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EXPLICIT ERROR ESTIMATES FOR MIXED AND NONCONFORMING FINITE ELEMENTS*

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Abstract

In this paper, we study the explicit expressions of the constants in the error estimates of the lowest order mixed and nonconforming finite element methods. We start with an explicit relation between the error constant of the lowest order Raviart-Thomas interpolation error and the geometric characters of the triangle. This gives an explicit error constant of the lowest order mixed finite element method. Furthermore, similar results can be extended to the nonconforming P_1 scheme based on its close connection with the lowest order Raviart-Thomas method. Meanwhile, such explicit a priori error estimates can be used as computable error bounds, which are also consistent with the maximal angle condition for the optimal error estimates of mixed and nonconforming finite element methods.

Mathematics subject classification: 65N12, 65N15, 65N30, 65N50.
Key words: Mixed finite element, Nonconforming finite element, Explicit error estimate, Maximal angle condition.

1. Introduction

Finite element methods for the accurate numerical solution of partial differential equations are of great practical interest in the engineering and scientific computing applications. Up to now, their mathematical theory such as a priori error estimates have been well established in the literature, see, e.g., [9, 14, 36]. Let u, u_h denote the exact solution of the model problem and the associated discretized solution, respectively. The convergence analysis of finite element method is typically of the form

$$\|u - u_h\| \leq Ch^k |u|, \tag{1.1}$$

where h denotes the maximal diameter of the triangulation, $\|\cdot\|$ and $|\cdot|$ stand for some appropriate norm and seminorm in certain function spaces, respectively.

Such a result may not be effective unless the dependence of the constant C is specified. The classical finite element theories, see, e.g., [9, 14], show that the constant C in (1.1) does not dependent on the function u, but may dependent on the sine of the minimal angle of the triangulation for the two dimensional case, which is equivalent to the well-known nondegenerate assumption or regular assumption of finite element meshes. In fact, the minimal angle condition for the finite elements can be relaxed, which results in the so-called degenerate elements. Error estimates for degenerate elements can go back to the works by Babuška and Aziz [5] and by Jamet [20]; both of them proved the optimal error estimate for the linear Lagrange triangular element under the assumption that the underlying meshes satisfy the maximal angle condition.

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Since late 1980's degenerate elements have been extensively studied; interested readers are referred to [2, 12, 22] and references therein.

As is known that there appear various constants in the process to derive the error estimates. It is good to evaluate these constants explicitly for a quantitative error bound purpose. Actually, there are some works on an explicit error estimate of the finite element methods, see, e.g., [3, 6, 7, 18, 21] for linear finite element methods and [25] for bilinear quadrilateral finite element methods. However, almost all of them are concentrated on the standard conforming finite element methods, which only involves an explicit interpolation error estimate. To the best of our knowledge, as far as other type finite element methods are concerned, for example, mixed elements and nonconforming elements, there seem no explicit error bounds are given. In order to obtain an explicit error bounds for such type elements, only having the interpolation error estimate is not enough. The mixed element methods and the nonconforming element methods need further an explicit bound of the discrete inf-sup constant and of the consistency error, respectively.

In this paper, we are aim to obtain an explicit error bound for the lowest order mixed finite element and nonconforming finite element for the second order problems ([33, 34]). Firstly, we prove some results on the error constants of the Raviart-Thomas interpolation, which plays an essential role in the a priori error estimates of finite element methods. The technical tool is an explicit trace theorem on the reference unit triangle. On the other hand, the Babuška-Brezzi condition is well-known to guarantee the stability of a mixed finite element and play a key role in the error estimates (cf. [10, 11]). It is also essential to give an explicit expression of the inf-sup constant. Based on these results we can derive a constructive error bound for the mixed finite element. Finally, we also obtain an explicit error estimate for the nonconforming Crouzeix-Raviart [16] element by its close relation to the mixed finite element method (cf. [4, 26]). Note that Kikuchi and Liu [21] recently derived an explicit interpolation error bounds for the nonconforming Crouzeix-Raviart element, but that can not implies an explicit bounds for the finite element error. The explicit a priori error estimates obtained in this paper provide computable error bounds and can serve as a posteriori error estimates for finite element methods [1, 35]. Furthermore, our explicit error estimates for the mixed and nonconforming elements are consistent with the maximal angle condition as the conforming linear Lagrange triangular element [5, 22].

The rest of the paper is organized as follows. In section 2, we introduce the set-up and approximation of the model problem along with some notations and preliminary results for subsequent use. Section 3 presents the an explicit priori error estimate for the lowest order Raviart-Thomas finite element. Similar estimates are extended to the nonconforming Crouzeix-Raviart element in section 4. Some numerical experiments are carried out in section 5. Finally, some comments and extensions of the results are given in section 6.

2. An Explicit Bound of the Inf-Sup Constant

In this section, after recalling the model formulation and some notation, we give a sharp Friedrichs' type inequality, based on which we obtain an explicit bound of the inf-sup constant.

Throughout this paper, we denote with small letters the scalar functions, with small bold fonts the vectorial ones. We will adopt the standard conventions for Sobolev norms and seminorms of a function v defined on an open set G:

$$\|v\|_{m,p,G} = \left(\int_G \sum_{|\alpha| \le m} |D^\alpha v|^p \right)^{\frac{1}{p}},$$

$$|v|_{m,p,G} = \left(\int_G \sum_{|\alpha| = m} |D^\alpha v|^p \right)^{\frac{1}{p}}.$$

We shall also denote by $P_l(G)$ the space of polynomials on G of degrees no more than l.

Without specific explanations, $\Omega \subset R^2$ is denoted a bounded convex polygonal domain in this paper. We consider the following second order elliptic equations: Find $u \in H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is known that for convex domain with $\partial\Omega \in C^2$, the above problem has a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, the following Miranda-Talenti estimate holds (cf. $[27-29]$),

$$|u|_{2,\Omega} \le \|f\|_{0,\Omega}, \tag{2.2}$$

which was extended to general convex polygonal domain in [7]. Note that (2.2) holds for any function belongs to $H_0^2(\Omega)$, so the results of this paper can be extended to more general second order problems.

