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LOW ORDER NONCONFORMING RECTANGULAR FINITE ELEMENT METHODS FOR DARCY-STOKES PROBLEMS*

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Abstract

In this paper, we consider lower order rectangular finite element methods for the singularly perturbed Stokes problem. The model problem reduces to a linear Stokes problem when the perturbation parameter is large and degenerates to a mixed formulation of Poisson's equation as the perturbation parameter tends to zero. We propose two 2D and two 3D nonconforming rectangular finite elements, and derive robust discretization error estimates. Numerical experiments are carried out to verify the theoretical results.

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Key words: Darcy-Stokes problem, Finite element, Uniformly stable.

1. Introduction

We consider the following model equations on a bounded connected polygonal domain $\Omega \subset \mathbb{R}^d, d = 2, 3$. The velocity $\mathbf{u}$ and the pressure p satisfy

$$-\epsilon^2 \Delta \mathbf{u} + \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = g \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

where $\epsilon \in (0, 1]$ is the perturbation parameter and the source term g is assumed to satisfy the solvability condition

$$\int_\Omega g \, d\Omega = 0. \tag{1.4}$$

Then the problem (1.1)-(1.3) admits a unique solution if the condition (1.4) is satisfied and the pressure p is determined only up to addition of a constant.

When ϵ is not too small, and $g \equiv 0$, the system (1.1)-(1.3) is simply a standard Stokes equation but with an additional nonharmful lower order term. On the other hand, when $\mathbf{f} \equiv \mathbf{0}$ and $\epsilon \to 0$, the model formally tends to a mixed formulation of Poisson's equation with

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homogeneous Neumann boundary conditions. When $\epsilon = 0$ the first equation (1.1) has the form of Darcy's law for flow in a homogeneous porous medium, with $\mathbf{u}$ a volume averaged velocity.

As we know, in order to make the discrete problem for the Darcy-Stokes system well posed, one has to take some care in the choice of the velocity/pressure approximation spaces. In particular, the naive choice of piecewise linear interpolations for both the velocity and pressure or piecewise linear interpolation for the velocity and piecewise constants for the pressure will result in ill-posed discrete problem. The remedy is either to enrich the velocity space, using high order interpolation or local bubble functions, or to stabilize the method using Galerkin/least-squares formulation. A vast number of discretization and stabilization techniques for the Stokes equation have been proposed in [1,3,8,9,11,12,15,17-19,21]. For the finite element methods treating the Darcy flow we refer to [23] and the references therein. In this work, we want to treat both porous media flow and open fluid flow, so it would be advantageous if the same element could be used in both the Stokes limit and the Darcy limit [14,16,20]. A seemingly promising candidate for such an element is the nonconforming Crouzeix-Raviart (CR) element [13,24], which has several useful properties. For example, in combination with piecewise constant pressure, it satisfies the *inf-sup* condition and is elementwise mass conserving; it is also easy to implement. However, Mardal, Tai and Winther [22] showed that the CR element does not converge when applied to Darcy problem (or the Darcy-Stokes problem with vanishing viscosity). By adding an edge stabilization term, Burman and Hansbo [10] proved that the simplest P_1/P_0 element can be used for both Darcy and Stokes problems, but the choice of stabilization parameter requires special care.

In [22], Mardal, Tai and Winther considered the singularly perturbed Stokes problem and proposed a triangular element which behaves uniformly with respect to the perturbation parameter. Later, they generalized it to 3D cases and proposed a robust tetrahedron element in [26]. Recently, Xie, Xu and Xue [27] discussed the Darcy-Stokes interface problems and concluded that a traditional stable and uniformly-consistent Stokes element is also uniformly stable for the Darcy-Stokes-Brinkman model if and only if the pressure space contains the divergence range of the velocity space. They also developed a class of low order simplex elements for both 2D and 3D cases which are uniform, stable with respect to the viscosity coefficient, zero-order term coefficient, and their jumps.

For rectangular element cases, it seems to be difficult to construct uniformly stable lower order $H(\mathrm{div})$-element and so far there is little related work. In this paper, we follow the idea of [22,26,27] and seek for uniform stable low order rectangular elements for the Darcy-Stokes problem (1.1)-(1.3) in both 2D and 3D cases.

The rest of this paper is organized as follows. In Section 2 we discuss the general assumptions for the construction of stable finite element methods for problem (1.1)-(1.3). Based on these assumptions, two new rectangular elements in 2D cases are constructed and analyzed in Section 3. Then, in Section 4, we derive uniform discretization error estimates with respect to the perturbation parameter. In Section 5 we discuss the case allowing the viscosity to be zero in the Darcy-Stokes equation. The extension to 3D cases is considered in Section 6. Finally, some numerical experiments are given in Section 7 to verify our theoretical results.

2. General Convergence Analysis for Nonconforming Elements

Let us introduce some notations. H^k denotes the Sobolev space of scalar function whose derivatives up to order k are square integrable, with the norm $\|\cdot\|_k$. The semi-norm derived from the partial derivatives of order equal to k is denoted by $|\cdot|_k$. Furthermore, $\|\cdot\|_{k,T}$, $|\cdot|_{k,T}$ denote respectively the norm $\|\cdot\|_k$ and the semi-norm $|\cdot|_k$ restricted to the domain T. The notation L_0^2 denotes the space of L^2 with mean value zero. Let T_h be a shape regular rectangular triangulation of the domain Ω with the mesh parameter $h = \max_{T\in T_h}\{\text{diameter of T}\}$. Let $E(T)$ $(F(T))$ denote the set of all edges (faces) of a rectangle T in 2D (3D) and $E(T_h)$ $(F(T_h))$ the set of all edges (faces) in the triangulation T_h in 2D (3D). We also denote by $P_k(T)$ the set of polynomials on T with degree at most k.

For simplicity, we use $X \lesssim Y (X \gtrsim Y)$ to denote that there exists a constant C, independent of the mesh size h, the perturbation parameter ϵ and the functions involved, such that $X \leq CY (X \geq CY)$.

Let us now introduce the variational formulation of the problem (1.1)-(1.3). Define the velocity and pressure spaces respectively as

$$V := H_0^1(\Omega)^d, \quad W := L_0^2(\Omega).$$

Let V' and W' be dual spaces of V and W respectively. Then the variational formulation reads as: given $\mathbf{f} \in V'$, $g \in W'$, find $(\mathbf{u}, p) \in V \times W$ such that

$$\begin{cases} a(\mathbf{u},\mathbf{v}) - (p, \mathrm{div}\mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle, & \forall\, \mathbf{v} \in V, \\ (\mathrm{div}\mathbf{u}, q) = \langle g, q\rangle, & \forall\, q \in W, \end{cases} \tag{2.1}$$

where

$$a(\mathbf{u},\mathbf{v}) = \epsilon^2(\nabla\mathbf{u}, \nabla\mathbf{v}) + (\mathbf{u},\mathbf{v}).$$

To obtain the uniform well-posedness of (2.1), define the norm in V

$$|||\mathbf{v}|||^2 = a(\mathbf{v},\mathbf{v}) + (\mathrm{div}\mathbf{v}, \mathrm{div}\mathbf{v}), \quad \mathbf{v} \in V. \tag{2.2}$$

According to the standard theory for saddle-point problems [5-7], the uniform well-posedness of (2.1) is guaranteed by the following continuity condition

$$(\mathrm{div}\mathbf{v}, q) \leq |||\mathbf{v}|||\, \|q\|_0, \quad \forall\, \mathbf{v} \in V,\ \forall\, q \in W, \tag{2.3}$$

and two stability conditions

$$a(\mathbf{v},\mathbf{v}) = |||\mathbf{v}|||^2, \qquad \forall\, \mathbf{v} \in Z := \{\mathbf{v} \in V : \mathrm{div}\mathbf{v} = 0\}, \tag{2.4}$$

$$\sup_{\mathbf{v}\in V} \frac{(\mathrm{div}\mathbf{v}, q)}{|||\mathbf{v}|||} \gtrsim \|q\|_0, \quad \forall\, q \in W. \tag{2.5}$$

The conditions (2.3) and (2.4) are trivially satisfied. The condition (2.5) is known with standard Stokes problem [17], which is also true for $0 < \epsilon \leq 1$ here. Thus the problem (2.1) has a unique solution such that the following estimate

$$|||\mathbf{u}||| + \|p\|_0 \lesssim |||\mathbf{f}|||_{V'} + \|g\|_{W'} \tag{2.6}$$

holds, where

$$|||\mathbf{f}|||_{V'} := \sup_{\mathbf{v}\in V} \frac{\langle \mathbf{f}, \mathbf{v}\rangle}{|||\mathbf{v}|||}.$$

Let $V_h (\subset \text{ or } \not\subset V)$ and $W_h \subset W$ be respectively the velocity and pressure finite element spaces. In this paper, we shall construct a few pairs of V_h and W_h satisfying the following assumptions.

(H1) $\mathrm{div}_h V_h \subset W_h$, where div_h (and ∇_h in what follows) is understood as piecewise divergence (gradient) on T_h.

(H2) There exists a linear interpolation operator $\Pi_h : V \to V_h$, such that for all $\mathbf{v} \in V$,

$$\begin{aligned}&\|\Pi_h \mathbf{v}\|_{1,h} \lesssim \|\mathbf{v}\|_1,\\ &\mathrm{div}_h \Pi_h \mathbf{v} = Q_h \mathrm{div}\mathbf{v},\end{aligned}$$

where $Q_h : L^2(\Omega) \to W_h$ is the orthogonal L^2-projection and the mesh dependent norm $\|\cdot\|_{k,h}$ is defined as, for $\mathbf{v} \in V_h$,

$$\|\mathbf{v}\|_{k,h}^2 = \sum_{T\in T_h} \|\mathbf{v}\|_{k,T}^2.$$

Then the discrete weak formulation reads as: find $(\mathbf{u}_h, p_h) \in V_h \times W_h$ such that

$$\left\{\begin{array}{ll} a_h(\mathbf{u}_h, \mathbf{v}) - (p_h, \mathrm{div}_h \mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall\, \mathbf{v} \in V_h, \\ (\mathrm{div}_h \mathbf{u}_h, q) = (g, q), & \forall\, q \in W_h, \end{array}\right. \tag{2.7}$$

where $a_h(\mathbf{u}_h, \mathbf{v})$ is defined by

$$a_h(\mathbf{u}_h, \mathbf{v}) = \epsilon^2 (\nabla_h \mathbf{u}_h, \nabla_h \mathbf{v}) + (\mathbf{u}_h, \mathbf{v}).$$

Similar to the continuous level, we define the discrete norm in V_h as follows

$$|||\mathbf{v}|||_h^2 = a_h(\mathbf{v}, \mathbf{v}) + (\mathrm{div}_h \mathbf{v}, \mathrm{div}_h \mathbf{v}), \quad \forall\, \mathbf{v} \in V_h. \tag{2.8}$$

Under this discrete norm for the velocity and L_0^2 norm for the pressure, we have the following uniform well-posedness result.

