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Review Article

FINITE ELEMENTS WITH LOCAL PROJECTION STABILIZATION FOR INCOMPRESSIBLE FLOW PROBLEMS*

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Abstract

In this paper we review recent developments in the analysis of finite element methods for incompressible flow problems with local projection stabilization (LPS). These methods preserve the favourable stability and approximation properties of classical residual-based stabilization (RBS) techniques but avoid the strong coupling of velocity and pressure in the stabilization terms. LPS-methods belong to the class of symmetric stabilization techniques and may be characterized as variational multiscale methods. In this work we summarize the most important a priori estimates of this class of stabilization schemes developed in the past 6 years. We consider the Stokes equations, the Oseen linearization and the Navier-Stokes equations. Furthermore, we apply it to optimal control problems with linear(ized) flow problems, since the symmetry of the stabilization leads to the nice feature that the operations "discretize" and "optimize" commute.

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1. Introduction

Among the most important points for the computation of flow problems is the choice of the underlying discretization in space and time. An appropriate discretization should be accurate and efficient (in terms of numerical costs). The accuracy is closely linked to their approximation and stability properties. Specifically, a finite element discretization for the Navier-Stokes equation has to deal with the stiff pressure-velocity coupling and with the advective terms for flows at higher Reynolds number.

Well established methods for steady problems are the Galerkin Least-Squares (GLS) techniques, where certain stabilization terms are added to the corresponding variational formulation. The basic idea is that those extra terms vanish for the exact (strong) solution, since they involve the strong residual. Due to this feature those methods are called *consistent*. The technique of streamline diffusion, see Johnson [1], to stabilize advective terms can also be accounted to these *residual-based* stabilization techniques.

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A drawback of the residual-based methods is that they introduce a large amount of additional couplings between the variables. This becomes troublesome especially for complex flows with compressible features or with additional variables as in reactive flow problems. Furthermore, an extension to non-steady problems is problematic since space-time finite elements have to be used. For a discussion of such problems, we refer to [2].

In this work we give a review on the *local projection stabilization* (or short LPS) which recently attracted the attention in numerical analysis. We collect the most important results for this kind of finite element stabilization and present them in an integrated framework. We mainly concentrate on the a priori results for pairs of finite elements of possibly different order for pressure and velocity. The principle idea of this symmetric stabilization technique is to project parts of the residual in order to obtain a stable scheme. Since not the full residual enters in the stabilization, the (strong) consistency is sacrificed. However, the method has a certain *weak consistency* property as explained later. As a consequence, this class of stabilization techniques is still of optimal order. Moreover, one does not need to resort to space-time finite elements for time stepping in order to stay consistent, but can apply any A-stable higher-order finite difference scheme for the discretization in time.

An outline of this paper is as follows: In Section 2, we introduce some notation and summarize basic material on the underlying finite element spaces. In Section 3, we consider the Stokes problem for the description of a viscous fluid in the domain $\Omega \subset \mathbb{R}^d$ with homogeneous boundary conditions for the velocity (no-slip):

$$-\Delta \mathbf{v} + \nabla p = f \quad \text{in } \Omega\,, \tag{1.1}$$
$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega\,, \tag{1.2}$$
$$\mathbf{v} = 0 \quad \text{on } \partial\Omega\,. \tag{1.3}$$

Here, $p : \Omega \to \mathbb{R}$ denotes the pressure and $\vec{u} : \Omega \to \mathbb{R}^d$ the velocity field. Basic ideas of the LPS approach will be explained for equal-order finite element pairs for velocity and pressure which do not pass the well-known discrete compatibility (or inf-sup stability) condition by Babuska-Brezzi.

In Section 4, we extend the approach to the generalized Oseen system

$$-\nu\Delta \mathbf{v} + (\mathbf{b}\cdot\nabla)\mathbf{v} + \sigma\mathbf{v} + \nabla p = f \quad \text{in } \Omega\,, \tag{1.4}$$
$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega\,, \tag{1.5}$$
$$\mathbf{v} = 0 \quad \text{on } \partial\Omega\,. \tag{1.6}$$

with constant parameters $\nu > 0$ and $\sigma \in \mathbb{R}^+$ and a given flow field $\mathbf{b} : \overline{\Omega} \to \mathbb{R}^d$. A unified analytical framework for the LPS method includes as well equal-order pairs for velocity and pressure as pairs for which the Babuska-Brezzi stability condition is valid. Moreover, the approach covers the so-called one-level and two-level variants of the LPS technique.

The Oseen system typically appears within the solution of the Navier-Stokes problem

$$\partial_t \mathbf{v} - \nu\Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = f \quad \text{in } \Omega\,, \tag{1.7}$$
$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega\,, \tag{1.8}$$
$$\mathbf{v} = 0 \quad \text{on } \partial\Omega\,. \tag{1.9}$$

as auxiliary problem if an A-stable implicit time discretization is applied first. This will be considered in Section 5 as a reasonable approach to laminar flows. Moreover, the link of the

LPS method to the concept of variational multiscale methods as a potential approach to large eddy simulation in turbulent flows will be addressed there.

Finally, in Section 6 we discuss the application of the LPS method to optimal control problems for the generalized Oseen problem. The symmetric form of the stabilization guarantees the favourable commutation property between the operations "discretize" and "optimize".

2. Notations and Preliminaries

Throughout this work, $\Omega \subset \mathbb{R}^d,\ d \in \{2,3\}$ will be a bounded, simply connected, polyhedral domain. Standard notations for Lebesgue and Sobolev spaces are used, e.g. $L^2(\Omega)$ for the space of square-integrable generalized functions over Ω with the inner product $(\cdot,\cdot)$, $H^r(\Omega)$ for the Sobolev space of order r, and $H^1_0(\Omega)$ for the Sobolev space with vanishing traces on $\partial\Omega$. The $L^2(\omega)$ inner product in $\omega \subset \Omega$ is denoted by $(\cdot,\cdot)_\omega$, and the corresponding norm by $\|\cdot\|_{0,\omega}$. The norm and seminorm on $H^l(\omega)$ will be denoted by $\|\cdot\|_{l,\omega}$ and $|\cdot|_{l,\omega}$. In case of $\omega = \Omega$, we usually omit the index Ω. In order to pronounce the vector-valued case, we write $\|\cdot\|_{[L^2(\omega)]^d}$ and $\|\cdot\|_{[H^l(\omega)]^d}$ for the norms in $[L^2(\omega)]^d$ and in $[H^l(\omega)]^d$, respectively. The natural space for the pressure p is

$$Q := L^2_0(\Omega) := \left\{ q \in L^2(\Omega) \;\middle|\; \int_\Omega q\,dx = 0 \right\}.$$

We denote the velocity field by $\mathbf{v} \in V := [H^1_0(\Omega)]^d$. Both variables are sampled together in $\mathbf{u} = (\mathbf{v}, p) \in \mathbf{U} := \mathbf{V} \times Q$.