We denote by $(\cdot,\cdot)_G$ the $L^2(G)$ inner product, and if $G = \Omega$, we will drop the notation Ω for simplicity. Let $V = H_0^1(\Omega)$. The standard variational form of (2.1) is:

$$\begin{cases} \text{Find } u \in V \quad \text{such} \quad \text{that} \\ (\nabla u, \nabla v) = (f, v), \ \forall\, v \in V. \end{cases} \tag{2.3}$$

A mixed formulation for (2.1) can be obtained by introducing a flux variable:

$$\mathbf{p} = -\nabla u. \tag{2.4}$$

Then the problem (2.1) is equivalent to seeking a pair $(\mathbf{p}, u)$ with $u \in V$ such that

$$\begin{cases} \mathbf{p} + \nabla u = 0, & \text{in } \Omega, \\ \quad \nabla \cdot \mathbf{p} = f, & \text{in } \Omega. \end{cases} \tag{2.5}$$

In order to derive a variational form for the system of linear equations (2.5), we introduce the following Hilbert space:

$$H(\mathrm{div};\Omega) := \{\mathbf{q} \in L^2(\Omega)^2; \nabla \cdot \mathbf{q} \in L^2(\Omega)\} \tag{2.6}$$

with the norm:

$$\|\mathbf{q}\|_{H(\mathrm{div};\Omega)} := \{\|\mathbf{q}\|_{0,\Omega}^2 + \|\nabla \cdot \mathbf{q}\|_{0,\Omega}^2\}^{\frac{1}{2}}. \tag{2.7}$$

Then the variational formulation for (2.5) is to seek $(\mathbf{p}, u) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$ such that

$$\begin{cases} (\mathbf{p}, \mathbf{q}) - (\nabla \cdot \mathbf{q}, u) = 0, \ \forall \mathbf{q} \in H(\mathrm{div};\Omega), \\ \quad (\nabla \cdot \mathbf{p}, v) = (f, v), \ \ \forall\, v \in L^2(\Omega), \end{cases} \tag{2.8}$$

The well-posedness of problem (2.8) has been well established in the literatures, see. e.g. [11]. The key ingredient in the process is the well known Babuška-Brezzi condition (or Ladyzhenskaja-Babuška-Brezzi condition) which reads as: there exists a constant $\beta(\Omega) > 0$ such that

$$\sup_{\mathbf{p}\in H(\mathrm{div};\Omega)\setminus 0} \frac{(\nabla\cdot\mathbf{p}, u)}{\|u\|_{0,\Omega}\|\mathbf{p}\|_{H(\mathrm{div};\Omega)}} \geq \beta(\Omega), \quad \forall \ \ u \in L^2(\Omega)\setminus 0. \tag{2.9}$$

The condition (2.9) is often called the inf-sup condition in view of the equivalent representation

$$\inf_{u\in L^2(\Omega)\setminus 0} \sup_{\mathbf{p}\in H(\mathrm{div};\Omega)\setminus 0} \frac{(\nabla\cdot\mathbf{p}, u)}{\|u\|_{0,\Omega}\|\mathbf{p}\|_{H(\mathrm{div};\Omega)}} \geq \beta(\Omega). \tag{2.10}$$

Generally speaking, the above inf-sup constant $\beta(\Omega)$ is only dependent on the domain Ω and the final error constants for the mixed finite element methods will dependent on $\beta(\Omega)$. Therefore, for some quantitative purposes, it is better to give an explicit bound of $\beta(\Omega)$. To this end, we firstly prove an explicit bound for the well-known Friedrichs' type inequality [19].

Lemma 2.1. *Suppose the domain Ω is star-shaped with respect to a point, which we just choose to be the origin for simplicity. Let the boundary of Ω be represented in the plane polar coordinates by $r = \rho(\theta)$. Then for any $u \in H_0^1(\Omega)$, we have*

$$\|u\|_{0,\Omega} \leq \max_{\theta} \rho(\theta)\|\nabla u\|_{0,\Omega}. \tag{2.11}$$

Proof. A density argument shows that we only need to prove (2.11) for smooth functions $u \in C_0^\infty(\Omega)$. For any fixed point $x = (r_1, \theta_1) \in \Omega$, there exists one another point $y = (\rho(\theta_1), \theta_1) \in \partial\Omega$. Assume $\nu(x) = (\nu_1(x), \nu_2(x))$ is the unit vector from the point $y$to x. Noticing that $u(\rho(\theta_1), \theta_1) = 0$, we have

$$\begin{aligned} & u(r_1,\theta_1) = u(r_1,\theta_1) - u(\rho(\theta_1),\theta_1) \\ & = \int_{r_1}^{\rho(\theta_1)} \frac{\partial u}{\partial r}(r,\theta_1)dr \\ & \leq \sqrt{\frac{\rho(\theta_1)}{r_1} - 1}\left(\int_{r_1}^{\rho(\theta_1)} \left|\frac{\partial u}{\partial r}(r,\theta_1)\right|^2 rdr\right)^{\frac{1}{2}}. \end{aligned} \tag{2.12}$$

Then an application of the Cauchy-Schwarz inequality yields that

$$\begin{aligned} & \|u\|_{0,\Omega}^2 = \int_0^{2\pi}\int_0^{\rho(\theta_1)} \big|u(r_1,\theta_1)\big|^2 r_1 dr_1 d\theta_1 \\ & \leq \int_0^{2\pi}\int_0^{\rho(\theta_1)} (\rho(\theta_1) - r_1)\int_{r_1}^{\rho(\theta_1)} \left|\frac{\partial u}{\partial r}(r,\theta_1)\right|^2 rdrdr_1d\theta_1 \\ & \leq \int_0^{2\pi}\left(\int_0^{\rho(\theta_1)} \left|\frac{\partial u}{\partial r}(r,\theta_1)\right|^2 rdr\right)\int_0^{\rho(\theta_1)} (\rho(\theta_1) - r_1)dr_1d\theta_1 \\ & \leq \frac{1}{2}\max_{\theta}\rho(\theta)^2 \int_0^{2\pi}\int_0^{\rho(\theta_1)} \left|\frac{\partial u}{\partial r}(r,\theta_1)\right|^2 rdrd\theta_1 \\ & \leq \max_{\theta}\rho(\theta)^2\|\nabla u\|_{0,\Omega}^2 \end{aligned} \tag{2.13}$$

which implies the desired assertion. $\square$

Remark 2.1. The constant $\max_\theta \rho(\theta)$ can be easily characterized in practice for some domains with simple structure. For example,

$$\max_\theta \rho(\theta) = \frac{d_\Omega}{2},$$

where d_Ω is the diameter of Ω if Ω is symmetric to a point, which is the case for the circles, triangles, rectangles, and so on. As for the convex polygonal domains which are interested the main concern of this paper, the value of $\max_\theta \rho(\theta)$ is also not difficult to bound. Moreover, in general, we have

$$\frac{d_\Omega}{2} \le \max_\theta \rho(\theta) < d_\Omega.$$

Remark 2.2. The estimate (2.11) provides a satisfactory explicit bound for the constant of the Friedrichs' inequality, which is particulary useful for some quantitative estimates in numerical analysis of partial differential equations. In the following part of this section, we will derive an explicit estimate for the inf-sup constant in (2.10).