Lemma 2.1. *Suppose the assumptions* (H1) *and* (H2) *are fulfilled. Then, the following discrete stability conditions hold*

$$a_h(\mathbf{v}, \mathbf{v}) = |||\mathbf{v}|||_h^2, \qquad \forall\, \mathbf{v} \in Z_h, \tag{2.9}$$

$$\sup_{\mathbf{v}\in V_h} \frac{(\mathrm{div}_h \mathbf{v}, q)}{|||\mathbf{v}|||_h} \gtrsim \|q\|_0, \qquad \forall\, q \in W_h, \tag{2.10}$$

where $Z_h = \big\{\mathbf{v} \in V_h : (\mathrm{div}_h \mathbf{v}, q) = 0, \quad \forall\, q \in W_h\big\}$.

Proof. The condition (2.9) is trivial by the definition of (2.8) and the assumption (H1). In what follows we will show (2.10) holds. In fact, for all $\mathbf{v} \in V$ and $q \in W_h$, by the definition of Q_h, and the assumptions (H1) and (H2), we have

$$\frac{(\mathrm{div}\mathbf{v}, q)}{\|\mathbf{v}\|_1} = \frac{(Q_h \mathrm{div}\mathbf{v}, q)}{\|\mathbf{v}\|_1} = \frac{(\mathrm{div}_h \Pi_h \mathbf{v}, q)}{\|\mathbf{v}\|_1} \lesssim \frac{(\mathrm{div}_h \Pi_h \mathbf{v}, q)}{\|\Pi_h \mathbf{v}\|_{1,h}}. \tag{2.11}$$

Notice that there holds the continuous *inf*-*sup* condition

$$\sup_{\mathbf{v}\in V} \frac{(\mathrm{div}\mathbf{v}, q)}{\|\mathbf{v}\|_1} \gtrsim \|q\|_0, \tag{2.12}$$

and the inequality

$$\|\Pi_h \mathbf{v}\|_{1,h} \gtrsim |||\Pi_h \mathbf{v}|||_h. \tag{2.13}$$

From (2.11), we then have

$$\sup_{\mathbf{v}\in V} \frac{(\mathrm{div}_h \Pi_h \mathbf{v}, q)}{|||\Pi_h \mathbf{v}|||_h} \gtrsim \sup_{\mathbf{v}\in V} \frac{(\mathrm{div}_h \Pi_h \mathbf{v}, q)}{\|\Pi_h \mathbf{v}\|_{1,h}} \gtrsim \|q\|_0.$$

Thus the discrete *inf-sup* condition (2.10) follows. □

Multiplying $\mathbf{v} \in V_h$ to Eq.(1.1) and using integration by parts, we have

$$a_h(\mathbf{u}, \mathbf{v}) - (\mathrm{div}_h \mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) + E_h(\mathbf{u}, p, \mathbf{v}), \tag{2.14}$$

where the consistency error term is

$$E_h(\mathbf{u}, p, \mathbf{v}) = \sum_{T\in T_h} \int_{\partial T} \big(\epsilon^2 \nabla \mathbf{u} - p\mathbf{I}\big)\mathbf{n}\cdot \mathbf{v}\, ds = \sum_{T\in T_h} \int_{\partial T} \sigma(\mathbf{u}, p)\mathbf{n}\cdot \mathbf{v}\, ds, \tag{2.15}$$

with the stress tensor $\sigma(\mathbf{u}, p) = \epsilon^2 \nabla \mathbf{u} - p\mathbf{I}$.

For simplicity, we introduce the energy norm in V_h as follows

$$\|\mathbf{v}\|_{a_h}^2 = a_h(\mathbf{v}, \mathbf{v}), \quad \forall\, \mathbf{v} \in V_h. \tag{2.16}$$

We are now in a position to state the following theorem.

Theorem 2.1. *Assume that* (H1) *and* (H2) *are fulfilled. Then problem* (2.7) *admits a unique solution* $(\mathbf{u}_h, p_h) \in V_h \times W_h$ *such that*

$$|||\mathbf{u} - \mathbf{u}_h|||_h \le 2\|\mathbf{u} - \Pi_h \mathbf{u}\|_{a_h} + \frac{|E_h(\mathbf{u}, p, \mathbf{u}_h - \Pi_h \mathbf{u})|}{\|\mathbf{u_h} - \Pi_h \mathbf{u}\|_{a_h}} + \|(I - Q_h)\mathrm{div}\mathbf{u}\|_0, \tag{2.17}$$

$$\|p - p_h\|_0 \le \|p - Q_h p\|_0 + 2\|\mathbf{u} - \Pi_h \mathbf{u}\|_{a_h} + 2 \sup_{\mathbf{v}\in V_h} \frac{|E_h(\mathbf{u}, p, \mathbf{v})|}{\|\mathbf{v}\|_{a_h}}, \tag{2.18}$$

$$\|\mathrm{div}\mathbf{u} - \mathrm{div}_h \mathbf{u}_h\|_0 = \|(I - Q_h)\mathrm{div}\mathbf{u}\|_0. \tag{2.19}$$

Proof. By (2.1), (2.7) and (H2), we obtain

$$\big(\mathrm{div}_h \mathbf{u}_h, q\big) = \big(\mathrm{div}\mathbf{u}, q\big) = \big(Q_h \mathrm{div}\mathbf{u}, q\big) = \big(\mathrm{div}_h \Pi_h \mathbf{u}, q\big), \quad \forall\, q \in W_h.$$

This implies that

$$\mathrm{div}_h \mathbf{u}_h = \mathrm{div}_h \Pi_h \mathbf{u}. \tag{2.20}$$

From (2.20) and the assumption (H2), we get the third estimate

$$\|\mathrm{div}\mathbf{u} - \mathrm{div}_h \mathbf{u}_h\|_0 = \|\mathrm{div}\mathbf{u} - \mathrm{div}_h \Pi_h \mathbf{u}\|_0 = \|(I - Q_h)\mathrm{div}\mathbf{u}\|_0. \tag{2.21}$$

By (2.14) and (2.7), we have

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) - \big(p - p_h, \mathrm{div}_h \mathbf{v}\big) = E_h(\mathbf{u}, p, \mathbf{v}), \quad \forall\, \mathbf{v} \in V_h. \tag{2.22}$$

Taking $\mathbf{v} = \mathbf{u}_h - \Pi_h \mathbf{u}$ in (2.22) gives

$$a\big(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_h - \Pi_h \mathbf{u}\big) = E_h\big(\mathbf{u}, p, \mathbf{u}_h - \Pi_h \mathbf{u}\big). \tag{2.23}$$

Using the triangle inequality, Cauchy-Schwartz inequality, and (2.23), we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{a_h} &\le \|\mathbf{u}-\Pi_h\mathbf{u}\|_{a_h} + \|\mathbf{u}_h-\Pi_h\mathbf{u}\|_{a_h}\\ &= \|\mathbf{u}-\Pi_h\mathbf{u}\|_{a_h} + \frac{a(\mathbf{u}_h-\Pi_h\mathbf{u},\mathbf{u}_h-\Pi_h\mathbf{u})}{\|\mathbf{u}_h-\Pi_h\mathbf{u}\|_{a_h}}\\ &= \|\mathbf{u}-\Pi_h\mathbf{u}\|_{a_h} + \frac{a(\mathbf{u}-\Pi_h\mathbf{u},\mathbf{u}_h-\Pi_h\mathbf{u})}{\|\mathbf{u}_h-\Pi_h\mathbf{u}\|_{a_h}} - \frac{a(\mathbf{u}-\mathbf{u}_h,\mathbf{u}_h-\Pi_h\mathbf{u})}{\|\mathbf{u}_h-\Pi_h\mathbf{u}\|_{a_h}}\\ &\le 2\|\mathbf{u}-\Pi_h\mathbf{u}\|_{a_h} + \frac{|E_h(\mathbf{u},p,\mathbf{u}_h-\Pi_h\mathbf{u})|}{\|\mathbf{u}_h-\Pi_h\mathbf{u}\|_{a_h}}.\end{aligned} \tag{2.24}$$

Then (2.19) and (2.24) imply (2.17). By the discrete *inf-sup* condition (2.10) and the assumption (H1),

$$\begin{aligned}\|Q_hp-p_h\|_0 &\lesssim \sup_{\mathbf{v}\in V_h}\frac{(\mathrm{div}_h\mathbf{v},Q_hp-p_h)}{\|\mathbf{v}\|_{a_h}}\\ &= \sup_{\mathbf{v}\in V_h}\frac{a(\mathbf{u}-\mathbf{u}_h,\mathbf{v})-E_h(\mathbf{u},p,\mathbf{v})}{\|\mathbf{v}\|_{a_h}}\\ &\le \sup_{\mathbf{v}\in V_h}\frac{|E_h(\mathbf{u},p,\mathbf{v})|}{\|\mathbf{v}\|_{a_h}} + \sup_{\mathbf{v}\in V_h}\frac{|a(\mathbf{u}-\mathbf{u}_h,\mathbf{v})|}{\|\mathbf{v}\|_{a_h}}\\ &\le \sup_{\mathbf{v}\in V_h}\frac{|E_h(\mathbf{u},p,\mathbf{v})|}{\|\mathbf{v}\|_{a_h}} + \|\mathbf{u}-\mathbf{u}_h\|_{a_h}.\end{aligned}$$

Combining the above two estimates, inequality (2.24), and the triangle inequality, we get the desired estimate for the pressure p. □

In addition to the assumptions (H1) and (H2), we assume that the operators Π_h and Q_h satisfy

(H3)

$$\|\mathbf{v}-\Pi_h\mathbf{v}\|_{j,h} \lesssim h^{k-j}|\mathbf{v}|_k, \quad 0\le j\le k,\ 1\le k,$$

(H4)

$$\|q-Q_hq\|_0 \lesssim h^m|q|_m.$$

Here k,m are two integers. For the consistency error $E_h(\mathbf{u},p,\mathbf{v})$, we also assume

(H5)

$$|E_h(\mathbf{u},p,\mathbf{v})| \lesssim h^l(|\mathbf{u}|_s+|p|_n)\|\mathbf{v}\|_{a_h}, \quad \forall\,\mathbf{v}\in V_h,$$

for integers l, s and n. If $V_h\subset H(\mathrm{div})$, there is no $|p|_n$ term in (H5).

Therefore, from Theorem 2.1, we have:

Theorem 2.2. *Under the assumptions* (H1)-(H5)*, problem* (2.7) *admits a unique solution* $(\mathbf{u}_h,p_h)\in V_h\times W_h$*, such that*

$$\begin{aligned}|||\mathbf{u}-\mathbf{u}_h|||_h &\lesssim h^{\min\{k-1,l,m\}}\big(|\mathbf{u}|_k+|\mathbf{u}|_s+|\mathrm{div}\mathbf{u}|_m+|p|_n\big),\\ \|\mathrm{div}\mathbf{u}-\mathrm{div}_h\mathbf{u}_h\|_0 &\lesssim h^m|\mathrm{div}\mathbf{u}|_m,\\ \|p-p_h\|_0 &\lesssim h^{\min\{k-1,l,m\}}\big(|\mathbf{u}|_k+|\mathbf{u}|_s+|p|_n+|p|_m\big).\end{aligned}$$

Remark 2.1. In fact, we cannot expect that all the norms of $\mathbf{u}$ and p in the above estimates are bounded independently of ϵ. As ϵ approaches zero $|\mathbf{u}|_2$ may blow up, especially in the case that the solution has boundary layers. In such a situation, higher order elements may not attain the expected higher order accuracy (see, e.g., Section 4).

3. Construction of 2D Nonconforming Rectangular Elements

In this section, we will construct 2D nonconforming rectangular elements based on the assumptions (H1)-(H5).