Let $\mathcal{T}_h$ be a shape-regular, admissible decomposition of Ω into d-dimensional simplices or quadrilaterals for $d = 2$ or hexahedra for $d = 3$. h_T is the diameter of a cell $T \in \mathcal{T}_h$ and $h = \max\{h_T : T \in \mathcal{T}_h\}$. Let $\hat{T}$ be a reference element of $\mathcal{T}_h$, $\mathbb{P}_k(\hat{T})$ the space of complete polynomials of degree k on $\hat{T}$, and $\mathbb{Q}_k(\hat{T})$ the space of all polynomials on $\hat{T}$ with maximal degree k in each coordinate direction.

We will use the discontinuous finite element spaces on $\mathcal{T}_h$ $(k \in \mathbb{N})$

$$\begin{aligned} P^{dc}_{\mathcal{T}_h,k} &:= \{v_h \in L^2(\Omega) \mid v_h|_T \circ F_T \in \mathbb{P}_k(\hat{T}), T \in \mathcal{T}_h\}, \\ Q^{dc}_{\mathcal{T}_h,k} &:= \{v_h \in L^2(\Omega) \mid v_h|_T \circ F_T \in \mathbb{Q}_k(\hat{T}), T \in \mathcal{T}_h\}\,, \end{aligned}$$

as well as the corresponding continuous conforming finite element spaces

$$P_{\mathcal{T}_h,k} := P^{dc}_{\mathcal{T}_h,k} \cap C(\overline{\Omega})\,, \quad Q_{\mathcal{T}_h,k} := Q^{dc}_{\mathcal{T}_h,k} \cap C(\overline{\Omega})\,.$$

These discontinuous spaces are mainly used for the definition of the stabilized scheme, whereas the discrete solution is sought in the conforming finite element spaces $(s, r \in \mathbb{N})$:

$$P_k(\mathcal{T}_h) := \begin{cases} P_{\mathcal{T}_h,k} & \text{(for simplicial meshes)}\,, \\ Q_{\mathcal{T}_h,k} & \text{(for quadrilateral/hexahedral meshes)}\,. \end{cases}$$

In particular, the finite element space for the pressure is

$$Q_{h,s} := P_s(\mathcal{T}_h) \cap Q\,,$$

and for the velocities is

$$\mathbf{V}_{h,r} := [P_r(\mathcal{T}_h)]^d \cap V\,,$$

with $r, s \in \mathcal{N}$, $1 \le s \le r$. With these notations, for the discrete pressure it holds $p_h \in Q_{h,s}$ and for the discrete velocity $\mathbf{v}_h \in \mathbf{V}_{h,r}$. The product space where the discrete solution $\mathbf{u}_h = (\mathbf{v}_h, p_h)$ is sought is given by

$$\mathbf{U}_{h,r,s} := \mathbf{V}_{h,r} \times Q_{h,s} \,.$$

For equal-order elements we will use the notation $\mathbf{U}_{h,r} := \mathbf{U}_{h,r,r}$. Furthermore, we will use the finite element space $R_{2h}^{dc} := Q_{\mathcal{T}_{2h},r-1}^{dc}$ in the case of quadrilateral meshes, and $R_{2h}^{dc} := P_{\mathcal{T}_{2h},r-1}^{dc}$ in the case of simplices, respectively.

In the analysis, local inverse inequalities on $T \in \mathcal{T}_h$ are used:

$$\begin{aligned} \|\operatorname{div} \mathbf{v}_h\|_{0,T} &\le \sqrt{d} \|\nabla \mathbf{v}_h\|_{[L^2(T)]^{d\times d}} \\ &\le \mu_{inv} r^2 h_T^{-1} \|\mathbf{v}_h\|_{[L^2(T)]^d} \quad \forall \mathbf{v}_h \in \mathbf{V}_{h,r}, \end{aligned} \tag{2.1}$$

with a constant μ_{inv} depending only on the shape-regularity.

The global Lagrange interpolants on the discrete spaces $P_{\mathcal{T}_h,r}$ or $Q_{\mathcal{T}_h,r}$ (depending on the type of mesh $\mathcal{T}_h$) are denoted by I_h^r. For $v \in H^k(T)$ with $k > d/2$ they satisfy the local estimate (cf. [3], Sect. 4)

$$\|v - I_h^r v\|_{m,T} \le C_I \frac{h_T^{l-m}}{r^{k-m}} \|v\|_{k,T}, \quad 0 \le m \le l = \min(r+1, k)\,. \tag{2.2}$$

Here C_I is a constant independent of h_T, r, v, T but dependent on m and k.

The notation $a \lesssim b$ is used for $a \le Cb$ with a suitable constant $C > 0$ independent of all relevant quantities, in particular independent of the local mesh sizes h_T and the parameters ν, β and σ.

We denote the Scott-Zhang quasi-interpolant ($t > \frac{1}{2}$) by

$$\mathcal{Z}_h^r : H^t(\Omega) \cap H_0^1(\Omega) \to H_0^1(\Omega) \cap P_{\mathcal{T}_h,r}\,.$$

This operator has the following interpolation property for $v|_{\omega_T} \in H^k(\omega_T)$, $k \ge t$, on the patches $\omega_T := \bigcup_{\overline{T'}\cap\overline{T}\neq\emptyset} T'$:

$$\|v - \mathcal{Z}_h^r v\|_{m,T} \lesssim \frac{h_T^{l-m}}{r^{k-m}} \|v\|_{k,\omega_T} \tag{2.3}$$

with $0 \le m \le l = \min(r+1, k)$. This quasi-interpolant can be extended to the vector-valued case and to quadrilateral meshes, $\mathcal{Z}_h^r : \mathbf{V} \to \mathbf{V}_{h,r}$. A similar interpolation operator $\mathcal{Z}_h^s : H^1(\Omega) \to P_s(\mathcal{T}_h)$ is defined for the pressure.