Lemma 2.2. *Suppose the domain Ω is star-shaped with respect to a point which we just choose to be the origin for simplicity. Let the boundary of Ω be represented in the plane polar coordinates by $r = \rho(\theta)$. Then we have the following inf-sup condition*

$$\inf_{u \in L^2(\Omega)\backslash 0} \sup_{\boldsymbol{p} \in H(div;\Omega)\backslash 0} \frac{(\nabla \cdot \boldsymbol{p}, u)}{\|u\|_{0,\Omega}\|\boldsymbol{p}\|_{H(div;\Omega)}} \ge \beta^*(\Omega) \tag{2.14}$$

with

$$\beta^*(\Omega) = \frac{1}{\sqrt{1 + \max_\theta \rho(\theta)}}.$$

Proof. For any fixed $u \in L^2(\Omega) \setminus 0$, there exists a unique $w \in H_0^1(\Omega)$ satisfying

$$\begin{cases} -\Delta w = u, & \text{in } \Omega \\ \quad w = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

From a simple variational analysis we conclude that the sup in (2.9) is attained at $\mathbf{p} = -\nabla w$ and

$$\sup_{\mathbf{p} \in H(\mathrm{div};\Omega)\backslash 0} \frac{(\nabla \cdot \mathbf{p}, u)}{\|u\|_{0,\Omega}\|\mathbf{p}\|_{H(\mathrm{div};\Omega)}} = \frac{\|u\|_{0,\Omega}}{(\|u\|_{0,\Omega}^2 + \|\nabla w\|_{0,\Omega}^2)^{\frac{1}{2}}}. \tag{2.16}$$

On the other hand, integrating both sides of the first identity of (2.15) gives that

$$(-\Delta w, w) = \|\nabla w\|_{0,\Omega}^2 = (u, w) \le \|u\|_{0,\Omega}\|w\|_{0,\Omega}, \tag{2.17}$$

which, together with (2.11), gives

$$\|\nabla w\|_{0,\Omega} \le \max_\theta \rho(\theta)\|u\|_{0,\Omega}. \tag{2.18}$$

Then the proof is completed by combining (2.16)-(2.18). □

Remark 2.3. From Lemma 2.2 we see that the inf-sup constant is independent of the aspect ratio of the domain. This is an advantage against the inf-sup constant of Stokes problem. It is known that the inf-sup constant deteriorates on domains with large aspect ratios which is well known for rectangles [13, 30]. This results in poor behaviors of the conjugate gradient method for the numerical solution of the discrete Schur complement operator, cf. [17, 23, 31].

3. Explicit Error Estimates for the Raviart-Thomas Element

In this section, we present the lowest order Raviart-Thomas element approximation of the mixed formulation (2.8) and aim to obtain an explicit error estimates. The main ingredients involve the accuracy bounds for the interpolation error and the discrete inf-sup constant.

We firstly give an introduction of the finite element space. To this end, let $\mathcal{J}_h$ be a finite element triangulation of Ω, with each element K being an open triangle of size h_K, $h = \max_{K \in \mathcal{J}_h} h_K$. For a general element K with its three vertexes $a_i, i = 1,2,3$, without lost of generality, assume the maximal angle of K is $\angle a_1 a_3 a_2 = \alpha_{M,K}$. Let $\mathbf{n}_i, \mathbf{v}_i$ and l_i be the unit exterior normal, direction and length of the edges from a_3 to a_2 and from a_3 to a_1, respectively, $i = 1, 2$. We denote by $\widehat{K}$ the reference unix simplex in the (ξ, η) space with vertices $\widehat{a}_1 = (0,0), \widehat{a}_2 = (1,0)$ and $\widehat{a}_3 = (0,1)$. Then for any K, let F_K be the affine mapping such that $F_K(\widehat{K}) = K$, where

$$F_K(\widehat{x}) = B\widehat{x} + a_3 \ \text{ with } \ B = (l_1\mathbf{v}_1, l_2\mathbf{v}_2),$$

see Fig. 1 as for an illustration.

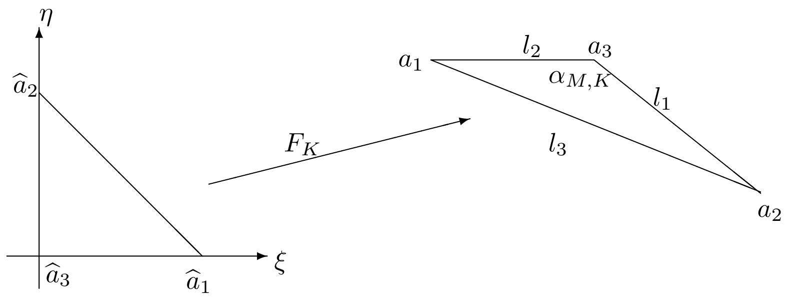

Fig. 1. An illustration of the affine mapping.