3.1. A low order finite element space

Denote $P_m^d(T)$ is the d-dimensional polynomial space of degree at most m on T. For each rectangle $T \in T_h$, we introduce the following affine invertible transformation

$$F_K : \widehat{K} \to K, \quad x = \frac{1}{2}h_x\xi + x_0, \quad y = \frac{1}{2}h_y\eta + y_0,$$

with the center (x_0, y_0), the horizonal and vertical edge lengthes h_x and h_y, respectively, and the reference element $\widehat{K} = [-1, 1]^2$.

The velocity polynomial space on the rectangle T, is defined by

$$V_T^{(1)} := \Big\{ \mathbf{v} = (v_1, v_2)^T : v_1 \in span\{1, x, y, y^2\}, v_2 \in span\{1, x, y, x^2\} \Big\}.$$

The dimension of $V_T^{(1)}$ is 8 and the 8 degrees of freedom are given below

$$\int_e \mathbf{v}\cdot\mathbf{n}\, ds, \quad \forall\, e \in E(T), \tag{3.1}$$

$$\int_e \mathbf{v}\cdot\mathbf{t}\, ds, \quad \forall\, e \in E(T). \tag{3.2}$$

Here $\mathbf{t}$ and $\mathbf{n}$ are the unit tangent and normal vectors on the edge e respectively. The element diagram is illustrated in Fig. 3.1.

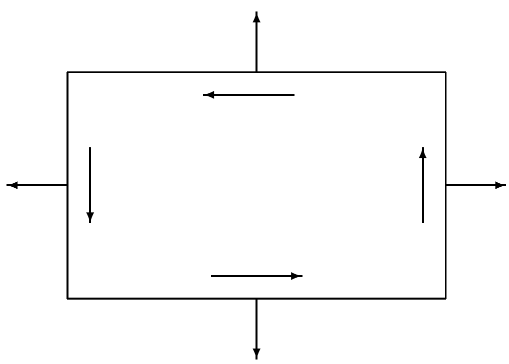

Fig. 3.1 The degrees of freedom of the first element.

Lemma 3.1. *For all* $\mathbf{v} \in V_T^{(1)}$ *is uniquely determined by the above* 8 *degrees of freedom.*

Proof. Let all degrees of freedom for $\mathbf{v} = (v_1, v_2)^T \in V_T^{(1)}$ be equal to zero, then it is enough to show $\mathbf{v} = \mathbf{0}$. Without losing the generality, we prove it on the reference element $\widehat{K}$. In this case, v_1 can be written as

$$v_1 = a_0 + a_1\xi + a_2\eta + a_3\eta^2,$$

with some interpolation constants a_0, a_1, a_2, a_3. By taking the above formulation in the related 4 of the 8 degrees, we can get

$$a_0 + a_1 + \frac{1}{3}a_3 = 0, \quad a_0 - a_1 + \frac{1}{3}a_3 = 0, \quad a_0 - a_2 + a_3 = 0, \quad a_0 + a_2 + a_3 = 0.$$

From the above equations we can easily derive $a_0 = a_1 = a_2 = a_3 = 0$ and then $v_1 = 0$. Owing to the symmetry, similar calculation gives $v_2 = 0$ and then $\mathbf{v} = \mathbf{0}$. □

The finite element spaces V_h and W_h are defined by

$$V_h^{(1)} := \left\{ \begin{array}{l} \mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_T \in V_T^{(1)}, \quad \forall T \in T_h, \\ \text{all the degrees on the boundary of } \Omega \text{ are equal to zero,} \\ \text{the moments (3.1) and (3.2) are continuous across mesh edges} \end{array} \right\},$$
$$W_h^{(1)} := \left\{ q \in W : q|_T \in P_0(T), \quad \forall T \in T_h \right\}.$$

It can be verified that $V_h^{(1)} \not\subset H(\mathrm{div}, \Omega)$ and so $V_h(1) \not\subset H_0^1(\Omega)$. This choice of spaces leads to a nonconforming finite element discretization of problem (2.7). By the construction of $V_h^{(1)}$, we easily have

$$\int_\Omega \mathrm{div}_h \mathbf{v}\, d\Omega = 0, \quad \forall \mathbf{v} \in V_h^{(1)}.$$

Then the assumption (H1) is satisfied. Define the interpolation operator $\Pi_h : \left(H^2(\Omega)\right)^2 \to V_h^{(1)}$ by

$$\int_e (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \mathbf{n}\, ds = 0, \quad \forall e \in E(T),$$
$$\int_e (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \mathbf{t}\, ds = 0, \quad \forall e \in E(T).$$

Let $\widehat{T} = [-1, 1]^2$ be the reference element and $\widehat{\Pi_h} : \left(H^2(\widehat{T})\right)^2 \to V_{\widehat{T}}^{(1)}$ the interpolation operator with respect to $\widehat{T}$. Then for all $\mathbf{v} \in \left(H^2(\widehat{T})\right)^2$, there holds

$$\|\widehat{\Pi_h}\mathbf{v}\|_{1,\widehat{T}} \lesssim \left(\|\mathbf{v}\cdot\mathbf{n}\|_{0,\partial\widehat{T}} + \|\mathbf{v}\cdot\mathbf{t}\|_{0,\partial\widehat{T}} \right) \lesssim \|\mathbf{v}\|_{0,\widehat{T}}^{\frac{1}{2}} \|\mathbf{v}\|_{1,\widehat{T}}^{\frac{1}{2}}.$$

Since the operator Π_h preserves the linear polynomials locally, it follows from a standard scaling argument, using the Bramble-Hilbert lemma, that the following estimates hold:

$$\|\mathbf{v} - \Pi_h \mathbf{v}\|_{j,h} \lesssim h^{2-j} |\mathbf{v}|_2, \quad j = 0, 1, \tag{3.3}$$
$$\|\Pi_h \mathbf{v}\|_{1,h} \lesssim \|\mathbf{v}\|_1. \tag{3.4}$$

This implies that Π_h is bounded and the assumption (H3) holds with $j = 0, 1, 2$ and $k = 2$. For all $\mathbf{v} \in V$ and $q \in P_0(T)$, by Green's formula we have

$$\begin{aligned}\int_T \operatorname{div}\Pi_h \mathbf{v} q \, d\Omega &= -\int_T \Pi_h \mathbf{v} \cdot \nabla q \, d\Omega + \int_{\partial T} \Pi_h \mathbf{v} \cdot \mathbf{n} \, q \, ds \\ &= \int_{\partial T} \Pi_h \mathbf{v} \cdot \mathbf{n} \, q \, ds = \int_{\partial T} \mathbf{v} \cdot \mathbf{n} \, q \, ds \\ &= -\int_T \mathbf{v} \cdot \nabla q \, d\Omega + \int_{\partial T} \mathbf{v} \cdot \mathbf{n} q \, ds \\ &= \int_T \operatorname{div}\mathbf{v} \, q \, d\Omega = \int_T Q_h \operatorname{div}\mathbf{v} \, q \, d\Omega.\end{aligned}$$

Thus, it follows from the relation $\operatorname{div}_h V_h^{(1)} \subset W_h^{(1)}$ that

$$\operatorname{div}_h \Pi_h \mathbf{v} = Q_h \operatorname{div}\mathbf{v}, \quad \forall T \in T_h.$$

This implies the commutativity property in the assumption (H2).

Since Q_h preserves constant locally, we have

$$\|q - Q_h q\|_0 \lesssim h|q|_1, \quad \forall q \in H^1 \cap L_0^2.$$

This implies that the assumption (H4) holds with $m = 1$.

We define the jump $[\mathbf{v} \cdot \mathbf{n}]|_e = \mathbf{v} \cdot \mathbf{n}|_{\partial T_1 \cap e} - \mathbf{v} \cdot \mathbf{n}|_{\partial T_2 \cap e}$ if e is an edge shared by two elements T_1 and T_2, and $[\mathbf{v} \cdot \mathbf{n}]|_e = \mathbf{v} \cdot \mathbf{n}|_e$ if $e \subset E(T_h) \bigcap \partial\Omega$. To estimate the consistency error, we need the following lemma.

Lemma 3.2. *For all $\mathbf{v} \in V_h^{(1)}$, there holds*

$$\sum_{e \in E(T_h)} \int_e q[\mathbf{v} \cdot \mathbf{n}] \, ds = 0, \quad \forall q \in C(\overline{\Omega}), \quad q|_T \in P_2(T). \tag{3.5}$$

Proof. Since

$$\int_e \mathbf{v} \cdot \mathbf{n} \, ds = \int_e \mathbf{v} \cdot \mathbf{t} \, ds = 0, \quad \forall \mathbf{v} \in V_h^{(1)}, \quad \forall e \subset E(T_h) \bigcap \partial\Omega,$$

the term $\sum\limits_{e \in E(T_h)} \int_e q[\mathbf{v} \cdot \mathbf{n}] \, ds$ can be written into the following form

$$\begin{aligned}\sum_{e \in E(T_h)} \int_e q[\mathbf{v} \cdot \mathbf{n}] \, ds &= \sum_{e \in E(T_h)} a_e \int_e \mathbf{v} \cdot \mathbf{n} \, ds + b_e \int_e \mathbf{v} \cdot \mathbf{t} \, ds \\ &= \sum_{e \in Ei(T_h)} a_e \int_e \mathbf{v} \cdot \mathbf{n} \, ds + b_e \int_e \mathbf{v} \cdot \mathbf{t} \, ds,\end{aligned} \tag{3.6}$$

where $Ei(T_h) := E(T_h) \setminus \big(E(T_h) \cap \partial\Omega\big)$ denote the set of all the interior edges in T_h, a_e and b_e are constants dependent on q and the basis functions of $V_h^{(1)}$. Then we only need to prove that all of the coefficients a_e and b_e are equal to zero.

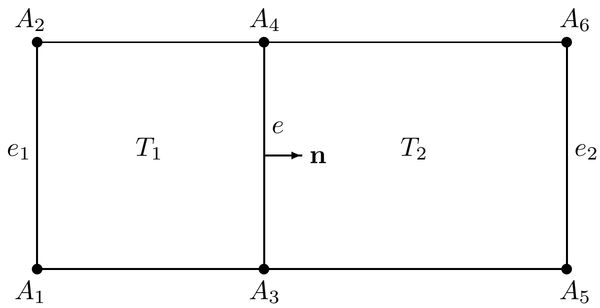

Fig. 3.2 An interior edge e shared by T_1 and T_2.

As shown in Fig. 3.2, we assume e is an interior edge shared by two rectangles T_1 and T_2 with element sizes $a_1 = |A_1A_3|, a_2 = |A_3A_5|, b = |A_3A_4|$. Let (x_0, y_0) be the center of T_1. Notice that e is vertical to the x-$axis$. In fact, in the case that e is parallel to the x-$axis$, the proof is similar.

