be derived based on error estimates for some Riemann initial data. This strategy is specifically suitable for linear schemes and linear equations and has been used in [5] to estimate lower error bounds for monotone difference schemes to the linear advection equation with a constant wave speed.

For Eqs. (1.11)-(1.13) with Riemann initial data, the L^1-error (upper bound) expressions for the immersed interface upwind scheme are binomial coefficient expressions. Therefore, estimating their upper bounds is equivalent to proving some inequalities on binomial coefficients, which are Theorems 1.1, 1.2 and 1.3 (Theorems 3.3, 3.4 and 3.5 in [6]). Thus these binomial coefficient inequalities are key ingredients in establishing the L^1-error estimates in [6] for the immersed interface upwind scheme to the linear advection equations (1.11)-(1.12) with a step function wave speed (1.13) and general interface conditions (1.14).

More recently, we have applied the L^1-error estimates established in [6] to study the L^1-error estimates for a Hamiltonian-preserving scheme developed in [1] to the Liouville equation with a piecewise constant potential [8]. The related work on the L^1-stability of the Hamiltonian-preserving scheme and the L^1-error estimates for the Hamiltonian-preserving scheme with perturbed initial data was studied in [7]. The Liouville equation with piecewise constant potentials is a linear hyperbolic equation with a measure-valued coefficient. The Hamiltonian-preserving scheme is designed by incorporating the particle behavior at the potential barrier into the numerical fluxes, see, e.g., [1–4]. By using the results in [6], we established in [8] the half-order L^1-error bounds with explicit coefficients for the Hamiltonian-preserving scheme with Dirichlet incoming boundary conditions and for a class of bounded initial data.

In Sections 2-4 we will prove Theorems 1.1-1.3 respectively. We conclude the paper in Section 5.

2. Proof of Theorem 1.1

We will split the binomial coefficient expression at the left hand side of (1.1) into two equivalent parts, for which the upper bound estimates are given.

Proof. We define

$$\sigma(n,m) = \sum_{l=0}^{[m]^-} \Gamma_{n,l}\left(1-\frac{m}{n}\right)(m-l), \qquad 0<m<n,\ n\in\mathbb{N},\ m\in\mathbb{R}. \tag{2.1}$$

Then we can show that

$$\sum_{l=0}^{n} \Gamma_{n,l}(\lambda)|n-n\lambda-l| = \sigma(n,n-n\lambda)+\sigma(n,n\lambda), \quad 0<\lambda<1, \tag{2.2}$$

$$\sigma(n,m) = \begin{cases} [m]^+\left(1-\frac{m}{n}\right)\Gamma_{n,[m]^+}\left(1-\frac{m}{n}\right), \\ [n-m]^+\frac{m}{n}\Gamma_{n,[m]^-}\left(1-\frac{m}{n}\right), \end{cases} \tag{2.3}$$

and

$$\sigma(n,m) \le \sqrt{\frac{1}{2e}}\sqrt{[m]^+\left(1-\frac{m}{n}\right)}, \tag{2.4}$$

$$\sigma(n,m) \le \sqrt{\frac{1}{2e}}\sqrt{[n-m]^+\frac{m}{n}}. \tag{2.5}$$

The result (2.2) can be verified directly. For (2.3), one has

$$\begin{aligned}
\sigma(n,m) &= \sum_{l=0}^{[m]^-} \Gamma_{n,l}\left(1-\frac{m}{n}\right)m - \sum_{l=1}^{[m]^-} \Gamma_{n,l}\left(1-\frac{m}{n}\right)l \\
&= \sum_{l=0}^{[m]^-} \Gamma_{n-1,l}\left(1-\frac{m}{n}\right)\left(1-\frac{m}{n}\right)m - \sum_{l=1}^{[m]^-} \Gamma_{n-1,l-1}\left(1-\frac{m}{n}\right)\frac{m}{n}(n-m) \\
&= [n-m]^+\Gamma_{n,[m]^-}\left(1-\frac{m}{n}\right)\frac{m}{n}.
\end{aligned}$$

One can then check the two parts of the right-hand side of (2.3) are equivalent. To prove (2.4)-(2.5), define

$$\alpha_{m,n}(x) = \sqrt{[m]^+\left(1-\frac{x}{n}\right)}\Gamma_{n,[m]^+}\left(1-\frac{x}{n}\right), \quad x \in \left([m]^+-1,[m]^+\right]. \tag{2.6}$$