The Raviart-Thomas element defined on K (cf. [33, 34]) reads

$$RT_0(K) = P_0(K)^2 \oplus \mathbf{x}P_0(K) \tag{3.1}$$

with its interpolation operator $RT_K : H^1(K)^2 \longrightarrow RT_0(K)$ defined by

$$\int_{l_i} RT_K\mathbf{w} \cdot \mathbf{v}_i ds = \int_{l_i} \mathbf{w} \cdot \mathbf{v}_i ds, \quad i = 1,2,3. \tag{3.2}$$

For the global versions of the interpolation operator, it is denoted as

$$RT_h|_K = RT_K, \ \ \forall K \in \mathcal{J}_h.$$

The Raviart-Thomas space $\mathbf{Q}_h \subset H(\text{div};\Omega)$ is defined as

$$\mathbf{Q}_h = \{\mathbf{q}_h \in H(\text{div};\Omega); \mathbf{q}_h|_K \in RT_0(K), \ \ \forall \ \ K \in \mathcal{J}_h\}. \tag{3.3}$$

Furthermore, let U_h be the space of piecewise constant functions, which is the discrete approximation of $L^2(\Omega)$. Define the operator $P_h : L^2(\Omega) \longrightarrow U_h$ by

$$P_h w|_K = P_{0K} w = \frac{1}{|K|} \int_K w dx dy,$$

for any $K \in \mathcal{J}_h$. It is well known that the above interpolation operator RT_h satisfies the following commutative diagram property:

$$\begin{array}{ccc} H^1(\Omega)^2 & \xrightarrow{\text{div}} & L^2(\Omega) \\ RT_h \downarrow & & \downarrow P_h \\ \mathbf{Q}_h & \xrightarrow{\text{div}} & U_h. \end{array} \tag{3.4}$$

The subsequent work we need to do is to prove an explicit local error estimates for the interpolation operators defined above. As a preparation, we firstly present the following classic Poincaré inequality which can be found in [8, 32].

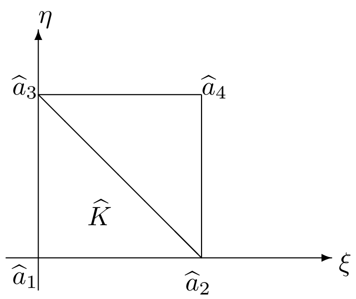

Fig. 2. The relation between the unit square domain $\hat{T}$ and the reference simplex $\hat{K}$.

Lemma 3.1. *Let Ω be a bounded convex domain and let $w \in H^1(\Omega)$ be a function with vanishing average. Then*

$$\|w\|_{0,\Omega} \le \frac{d_\Omega}{\pi}|w|_{1,\Omega}. \tag{3.5}$$

Remark 3.1. This above inequality was firstly proved by Payne and Weinberger in [32]. However, the proof contains an error, and recently [8] gave a modified proof. Fortunately, the optimal constant d/π in the Poincaré inequality remains valid.

For the convenient of the subsequent use, we will prove a sharp trace theorem on the reference element $\widehat{K}$.

Lemma 3.2. *For a general element K, let $\widehat{l}_i = F_K^{-1}(l_i), i = 1, 2$. Then $\forall \widehat{w} \in H^1(\widehat{K})$ and for all $\epsilon > 0$, we have*

$$\|\widehat{w}\|_{0,\widehat{l}_i}^2 \le \left(2 + \frac{2}{\epsilon^2}\right)\|\widehat{w}\|_{0,\widehat{K}}^2 + \epsilon^2|\widehat{w}|_{1,\widehat{K}}^2, \quad i = 1, 2. \tag{3.6}$$

Proof. We will use a direct argument. Let $\widehat{T}$ be the unit square domain containing $\widehat{K}$, i.e.,

$$\widehat{T} = \{\widehat{x} = (\xi, \eta); 0 < \xi, \eta < 1\},$$

see Fig. 2. Then we can extend $\widehat{w}$ to $\widehat{T}$ by reflection with respect to the line $\xi + \eta = 1$ as the following:

$$\widehat{w}^* = \begin{cases} \widehat{w}(\xi, \eta), & \text{if } (\xi, \eta) \in \widehat{K}, \\ \widehat{w}(1-\eta, 1-\xi), & \text{if } (\xi, \eta) \in \widehat{T}\backslash\widehat{K}. \end{cases} \tag{3.7}$$

Due to the symmetry of $\widehat{w}^*$, we have

$$\|\widehat{w}^*\|_{0,\widehat{T}}^2 = 2\|\widehat{w}\|_{0,\widehat{K}}^2, \quad |\widehat{w}^*|_{1,\widehat{T}}^2 = 2|\widehat{w}|_{1,\widehat{K}}^2. \tag{3.8}$$

Consider a smooth function $\widehat{w}^* \in C^1(\overline{\widehat{T}})$. We have

$$\begin{aligned}\widehat{w}^*(\xi,\eta)^2 &= \widehat{w}^*(\xi,0)^2 + \int_0^\eta \frac{\partial(\widehat{w}^*(\xi,t)^2)}{\partial t}dt \\ &= \widehat{w}^*(\xi,0)^2 + 2\int_0^\eta \widehat{w}^*(\xi,t)\frac{\partial \widehat{w}^*(\xi,t)}{\partial t}dt, \end{aligned} \tag{3.9}$$

which implies that

$$\widehat{w}^*(\xi,0)^2 \le \widehat{w}^*(\xi,\eta)^2 + 2\int_0^\eta |\widehat{w}^*(\xi,t)| \left|\frac{\partial \widehat{w}^*(\xi,t)}{\partial t}\right| dt. \tag{3.10}$$

Integrating both sides of (3.10) over ξ and η on $\widehat{T}$, we get

$$\begin{aligned}\|\widehat{w}^*\|^2_{0,\widehat{l}_2} &\le \|\widehat{w}^*\|^2_{0,\widehat{T}} + 2\|\widehat{w}^*\|_{0,\widehat{T}} \left\|\frac{\partial \widehat{w}^*}{\partial \eta}\right\|_{0,\widehat{T}} \\ &\le \left(1+\frac{1}{\epsilon^2}\right)\|\widehat{w}^*\|^2_{0,\widehat{T}} + \epsilon^2 \left\|\frac{\partial \widehat{w}^*}{\partial \eta}\right\|^2_{0,\widehat{T}} \\ &= \left(2+\frac{2}{\epsilon^2}\right)\|\widehat{w}\|^2_{0,\widehat{K}} + \epsilon^2 |\widehat{w}|^2_{1,\widehat{K}}. \end{aligned} \tag{3.11}$$

This is just (3.6) for the case $i=2$. The other case can be proved by the same argument. The proof of the lemma is complete. □

The following result can be regarded as a generalization of the Poincaré inequality, which will play an important role in the interpolation error estimate of the Raviart-Thomas element.

Lemma 3.3. *Let $w \in H^1(K)$ be a function with vanishing average on the side $l_i \boldsymbol{v}_i, i = 1$ or 2. Then we have*

$$\|w\|_{0,K} \le \frac{\sqrt{4+2\sqrt{2}\pi}}{\pi}\left\|\sum_{j=1}^2 l_j \frac{\partial w}{\partial v_j}\right\|_{0,K}, \tag{3.12}$$

where $\partial/\partial v_j$ denotes the derivative along the direction of $\boldsymbol{v}_i$.