For $\mathbf{v} \in V_h^{(1)}$, we can write, for $i = 1, 2$,

$$\mathbf{v}|_{T_i} = \mathbf{N_i} \int_e \mathbf{v} \cdot \mathbf{n} ds + \sum_{e' \in E(T_i) \backslash e} \mathbf{N}_{\mathbf{e'}}^{(\mathbf{i})} \int_{e'} \mathbf{v} \cdot \mathbf{n} ds + \sum_{e' \in E(T_i)} \bar{\mathbf{N}}_{\mathbf{e'}}^{(\mathbf{i})} \int_{e'} \mathbf{v} \cdot \mathbf{t} ds, \tag{3.7}$$

where $\mathbf{N_i}$, $\mathbf{N}_{\mathbf{e'}}^{(\mathbf{i})}, \bar{\mathbf{N}}_{\mathbf{e'}}^{(\mathbf{i})}$ are the corresponding nodal basis functions on the elements T_i, and

$$\mathbf{N_1} = \frac{2}{b}\left[\frac{3}{8} + \frac{1}{2a_1}(x - x_0) - \frac{3}{2b^2}(y - y_0)^2, 0\right]^T =: [N_1, 0]^T,$$
$$\mathbf{N_2} = \frac{2}{b}\left[\frac{3}{8} - \frac{1}{2a_1}(x - x_0 - \frac{a_1 + a_2}{2}) - \frac{3}{2b^2}(y - y_0)^2, 0\right]^T =: [N_2, 0]^T.$$

Thus only the first term on the right side of (3.7) contributes to $a_e =: a_e^{(1)} + a_e^{(2)}$ with

$$a_e^{(i)} = \sum_{e' \in E(T_i)} \int_{e'} q\mathbf{N_i} \cdot \mathbf{n} ds.$$

Along the four edges A_1A_3, A_2A_4, A_3A_5 and A_4A_6, the unit normal vectors are of the form $[0, 1]^T$ or $[0, -1]^T$, so $\mathbf{N_1} \cdot \mathbf{n}$ and $\mathbf{N_2} \cdot \mathbf{n}$ are equal to zero. This leads to

$$a_e = \int_{e_2} qN_2\, ds - \int_{e_1} qN_1\, ds + \int_e qN_2\, ds - \int_e qN_1\, ds. \tag{3.8}$$

Along the edge e, we have $x = x_0 + a_1/2$. This indicates $N_1 - N_2 \equiv 0$ on e. Thus, by the continuity of q we obtain

$$\int_e qN_2\, ds - \int_e qN_1\, ds = 0. \tag{3.9}$$

Through translation $\eta = 2(y - y_0)/b$, we have

$$\int_{e_2} N_2\, ds = \int_{-1}^{1} \left(\frac{1}{8} - \frac{3}{8}\eta^2\right) d\eta = 0,$$
$$\int_{e_2} yN_2\, ds = \int_{-1}^{1} \left(\frac{1}{8} - \frac{3}{8}\eta^2\right)\left(\frac{b}{2}\eta + y_0\right) d\eta = 0.$$

Consequently,

$$\int_{e_2} qN_2\, ds = 0, \quad \forall\, q \in P_1(e_2). \tag{3.10}$$

Similarly, we have

$$\int_{e_1} qN_1\, ds = 0, \quad \forall\, q \in P_1(e_1). \tag{3.11}$$

For $q|_T \in P_2(T), \forall\, T \in T_h$, we can suppose that

$$\begin{aligned} q|_{T_1} &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_6 y^2, \\ q|_{T_2} &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy + \beta_5 x^2 + \beta_6 y^2. \end{aligned}$$

By the continuity of q, we know that $q|_{T_1}$ is equal to $q|_{T_2}$ along the edge e: $x = x_0 + a_1/2$. Hence we easily get $\alpha_6 = \beta_6$. From (3.8)-(3.11) we have

$$\begin{aligned} a_e &= \int_{e_2} \beta_6 y^2 N_2\, ds - \int_{e_1} \alpha_6 y^2 N_1\, ds \\ &= \int_{-1}^{1} (\beta_6 - \alpha_6)\left(\frac{b}{2}\eta + y_0\right)^2 \left(\frac{1}{8} - \frac{3}{8}\eta^2\right) d\eta = 0. \end{aligned}$$

Similarly we can derive that $b_e = 0$. This completes the proof. □

Remark 3.1. Similar to the proof of Lemma 3.2, we can show that for all $\mathbf{v} \in V_h^{(1)}$, there holds

$$\sum_{e\in E(T_h)} \int_e q[\mathbf{v}\cdot\mathbf{t}]\, ds = 0, \quad \forall\, q \in C(\overline{\Omega}),\ q|_T \in P_2(T). \tag{3.12}$$

From the proof of Lemma 3.2, we can derive the following stronger conclusion.

Corollary 3.1. *For all $\mathbf{v} \in V_h^{(1)}$, there holds*

$$\sum_{e\in E(T_h)} \int_e q[\mathbf{v}\cdot\mathbf{n}]\, ds = 0, \quad \forall\, q \in \widetilde{S}, \tag{3.13}$$

where

$$\widetilde{S} = \Big\{ q \in C(\overline{\Omega}) :\ q|_T \in P_2(T) \cup span\{x^3, y^3, x^4, y^4, x^5, y^5 \cdots\}, \quad \forall\, T \in T_h \Big\}.$$

We rewrite the consistency error (2.15) as following

$$E_h(\mathbf{u}, p, \mathbf{v}) = \sum_{e\in E(T_h)} \int_e \sigma(\mathbf{u}, p)\mathbf{n}\cdot[\mathbf{v}]ds.$$

On the edge e, we decompose the vector $\sigma(\mathbf{u}, p)\mathbf{n}$ and $\mathbf{v}$ along the normal direction $\mathbf{n}$ and the tangential direction $\mathbf{t}$, i.e.,

$$\begin{aligned} &\sigma(\mathbf{u}, p)\mathbf{n} = (\sigma(\mathbf{u}, p)\mathbf{n}\cdot\mathbf{n})\mathbf{n} + (\sigma(\mathbf{u}, p)\mathbf{n}\cdot\mathbf{t})\mathbf{t}, \\ &\mathbf{v} = (\mathbf{v}\cdot\mathbf{n})\mathbf{n} + (\mathbf{v}\cdot\mathbf{t})\mathbf{t}, \end{aligned}$$

This leads to

$$E_h(\mathbf{u}, p, \mathbf{v}) = \sum_{e\in E(T_h)} \int_e (\sigma(\mathbf{u}, p)\mathbf{n}\cdot\mathbf{t})[\mathbf{v}\cdot\mathbf{t}]ds + \sum_{e\in E(T_h)} \int_e (\sigma(\mathbf{u}, p)\mathbf{n}\cdot\mathbf{n})[\mathbf{v}\cdot\mathbf{n}]ds. \tag{3.14}$$

Lemma 3.3. *For* $\mathbf{u} \in H_0^1 \cap H^3$, $p \in H^k \cap L_0^2, k = 2, 3,$

$$|E_h(\mathbf{u}, p, \mathbf{v})| \lesssim (h\epsilon|\mathbf{u}|_3 + h^{k-1}|p|_k)\|\mathbf{v}\|_{a_h}, \quad \forall\, \mathbf{v} \in V_h^{(1)}. \tag{3.15}$$

Proof. First we have

$$\begin{aligned}
&\Big|\sum_{e\in E(T_h)} \int_e (\sigma(\mathbf{u},p)\mathbf{n}\cdot\mathbf{t})\,[\mathbf{v}\cdot\mathbf{t}]\,ds\Big| \\
&= \Big|\sum_{e\in E(T_h)} \int_e (\epsilon^2\nabla\mathbf{u}\mathbf{n}\cdot\mathbf{t})[\mathbf{v}\cdot\mathbf{t}]\,ds - \int_e ((p\mathbf{I})\mathbf{n}\cdot\mathbf{t})\,[\mathbf{v}\cdot\mathbf{t}]\,ds\Big| \\
&= \Big|\sum_{e\in E(T_h)} \int_e (\epsilon^2\nabla\mathbf{u}\mathbf{n}\cdot\mathbf{t})[\mathbf{v}\cdot\mathbf{t}]\,ds\Big| = \inf_{q\in\tilde{P}_1}\Big|\epsilon^2 \sum_{e\in E(T_h)} \int_e (\nabla\mathbf{u}\mathbf{n}\cdot\mathbf{t} - q)[\mathbf{v}\cdot\mathbf{t}]\,ds\Big| \\
&= \inf_{q\in\tilde{P}_1}\Big|\epsilon^2 \sum_{T\in T_h} \int_{\partial T} (\nabla\mathbf{u}\mathbf{n}\cdot\mathbf{t} - q)\mathbf{v}\cdot\mathbf{t}\,ds\Big| \\
&= \epsilon^2 \inf_{q\in\tilde{P}_1}\Big|\sum_{T\in T_h} \int_T \nabla(\nabla\mathbf{u}\mathbf{n}\cdot\mathbf{t} - q)\times\mathbf{v}\,d\Omega + \int_T (\nabla\mathbf{u}\mathbf{n}\cdot\mathbf{t} - q)\mathrm{rot}\mathbf{v}\,d\Omega\Big| \\
&\le \epsilon^2 \inf_{q\in\tilde{P}_1} \sum_{T\in T_h} \|\nabla(\nabla\mathbf{u}\mathbf{n}\cdot\mathbf{t} - q)\|_{0,T}\|\mathbf{v}\|_{0,T} + \|(\nabla\mathbf{u}\mathbf{n}\cdot\mathbf{t} - q)\|_{0,T}\|\mathrm{rot}\mathbf{v}\|_{0,T} \\
&\lesssim \epsilon \sum_{T\in T_h} h|u|_{3,T}(\epsilon\|\mathbf{v}\|_{0,T} + \epsilon\|\mathrm{rot}\mathbf{v}\|_{0,T}) \lesssim \epsilon h|\mathbf{u}|_{3,h}\|\mathbf{v}\|_{a_h},
\end{aligned} \tag{3.16}$$

where $\tilde{P}_1 := \big\{q \in C(\overline{\Omega}) :\ q|_T \in P_1(T),\ \forall\, T \in T_h\big\}$, and we have used (3.12) to obtain the third equality above.