Using (2.3) one has

$$\begin{aligned}
\frac{\sigma(n,m)}{\sqrt{[m]^+\left(1-\frac{m}{n}\right)}} &= \alpha_{m,n}(m) \\
&\le \alpha_{m,n}\left(\frac{n}{n+\frac{1}{2}}[m]^+\right) \\
&\le \frac{1}{\sqrt{2}}\left(1-\frac{1}{2\left(n+\frac{1}{2}\right)}\right)^{n+\frac{1}{2}} \le \sqrt{\frac{1}{2e}},
\end{aligned}$$

which gives (2.4). To prove (2.5), observe that (2.3) implies $\sigma(n,m)=\sigma(n,n-m)$. Applying (2.4) to $\sigma(n,n-m)$ gives (2.5).

By (2.3)-(2.5), for $0<\lambda<1, n\in\mathbb{N}$,

$$\begin{aligned}
\sigma(n,n-n\lambda) &= \sigma(n,n\lambda) \\
&\le \sqrt{\frac{1}{2e}}\sqrt{\min\{[\lambda n]^+(1-\lambda),[(1-\lambda)n]^+\lambda\}}.
\end{aligned} \tag{2.7}$$

Then applying (2.2) gives

$$\begin{aligned}
&\sum_{l=0}^{n}\Gamma_{n,l}(\lambda)|n-n\lambda-l| \\
&= \sigma(n,n-n\lambda)+\sigma(n,n\lambda) \\
&\le \sqrt{\frac{2}{e}}\sqrt{\min\{[\lambda n]^+(1-\lambda),[(1-\lambda)n]^+\lambda\}} \\
&\le \sqrt{\frac{2}{e}}\sqrt{\lambda(1-\lambda)(n+1)}.
\end{aligned} \tag{2.8}$$

This completes the proof of this theorem. □

3. Proof of Theorem 1.2

Define

$$\theta(n,p,q)=\sum_{l=n-p+q}^{n}\Gamma_{n,l}(\lambda^-)(l-n+p-q+1)+\sum_{l=p+1}^{n}\Gamma_{n,l}(\lambda^+)(l-p)$$
$$+\sum_{l=q}^{n-1}\sum_{j=\max(p+1-l,1)}^{\min(n-l,p-q+1)}\sum_{k=0}^{l}\Lambda^n_{j,k,l},\qquad 0\le q\le p\le n-1, \tag{3.1}$$

$$\theta(n,p,q)=\sum_{l=p+1}^{n}\Gamma_{n,l}(\lambda^+)(l-p),\qquad \text{for } q\ge p+1, 0\le p\le n-1, \tag{3.2}$$

$$\theta(n,p,q)=\sum_{l=q}^{n}\Gamma_{n,l}(\lambda^-)(l-q+1),\qquad \text{for } 0\le q\le n, p=n, \tag{3.3}$$

$$\theta(n,p,q)=0,\qquad \text{for } q\ge n+1, p=n. \tag{3.4}$$

From the definition (1.3)-(1.4), one can check that

$$T_1=\theta(n,n-[K]^-,n+J+1-[K]^-). \tag{3.5}$$

We will focus on upper bound estimates for $\theta(n,p,q)$ when $0<\lambda^-\neq\lambda^+<1$ and

$$(n,p,q)\in\Omega_1\equiv\left\{(n,p,q)\in\mathbb{N}\times\mathbb{Z}^2|0\le p\le n, 0\le q\le p+1\right\}, \tag{3.6}$$

and consequently prove Theorem 1.2. We use the recurrence relations for $\theta(n,p,q)$ when (n,p,q) are restricted in Ω_1. We establish the upper bound estimates for $\theta(n,p,q)$ when (n,p,q) are at boundaries of Ω_1 where the recurrence relations do not apply. Together with these results we can prove upper bound estimates for $\theta(n,p,q)$ in Ω_1. Finally we give the proof of Theorem 1.2.