Proof. Let $P_{0\widehat{l}_i}$ be the mean value interpolation operator over $\widehat{l}_i$. Then on the reference element, we can derive $\widehat{K}$

$$\begin{aligned}\|\widehat{w}\|^2_{0,\widehat{K}} &= \left\|\widehat{w} - P_{0\widehat{l}_i}\widehat{w}\right\|^2_{0,\widehat{K}} \\ &= \left\|(\widehat{w} - P_{0\widehat{K}}\widehat{w}) - P_{0\widehat{l}_i}(\widehat{w} - P_{0\widehat{K}}\widehat{w})\right\|^2_{0,\widehat{K}} \\ &\le \left\|(\widehat{w} - P_{0\widehat{K}}\widehat{w})\right\|^2_{0,\widehat{K}} + \frac{1}{2}\left\|(\widehat{w} - P_{0\widehat{K}}\widehat{w})\right\|^2_{0,\widehat{l}_i} \\ &\le (2+\frac{1}{\epsilon^2})\left\|(\widehat{w} - P_{0\widehat{K}}\widehat{w})\right\|^2_{0,\widehat{K}} + \epsilon^2\left\|\widehat{\nabla}\widehat{w}\right\|^2_{0,\widehat{K}} \\ &\le \left(\frac{4+\frac{2}{\epsilon^2}}{\pi^2}+\epsilon^2\right)\left\|\widehat{\nabla}\widehat{w}\right\|^2_{0,\widehat{K}} \\ &\le \frac{4+2\sqrt{2}\pi}{\pi^2}\left\|\widehat{\nabla}\widehat{w}\right\|^2_{0,\widehat{K}}, \end{aligned} \tag{3.13}$$

where we have used Lemmas 3.1 and 3.2 in the above steps. Noticing that

$$\widehat{\nabla}\widehat{w} = B^\top \nabla w = \sum_{j=1}^{2} l_j \frac{\partial w}{\partial v_j}.$$

Then the desired result can be obtained by a scaling argument. $\square$

Remark 3.2. It is noted that the constant in the Poincaré inequality and its generalization in Lemma 3.3 can be taken explicitly. This allows us to give quantitative and accuracy estimates in numerical analysis of PDEs.

Now, we are in the position to bound the Raviart-Thomas interpolation error in an explicit manner.

Lemma 3.4. *Let K be a general element and $\boldsymbol{w} \in H^1(K)^2$. Then we have*

$$\left\|\boldsymbol{w} - RT_K\boldsymbol{w}\right\|_{0,K} \leq \frac{2\sqrt{2+\sqrt{2}\pi}}{\pi} h_K \left(\frac{2}{\sin\alpha_{M,K}} \|\nabla \boldsymbol{w}\|_{0,K} + \left\|\nabla\cdot\boldsymbol{w}\right\|_{0,K} \right). \tag{3.14}$$

Proof. Set N be the matrix with $\mathbf{n}_1$ and $\mathbf{n}_2$ as its rows. Since $\mathbf{n}_i, i = 1,2$ are unit vectors, it follows that $|\det(N)| = \sin\alpha_{M,K}$. By using of the decomposition

$$\mathbf{w} - RT_K\mathbf{w} = N^{-1}\big((\mathbf{w} - RT_K\mathbf{w})\cdot\mathbf{n}_1, (\mathbf{w} - RT_K\mathbf{w})\cdot\mathbf{n}_2\big)^\top$$

together with a simple calculation, we have

$$\begin{aligned}\left\|\mathbf{w} - RT_K\mathbf{w}\right\|_{0,K}^2 &= \frac{1}{\sin^2\alpha_{M,K}} \left(\sum_{i=1}^{2} \|(\mathbf{w} - RT_K\mathbf{w})\cdot\mathbf{n}_i\|_{0,K}^2 \right. \\ &\qquad \left. -2\mathbf{n}_1\cdot\mathbf{n}_2 \int_K (\mathbf{w} - RT_K\mathbf{w})\cdot\mathbf{n}_1 (\mathbf{w} - RT_K\mathbf{w})\cdot\mathbf{n}_2 \right) \\ &\leq \frac{2}{\sin^2\alpha_{M,K}} \sum_{i=1}^{2} \|(\mathbf{w} - RT_K\mathbf{w})\cdot\mathbf{n}_i\|_{0,K}^2. \end{aligned} \tag{3.15}$$

Since $(\mathbf{w} - RT_K\mathbf{w})\cdot\mathbf{n}_i$ has zero mean value on the edge $l_i\mathbf{n}_i$, by Lemma 3.3 we obtain

$$\begin{aligned}&\left\|\mathbf{w} - RT_K\mathbf{w}\right\|_{0,K}^2 \\ &\leq \frac{4(2+\sqrt{2}\pi)}{\pi^2\sin^2\alpha_{M,K}} \sum_{i=1}^{2} \left\| \sum_{j=1}^{2} l_j \frac{\partial(\mathbf{w} - RT_K\mathbf{w})\cdot\mathbf{n}_i}{\partial v_j} \right\|_{0,K}^2 \\ &= \frac{4(2+\sqrt{2}\pi)}{\pi^2\sin^2\alpha_{M,K}} \sum_{i=1}^{2} \left\| B^\top\nabla\mathbf{w}\cdot\mathbf{n}_i - \sum_{j=1}^{2} l_j \frac{\mathbf{v}_j\cdot\mathbf{n}_i}{2} P_{0K}\nabla\cdot\mathbf{w} \right\|_{0,K}^2 \\ &\leq \frac{8(2+\sqrt{2}\pi)}{\pi^2\sin^2\alpha_{M,K}} \sum_{i=1}^{2} \left\|B^\top\nabla\mathbf{w}\cdot\mathbf{n}_i\right\|_{0,K}^2 + \frac{2(2+\sqrt{2}\pi)}{\pi^2} \sum_{j=1}^{2} l_j^2 \left\|P_{0K}\nabla\cdot\mathbf{w}\right\|_{0,K}^2 \\ &\leq \frac{16(2+\sqrt{2}\pi)}{\pi^2\sin^2\alpha_{M,K}} h_K^2 \left\|\nabla\mathbf{w}\right\|_{0,K}^2 + \frac{4(2+\sqrt{2}\pi)}{\pi^2} h_K^2 \left\|\nabla\cdot\mathbf{w}\right\|_{0,K}^2, \end{aligned} \tag{3.16}$$