3. Local Projection for the Stokes System with Equal-Order Elements

The Stokes problem (1.1)-(1.3) is the simplest model of an incompressible viscous flow. It allows to present one of the basic difficulties of finite element methods for such flows: the coupling problem between velocity and pressure.

3.1. Galerkin formulation of the Stokes problem

The standard variational formulation of the Stokes problem (1.1)-(1.3) is: Find $\mathbf{u} \in \mathbf{U}$ such that

$$A(\mathbf{u}, \mathbf{z}) = (\mathbf{f}, \mathbf{w}) \quad \forall \mathbf{z} = (\mathbf{w}, q) \in \mathbf{U}\,, \tag{3.1}$$

where $A(\mathbf{u},\mathbf{z})$ is defined by

$$A(\mathbf{u},\mathbf{z}) := (\nabla\mathbf{v},\nabla\mathbf{w}) - (p,\operatorname{div}\mathbf{w}) + (\operatorname{div}\mathbf{v},q)\,.$$

The pure Galerkin approximation of (3.1) is not stable for equal-order elements, i.e., for $\mathbf{u}_h \in \mathbf{U}_{h,r}$. This instability stems from the violation of the discrete inf-sup condition for $\mathbf{U}_{h,r}$: There exists $\gamma > 0$ independent of h with

$$\inf_{p_h\in Q_h}\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{(p_h,\operatorname{div}\mathbf{v}_h)}{\|\mathbf{v}_h\|_1\|p_h\|_0} \geq \gamma\,.$$

One possibility to overcome this so called LBB condition (**L**adyshenskaya, **B**abuska, **B**rezzi) is the introduction of stabilization terms:

$$\mathbf{u}_h \in \mathbf{U}_h:\; A(\mathbf{u}_h,\mathbf{z}_h) + S_h(\mathbf{u}_h,\cdot;\mathbf{z}_h) = (\mathbf{f},\mathbf{w}_h) \quad \forall \mathbf{z}_h \in \mathbf{U}_{h,r}\,. \tag{3.2}$$

Several possibilities of those stabilization terms are investigated and analyzed. Depending on the specific choice, the term $S_h(\mathbf{u}_h,\cdot;\mathbf{z}_h)$ may depend on $\mathbf{u}_h$ and on further quantities, e.g. on the right hand side $\mathbf{f}$. Therefore, we use the symbol "·" to indicate this. The among most popular one is presented shortly in the following subsection.

3.2. Pressure stabilization by Petrov-Galerkin

The classical pressure stabilization Petrov-Galerkin (PSPG) scheme goes back to Hughes and Franca [4]. In this method, the following stabilization terms to the Galerkin formulation are introduced:

$$S_h^{pspg}(\mathbf{u}_h,\mathbf{f};q_h) := \sum_{T\in\mathcal{T}_h} \alpha_T(-\Delta\mathbf{v}_h + \nabla p_h - \mathbf{f}, \nabla q_h)_T\,. \tag{3.3}$$

Taking a $\alpha_T = \alpha_0 h_T^2$ with $\alpha_0 > 0$ large enough results in a stable discretization. This discretization is "fully consistent" in the sense that if the continuous solution u is smooth enough, for instance if $\mathbf{v}_h \in C^2(\Omega)\cap C(\overline{\Omega})$ and $p_h \in C^1(\Omega)$, the added terms vanish. This is an important property of PSPG. However, for the discrete solution $\mathbf{u}_h$ with low-order finite elements, the term $|(-\Delta\mathbf{v}_h + \nabla p_h - \mathbf{f})|_K$ may be a rough approximation of the residual on element K. Especially, the introduction of the stabilizing diffusive term for the pressure implies an artificial pressure boundary condition of the form

$$\alpha(\partial_n p_h - \mathbf{f}\cdot n) \approx 0,$$

which leads to a decrease of accuracy close to the boundary. We refer to the analysis of Rannacher et al. [5]. Further drawbacks of this method are as follows:

- The test function ∇q_h acts on the full residual of the equation. Therefore, second derivatives have to be evaluated in the case of $r \geq 2$. This is also the case for $r = 1$ on non-parallelograms, since the pure second derivatives do not vanish on the iso-parametric transformed cells even if they do on the reference cell. This requires high numerical costs.
- The right-hand side enters also into the stabilization. This becomes troublesome in the time-dependent case, because then the right-hand side contains information of the previous time step. This requires the usage of space-time elements.
- The construction of efficient algebraic solvers is not trivial.

• The extension of those residual-based stabilization techniques for more complicated flow problems (for instance reactive flows) introduces a large set of additional couplings between the different variables.

In order to allow even discontinuous pressure, a variant of this PSPG method was proposed by Hughes and Franca [6] and a modified version was developed independently by Douglas and Wang, [7]. Additional to (3.3), jump terms of the pressure across element edges (resp. faces in three dimensions) are also added to the Galerkin term. A revival of such jump terms have emerged due to the work of Burman, Hansbo and co-workers on edge oriented stabilization, see, e.g., [2, 8–10].