3.1. Some lemmas

Lemma 3.1. *For ν defined by* (1.5)*, one has*

$$\nu(n,q,z_2)\le\nu(n,q_0,z_1),\quad \text{for } 0<z_1<z_2<1, \tag{3.7}$$

where

$$q_0=\left[q+(n-q+1)\left(1-z_1/z_2\right)\right]^-. \tag{3.8}$$

Proof. Define $\widehat{m}=\left[(n-q+1)z_1/z_2\right]^+$. Then $q_0=n-\widehat{m}+1$. Using the monotonically decreasing property of $\nu(n,q_0,z)$ with respect to z gives

$$\nu(n,q_0,z_1)\ge\nu(n,q_0,\widehat{m}z_2/(n-q+1)). \tag{3.9}$$

We will use the following inequality

$$\nu(n,n-m+1,\lambda m/(m+1))\ge\nu(n,n-m,\lambda),\quad 1\le m\le n,\ \ 0<\lambda<1 \tag{3.10}$$

which is proved as follows

$$
\begin{aligned}
&\nu(n, n-m+1, \lambda m/(m+1))\lambda \\
&= \frac{m+1}{m} \sum_{l=n-m+1}^{n} C_n^l \left(\lambda \frac{m}{m+1}\right)^{n-l} \left((1-\lambda) + \frac{1}{m+1}\lambda\right)^l (l-n+m) \\
&= \frac{m+1}{m} \sum_{l=n-m+1}^{n} C_n^l \left(\lambda \frac{m}{m+1}\right)^{n-l} \sum_{k=0}^{l} C_l^k (1-\lambda)^k \left(\frac{1}{m+1}\lambda\right)^{l-k} (l-n+m) \\
&= \frac{m+1}{m} \sum_{k=0}^{n} \sum_{l=\max(n-m+1,k)}^{n} C_n^l C_l^k \lambda^{n-k} (1-\lambda)^k \left(\frac{m}{m+1}\right)^{n-l} \left(\frac{1}{m+1}\right)^{l-k} (l-n+m) \\
&\geq \frac{m+1}{m} \sum_{k=n-m}^{n} \sum_{l=k}^{n} C_n^l C_l^k \lambda^{n-k} (1-\lambda)^k \left(\frac{m}{m+1}\right)^{n-l} \left(\frac{1}{m+1}\right)^{l-k} (l-n+m) \\
&= \frac{m+1}{m} \sum_{k=n-m}^{n} \Gamma_{n,k}(\lambda) \sum_{l=k}^{n} \Gamma_{n-k,l-k} \left(\frac{m}{m+1}\right) (l-n+m) \\
&= \sum_{k=n-m}^{n} \Gamma_{n,k}(\lambda) \left[(m+1) \sum_{l=k}^{n} \Gamma_{n-k,l-k} \left(\frac{m}{m+1}\right) - \sum_{l=k}^{n} \Gamma_{n-k,l-k} \left(\frac{m}{m+1}\right) \frac{m+1}{m}(n-l)\right] \\
&= \sum_{k=n-m}^{n} \Gamma_{n,k}(\lambda) \left[m+1-(n-k) \sum_{l=k}^{n-1} \Gamma_{n-k-1,l-k} \left(\frac{m}{m+1}\right)\right] \\
&= \sum_{k=n-m}^{n} \Gamma_{n,k}(\lambda)\,(k-n+m+1)\,.
\end{aligned}
$$

Therefore, by repeatedly using (3.10) one obtains

$$\nu(n, n-\widehat{m}+1, \widehat{m} z_2/(n-q+1)) \geq \nu(n, n-(n-q+1)+1, z_2). \tag{3.11}$$

Combining (3.9) and (3.11) gives (3.7). □

Lemma 3.2. *For $(n,p,q) \in \Omega_1$, $\theta(n,p,q)$ defined in* (3.1)-(3.4) *satisfies*

$$\theta(n,p,q) \leq \begin{cases} \nu\left(n, \Phi_{p,q}^{n,1}, \lambda^-\right)\lambda^+, & 0<\lambda^-<\lambda^+<1, \\ \nu\left(n, \Phi_{p,q}^{2}, \lambda^+\right)\lambda^-, & 0<\lambda^+<\lambda^-<1, \end{cases} \tag{3.12}$$

where

$$\Phi_{p,q}^{n,1} = \left[q+(n-p)\left(1-\lambda^-/\lambda^+\right)\right]^-, \quad \Phi_{p,q}^{2} = \left[q+(p+1-q)\left(1-\lambda^+/\lambda^-\right)\right]^-. \tag{3.13}$$

Proof. We give proof for the first part of (3.12). The second part can be proved in the same spirit. Firstly we check that $\theta(n,p,q)$ defined in (3.1) and (3.3) satisfies

$$\theta(n,p,0) \leq \nu\left(n, \left[(n-p)\left(1-\lambda^-/\lambda^+\right)\right]^-, \lambda^-\right)\lambda^+, \quad \text{for } 0<\lambda^-<\lambda^+<1. \tag{3.14}$$