Secondly, we have

$$\begin{aligned}
&\sum_{e\in E(T_h)} \int_e \big(\sigma(\mathbf{u},p)\mathbf{n}\cdot\mathbf{n}\big)[\mathbf{v}\cdot\mathbf{n}]\,ds = \sum_{e\in E(T_h)} \int_e \big((\epsilon^2\nabla\mathbf{u} - p\mathbf{I})\mathbf{n}\cdot\mathbf{n}\big)[\mathbf{v}\cdot\mathbf{n}]\,ds \\
&= \sum_{e\in E(T_h)} \int_e \big((\epsilon^2\nabla\mathbf{u})\mathbf{n}\cdot\mathbf{n}\big)[\mathbf{v}\cdot\mathbf{n}]\,ds - \sum_{e\in E(T_h)} \int_e \big((p\mathbf{I})\mathbf{n}\cdot\mathbf{n}\big)[\mathbf{v}\cdot\mathbf{n}]\,ds.
\end{aligned} \tag{3.17}$$

For the first term on the right side of (3.17), using a trick similar to the proof of (3.16) gives

$$\Big|\sum_{e\in E(T_h)} \int_e \big((\epsilon^2\nabla\mathbf{u})\mathbf{n}\cdot\mathbf{n}\big)[\mathbf{v}\cdot\mathbf{n}]\,ds\Big| \lesssim \epsilon h|\mathbf{u}|_3\|\mathbf{v}\|_{a_h}. \tag{3.18}$$

For the second term on the right side of (3.17), we have

$$
\begin{aligned}
&\Big|\sum_{e\in E(T_h)}\int_e \big((p\mathbf{I})\mathbf{n}\cdot\mathbf{n}\big)[\mathbf{v}\cdot\mathbf{n}]\,ds\Big| = \Big|\sum_{e\in E(T_h)}\int_e p[\mathbf{v}\cdot\mathbf{n}]\,ds\Big| \\
&=\Big|\inf_{q\in\tilde{P}_2}\sum_{e\in E(T_h)}\int_e (p-q)[\mathbf{v}\cdot\mathbf{n}]\,ds\Big| = \Big|\inf_{q\in\tilde{P}_2}\sum_{T\in T_h}\int_{\partial T}(p-q)\mathbf{v}\cdot\mathbf{n}\,ds\Big| \\
&=\Big|\inf_{q\in\tilde{P}_2}\sum_{T\in T_h}\int_T \nabla(p-q)\mathbf{v}\,d\Omega + \int_T (p-q)\mathrm{div}\mathbf{v}\,d\Omega\Big| \\
&\le \inf_{q\in\tilde{P}_2}\sum_{T\in T_h}\|\nabla(p-q)\|_{0,T}\|\mathbf{v}\|_{0,T} + \inf_q \|p-q\|_{0,T}\|\mathrm{div}\mathbf{v}\|_{0,T} \\
&\lesssim \sum_{T\in T_h} h^{k-1}|p|_{k,T}\big(\|\mathbf{v}\|_{0,T}+\|\mathrm{div}\mathbf{v}\|_{0,T}\big) \\
&\lesssim h^{k-1}|p|_k\|\mathbf{v}\|_{a_h},
\end{aligned} \tag{3.19}
$$

where $\tilde{P}_2 := \big\{q\in C(\overline{\Omega}):\ q|_T\in P_2(T),\ \forall\, T\in T_h\big\}$, and we have used (3.5) to obtain the second equality above. The desired estimate follows from (3.16)-(3.19). □

From Lemma 3.3, the assumption (H5) holds with $l=1, s=3, n=2,3$. Taking $V_h=V_h^{(1)}$, $W_h=W_h^{(1)}$ in the problem (2.7), and applying Theorem 2.1, we then get the following error estimates.

Theorem 3.1. *Let $(\mathbf{u}_h,p_h)\in V_h^{(1)}\times W_h^{(1)}$ be the solution of the problem (2.7). If $\mathbf{u}\in H_0^1\cap H^3$ and $p\in L_0^2\cap H^k,\ k=2,3$, then the following error estimates hold:*

$$
\begin{aligned}
|||\mathbf{u}-\mathbf{u}_h|||_h &\lesssim h(h+\epsilon)|\mathbf{u}|_3 + h^{k-1}|p|_k, \\
\|\mathrm{div}(\mathbf{u}-\mathbf{u}_h)\|_0 &\lesssim h|\mathrm{div}\mathbf{u}|_1, \\
\|p-p_h\|_0 &\lesssim h\big(|p|_1+(h+\epsilon)|\mathbf{u}|_3+h^{k-2}|p|_k\big).
\end{aligned}
$$

Remark 3.2. If $p\in\widetilde{S}$, from (3.13) and the analysis of the consistency error, we can see that the $|p|_k$ term will not appear in Lemma 3.3 and Theorem 3.1.

3.2. A higher order element

Define

$$
\begin{aligned}
V_T^{(2)} := \Big\{\mathbf{v}=(v_1,v_2)^T : v_1\in span\{1,x,y,xy,x^2,y^2,y^3\}, \\
v_2\in span\{1,x,y,xy,x^2,y^2,x^3\}\Big\}.
\end{aligned}
$$

The dimension of this space is 14 and the 14 degree of freedoms are:

$$
\int_e \mathbf{v}\cdot\mathbf{n}q\,ds,\quad \forall\, q\in P_1(e),\quad \forall\, e\in E(T), \tag{3.20}
$$

$$
\int_e \mathbf{v}\cdot\mathbf{t}\,ds,\qquad \forall\, e\in E(T), \tag{3.21}
$$

$$
\int_T \mathbf{v}\cdot\mathbf{q}\,d\Omega,\quad \forall\,\mathbf{q}\in P_0^2(T). \tag{3.22}
$$

The element diagram is give in Fig. 3.3. Let us define the following finite element spaces:

$$V_h^{(2)} := \left\{ \begin{array}{l} \mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_T \in V_T^{(2)}, \quad \forall T \in T_h, \\ \text{all the degrees on the boundary of } \Omega \text{ are equal to zero,} \\ \text{the moments (3.20) and(3.21) are continuous across mesh edges} \end{array} \right\},$$
$$W_h^{(2)} := \left\{ q \in W : \ q|_T \in P_1(T), \quad \forall T \in T_h \right\}.$$

Following the discussions similar to those in Section 3.1, we can prove that the above element is unisolvent and satisfies the assumptions (H1)-(H5) for some suitable integers. So Theorem 2.2 holds for this element and we have the following error estimate.

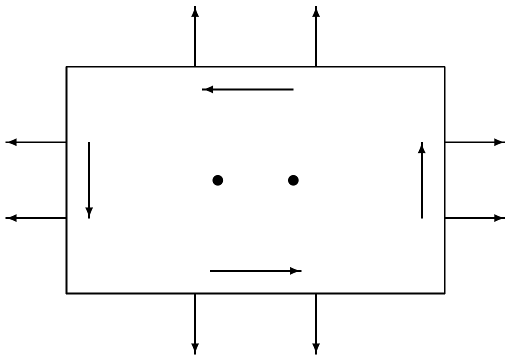

Fig. 3.3 The degrees of freedom of the second element.

Theorem 3.2. *Let* $(\mathbf{u}_h, p_h) \in V_h^{(2)} \times W_h^{(2)}$ *be the solution of the problem* (2.7). *If* $\mathbf{u} \in H_0^1 \cap H^4$ *and* $p \in L_0^2 \cap H^k, k = 3, 4$*, then the following error estimates hold*

$$|||\mathbf{u} - \mathbf{u}_h|||_h \lesssim h^2(h+\epsilon)|\mathbf{u}|_4 + h^{k-1}|p|_k,$$
$$\|\mathrm{div}(\mathbf{u} - \mathbf{u}_h)\|_0 \lesssim h^m|\mathrm{div}\mathbf{u}|_m, \quad m = 1, 2,$$
$$\|p - p_h\|_0 \lesssim h^2\Big(|p|_2 + (h+\epsilon)|\mathbf{u}|_4 + h^{k-3}|p|_k\Big).$$

Remark 3.3. Let

$$\widetilde{S1} := \Big\{q \in C(\overline{\Omega}) : \ q|_T \in P_3(T) \cup span\{x^4, y^4, x^5, y^5, x^6, y^6 \cdots\}, \quad \forall T \in T_h\Big\},$$

similar to the lower order element, if $p \in \widetilde{S1}$, then there is no $|p|_k$ term in Theorem 3.2 because

$$\sum_{e \in E(T_h)} \int_e q[\mathbf{v} \cdot \mathbf{n}]\, ds = 0, \quad \forall q \in \widetilde{S1}, \quad \forall \mathbf{v} \in V_h^{(2)}.$$

Remark 3.4. For $\mathbf{u} \in H_0^1 \cap H^4, \ p \in H^k, \ k = 3, 4$, from a consistency error analysis similar to that in Section 3.1, we can easily obtain

$$|E_h(\mathbf{u}, p, \mathbf{v})| \lesssim \big(h^2\epsilon|\mathbf{u}|_4 + h^{k-1}|p|_k\big)\|\mathbf{v}\|_{a_h}, \quad \forall \mathbf{v} \in V_h^{(2)}$$

as $\widetilde{S1}$ contains piecewise polynomials of degree not more than three.

4. Boundary Layers and Uniform Estimates

In general we cannot expect the norm $|\mathbf{u}|_2$ and $\|p\|_1$ are bounded independently of ϵ. As ϵ approaches zero $|\mathbf{u}|_2$ may blow up, especially in the case that the solution has boundary layers. So the previous estimates do not imply uniform convergence with respect to ϵ. Let

$$H(\text{rot}) = \left\{\mathbf{f} \in L^2(\Omega)^2 | \quad \text{rot}\,\mathbf{f} \in L^2(\Omega)\right\}, \qquad \text{where rot}\mathbf{v} = \frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x},$$
$$H^1_+ = \left\{g \in H^1 \cap L^2_0 : \int_\Omega \frac{|g(x)|^2}{|x - x_j|^2} d\Omega < \infty, \quad j = 1, 2, \cdots, N\right\},$$

with an associated norm

$$\|g\|^2_{1,+} := \|g\|^2_1 + \sum_{j=1}^{N} \int_\Omega \frac{|g(x)|^2}{|x - x_j|^2} d\Omega,$$

where $x_1, x_2, \cdots, x_N$ denote the vertices of Ω. The following lemmas by Mardal, Tai and Winther [22], will be useful in our analysis.

Lemma 4.1. *Assume that* $\mathbf{f} \in H(\text{rot})$*,* $g \in H^1_+$*, and let* $(\mathbf{u}, p)$ *be the corresponding solution of* (1.1)-(1.3)*. Then*

$$\epsilon^{\frac{1}{2}} \|\text{rot}\mathbf{u}\|_0 + \epsilon^{\frac{3}{2}} \|\text{rot}\mathbf{u}\|_1 \lesssim \|\text{rot}\mathbf{f}\|_0 + \|g\|_{1,+}. \tag{4.1}$$

Lemma 4.2. *Assume that* $\mathbf{f} \in H(\text{rot})$*,* $g \in H^1_+$ *and* $(\mathbf{u}, p)$ *is the solution of* (1.1)-(1.3)*. Then*

$$\|\mathbf{u} - \mathbf{u}^0\|_0 + \|p - p^0\|_1 \lesssim \epsilon^{\frac{1}{2}} \left(\|\mathbf{f}\|_{\text{rot}} + \|g\|_{1,+}\right), \tag{4.2}$$

where $(\mathbf{u}^0, p^0)$ *is the solution of the following reduced system:*

$$\begin{aligned} &\mathbf{u}^0 - \nabla p^0 = \mathbf{f} && \textit{in} \quad \Omega, \\ &\text{div}\mathbf{u}^0 = g && \textit{in} \quad \Omega, \\ &\mathbf{u}^0 \cdot \mathbf{n} = 0 && \textit{on} \quad \partial\Omega. \end{aligned} \tag{4.3}$$

Now we rewrite the consistency error (3.14) as

$$E_h(\mathbf{u}, p, \mathbf{v}) = \epsilon^2 \sum_{e \in E(T_h)} \int_e (\nabla \mathbf{u} \mathbf{n} \cdot \mathbf{t})[\mathbf{v} \cdot \mathbf{t}]\, ds + \sum_{e \in E(T_h)} \int_e (\sigma(\mathbf{u}, p)\mathbf{n} \cdot \mathbf{n})[\mathbf{v} \cdot \mathbf{n}]\, ds. \tag{4.4}$$

We suppose the exact solution p satisfies the following condition

$$p \in \widetilde{S}. \tag{4.5}$$

Then the consistency error can be further simplified as

$$E_h(\mathbf{u}, \mathbf{v}) = \epsilon^2 \sum_{e \in E(T_h)} \int_e (\nabla \mathbf{u} \mathbf{n} \cdot \mathbf{t})[\mathbf{v} \cdot \mathbf{t}]\, ds + \epsilon^2 \sum_{e \in E(T_h)} \int_e (\nabla \mathbf{u} \mathbf{n} \cdot \mathbf{n})[\mathbf{v} \cdot \mathbf{n}]\, ds. \tag{4.6}$$

Remark 4.1. To simplify the analysis, we impose here the condition (4.5) to get uniform error estimates. In some cases, the solution p may satisfy condition (4.5) or at least can be approximated well by functions in the space $\widetilde{S}$, so the assumption and the following analysis make sense in such situations.