3.3. Global projection schemes

Codina & Blasco proposed in [11] a global orthogonal projection in order to overcome some of the above mentioned short-comings. The principal idea is to add to the divergence free condition the difference of the pressure gradient and its global projection. The projected space is the discrete velocity space without Dirichlet conditions:

$$\widetilde{\mathbf{V}}_{h,r} := [P_{\mathcal{T}_h,r}]^d \,.$$

The L^2-projection $\widetilde{\pi}_h : [L^2(\Omega)]^d \to \widetilde{\mathbf{V}}_{h,r}$ is for $\mathbf{v} \in [L^2(\Omega)]^d$ defined by

$$(\widetilde{\pi}_h \mathbf{v}, \widetilde{\mathbf{w}}) = (\mathbf{v}, \widetilde{\mathbf{w}}) \quad \forall \widetilde{\mathbf{w}} \in \widetilde{\mathbf{V}}_{h,r} \,.$$

The fluctuation operator $\widetilde{\vartheta}_h$, given by

$$\widetilde{\vartheta}_h := I - \widetilde{\pi}_h \,,$$

where I stands for the identity mapping, enters in the stabilization term:

$$S_h^{gps}(p_h, q_h) := (\alpha \widetilde{\vartheta}_h \nabla p_h, \nabla q_h) \,,$$

which itself enters in the discrete equation (3.2). Since $\vartheta_h \nabla p_h$ can not be computed locally, we get a coupled system:

$$A(\mathbf{u}_h, \mathbf{z}_h) + S_h^{gps}(p_h, q_h) = (\mathbf{f}, \mathbf{w}_h) \quad \forall \mathbf{z}_h = (\mathbf{w}_h, q_h) \in \mathbf{U}_{h,r} \,, \tag{3.4}$$

$$(\widetilde{\vartheta}_h \nabla p_h, \widetilde{\mathbf{v}}_h) = 0 \qquad \forall \widetilde{\mathbf{v}}_h \in \widetilde{\mathbf{V}}_{h,r} \,. \tag{3.5}$$

The expression $\widetilde{\vartheta}_h \nabla p_h$ can be considered as an additional variable which can not be eliminated locally. However, it is possible to avoid this additional variable by some defect correction method. We refer to Codina et al. [12] for details. The use of a local projection instead of a global one avoids such an additional variable completely. This will be the matter of the remaining part of this work.

Note, that due to the orthogonality condition (3.5), the stabilization term is symmetric:

$$\begin{aligned} S_h^{gps}(p_h, q_h) &= (\alpha \widetilde{\vartheta}_h \nabla p_h, \nabla q_h - \widetilde{\pi}_h \nabla q_h) \\ &= (\alpha^{1/2} \widetilde{\vartheta}_h \nabla p_h, \alpha^{1/2} \widetilde{\vartheta}_h \nabla q_h) \,. \end{aligned}$$

Somehow related is the method of Cai, Manteuffel and McCormick [13]. With help of additional variables, the second-order Stokes system is transformed into a first-order system. This also leads to stable discretizations. This is an overall concept which may also apply to other systems of PDEs, e.g., [14].

3.4. Principle idea of local projection

Motivated by the global projection just presented, we want to use a local projection of the pressure or its gradient. We will consider symmetric stabilization terms $S_h(p_h, q_h)$ of the form:

$$S_h^{lps}(p,q) := (\vartheta_h(p), \vartheta_h(q)) \qquad \forall p, q \in H^1(\Omega)\,, \tag{3.6}$$

with a local fluctuation operator

$$\vartheta_h : H^1(\Omega) \to L^2(\Omega)\,.$$

This local property is linked to a possibly coarser non-overlapping, shape-regular mesh $\mathcal{M}_h = \{M_i\}_{i\in I}$ constructed by coarsening $\mathcal{T}_h$ s. t. each $M \in \mathcal{M}_h$ with diameter h_M consists of one or more neighboring cells $T \in \mathcal{T}_h$. Moreover, suppose that there exists $C > 0$ s. t. $h_M \leq Ch_T$ for all $T \in \mathcal{T}_h, M \in \mathcal{M}_h$ with $T \subset M$. Here, we still allow for $\mathcal{M}_h = \mathcal{T}_h$.

Before we present possible realizations of the stabilization terms $S_h^{lps}(p_h, q_h)$ based on local projections, we specify some abstract conditions originally formulated in [15], but here specified for the form (3.6):

Assumption 3.1. *It exists a discrete pressure space $\widetilde{Q}_h$, so that the pair $\mathbf{V}_{h,r} \times \widetilde{Q}_h$ is LBB stable.*

Assumption 3.2. *It exists a continuous* (*not necessarily local*) *projection operator*

$$\Pi_h : Q_{h,r} \to \widetilde{Q}_h\,,$$

i.e., $\|\Pi_h p_h\| \leq C\|p_h\|$ for all $p_h \in Q_{h,r}$ with a suitable constant C.

Assumption 3.3. *The fluctuations with respect to Π_h can be controlled by the fluctuation operator:*

$$\|p_h - \Pi_h p_h\|_0 \lesssim \|\vartheta_h(p_h)\|_0 \qquad \forall p_h \in Q_{h,r}\,.$$

Assumption 3.4. *It should hold for $0 \leq l \leq r$*

$$\|\vartheta_h(p)\|_{0,M} \lesssim \frac{h^l}{r^l} |p|_{H^l(M)} \qquad \forall p \in H^l(M) \quad \forall M \in \mathcal{M}_h\,.$$

In order to show stability. we will use the triple norm

$$|\!|\!|\mathbf{u}|\!|\!|_{S_h^{lps}} := (|\mathbf{v}|_1^2 + \|p\|^2 + S_h^{lps}(p,p))^{1/2}\,.$$

Lemma 3.1. *As a consequence of the three Assumptions* 3.1*,* 3.2 *and* 3.3*, it exists a constant $\gamma > 0$, so that*

$$\sup_{\mathbf{z}_h \in \mathbf{U}_{h,r}} \left\{ \frac{A(\mathbf{u}_h, \mathbf{z}_h) + S_h^{lps}(p_h, q_h)}{|\!|\!|\mathbf{z}_h|\!|\!|_{S_h^{lps}}} \right\} \geq \gamma |\!|\!|\mathbf{u}_h|\!|\!|_{S_h^{lps}} \qquad \forall \mathbf{u}_h \in \mathbf{U}_h\,.$$

Proof. Let $\mathbf{u}_h = (\mathbf{v}_h, p_h) \in \mathbf{U}_h$ be given. The desired bound is obtained by constructing an appropriate test function of the form $\mathbf{z}_h := \mathbf{u}_h + \epsilon \mathbf{y}_h$ with $\epsilon > 0$. Diagonal testing gives:

$$A(\mathbf{u}_h, \mathbf{u}_h) + S_h^{lps}(p_h, p_h) = |\mathbf{v}_h|_1^2 + S_h^{lps}(p_h, p_h)\,.$$