One can check the following equality

$$\theta(n,p,0) = \sum_{l=0}^{n} \Gamma_{n,l}(\lambda^-)(l+1) - \left(\lambda^+/\lambda^- - 1\right) \sum_{l=0}^{n-p-2} \Gamma_{n,l}(\lambda^-)(n-p-1-l). \tag{3.15}$$

Denote $\widehat{m} = [(n-p)(1-\lambda^-/\lambda^+)]^-$. From (3.15) one can check (3.14) as follows

$$\begin{aligned}
&\sum_{l=\widehat{m}}^{n} \Gamma_{n,l}(\lambda^-)\,(l-\widehat{m}+1)\,\lambda^+/\lambda^- - \theta\,(n,p,0)\\
&\geq \sum_{l=\widehat{m}}^{n} \Gamma_{n,l}(\lambda^-)\,(l-\widehat{m}+1)\,\lambda^+/\lambda^- - \sum_{l=0}^{n} \Gamma_{n,l}(\lambda^-)\,(l+1)\,\lambda^+/\lambda^-\\
&\quad + \left(\lambda^+/\lambda^- - 1\right)\sum_{l=0}^{n} \Gamma_{n,l}(\lambda^-)(l+1) + \left(\lambda^+/\lambda^- - 1\right)\sum_{l=0}^{n} \Gamma_{n,l}(\lambda^-)(n-p-1-l)\\
&\geq \left(\lambda^+/\lambda^- - 1\right)(n-p) - \widehat{m}\lambda^+/\lambda^-\\
&= \lambda^+/\lambda^- \left[(n-p)\left(1-\lambda^-/\lambda^+\right) - \widehat{m}\right] \geq 0.
\end{aligned}$$

The first part of (3.12) is equivalent to

$$\nu\left(n, \Phi^{n,1}_{p,q}, \lambda^-\right)\lambda^+ - \theta(n,p,q) \equiv \theta^-(n,p,q) \geq 0, \quad \text{if } 0 < \lambda^- < \lambda^+ < 1. \tag{3.16}$$

From (3.14) and Lemma 3.1 one has

$$\theta^-\,(n,p,0) \geq 0, \quad \theta^-(n,p,p+1) \geq 0, \qquad \text{if } 0 < \lambda^- < \lambda^+ < 1, n \in \mathbb{N}. \tag{3.17}$$

We will prove (3.16) by induction. First, (3.16) holds for $n = 1$. Now suppose (3.16) holds for n, we will prove it is also true for $n+1$. It can be checked that $\theta(n,p,q)$ defined in (3.1)-(3.4) satisfies the following recurrence relation for $1 \leq q \leq p+1$, $0 \leq p \leq n$

$$\theta(n+1,p+1,q) = \lambda^-\theta(n,p,q) + (1-\lambda^-)\theta(n,p,q-1). \tag{3.18}$$

From (3.18) one can deduce

$$\theta^-(n+1,p+1,q) = \lambda^-\theta^-(n,p,q) + (1-\lambda^-)\theta^-(n,p,q-1). \tag{3.19}$$

From (3.19), applying the assumption that (3.16) holds for n, one has

$$\theta^-(n+1,p+1,q) \geq 0, \quad \text{if } 0 < \lambda^- < \lambda^+ < 1, \;\; 1 \leq q \leq p+1, \;\; 0 \leq p \leq n. \tag{3.20}$$

Combining (3.20) and (3.17) implies that (3.16) holds for $n+1$. $\square$

3.2. Proof of Theorem 1.2

Proof. From the relation (3.5) and applying the first part of Lemma 3.2, for $0 < \lambda^- < \lambda^+ < 1$ one has

$$T_1 \leq \nu(n, q1, \lambda^-)\lambda^+, \tag{3.21}$$

where

$$q_1 = \left[n + J + 1 - [K]^- + [K]^-\left(1 - \frac{\lambda^-}{\lambda^+}\right)\right]^- \geq n - [n\lambda^-]^+ + 1. \tag{3.22}$$

Thus

$$T_1 \leq \nu\left(n, n - \left[n\lambda^-\right]^+ + 1, \lambda^-\right)\lambda^+, \quad \text{if } 0 < \lambda^- < \lambda^+ < 1. \tag{3.23}$$