In order to get uniform error estimates for our new elements, we need the following consistency error estimate.

Lemma 4.3. *For* $\mathbf{u} \in H^2 \cap H_0^1$, $\forall\, \mathbf{v} \in V_h$, *we have*

$$|E_h(\mathbf{u},\mathbf{v})| \lesssim \epsilon h^{\frac{1}{2}} \|\mathbf{u}\|_1^{\frac{1}{2}}\ \|\mathbf{u}\|_2^{\frac{1}{2}}\ \|\mathbf{v}\|_{a_h}. \tag{4.7}$$

Proof. Following Lemma 3.3, we get

$$\int_e (\nabla \mathbf{un}\cdot \mathbf{t})\cdot [\mathbf{v}\cdot \mathbf{t}]\, ds \lesssim \inf_{\lambda\in\mathbf{R}} \|\nabla \mathbf{un}\cdot\mathbf{t}-\lambda\|_{0,e} \inf_{\mu\in\mathbf{R}} \|[\mathbf{v}\cdot\mathbf{t}-\mu]\|_{0,e}.$$

Using the standard scaling argument and the trace theorem, we have

$$\inf_{\lambda\in\mathbf{R}} \|\nabla \mathbf{un}\cdot\mathbf{t}-\lambda\|_{0,e} \inf_{\mu\in\mathbf{R}} \|[\mathbf{v}\cdot\mathbf{t}-\mu]\|_{0,e} \lesssim h^{\frac{1}{2}} |\mathbf{u}|^{\frac{1}{2}}_{1,T_e^+\cup T_e^-} |\mathbf{u}|^{\frac{1}{2}}_{2,T_e^+\cup T_e^-} |\mathbf{v}|_{1,T_e^+\cup T_e^-}.$$

Combining the above two estimates, we see from (4.6) the first part of $E_h(\mathbf{u},\mathbf{v})$ satisfies the lemma. The proof for the second part is similar. □

Applying Lemmas 4.1 and 4.3, Theorems 3.1 and 3.2, we have the following uniform error estimates for our new elements.

Theorem 4.1. *Let* $(\mathbf{u}_h, p_h) \in V_h^{(i)} \times W_h^{(i)}, i = 1,2,$ *be the solution of the problem* (2.7). *If* $\mathbf{f} \in H(\mathrm{rot})$, $g \in H^1_+$ *and condition* (4.5) *holds, then*

$$|||\mathbf{u}-\mathbf{u_h}|||_h + \|p-p_h\|_0 \lesssim h^{\frac{1}{2}}\big(\|\mathrm{rot}\,\mathbf{f}\|_0 + \|g\|_{1,+}\big). \tag{4.8}$$

5. Vanishing Viscosity

In what follows we will show that Theorems 3.1, 3.2 and 4.1 are also valid in the case of $\epsilon = 0$, where the velocity space V is taken as

$$V = H_0(\mathrm{div},\Omega) := \big\{\mathbf{v}\in H(\mathrm{div},\Omega)|\ \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\big\}.$$

The finite dimensional space $V_h \subset V$, the norms on V and V_h reduce to

$$|||\mathbf{v}|||^2 = (\mathrm{div}\mathbf{v}, \mathrm{div}\mathbf{v}) + (\mathbf{v},\mathbf{v}), \qquad \mathbf{v}\in V, \tag{5.1}$$

$$|||\mathbf{v}|||_h = |||\mathbf{v}|||, \quad \|\mathbf{v}\|_{a_h} = \|\mathbf{v}\|_0, \quad \mathbf{v}\in V_h, \tag{5.2}$$

and the boundary condition (1.3) becomes $\mathbf{u}\cdot\mathbf{n} = 0$.

With the above modifications, by a similar argument to that in the case of $\epsilon \in (0,1]$ in Section 2, we easily know that the two stability conditions (2.4) and (2.5) trivially hold in the case $\epsilon = 0$. In addition, we can see that Lemma 2.1 also holds. As a result, we have the following simplified version of Theorem 2.1.

Theorem 5.1. *Let* $(\mathbf{u}_h, p_h) \in V_h^{(i)} \times W_h^{(i)}$ $(i = 1,2)$ *be the solution of the problem* (2.7) *in the case of* $\epsilon = 0$. *Assume* (H1) *and* (H2) *are fulfilled. Then we have*

$$\begin{aligned} |||\mathbf{u}-\mathbf{u}_h|||_h &\le 2\|\mathbf{u}-\Pi_h\mathbf{u}\|_0 + \|(I-Q_h)\mathrm{div}\mathbf{u}\|_0,\\ \|p-p_h\|_0 &\le \|p-Q_hp\|_0 + 2\|\mathbf{u}-\Pi_h\mathbf{u}\|_0 + \|(I-Q_h)\mathrm{div}\mathbf{u}\|_0,\\ \|\mathrm{div}(\mathbf{u}-\mathbf{u}_h)\|_0 &= \|(I-Q_h)\mathrm{div}\mathbf{u}\|_0. \end{aligned}$$

Thus, in the case of $\epsilon = 0$ the same error estimates as in Theorems 3.1 and 3.2 also hold for the two elements.

6. Construction of 3D Nonconforming Rectangular Elements

In this Section we will construct 3D finite elements based on the assumptions (H1)-(H5).

6.1. A low order finite element space

For each 3-rectangle $T \in T_h$, we introduce the similar affine invertible transformation as

$$F_K : \widehat{K} \to K, \quad x = \frac{1}{2}h_x\xi + x_0, \quad y = \frac{1}{2}h_y\eta + y_0, \quad z = \frac{1}{2}h_z\zeta + z_0,$$

with the center (x_0, y_0, z_0), the three axes edge lengthes h_x, h_y and h_z, respectively, and the reference element $\widehat{K} = [-1, 1]^3$.

The velocity polynomial space on the 3-rectangle T is defined by

$$V_T^{(3)} = \left\{ \begin{array}{ll} \mathbf{v} = (v_1, v_2, v_3)^T : & v_1 \in span\{1, x, y, z, y^2, z^2\}, \\ & v_2 \in span\{1, x, y, z, x^2, z^2\}, \\ & v_3 \in span\{1, x, y, z, x^2, y^2\} \end{array} \right\}.$$

The dimension of $V_T^{(3)}$ is 18 and the 18 degrees of freedom are given below

$$\int_f \mathbf{v} \cdot \mathbf{n}\, ds, \quad \forall f \in F(T), \tag{6.1}$$

$$\int_f (\mathbf{v} \times \mathbf{n}) \cdot \mathbf{q}\, ds, \quad \mathbf{q} \in P_0^2(f), \quad \forall f \in F(T). \tag{6.2}$$

Lemma 6.1. *For any $\mathbf{v} \in V_T^{(3)}$, it is uniquely determined by the above 18 degrees of freedom.*

Proof. Let all degrees of freedom for $\mathbf{v} = (v_1, v_2, v_3)^T \in V_T^{(3)}$ be equal to zero, then it is enough to show $\mathbf{v} = \mathbf{0}$. Without losing the generality, we prove it on the reference element $\widehat{K}$. In this case, v_1 can be written as

$$v_1 = a_0 + a_1\xi + a_2\eta + a_3\zeta + a_4\eta^2 + a_5\zeta^2,$$

with some interpolation constants $a_0, a_1, a_2, a_3, a_4, a_5$. By taking the above formulation in the related 6 of the 18 degrees, we can get

$$\begin{aligned} &a_0 + a_1 + \frac{1}{3}a_4 + \frac{1}{3}a_5 = 0, \quad && a_0 - a_1 + \frac{1}{3}a_4 + \frac{1}{3}a_5 = 0, \\ &a_0 - a_2 + a_4 + \frac{1}{3}a_5 = 0, && a_0 + a_2 + a_4 + \frac{1}{3}a_4 = 0, \\ &a_0 + a_3 + \frac{1}{3}a_4 + a_5 = 0, && a_0 - a_3 + \frac{1}{3}a_4 + a_5 = 0. \end{aligned}$$

From the above equations we can easily derive that $a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = 0$ and then $v_1 = 0$. Owing to the symmetry, similar calculation gives $v_2 = 0, v_3 = 0$ and then $\mathbf{v} = \mathbf{0}$. □

The finite element spaces $V_h^{(3)}$ and $W_h^{(3)}$ are defined by

$$\begin{aligned} V_h^{(3)} &:= \left\{ \begin{array}{l} \mathbf{v} \in L^2(\Omega)^3 : \mathbf{v}|_T \in V_T^{(3)}, \quad \forall T \in T_h, \\ \text{all the degrees on the boundary of } \Omega \text{ are equal to zero,} \\ \text{the moments (6.1) and (6.2) are continuous across mesh edges} \end{array} \right\}, \\ W_h^{(3)} &:= \{q \in W : \ q|_T \in P_0(T), \ \forall T \in T_h\}. \end{aligned}$$

It can be verified that $V_h^{(3)} \not\subset H(\mathrm{div},\Omega)$ and so $V_h^{(3)} \not\subset H_0^1(\Omega)$. The assumptions in (H1) are satisfied by construction. The interpolation operator $\Pi_h : V \to V_h^{(3)}$ is defined by

$$\int_f (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \mathbf{q}\, ds = 0, \quad \forall\, \mathbf{q} \in P_0^3(f), \quad \forall\, f \in F(T).$$

For any $\mathbf{v} \in V$ and $p \in P_0(T)$, using Green's formula, we have

$$\begin{aligned}\int_T \mathrm{div}\Pi_h \mathbf{v} p\, d\Omega &= -\int_T \Pi_h \mathbf{v} \cdot \nabla p\; d\Omega + \int_{\partial T} \Pi_h \mathbf{v} \cdot \mathbf{n} p\; ds \\ &= \int_{\partial T} \Pi_h \mathbf{v} \cdot \mathbf{n} p\; ds = \int_{\partial T} \mathbf{v} \cdot \mathbf{n} p\; ds \\ &= -\int_T \mathbf{v} \cdot \nabla p\; d\Omega + \int_{\partial T} \mathbf{v} \cdot \mathbf{n} p\; ds \\ &= \int_T \mathrm{div}\mathbf{v} p\; d\Omega = \int_T Q_h \mathrm{div}\mathbf{v} p\; d\Omega.\end{aligned}$$

Taking $p = \mathrm{div}\Pi_h \mathbf{v} - Q_h \mathrm{div}\mathbf{v}$, we get $\mathrm{div}_h \Pi_h \mathbf{v} = Q_h \mathrm{div}\mathbf{v}$ for all $T \in T_h$. This verifies the assumption (H2). Similar to (3.3) and (3.4) in Section 3.1, in 3D cases the assumptions (H3) and (H4) hold as

$$\begin{aligned}&\|\Pi_h \mathbf{v}\|_{1,h} \lesssim \|\mathbf{v}\|_1, \\ &\|v - \Pi_h \mathbf{v}\|_{j,h} \lesssim h^{2-j} |\mathbf{v}|_2, \quad j = 0, 1.\end{aligned}$$