Due to Assumption 3.1 the pair $\mathbf{V}_{h,r} \times \widetilde{Q}_h$ fulfills a LBB condition. This is equivalent to the existence of a constant $\widetilde{\gamma} > 0$ with the following property: For each $\widetilde{p}_h \in \widetilde{Q}_h$ exists a velocity $\mathbf{w}_h \in \mathbf{V}_{h,r}$ with

$$|\mathbf{w}_h|_1 = 1 \quad \text{and} \quad (\widetilde{p}_h, \nabla \cdot \mathbf{w}_h) \geq \widetilde{\gamma} \|\widetilde{p}_h\|_0 .$$

We choose $\widetilde{p}_h := \Pi_h p_h$ and $\mathbf{y}_h := (-\mathbf{w}_h \|p_h\|, 0)$ and use Assumptions 3.2 and 3.3:

$$\begin{aligned} A(\mathbf{u}_h, \mathbf{y}_h) + S_h^{lps}(p_h, 0) &= \epsilon\big(- (\nabla \mathbf{v}_h, \nabla \mathbf{w}_h) + (p_h, \nabla \cdot \mathbf{w}_h)\big) \|p_h\|_0 \\ &\geq \epsilon\big(- |\mathbf{v}_h|_1 |\mathbf{w}_h|_1 + (\widetilde{p}_h, \nabla \cdot \mathbf{w}_h) + (p_h - \widetilde{p}_h, \nabla \cdot \mathbf{w}_h)\big) \|p_h\|_0 \\ &\geq \big(- |\mathbf{v}_h|_1 + \widetilde{\gamma} \|\widetilde{p}_h\|_0 - \|p_h - \widetilde{p}_h\|\big) \|p_h\|_0 \\ &\geq \big(- |\mathbf{v}_h|_1 + c_1 \widetilde{\gamma} \|p_h\|_0 - c_2 \|\vartheta(p_h)\|\big) \|p_h\|_0 . \end{aligned}$$

The application of the inequality of Young leads to

$$A(\mathbf{u}_h, \mathbf{y}_h) + S_h^{lps}(p_h, 0) \geq c_3 \|p_h\|_0^2 - c_4 |\mathbf{v}_h|_1^2 - c_5 S_h^{lps}(p_h, p_h) ,$$

with positive constants $c_3, c_4, c_5 > 0$. With $\epsilon > 0$ small enough, we obtain the assertion. □

Now it is easy to show the a priori estimate for the solution of (3.2) with the stabilization term $S_h^{lps}(\cdot, \cdot)$:

Theorem 3.1. *Let the fluctuation operator $\vartheta_h : H^1(\Omega) \to L^2(\Omega)$ fulfill the Assumptions* 3.1-3.4*, and let $\mathbf{u}_h = (\mathbf{v}_h, p_h)$ be the discrete solution of the stabilized problem* (3.2) *with S_h^{lps} given by* (3.6)*. The continuous solution $\mathbf{u}$ is assumed to be in $[H^{k+1}(\Omega)]^d \times H^k(\Omega)$. Then it holds on quasi-uniform meshes with $m = \min(r, k)$:*

$$|\mathbf{v} - \mathbf{v}_h|_1 + \|p - p_h\|_0 \leq C \left(\frac{h}{r}\right)^m (\|\mathbf{v}\|_{m+1} + \|p\|_m) . \tag{3.7}$$

Proof. We give only the principal idea of the proof, because all details can be found in [15]. The a priori estimate is obtained by splitting the error in an interpolation part $\mathbf{u} - \mathcal{Z}_h^r \mathbf{u}$ and a projection part $\mathcal{Z}_h^r \mathbf{u} - \mathbf{u}_h$ by use of the Scott-Zhang operator $\mathcal{Z}_h^r$. The bound for the interpolation error is a simple consequence of the standard interpolation results of the Clement interpolant. The bound for the projection error is obtained by the coercivity result of Lemma 3.1 in the triple norm $|\!|\!|\cdot|\!|\!|_{S_h^{lps}}$ with the stability constant $\gamma > 0$. Now, we obtain with a suitable $\mathbf{z}_h \in \mathbf{U}_{h,r}$, $|\!|\!|\mathbf{z}_h|\!|\!|_{S_h^{lps}} = 1$ with the perturbed Galerkin orthogonality:

$$\begin{aligned} |\mathcal{Z}_h^r \mathbf{v} - \mathbf{v}_h|_1 + \|\mathcal{Z}_h^r p - p_h\|_0 &\leq |\!|\!|\mathcal{Z}_h^r \mathbf{u} - \mathbf{u}_h|\!|\!|_{S_h^{lps}} \\ &\leq \frac{1}{\gamma} (A(\mathcal{Z}_h^r \mathbf{u} - \mathbf{u}_h, \mathbf{z}_h) + S_h^{lps}(\mathcal{Z}_h^r p - p_h, q_h)) \\ &= \frac{1}{\gamma} (A(\mathcal{Z}_h^r \mathbf{u} - \mathbf{u}, \mathbf{z}_h) + S_h^{lps}(\mathcal{Z}_h^r p, q_h)) . \end{aligned}$$

The Galerkin term $A(\mathcal{Z}_h^r \mathbf{u} - \mathbf{u}, \mathbf{z}_h)$ is bounded properly due to the standard interpolation results of $\mathcal{Z}_h^r$. The stabilization part $S_h^{lps}(\mathcal{Z}_h^r p, q_h)$ is controlled by:

$$\begin{aligned} |S_h^{lps}(\mathcal{Z}_h^r p, q_h)| &\leq \|\vartheta_h(\mathcal{Z}_h^r p)\|_0 \; S_h(q_h, q_h)^{1/2} \\ &\leq (\|\vartheta_h(\mathcal{Z}_h^r p - p)\|_0 + \|\vartheta_h(p)\|_0) \; S_h(q_h, q_h)^{1/2} . \end{aligned}$$

Using Assumption 3.4, the interpolation properties (2.3) of $\mathcal{Z}_h^r$ and

$$|S_h(q_h, q_h)|^{1/2} \le \|\mathbf{z}_h\|_{S_h^{lps}} = 1\,.$$

gives the assertion. □

Now, we will give some possibilities of fluctuation operators ϑ_h which satisfy Assumptions 3.1-3.4.