From the relation (3.5) and applying the second part of Lemma 3.2, for $0 < \lambda^+ < \lambda^- < 1$ one has

$$T_1 \le \nu(n, q2, \lambda^+)\lambda^-, \tag{3.24}$$

where

$$q_2 = \left[n + J + 1 - [K]^- - J\left(1 - \frac{\lambda^+}{\lambda^-}\right)\right]^- \ge n - [n\lambda^+]^+ + 1. \tag{3.25}$$

Thus

$$T_1 \le \nu\left(n, n - \left[n\lambda^+\right]^+ + 1, \lambda^+\right)\lambda^-, \quad \text{if } 0 < \lambda^+ < \lambda^- < 1. \tag{3.26}$$

Combining (3.23), (3.26) and checking that (1.7) holds for $0 < \lambda^- = \lambda^+ < 1$ complete the proof of Theorem 1.2. □

4. Proof of Theorem 1.3

Define

$$\chi(n, p, q) = \sum_{l=0}^{p} \sum_{j=1+q}^{p+1+q-l} \sum_{k=0}^{l} \Lambda^n_{j,k,l}, \quad 0 \le p, q, p+q \le n-1. \tag{4.1}$$

From the definition (1.8), one can check that

$$T_2 = \chi(n, n + J - [K]^+ - 1, -J). \tag{4.2}$$

The idea of the proof is similar to that of Theorem 1.2. More precisely, we will focus on upper bound estimates for $\chi(n, p, q)$ when $0 < \lambda^- \neq \lambda^+ < 1$ and

$$(n, p, q) \in \Omega_2 \equiv \left\{(n, p, q) \in \mathbb{N} \times \mathbb{Z}^2 | 0 \le p, q, p+q \le n-1\right\}, \tag{4.3}$$

using the recurrence relations for $\chi(n, p, q)$ when (n, p, q) are restricted in Ω_2 and upper bound estimates for $\chi(n, p, q)$ when (n, p, q) are at boundaries of Ω_2 where the recurrence relations do not apply. Based on these estimates we then give the proof of Theorem 1.3.

Proof. We will use the following estimates. For $(n, p, q) \in \Omega_2$, $\chi(n, p, q)$ defined in (4.1) satisfies

$$\chi(n, p, q) \le \begin{cases} \eta\left(n, \Psi^{n,1}_{p,q}, \lambda^-\right)\lambda^+, & 0 < \lambda^- < \lambda^+ < 1, \\ \eta\left(n, \Psi^2_{p,q}, \lambda^+\right)\lambda^-, & 0 < \lambda^+ < \lambda^- < 1, \end{cases} \tag{4.4}$$

where

$$\Psi^{n,1}_{p,q} = \left[p + (n - 1 - p - q)\left(1 - \lambda^-/\lambda^+\right)\right]^+, \quad \Psi^2_{p,q} = \left[p + q\left(1 - \lambda^+/\lambda^-\right)\right]^+. \tag{4.5}$$

We check the first part of (4.4) which is equivalent to

$$\eta\left(n, \Psi^{n,1}_{p,q}, \lambda^-\right)\lambda^+ - \chi(n, p, q) \equiv \chi^-(n, p, q) \ge 0, \quad \text{if } 0 < \lambda^- < \lambda^+ < 1, \tag{4.6}$$

and omit the proof for the second part being in similar spirit. Firstly we check that

$$\chi^-(n, 0, q) \ge 0, \quad \chi^-(n, p, 0) \ge 0, \qquad \text{if } 0 < \lambda^- < \lambda^+ < 1, n \in \mathbb{N}. \tag{4.7}$$

In a similar spirit for proving Lemma 3.1 one can prove the following inequality

$$\eta(n, p, z_2) \le \eta(n, q_0, z_1), \quad \text{for } 0 < z_1 < z_2 < 1, \tag{4.8}$$

where

$$q_0 = [p + (n-1-p)(1 - z_1/z_2)]^+ . \tag{4.9}$$

Applying (4.8) in the case $p = 0$ one can check the first part of (4.7)

$$\begin{aligned}\chi(n,0,q) &\le (\lambda^-)^q \eta\left(n-q, \left[(n-q-1)\left(1-\lambda^-/\lambda^+\right)\right]^+, \lambda^-\right)\lambda^+ \\ &\le \eta\left(n, \left[(n-1-q)\left(1-\lambda^-/\lambda^+\right)\right]^+, \lambda^-\right)\lambda^+.\end{aligned} \tag{4.10}$$

Define

$$\varphi(n,p) = \sum_{l=0}^{p} \sum_{j=p+1-l} \sum_{k=0}^{l} \Lambda_{j,k,l}^n. \tag{4.11}$$

From the definition (4.1) and applying Lemma 3.1 in [6] one has

$$\chi(n,p,0) = \sum_{i=n-p}^{n} \varphi(n, n-i) = \eta(n,p,\lambda^+)\lambda^+. \tag{4.12}$$

Then applying (4.8) gives the second part of (4.7).