In 3D, the consistency error term is

$$E_h(\mathbf{u}, p, \mathbf{v}) = \epsilon^2 \sum_{f \in F(T_h)} \int_f (\nabla \mathbf{u}\mathbf{n}) \cdot [\mathbf{v}]\, ds + \sum_{f \in F(T_h)} \int_f p[\mathbf{v} \cdot \mathbf{n}]\, ds. \tag{6.3}$$

By decomposing vectors $\nabla \mathbf{u}\mathbf{n}$ and $\mathbf{v}$ on the face f into their normal and tangential components, i.e.,

$$\nabla \mathbf{u}\mathbf{n} = (\nabla \mathbf{u}\mathbf{n} \times \mathbf{n}) + (\nabla \mathbf{u}\mathbf{n} \cdot \mathbf{n})\mathbf{n}, \qquad \mathbf{v} = (\mathbf{v} \times \mathbf{n}) + (\mathbf{v} \cdot \mathbf{n})\mathbf{n}.$$

Then we can further write the consistency error term as

$$\begin{aligned}&E_h(\mathbf{u}, p, \mathbf{v}) \\ =\;& \epsilon^2 \sum_{f \in F(T_h)} \Big(\int_f (\nabla \mathbf{u}\mathbf{n} \cdot \mathbf{n})[\mathbf{v} \cdot \mathbf{n}]\, ds + \int_f (\nabla \mathbf{u}\mathbf{n} \times \mathbf{n}) \cdot [\mathbf{v} \times \mathbf{n}]\, ds \Big) + \sum_{f \in F(T_h)} \int_f p[\mathbf{v} \cdot \mathbf{n}]\, ds.\end{aligned}$$

Similar to the discussion in Section 3.1, we can derive the following estimates

$$E_h(\mathbf{u}, p, \mathbf{v}) \lesssim \big(h\epsilon |\mathbf{u}|_3 + h^2 |p|_3\big) \|\mathbf{v}\|_{a_h}. \tag{6.4}$$

Moreover, if $\mathbf{u} \in H^{m+1}$, $m = 1$, we have

$$\|\mathrm{div}(\mathbf{u} - \mathbf{u}_h)\|_0 \lesssim h^m |\mathrm{div}\mathbf{u}|_m.$$

Then we can immediately get the following theorem.

Theorem 6.1. *Let* $(\mathbf{u}_h, p_h) \in V_h^{(3)} \times W_h^{(3)}$ *be the solution of the problem* (2.7). *If* $\mathbf{u} \in H_0^1 \cap H^3$ *and* $p \in L_0^2 \cap H^k, k = 2, 3$, *then the following error estimates hold:*

$$\begin{aligned}|||\mathbf{u} - \mathbf{u}_h|||_h &\lesssim h(h + \epsilon)|\mathbf{u}|_3 + h^{k-1}|p|_k, \\ \|\mathrm{div}\mathbf{u} - \mathrm{div}_h \mathbf{u}_h\|_0 &\lesssim h|\mathrm{div}\mathbf{u}|_1, \\ \|p - p_h\|_0 &\lesssim h\big(|p|_1 + (\epsilon + h)|\mathbf{u}|_3 + h^{k-2}|p|_k\big).\end{aligned}$$

6.2. A higher order element

Define

$$V_T^{(4)} = \left\{ \begin{array}{ll} \mathbf{v} = (v_1, v_2, v_3)^T : & v_1 \in span\{1, x, y, z, x^2, y^2, z^2, xy, xz, y^3, z^3\} \\ & v_2 \in span\{1, x, y, z, x^2, y^2, z^2, xy, yz, x^3, z^3\} \\ & v_3 \in span\{1, x, y, z, x^2, y^2, z^2, xz, yz, x^3, y^3\} \end{array} \right\}.$$

The dimension of $V_T^{(4)}$ is 33 and we give the following 33 degrees of freedom as

$$\int_f \mathbf{v} \cdot \mathbf{n} q \, ds, \qquad q \in P_1(f), \tag{6.5}$$

$$\int_f (\mathbf{v} \times \mathbf{n}) \cdot \mathbf{q} \, ds, \quad \mathbf{q} \in P_0^2(f), \tag{6.6}$$

$$\int_T \mathbf{v} \cdot \mathbf{q} \, d\Omega, \qquad \mathbf{q} \in P_0^3(T). \tag{6.7}$$

Let us define the following finite element spaces

$$\begin{aligned} V_h^{(4)} &:= \left\{ \begin{array}{l} \mathbf{v} \in L^2(\Omega)^3 : \ \mathbf{v}|_T \in V_T^{(4)}, \quad \forall T \in T_h, \\ \text{all the degrees on the boundary of } \Omega \text{ are equal to zero,} \\ \text{the moments (6.5) and (6.6) are continuous across mesh edges} \end{array} \right\}, \\ W_h^{(4)} &:= \ \left\{ q \in W : \ q|_T \in P_1(T), \ \forall T \in T_h \right\}. \end{aligned}$$

Following some discussions similar to those in Section 6.1, we can show that this element is unisolvent and satisfies the assumptions (H1) to (H5) for some suitable integers. So Theorem 2.2 holds for this element and we have the following error estimate.

Theorem 6.2. *Let* $(\mathbf{u}_h, p_h) \in V_h^{(4)} \times W_h^{(4)}$ *be the solution of the problem* (2.7). *If* $\mathbf{u} \in H_0^1 \cap H^4$ *and* $p \in L_0^2 \cap H^k, k = 3, 4$*, then the following error estimates hold:*

$$\begin{aligned} |||\mathbf{u} - \mathbf{u}_h|||_h &\lesssim h^2(h + \epsilon)|\mathbf{u}|_4 + h^{k-1}|p|_k, \\ \|\text{div}\mathbf{u} - \text{div}_h \mathbf{u}_h)\|_0 &\lesssim h^m |\text{div}\mathbf{u}|_m, m = 1, 2, \\ \|p - p_h\|_0 &\lesssim h^2\big(|p|_1 + (\epsilon + h)|\mathbf{u}|_4 + h^{k-3}|p|_k\big). \end{aligned}$$

6.3. Uniform estimates and vanishing viscosity

Similar to the 2D cases, by applying the regularity results by Tai and Winther [26] and assuming the source term $g = 0$, we can derive the following uniform error estimates.

Theorem 6.3. *Let* $(\mathbf{u}_h, p_h) \in V_h^{(i)} \times W_h^{(i)}, i = 3, 4,$ *be the solution of the problem* (2.7). *If* $\mathbf{f} \in H^1$*, then*

$$|||\mathbf{u} - \mathbf{u}_h|||_h + \|p - p_h\|_0 \lesssim h^{\frac{1}{2}} \|\mathbf{f}\|_1.$$

For the vanishing viscosity $\epsilon = 0$, similar to Theorem 5.1, we have

Theorem 6.4. *Let* $(\mathbf{u}_h, p_h) \in V_h^{(i)} \times W_h^{(i)}, i = 3, 4,$ *be the solution of the problem* (2.7) *in the case of* $\epsilon = 0$. *If* (H1) *and* (H2) *are fulfilled, then*

$$\begin{aligned} |||\mathbf{u} - \mathbf{u}_h|||_h &\le 2\|\mathbf{u} - \Pi_h \mathbf{u}\|_0 + \|(I - Q_h)\text{div}\mathbf{u}\|_0, \\ \|p - p_h\|_0 &\le \|p - Q_h p\|_0 + 2\|\mathbf{u} - \Pi_h \mathbf{u}\|_0 + \|(I - Q_h)\text{div}\mathbf{u}\|_0, \\ \|\text{div}(\mathbf{u} - \mathbf{u}_h)\|_0 &= \|(I - Q_h)\text{div}\mathbf{u}\|_0. \end{aligned}$$

7. Numerical Tests

In this section, we give some 2D numerical tests to verify the theoretical analysis.

7.1. An example without boundary layers

We consider the problem (1.1)-(1.3) with Ω taken as the unit square. The domain is triangulated into $n \times n$ squares. The exact solution $p = \sin(\pi x) - 2/\pi$ and $\mathbf{u} = \mathbf{curl}\sin^2(\pi x)\sin^2(\pi y)$, while $\mathbf{f} = \mathbf{u} - \epsilon^2\Delta\mathbf{u} - \mathbf{grad}p$. So the function $g = 0$ and the solution is independent of the perturbation parameter ϵ. This is the same example as used in [22].

In Tables 7.1 and 7.2, we give the computational results of the new low order element for the absolute error for velocity measured in L^2 norm and energy norm for different values of ϵ and $h = 1/n$, respectively. In Table 7.3 we give the results of the absolute error for pressure measured in L^2 norm. The corresponding results for the higher order element are listed in Tables 7.4-7.6. For each element and each fixed ϵ the convergence rate with respect to h is estimated by computing the average of the usual log-linear relation. From the tables we can see that the computational results are in good agreement with our theoretical analysis.

Table 7.1: The absolute error for velocity in L^2 norm for the new low order element.

$h \setminus \epsilon$	1	2^{-2}	2^{-4}	2^{-8}	2^{-10}	0
2^{-2}	3.12e-1	3.04e-1	2.92e-1	2.91e-1	2.91e-1	2.86e-1
2^{-3}	8.40e-2	8.06e-2	7.52e-2	7.44e-2	7.44e-2	7.39e-2
2^{-4}	2.14e-2	2.05e-2	1.89e-2	1.86e-2	1.86e-2	1.86e-2
rate	1.93	1.95	1.97	1.98	1.98	1.97

Table 7.2: The absolute error for velocity in energy norm for the new low order element.

$h \setminus \epsilon$	1	2^{-2}	2^{-4}	2^{-8}	2^{-10}	0
2^{-2}	5.47	1.39	4.47e-1	2.91e-1	2.91e-1	2.86e-1
2^{-3}	2.74	6.89e-1	1.87e-1	7.52e-2	7.45e-2	7.39e-2
2^{-4}	1.37	3.43e-1	8.76e-2	1.94e-2	1.87e-2	1.86e-2
rate	1.00	1.01	1.18	1.95	1.98	1.97

Table 7.3: The absolute error for pressure in L^2 norm for the new low order element.

$h \setminus \epsilon$	1	2^{-2}	2^{-4}	2^{-8}	2^{-10}	0
2^{-2}	9.15e-1	1.72e-1	1.60e-1	1.59e-1	1.59e-1	1.59e-1
2^{-3}	3.59e-1	8.41e-2	8.01e-2	8.00e-2	8.00e-2	8.00e-2
2^{-4}	1.04e-1	4.07e-2	4.01e-2	4.01e-2	4.01e-2	4.01e-2
rate	1.57	1.04	0.99	0.99	0.99	0.99

Table 7.4: The absolute error for velocity in L^2 norm for the new higher order element.

$h \setminus \epsilon$	1	2^{-2}	2^{-4}	2^{-8}	2^{-10}	0
2^{-2}	1.13e-1	1.12e-1	1.07e-1	1.04e-1	1.04e-1	1.02e-1
2^{-3}	1.17e-2	1.16e-2	1.13e-2	1.09e-2	1.09e-2	1.08e-2
2^{-4}	1.30e-3	1.30e-3	1.30e-3	1.20e-3	1.20e-3	1.20e-3
rate	3.22	3.21	3.18	3.22	3.22	3.20

Table 7.5: The absolute error for velocity in energy norm for the new higher order element.