where we have used the fact that

$$\frac{\partial (RT_K \mathbf{w})\cdot \mathbf{n}_i}{\partial v_j} = \frac{1}{2}(\nabla \cdot RT_K \mathbf{w})\mathbf{v}_j \cdot \mathbf{n}_i = \frac{\mathbf{v}_j \cdot \mathbf{n}_i}{2} P_{0K} \nabla \cdot \mathbf{w}$$

which can be proved by a simple computation. Then the proof is complete. □

Remark 3.3. If $\alpha_{M,K} \geq \frac{\pi}{2}$, i.e., the triangle is an obtuse-angled (or right-angled) triangle, by the Cosine Theorem we have

$$\sum_{j=1}^{2} l_j^2 \leq h_K^2.$$

Then the constant of the term $\left\|\nabla \cdot \mathbf{w}\right\|_{0,K}^2$ of (3.16) can be improved by $2(2+\sqrt{2}\pi)h_K^2/\pi^2$.

A direct corollary of Lemma 3.4 is the following global interpolation error estimate.

Theorem 3.1. *Let $\alpha = \max_{K\in\mathcal{J}_h} \alpha_{M,K}$ and $\boldsymbol{w} \in H^1(\Omega)^2$. Then we have*

$$\left\|\boldsymbol{w} - RT_h \boldsymbol{w}\right\|_{0,\Omega} \leq \frac{2\sqrt{2+\sqrt{2}\pi}}{\pi} h \left(\frac{2}{\sin\alpha} \left|\boldsymbol{w}\right|_{1,\Omega} + \left\|\nabla \cdot \boldsymbol{w}\right\|_{0,\Omega} \right). \tag{3.17}$$

Now, let us consider the mixed finite element formulation of (2.8), which is to seek $(\mathbf{p}_h, u_h) \in \mathbf{Q}_h \times U_h$ such that

$$\begin{cases} (\mathbf{p}_h, \mathbf{q}_h) - (\nabla \cdot \mathbf{q}_h, u_h) = 0, \quad \forall \mathbf{q}_h \in \mathbf{Q}_h, \\ \qquad\qquad (\nabla \cdot \mathbf{p}_h, v_h) = (f, v_h), \quad \forall\, v_h \in U_h. \end{cases} \tag{3.18}$$

For the purpose of the examination of the existence and uniqueness of the discrete problem (3.18), we need to give a characterization of the discrete inf-sup constant, which is the task of the following lemma.

Lemma 3.5. *Let Ω be a convex polygonal and its boundary is represented in the plane polar coordinates by $r = \rho(\theta)$. For the Raviart-Thomas finite element space, we have the following discrete inf-sup condition*

$$\inf_{v_h \in U_h \backslash 0} \sup_{\boldsymbol{q}_h \in \boldsymbol{Q}_h} \frac{(\nabla \cdot \boldsymbol{q}_h, v_h)}{\|v_h\|_{0,\Omega} \|\boldsymbol{q}_h\|_{H(div;\Omega)}} \geq \beta^0(\Omega) \tag{3.19}$$

with

$$\beta^0(\Omega) = \left(\frac{8(2+\sqrt{2}\pi)}{\pi^2} \left(4 \max_{K\in\mathcal{J}_h} R_K^2 + h^2 \right) + 2 \max_{\theta} \rho(\theta) + 1 \right)^{-\frac{1}{2}},$$

where R_K is the circumcircle diameter of the element K.

Proof. We proceeding along the same lines of Lemma 2.2. For any fixed $v_h \in U_h \setminus 0$, there exists a unique $w \in H_0^1(\Omega)$ satisfying

$$\begin{cases} -\Delta w = v_h, & \text{in } \Omega \\ \quad w = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

Set $\mathbf{q} = -\nabla w$ and let $\mathbf{q}_h = RT_h\mathbf{q} \in \mathbf{Q}_h$, Using (2.17), (2.2) Theorem 3.5, we have

$$
\begin{aligned}
&\left\|RT_h\mathbf{q}\right\|_{H(\mathrm{div};\Omega)} \\
&\le \left(2\left\|\mathbf{q} - RT_h\mathbf{q}\right\|_{0,\Omega}^2 + 2\left\|\mathbf{q}\right\|_{0,\Omega}^2 + \left\|\nabla\cdot RT_h\mathbf{q}\right\|_{0,\Omega}^2\right)^{\frac{1}{2}} \\
&\le \left(\frac{8(2+\sqrt{2}\pi)}{\pi^2}\left(\max_{K\in\mathcal{J}_h}\frac{4h_K^2}{\sin^2\alpha_{M,K}} + h^2\right) + 2\max_{\theta}\rho(\theta) + 1\right)^{\frac{1}{2}}\left\|v_h\right\|_{0,\Omega},
\end{aligned} \tag{3.21}
$$

which, together with the fact that

$$
(\nabla\cdot\mathbf{q}_h, v_h) = (\nabla\cdot RT_h\mathbf{q}, v_h) = (\nabla\cdot\mathbf{q}, v_h) = \left\|v_h\right\|_{0,\Omega}^2 \tag{3.22}
$$

yields the desired result. □

Remark 3.4. From Lemma 3.6 we can see that the discrete inf-sup constant is bounded if there exists a general constant σ such that

$$
\max_{K\in\mathcal{J}_h} R_K \le \sigma.
$$

This is the case when the triangulation satisfies the maximal angle condition. Moreover, the above condition is weaker than the maximal angle condition since it admits the maximal triangle $\alpha_{M,K} = \pi - \mathcal{O}(h_K)$, which may be very close to π when $h_K \to 0$.

Having established the inf-sup condition of the mixed finite element method, we conclude that problem (3.18) has a unique solution pair $(\mathbf{p}_h, u_h) \in \mathbf{Q}_h \times U_h$ with the following error estimate

$$
\begin{aligned}
&\|\mathbf{p} - \mathbf{p}_h\|_{H(\mathrm{div};\Omega)} + \|u - u_h\|_{0,\Omega} \\
&\le C\left(\inf_{q_h\in\mathbf{Q}_h}\|\mathbf{p} - \mathbf{q}_h\|_{H(\mathrm{div};\Omega)} + \inf_{v_h\in U_h}\|u - v_h\|_{0,\Omega}\right).
\end{aligned} \tag{3.23}
$$

In general, the above constant C is dependent of the discrete inf-sup constant. However, for the purpose of a more precise error estimates, we will not utilize (3.23) directly.