3.5. Patch-wise local projection of the pressure gradient

For the formulation of patch-wise local projection, we restrict ourselves for a while to a certain class of meshes. We assume that the mesh $\mathcal{T}_h$ results from a coarser mesh $\mathcal{M}_h = \mathcal{T}_{2h}$ by one global refinement. Hence, the mesh $\mathcal{M}_h$ consists of patches of elements; for instance in two dimensions, four quadrilaterals can be grouped together in order to form one quadrilateral of $\mathcal{M}_h$, see Fig. 3.1. We introduce the L^2-projections

$$\overline{\pi}_{2h} : L^2(\Omega) \to R_{2h}^{dc}\,.$$

This projection is characterized by

$$(q - \overline{\pi}_{2h} q, \psi) \;=\; 0 \quad \forall q \in L^2(\Omega) \quad \forall \psi \in R_{2h}^{dc}\,.$$

The corresponding fluctuation operator proposed in [15] is

$$\vartheta_h(p) \;:=\; \alpha^{1/2}(\nabla p - \overline{\pi}_{2h}\nabla p)\,, \tag{3.8}$$

with patch-wise constant parameters α depending on the local mesh size similar to the PSPG method, $\alpha|_M \sim h_M^2$ for all $M \in \mathcal{M}_h$.

Lemma 3.2. *The local fluctuation operator on patches* (3.8) *fulfills the Assumptions* 3.1-3.4. *Hence, the a priori estimate of Theorem* 3.1 *holds.*

Proof. In order to verify Assumptions 3.1-3.4 we specify the space $\widetilde{Q}_h := Q_{2h,r}$. The pair $V_{h,r} \times Q_{2h,r}$ is well-known to be LBB stable, i.e. Assumption 3.1 holds. The projection $\Pi_h := I_{2h}^r$ is the nodal interpolation on $Q_{2h,r}$, which is continuous (Assumption 3.2). Assumption 3.3 is obtained by a standard scaling argument: We denote the quantities transformed onto the reference patch $\widehat{M}$ by $\widehat{\ }$. The nodal interpolator and the L^2-projection onto $\widehat{M}$ are denoted by $\widehat{I}^r$ and $\widehat{\pi}$, respectively. For $p_h \in Q_{h,r}$ we estimate:

$$\begin{aligned}
\|p_h - I_{2h}^r p_h\|_{0,M} &\lesssim h_M \|\widehat{p}_h - \widehat{I}^r p_h\|_{0,\widehat{M}} \\
&\lesssim h_M \|\widehat{\nabla}\widehat{p}_h - \widehat{\pi}(\widehat{\nabla}\widehat{p}_h)\|_{0,\widehat{M}} \\
&\lesssim h_M \|\nabla p_h - \overline{\pi}_{2h}\nabla p_h\|_{0,M} \\
&\lesssim \alpha_M^{1/2} \|\nabla p_h - \overline{\pi}_{2h}\nabla p_h\|_{0,M} \\
&= \|\vartheta_h(p_h)\|_{0,M}\,.
\end{aligned}$$

Finally, Assumption 3.4 is a simple consequence of the definition of S_h^{lps}. □

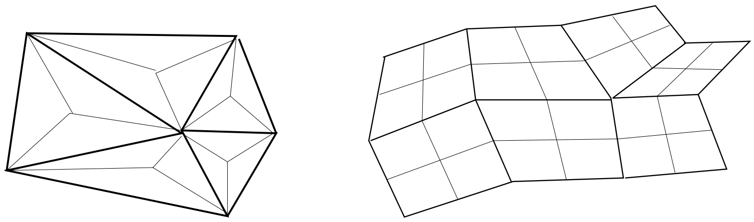

Fig. 3.1. Type of meshes needed for patch-wise local projection for triangular meshes (left) or quadrilateral meshes (right) in Section 3.5. The bold lines indicate the mesh $\mathcal{T}_{2h}$, the fine lines $\mathcal{T}_h$.

3.6. Local projections of the pressure itself

Instead of projecting the pressure gradient, one may project the pressure itself on finite elements on a coarser mesh,

$$\vartheta_h(p_h) := \alpha^{1/2}\nabla(p_h - I_{2h}^r p_h)\,, \tag{3.9}$$

or, as an alternative in the case of $r \geq 2$, on finite elements of lower order and build the gradient on the difference:

$$\vartheta_h(p_h) := \alpha^{1/2}\nabla(p_h - I_h^{r-1} p_h)\,. \tag{3.10}$$

For the verification of Assumptions 3.1-3.4 one may take $\widetilde{Q}_h := Q_{2h,r}$ or $\widetilde{Q}_h := Q_{h,r-1}$ as stable pressure space. From implementational aspects, (3.10) is very attractive, because the projection acts very local only on cells and not on patches of elements. While such a projection remains optimal for the Stokes problem (Assumptions 3.1-3.4 are fulfilled), the extension to the Oseen system is not any more optimal.

These stabilizations can be considered as the subgrid modelling proposed by Guermond for convection-diffusion problems [16] extended to the Stokes problem.

3.7. Relation to continuous interior penalty stabilization

A further discrete equivalent form proposed in [15] consists of the jumps of the pressure gradients in normal direction across the interior edges of the patches. This stabilization is not of the form (3.6) because it involves integrals along edges, but it remains symmetric.