$h \setminus \epsilon$	1	2^{-2}	2^{-4}	2^{-8}	2^{-10}	0
2^{-2}	2.71	6.85e-1	1.98e-1	1.05e-1	1.04e-1	1.02e-1
2^{-3}	6.62e-1	1.66e-1	4.27e-2	1.12e-2	1.09e-2	1.08e-2
2^{-4}	1.58e-1	3.94e-2	9.90e-3	1.40e-3	1.20e-3	1.20e-3
rate	2.05	2.06	2.16	3.11	3.22	3.20

Table 7.6: The absolute error for pressure in L^2 norm for the new higher order element.

$h \setminus \epsilon$	1	2^{-2}	2^{-4}	2^{-8}	2^{-10}	0
2^{-2}	1.01	6.70e-2	1.72e-2	1.63e-2	1.63e-2	1.63e-2
2^{-3}	1.87e-1	1.24e-2	4.10e-3	4.10e-3	4.10e-3	4.10e-3
2^{-4}	2.51e-2	1.90e-3	1.01e-3	1.01e-3	1.01e-3	1.01e-3
rate	2.66	2.57	2.05	2.01	2.01	2.01

Table 7.7: The absolute error for velocity in L^2 norm for the new low order element for case 1.

$h \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
2^{-1}	5.67e-2	7.46e-2	1.46e-1	1.93e-1	1.98e-1	1.99e-1
2^{-2}	1.65e-2	2.60e-2	7.75e-2	1.28e-1	1.42e-1	1.43e-1
2^{-3}	4.30e-3	7.71e-3	3.20e-2	7.46e-2	9.88e-2	1.01e-1
2^{-4}	1.12e-3	2.12e-3	9.13e-3	3.85e-2	6.39e-2	7.09e-2
rate	1.90	1.72	1.33	0.78	0.54	0.50

Table 7.8: The absolute error for velocity in energy norm for the new low order element for case 1.

$h \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
2^{-1}	1.40e-1	1.04e-1	1.50e-1	1.94e-1	1.98e-1	1.99e-1
2^{-2}	7.02e-2	4.79e-2	8.81e-2	1.29e-1	1.42e-1	1.43e-1
2^{-3}	3.50e-2	2.05e-2	4.47e-2	7.67e-2	9.89e-2	1.01e-1
2^{-4}	1.74e-2	9.20e-3	1.84e-2	4.37e-2	6.41e-2	7.09e-2
rate	1.00	1.17	1.01	0.72	0.54	0.50

7.2. Examples with boundary layers

In this section we give two examples to verify the theoretical analysis for boundary layers. Let $\mathbf{u} = \epsilon\mathbf{curl}e^{-xy/\epsilon}$ and then $g = 0$. In fact, $\mathbf{u}$ is not the solution of the corresponding system (1.1)-(1.3) because the boundary conditions are not satisfied. However, the adaptation of our new methods to nonhomogeneous boundary conditions is straightforward.

The significance of the solution $\mathbf{u}$ just given is related to the fact that the quantities of $\|\mathrm{rot}\mathbf{u}\|_0$ and $\epsilon\|\mathrm{rot}\mathbf{u}\|_1$ are both of order $\epsilon^{-1/2}$ as ϵ tends to zero. As we have seen in the former analysis, this behavior is typical for solutions of singular perturbation problem (1.1)-(1.3).

We select p in the following two cases:

$$\begin{aligned}&\text{case 1}: p = \epsilon e^{-x/\epsilon} + \epsilon^2(e^{-\frac{1}{\epsilon}} - 1),\\ &\text{case 2}: p = \epsilon e^{-(x+y)/\epsilon} - \epsilon^3(e^{-\frac{1}{\epsilon}} - 1)^2,\end{aligned}$$

of which the former is the same as that in [19] and satisfies (4.5), while the other one does not satisfy it.

The computational results for case 1 are listed in Table 7.7-7.12. we can see that they are in good agreement with our theoretical analysis as the condition (4.5) holds. In Table 7.13-7.18, we give the computational results for case 2. Although condition (4.5) is not satisfied, the results are similar to the former as p can be approximated well by functions in the space $\widetilde{S}$ in Lemma 4.

Table 7.9: The absolute error for pressure in L^2 norm for the new low order element for case 1.

$h \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
2^{-1}	4.70e-2	1.22e-2	8.30e-3	9.91e-3	1.07e-2	1.40e-2
2^{-2}	2.51e-2	8.20e-3	3.41e-3	5.50e-3	6.40e-3	6.94e-3
2^{-3}	1.28e-2	5.41e-3	1.51e-3	2.41e-3	3.31e-3	3.61e-3
2^{-4}	6.40e-3	3.00e-3	1.01e-3	8.60e-4	1.52e-3	1.80e-3
$rate$	0.96	0.68	1.02	1.18	0.95	0.97

Table 7.10: The absolute error for velocity in L^2 norm for the new higher order element for case 1.

$h \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
2^{-1}	1.53e-2	3.12e-2	9.05e-2	1.46e-1	1.53e-1	1.53e-1
2^{-2}	2.20e-3	8.40e-3	3.64e-2	9.10e-2	1.07e-1	1.08e-1
2^{-3}	2.79e-4	1.61e-3	1.04e-2	4.54e-2	7.29e-2	7.54e-2
2^{-4}	3.43e-5	2.31e-4	1.92e-3	1.76e-2	4.47e-2	5.25e-2
$rate$	2.93	2.36	1.86	1.02	0.59	0.52

Table 7.11: The absolute error for velocity in energy norm for the new higher order element for case 1.

$h \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
2^{-1}	5.96e-2	5.59e-2	9.43e-2	1.46e-1	1.53e-1	1.53e-1
2^{-2}	1.70e-2	2.15e-2	4.50e-2	9.16e-2	1.07e-1	1.08e-1
2^{-3}	4.41e-3	6.80e-3	1.82e-2	4.70e-2	7.31e-2	7.55e-2
2^{-4}	1.10e-3	1.90e-3	5.21e-3	2.17e-2	4.50e-2	5.26e-2
$rate$	1.92	1.63	1.39	0.92	0.59	0.51

Table 7.12: The absolute error for pressure in L^2 norm for the new higher order element for case 1.

$h \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
2^{-1}	3.08e-2	7.71e-3	5.12e-3	7.70e-3	8.90e-3	9.20e-3
2^{-2}	1.54e-2	4.70e-3	1.61e-3	3.32e-3	4.31e-3	4.60e-3
2^{-3}	7.70e-3	2.30e-3	8.51e-4	1.13e-3	1.82e-3	2.12e-3
2^{-4}	3.81e-3	1.02e-3	4.80e-4	2.93e-4	7.39e-4	9.59e-4
$rate$	1.01	0.98	1.14	1.57	1.20	1.09

Table 7.13: The absolute error for velocity in L^2 norm for the new low order element for case 2.

$h \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
2^{-1}	5.68e-2	7.46e-2	1.46e-1	1.93e-1	1.98e-1	1.99e-1
2^{-2}	1.65e-2	2.61e-2	7.75e-2	1.28e-1	1.42e-1	1.43e-1
2^{-3}	4.30e-3	7.80e-3	3.20e-2	7.46e-2	9.88e-2	1.01e-1
2^{-4}	1.11e-3	2.12e-3	9.13e-3	3.85e-2	6.39e-2	7.09e-2
$rate$	1.90	1.72	1.33	0.78	0.54	0.50

Table 7.14: The absolute error for velocity in energy norm for the new low order element for case 2.

$h \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
2^{-1}	1.40e-1	1.04e-1	1.50e-1	1.94e-1	1.98e-1	1.99e-1
2^{-2}	7.02e-2	4.81e-2	8.81e-2	1.29e-1	1.42e-1	1.43e-1
2^{-3}	3.50e-2	2.07e-2	4.47e-2	7.67e-2	9.89e-2	1.01e-1
2^{-4}	1.74e-2	9.30e-3	1.84e-2	4.37e-2	6.41e-2	7.09e-2
$rate$	1.00	1.16	1.01	0.72	0.54	0.50

Table 7.15: The absolute error for pressure in L^2 norm for the new low order element for case 2.

$h \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
2^{-1}	2.79e-2	7.90e-3	8.20e-3	9.91e-3	1.07e-2	1.40e-2
2^{-2}	1.36e-2	2.61e-3	3.21e-3	5.50e-3	6.40e-3	6.94e-3
2^{-3}	6.60e-3	1.42e-3	8.93e-4	2.41e-3	3.31e-3	3.61e-3
2^{-4}	3.20e-3	7.64e-4	2.05e-4	8.48e-4	1.52e-3	1.80e-3
$rate$	1.04	1.12	1.77	1.18	0.95	0.97

Table 7.16: The absolute error for velocity in L^2 norm for the new higher order element for case 2.

$h \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
2^{-1}	1.53e-2	3.12e-2	9.05e-2	1.46e-1	1.53e-1	1.53e-1
2^{-2}	2.20e-3	8.40e-3	3.64e-2	9.10e-2	1.07e-1	1.08e-1
2^{-3}	2.79e-4	1.63e-3	1.04e-2	4.54e-2	7.29e-2	7.54e-2
2^{-4}	3.43e-5	2.32e-4	1.92e-3	1.76e-2	4.47e-2	5.25e-2
$rate$	2.93	2.36	1.86	1.02	0.59	0.52

Table 7.17: The absolute error for velocity in energy norm for the new higher order element for case 2.

$h \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
2^{-1}	5.96e-2	5.59e-2	9.43e-2	1.46e-1	1.53e-1	1.53e-1
2^{-2}	1.70e-2	2.15e-2	4.50e-2	9.16e-2	1.07e-1	1.08e-1
2^{-3}	4.41e-3	6.80e-3	1.82e-2	4.70e-2	7.31e-2	7.55e-2
2^{-4}	1.10e-3	1.90e-3	5.21e-3	2.17e-2	4.50e-2	5.26e-2
$rate$	1.92	1.63	1.39	0.92	0.59	0.51

Table 7.18: The absolute error for pressure in L^2 norm for the new higher order element for case 2.

$h \setminus \epsilon$	2^{-2}	2^{-4}	2^{-6}	2^{-8}	2^{-10}	2^{-12}
2^{-1}	1.26e-2	2.31e-3	4.92e-3	7.70e-3	8.90e-3	9.20e-3
2^{-2}	4.70e-3	1.30e-3	1.30e-3	3.32e-3	4.31e-3	4.60e-3
2^{-3}	1.90e-3	5.58e-4	2.28e-4	1.13e-3	1.82e-3	2.12e-3
2^{-4}	8.96e-4	1.76e-4	7.90e-5	2.68e-4	7.38e-4	9.59e-4
$rate$	1.27	1.24	1.98	1.61	1.20	1.09

8. Conclusions

In this article, we have proposed two lower order 2D rectangular elements which are uniformly stable for the Darcy-Stokes problem. We have also suggested two 3D uniformly stable rectangular elements. However, it seems not easy to extend these elements to arbitrary quadrilaterals and hexagons. We will consider this issue in future work.

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