It can be easily verified that (4.6) holds for $n = 1$. Now suppose (4.6) holds for n, we will prove it is also true for $n+1$. It can be checked that $\chi(n,p,q)$ defined in (4.1) satisfies the following recurrence relation for $1 \le p, q, p+q \le n$

$$\chi(n+1,p,q) = \lambda^-\chi(n,p,q-1) + (1-\lambda^-)\chi(n,p-1,q). \tag{4.13}$$

From (4.13) one can deduce

$$\chi^-(n+1,p,q) = \lambda^-\chi^-(n,p,q-1) + (1-\lambda^-)\chi^-(n,p-1,q). \tag{4.14}$$

From (4.14), applying the assumption that (4.6) holds for n, one has

$$\chi^-(n+1,p,q) \ge 0, \quad \text{if } 0 < \lambda^- < \lambda^+ < 1, \;\; 1 \le p, q, p+q \le n. \tag{4.15}$$

Combining (4.15) and (4.7) implies that (4.6) holds for $n+1$.

From the relation (4.2) and applying the first part of (4.4), for $0 < \lambda^- < \lambda^+ < 1$ one has

$$T_2 \le \eta(n, q1, \lambda^-)\lambda^+, \tag{4.16}$$

where

$$q_1 = \left[n + J - [K]^+ - 1 + [K]^+\left(1 - \frac{\lambda^-}{\lambda^+}\right)\right]^+ \le n - [n\lambda^-]^- - 1. \tag{4.17}$$

Thus

$$T_2 \le \eta\left(n, n - \left[n\lambda^-\right]^- - 1, \lambda^-\right)\lambda^+, \quad \text{if } 0 < \lambda^- < \lambda^+ < 1. \tag{4.18}$$

From the relation (4.2) and applying the second part of (4.4), for $0 < \lambda^+ < \lambda^- < 1$ one has

$$T_2 \le \eta(n, q2, \lambda^+)\lambda^-, \tag{4.19}$$

where

$$q_2 = \left[n + J - [K]^+ - 1 - J\left(1 - \frac{\lambda^+}{\lambda^-}\right)\right]^+ \le n - [n\lambda^+]^- - 1. \tag{4.20}$$

Thus

$$T_2 \le \eta \left(n, n - \left[n\lambda^+\right]^- - 1, \lambda^+\right) \lambda^-, \quad \text{if } 0 < \lambda^+ < \lambda^- < 1. \tag{4.21}$$

Combining (4.18), (4.21), and checking that (1.9) holds for $0 < \lambda^- = \lambda^+ < 1$ complete the proof of Theorem 1.3. □

5. Conclusion

In this paper, three binomial coefficient inequalities were proved, which are key ingredients used in [6] to establish the L^1-error estimates for the upwind difference scheme to the linear advection equations with piecewise constant wave speeds and a general interface condition. More recently based on the work [6] we have established the L^1-error estimates for a Hamiltonian-preserving scheme developed in [1] to the Liouville equation with a piecewise constant potential [8].

In proving the first binomial coefficient inequality (Theorem 1.1), we split the binomial coefficient expression into two equivalent parts, for which upper bound estimates were derived.

In proving the next two binomial coefficient inequalities (Theorems 1.2 and 1.3), we obtained upper bound estimates for the binomial coefficient expressions when their parameters taking values in larger domains than considered in the theorems. We used the recurrence relations for the binomial coefficient expressions in the extended domains and established upper bound estimates for the binomial coefficient expressions at boundaries of the extended domains where the recurrence relations do not apply. Together with these results we proved upper bound estimates for the binomial coefficient expressions in the extended domains.

Acknowledgment. The research of the author is supported in part by the Knowledge Innovation Project of the Chinese Academy of Sciences grants K5501312S1, K5502212F1, K7290312G7 and K7502712F7, and NSFC grant 10601062.

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