We denote the set of internal edges of a patch $T \in \mathcal{T}_{2h}$ by E_T. The jump of the derivative in normal direction n in a point $x \in e$ is defined by:

$$[\partial_n p_h](x) := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(p_h(x+n\epsilon) - p_h(x-n\epsilon)\right).$$

A possible stabilization term is:

$$S_h(p_h, q_h) := \sum_{T\in\mathcal{T}_{2h}} \sum_{e\in E_T} \alpha_e \int_e [\partial_n p_h][\partial_n q_h]ds\,.$$

The free parameter has to be chosen as $\alpha_e \sim h_e^3$. In order to show the estimate (3.7), the Assumptions 3.3 and 3.4 have to be replaced by:

$$\|p_h - \Pi_h p_h\|_0^2 \lesssim S_h(p_h, p_h) \qquad \forall p_h \in Q_{h,r}\,,$$

$$|S_h(p,q)| \lesssim \frac{h^l}{r^l}|p|_l S_h(q,q)^{1/2} \qquad \forall p \in H_1(\Omega) \;\; \forall q \in H_1(\Omega),\; 0 \leq l \leq r\,.$$

Alternatively, one may build the sum over all interior edges of the triangulation, instead of considering only the inner edges corresponding to patches of elements. This method on triangular meshes is called continuous interior penalty (CIP) stabilization and has been analyzed by Burman and Hansbo in [8] for convection-diffusion problems.

Remark 3.1. The requirement of the local projection (3.8) is the presence of the patch mesh $\mathcal{T}_{2h}$ and the corresponding data structure to be able to assemble the local projection. A severe drawback from the computational point of view is the increased finite element stencil compared to the pure Galerkin formulation. This can be circumvented by applying a preconditioner with the Galerkin stencil. One possibility is to use as preconditioner the matrix of the bilinear form

$$(\mathbf{u}, \mathbf{z}_h) \ \mapsto (A(\mathbf{u}, \mathbf{z}_h) + (\nabla p_h, \alpha \nabla q_h)\,.$$

3.8. Numerical results

Numerical comparisons of the local projection related to (3.6) and (3.8) show exactly the same order of convergence as PSPG. However, the sensitivity of the error with respect to the stabilization parameter α_0 is much less severe with LPS compared to PSPG. This feature is illustrated in Fig. 3.2 for a two-dimensional Stokes flow with analytical solution. Whereas the error is quite comparable for both methods when α_0 is small ($\alpha_0 \leq 0.1$). For large α_0 the error for PSPG becomes unbounded while the error with LPS is bounded on a quite small level. We refer to [15] for the details.

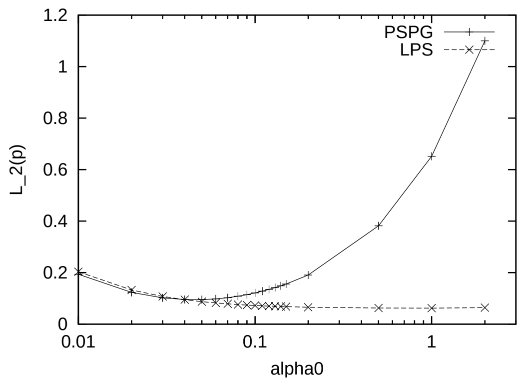

Fig. 3.2. Comparison of the error $\|p - p_h\|_0$ in dependence of the stabilization constant α_0 for LPS and pressure stabilized Petrov-Galerkin stabilization (PSPG).

4. Oseen Linearization

In this section, we extend the LPS approach of the previous sections to a more general framework for the Oseen problem (1.4)-(1.6). Additionally to the compatibility problem between the discrete pressure and velocity problems, the main new problem is a proper treatment of (locally) dominating advective terms or of an reaction term (which may stem from an implicit time discretization of the Navier-Stokes problem).

Originally, Braack and Burman analyzed the LPS technique for the Oseen problem (1.4)-(1.6) in [17] for low-order equal-order element pairs. Matthies et al. [18] developed a more general framework for equal-order interpolation of velocity and pressure. Very recently, an extension to inf-sup stable interpolation pairs has been considered by Rapin / Löwe [19] and by Matthies / Tobiska [20]. The presentation in this section follows the lines of Lube et al. [21] where a unified approach to both cases, equal-order and inf-sup stable pairs, is given.

4.1. Variational formulation

The weak formulation for the Oseen problem (1.4)-(1.6) with homogeneous Dirichlet data reads: Find $\mathbf{u} = (\mathbf{v}, p) \in \mathbf{U}$, such that

$$A(\mathbf{u}, \mathbf{z}) = (\mathbf{f}, \mathbf{w}) \qquad \forall \mathbf{z} = (\mathbf{w}, q) \in \mathbf{U}\,, \tag{4.1}$$

with the bilinear form

$$A(\mathbf{u}, \mathbf{z}) := (\nu \nabla \mathbf{v}, \nabla \mathbf{w}) + ((\mathbf{b} \cdot \nabla)\mathbf{v} + \sigma \mathbf{v}, \mathbf{w}) - (p, \nabla \cdot \mathbf{w}) + (\nabla \cdot \mathbf{v}, q)\,. \tag{4.2}$$

Let the viscosity $\nu \in \mathbb{R}^+$ and the reaction term $\sigma \in \mathbb{R}^+$ be constant and the right-hand side $\mathbf{f} \in [L^2(\Omega)]^d$ as before, For the advection field we suppose $\mathbf{b} \in [L^\infty(\Omega) \cap H^1(\Omega)]^d$ with $\operatorname{div} \vec{\mathbf{b}} = 0$. Usually, $\mathbf{b}$ is a FE solution of (1.4)-(1.6) with $(\operatorname{div} \mathbf{b}, q_h) = 0$ for some q_h but $\operatorname{div} \mathbf{b}$ does not vanish pointwise. A remedy is to write the advective term in skew-symmetric form.

Beside a possible violation of the discrete inf-sup (or Babuška-Brezzi) stability condition (e.g., in the case of $r = s$), the standard Galerkin finite element method for (1.4)-(1.6) may suffer from dominating advection (and reaction) on relatively coarse meshes, i.e. in the case of $0 < \nu \ll h_T \|\mathbf{b}\|_{L^\infty(T)}$ (and/or $\sigma \gg h_T^{-1} \max(\nu, \|\mathbf{b}\|_{L^\infty(T)})$) on certain cells $T \in \mathcal{T}_h$.