Theorem 3.2. *Let* $(\boldsymbol{p}, u) \in H(div;\Omega) \times L^2(\Omega)$, $(\boldsymbol{p}_h, u_h) \in \boldsymbol{Q}_h \times U_h$ *be the solutions of* (2.8) *and* (3.18), *respectively. Then we have*

$$
\left\|\boldsymbol{p} - \boldsymbol{p}_h\right\|_{0,\Omega} \le \frac{2\sqrt{2+\sqrt{2}\pi}}{\pi}h\left(\frac{2}{\sin\alpha} + 1\right)\left\|f\right\|_{0,\Omega} \tag{3.24}
$$

and

$$
\left\|u - u_h\right\|_{0,\Omega} \le \frac{h}{\pi}\left(\frac{2\sqrt{2+\sqrt{2}\pi}}{\beta^0(\Omega)}\left(\frac{2}{\sin\alpha} + 1\right) + \max_{\theta}\rho(\theta)\right)\left\|f\right\|_{0,\Omega}. \tag{3.25}
$$

If in addition $f \in H^1(\Omega)$, *then we have*

$$
\left\|\nabla\cdot\boldsymbol{p} - \nabla\cdot\boldsymbol{p}_h\right\|_{0,\Omega} \le \frac{h}{\pi}\left|f\right|_{1,\Omega}. \tag{3.26}
$$

Proof. Taking $\mathbf{q} = \mathbf{q}_h$ in (2.8) and subtracting (3.18) from (2.8), we get

$$\begin{cases} (\mathbf{p} - \mathbf{p}_h, \mathbf{q}_h) - (\nabla \cdot \mathbf{q}_h, u - u_h) = 0, & \forall \mathbf{q}_h \in \mathbf{Q}_h, \\ (\nabla \cdot (p - \mathbf{p}_h), v_h) = 0, & \forall\, v_h \in U_h. \end{cases} \tag{3.27}$$

The second equation of (3.27) implies immediately that

$$\nabla \cdot \ \mathbf{p}_h = P_{0h} \nabla \cdot \mathbf{p} = \nabla \cdot RT_h \mathbf{p}, \tag{3.28}$$

which together with Lemma 2.1, gives

$$\begin{aligned} &\left\| \nabla \cdot \mathbf{p} - \nabla \cdot \mathbf{p}_h \right\|_{0,\Omega} = \left\| \nabla \cdot \mathbf{p} - P_{0h} \nabla \cdot \mathbf{p} \right\|_{0,\Omega} \\ &= \left(\sum_{K \in \mathcal{J}_h} \left\| \nabla \cdot \mathbf{p} - P_{0K} \nabla \cdot \mathbf{p} \right\|_{0,K}^2 \right)^{\frac{1}{2}} \le \frac{h}{\pi} \left| \nabla \cdot \mathbf{p} \right|_{1,\Omega} = \frac{h}{\pi} \left| f \right|_{1,\Omega}. \end{aligned} \tag{3.29}$$

Recalling the first equation of (3.27) and noticing (3.28), we can derive

$$\begin{aligned} \left\| \mathbf{p} - \mathbf{p}_h \right\|_{0,\Omega}^2 &= (\mathbf{p} - \mathbf{p}_h, \mathbf{p} - RT_h \mathbf{p}) + (\mathbf{p} - \mathbf{p}_h, RT_h \mathbf{p} - \mathbf{p}_h) \\ &= (\mathbf{p} - \mathbf{p}_h, \mathbf{p} - RT_h \mathbf{p}) + (\nabla \cdot (RT_h \mathbf{p} - \mathbf{p}_h), u - u_h) \\ &= (\mathbf{p} - \mathbf{p}_h, \mathbf{p} - RT_h \mathbf{p}). \end{aligned} \tag{3.30}$$

Then by the interpolation error estimate of Theorem 3.5, we have

$$\begin{aligned} \left\| \mathbf{p} - \mathbf{p}_h \right\|_{0,\Omega} &\le \left\| \mathbf{p} - RT_h \mathbf{p} \right\|_{0,\Omega} \\ &\le \frac{2\sqrt{2 + \sqrt{2}\pi}}{\pi} h \left(\frac{2}{\sin \alpha} \left| \mathbf{p} \right|_{1,\Omega} + \left\| \nabla \cdot \mathbf{p} \right\|_{0,\Omega} \right) \\ &= \frac{2\sqrt{2 + \sqrt{2}\pi}}{\pi} h \left(\frac{2}{\sin \alpha} \left| u \right|_{2,\Omega} + \left\| f \right\|_{0,\Omega} \right), \end{aligned} \tag{3.31}$$

which, together with Miranda-Talenti estimate (2.2), implies (3.24).

Now, we are in the position to estimate $\left\| u - u_h \right\|_{0,\Omega}$. To this end, we only need to estimate $\left\| P_{0h} u - u_h \right\|_{0,\Omega}$. Let us revisit the problem (3.20) with $v_h = P_{0h} u - u_h$. By (3.22) and the first equation of (3.27), we obtain

$$\begin{aligned} &\left\| P_{0h} u - u_h \right\|_{0,\Omega}^2 \\ &= \nabla \cdot RT_h \mathbf{q}, P_{0h} u - u_h) = (\nabla \cdot RT_h \mathbf{q}, u - u_h) \\ &= (\mathbf{p} - \mathbf{p}_h, RT_h \mathbf{q}) \le \left\| \mathbf{p} - \mathbf{p}_h \right\|_{0,\Omega} \left\| RT_h \mathbf{q} \right\|_{0,\Omega} \\ &\le \frac{1}{\beta^0(\Omega)} \left\| \mathbf{p} - \mathbf{p}_h \right\|_{0,\Omega} \left\| P_{0h} u - u_h \right\|_{0,\Omega}. \end{aligned} \tag{3.32}$$

Consequently,

$$\left\| P_{0h} u - u_h \right\|_{0,\Omega} \le \frac{1}{\beta^0(\Omega)} \left\| \mathbf{p} - \mathbf{p}_h \right\|_{0,\Omega}, \tag{3.33}$$

which, together with (3.24) and the fact

$$\left\| u - P_{0h} u \right\|_{0,\Omega} \le \frac{h}{\pi} \left\| \nabla u \right\|_{1,\Omega} = \frac{\max_\theta \rho(\theta)}{\pi} h \left\| f \right\|_{0,\Omega}, \tag{3.34}$$

yields the desired result. The proof is complete. □

4. Error Bounds for Nonconforming Crouzeix-Raviart Element

Let V_h^n denote the nonconforming P_1 finite element space (Crouzeix-Raviart element [16] over $\mathcal{J}_h$), which is given by

$$V_h^n := \left\{ v_h^n \in L^2(\Omega); \forall K \in \mathcal{J}_h, v_h^n|_K \in P_1(K); \forall E \in \mathcal{E}_h, \int_E [v_h^n]_E ds = 0 \right\}. \tag{4.1}$$

Here $[v_h^n]_E$ stands for the jump of v_h^n across E and vanishes when $E \subset \partial\Omega$.

The finite element approximation of (2.3) reads

$$\begin{cases} \text{Find } u_h^n \in V_h^n, \text{ such that} \\ a_h(u_h^n, v_h^n) = f(v_h^n), \ \forall \ v_h^n \in V_h^n \end{cases} \tag{4.2}$$

with

$$a_h(u_h^n, v_h^n) = \sum_{K \in \mathcal{J}_h} \int_K \nabla u_h^n \nabla v_h^n dxdy.$$

Set

$$\| \cdot \|_h = \left(\sum_{K \in \mathcal{J}_h} | \cdot |_{1,K}^2 \right)^{\frac{1}{2}}. \tag{4.3}$$

Then it is easy to see that $\| \cdot \|_h$ is a norm over V_h^n.

It is known that the nonconforming finite element method has a close relationship with the mixed finite element [26], which is stated as follows:

Lemma 4.1. *Let $\boldsymbol{p}_h \in \boldsymbol{Q}_h$ and $u_h^n \in V_h^n$ be the unique solution of* (3.18) *and* (4.2)*, respectively. Then we have*

$$\nabla u_h^n|_K(\boldsymbol{x}) = \boldsymbol{p}_h(\boldsymbol{x})|_K + \frac{1}{2} f_K(\boldsymbol{x} - \boldsymbol{x}_K), \quad \forall \quad \boldsymbol{x} \in K, \quad \forall K \in \mathcal{J}_h, \tag{4.4}$$

where $f_K = \int_K f dx/|K|$, $\boldsymbol{x} = (x, y)$ is a point contained in K, $\boldsymbol{x}_K$ is the barycenter of K.

Then a combination of Lemma 4.1 and Theorem 3.7 yields

Theorem 4.1. *Let u and u_h^n be the solution of* (2.8) *and* (4.2)*, respectively. Then we have the following error estimate*

$$\|u - u_h^n\|_h \leq \left(\frac{2\sqrt{2+\sqrt{2}\pi}}{\pi} \left(\frac{2}{\sin\alpha} + 1 \right) + \frac{\sqrt{3}}{12} \right) h \|f\|_{0,\Omega}. \tag{4.5}$$

Proof. By (4.4) and Young's inequality, for any $\epsilon > 0$, we have

$$\|u - u_h^n\|_h^2 \leq (1+\epsilon)\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega}^2 + \left(1 + \frac{1}{\epsilon}\right) \frac{h^2}{48} \|f\|_{0,\Omega}^2. \tag{4.6}$$

Then by (3.24) and taking a proper value of ϵ, we can obtain the desired result. □

5. Numerical Experiment

In this section, we test the error bounds studied in the above sections by numerical computation. Consider the following Dirichlet problem:

$$\begin{cases} -\triangle u = f, & \text{in } \Omega = [-1,1]\times[-1,1], \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with $f(x,y) = 4 - 2x^2 - 2y^2$. The exact solution of this problem is $u(x,y) = (1-x^2)(1-y^2)$.

We first divide Ω into n^2 equal squares. which are further divided into triangles by the diagonals parallel to $x + y = 1$, except in the top right and bottom left squares which are divided by the diagonals parallel to $x - y = 0$.

Numerical calculations are carried out by employing the lowest order mixed finite element method and nonconforming P_1 finite element method, respectively. Numerical results for both methods are listed in Table 5.1 and Table 5.2, respectively. Herein, the constants C_M and C_N denotes the error constants defined by

$$C_M = \frac{\|\mathbf{p} - \mathbf{p}_h\|_{0,\Omega}}{h\|f\|_{0,\Omega}}$$

and

$$C_N = \frac{\|u - u_h\|_h}{h\|f\|_{0,\Omega}},$$

respectively. From the numerical results one can easily see that the experimentally determined constants are below the theoretical estimates.

Table 5.1: Numerical results of the lowest order mixed finite element method

n	8	16	32	64	128
$\|\mathbf{p}-\mathbf{p}_h\|_{0,\Omega}$	0.5507207331	0.2763366501	0.1382911861	0.0691609781	0.0345824130
C_M	0.2784718819	0.2794591972	0.2797076960	0.2797699314	0.2797854963
$\|f\|_{0,\Omega}$	5.5936471902	5.5936471902	5.5936471902	5.5936471902	5.5936471902
h	0.3535533906	0.1767766953	0.0883883476	0.0441941738	0.0220970869

Table 5.2: Numerical results of nonconforming P_1 finite element method

n	8	16	32	64	128
$\|u-u_h\|_h$	0.3765794544	0.1889822268	0.0945784952	0.0473001982	0.0236514689
C_N	0.1904173623	0.1911176869	0.1912944253	0.1913387229	0.1913498046
$\|f\|_{0,\Omega}$	5.5936471902	5.5936471902	5.5936471902	5.5936471902	5.5936471902
h	0.3535533906	0.1767766953	0.0883883476	0.0441941738	0.0220970869

6. Conclusion

We have developed explicit error estimates for the lowest-order mixed and nonconforming finite elements based on a careful exploration. The explicit constants of some inequalities, together with the continue and discrete inf-sup constants, have been obtained. These estimates allow us to derive some computable upper bound of the finite element errors, which can serve as a posteriori error estimate. Another feature of our error estimates is that we do not need to assume any mesh conditions on the triangulation.

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