An established finite element method for the Oseen system is the well-known *streamline upwind/ Petrov-Galerkin (SUPG) method* introduced in [22], and the *pressure-stabilization/Petrov-Galerkin* (PSPG) method, introduced in [1,4]. The corresponding equation is of the form (3.2) with the bilinear form $A(\cdot,\cdot)$ given by (4.2) and the stabilization term

$$\begin{aligned} S_h(\mathbf{u}_h, \mathbf{f}; \mathbf{z}_h) := \sum_{T \in \mathcal{T}_h} \Big\{ &((\mathbf{b} \cdot \nabla)\mathbf{v}_h + \sigma \mathbf{v}_h - \nu \Delta \mathbf{v}_h + \nabla p_h - \mathbf{f}, \alpha_T \nabla q_h + \tau_T (\mathbf{b} \cdot \nabla)\mathbf{w}_h)_T \\ &+ \ (\operatorname{div} \mathbf{v}_h, \mu_T \operatorname{div} \mathbf{w}_h)_T \Big\}, \end{aligned}$$

with cell-wise constant parameters α_T, τ_T, μ_T.

As an alternative we extend the idea of local projection as discussed previously for Stokes for the Oseen system. In LPS-methods the discrete function spaces are split into small and large scales and stabilization terms of diffusion-type acting only on the small scales are added. Here, we have to control fluctuations of the velocity gradient or alternatively its directional derivative in streamline direction together with additional control of fluctuations of the divergence-free condition.

4.2. General local projection formulation

The local projection scheme reads: Find $\mathbf{u}_h = (\mathbf{v}_h, p_h) \in \mathbf{U}_{h,r,s}$ such that

$$A(\mathbf{u}_h, \mathbf{z}_h) + S_h^{lps}(\mathbf{u}_h, \mathbf{z}_h) = (\mathbf{f}, \mathbf{w}_h), \qquad \forall \mathbf{z}_h = (\mathbf{w}_h, q_h) \in \mathbf{U}_{h,r,s}\,. \tag{4.3}$$

For the definition of the stabilization term we follow [18]. At first, we define a FE space $D_{h,k}, k \in \mathbb{N}$. Possible choices will be discussed later in Remark 4.3. The subscript k indicates that this space will have a certain relation to $P_k(\mathcal{T}_h)$. The restriction to $M \in \mathcal{M}_h$ is denoted by

$$D_{h,k}(M) = \{v_h|_M \mid v_h \in D_{h,k}\}.$$

At second, we use the local L^2-projection $\pi_{M,k} : L^2(M) \to D_{h,k}(M), k \in \mathbb{N}$ which defines the global projection

$$\pi_{h,k} : L^2(\Omega) \to D_{h,k}\,, \quad (\pi_{h,k}v)|_M := \pi_{M,k}(v|_M)$$

for all $M \in \mathcal{M}_h$. Denoting the identity on $L^2(\Omega)$ by I we define the associated fluctuation operator $\kappa_{h,k} := I - \pi_{h,k}$:

$$\kappa_{h,k} : L^2(\Omega) \to L^2(\Omega)\,, \quad u \mapsto u - \pi_{h,k}u\,.$$

These operators are applied to vector-valued functions in a component-wise manner, e.g., $\kappa_{h,k} : [L^2(\Omega)]^d \to [L^2(\Omega)]^d$. The vector valued FE space $D^v_{h,k}$ is defined as the tensor product $D^v_{h,k} := [D_{h,k}]^d$. With these notations, we are able to formulate the additional stabilization term

$$\begin{aligned} S_h^{lps}(\mathbf{u},\mathbf{z}) := {}& (\kappa_{h,r}(\nabla p), \alpha\kappa_{h,r}(\nabla q)) + (\kappa_{h,r}(\mathbf{b}\cdot\nabla\mathbf{v}), \tau\kappa_{h,r}(\mathbf{b}\cdot\nabla\mathbf{w})) \\ &+ (\kappa_{h,s}(\nabla\cdot\mathbf{v}), \mu\kappa_{h,s}(\nabla\cdot\mathbf{w}))\,. \end{aligned} \tag{4.4}$$

The parameter α, τ and μ are patch-wise constants on $M \in \mathcal{M}_h$. An immediate consequence of the special form (4.4) of the stabilization part is formulated in the following lemma:

Lemma 4.1. *For all* $\mathbf{u}, \mathbf{z} \in [H^1(\Omega)]^{d+1}$ *it holds:*

$$S_h^{lps}(\mathbf{u},\mathbf{u}) \ge 0\,, \quad |S_h^{lps}(\mathbf{u},\mathbf{z})| \le S_h^{lps}(\mathbf{u},\mathbf{u})^{1/2} S_h^{lps}(\mathbf{z},\mathbf{z})^{1/2}\,.$$

Proof. The results are obvious. □

Since $S_h^{lps}(\mathbf{u},\mathbf{u})$ is non-negative, we may associate it to the discrete system (4.3) the semi-norm

$$|\!|\!|\mathbf{u}|\!|\!|_{lps} := \left(\nu\|\nabla\mathbf{v}\|_0^2 + \sigma\|\mathbf{v}\|_0^2 + S_h^{lps}(\mathbf{u},\mathbf{u})\right)^{1/2}\,.$$

Lemma 4.2. *If the discrete scheme* (4.3) *has a solution* $\mathbf{u}_h = (\mathbf{v}_h, p_h)$, *then the velocity* $\mathbf{v}_h \in \mathbf{V}_{h,r}$ *is unique.*

Proof. Using the solution as a test function and integration by parts leads to:

$$(\mathbf{f}, \mathbf{v_h}) = A(\mathbf{u}_h, \mathbf{u}_h) + S_h^{lps}(\mathbf{u}_h, \mathbf{u}_h) = |\!|\!|\mathbf{u}_h|\!|\!|_{lps}.$$

Hence, for the homogeneous system, $\mathbf{f} = 0$, the discrete velocity vanishes. This implies uniqueness for the inhomogeneous case. □

Lemma 4.3. *For the error* $\mathbf{u} - \mathbf{u}_h$ *of the continuous solution* $\mathbf{u} \in \mathbf{U}$ *of* (4.1) *and the discrete solution* $\mathbf{u}_h \in \mathbf{U}_h$ *of* (4.3) *we have the perturbed Galerkin orthogonality:*

$$A(\mathbf{u} - \mathbf{u}_h, \mathbf{z}_h) = S_h^{lps}(\mathbf{u}_h, \mathbf{z}_h) \qquad \forall \mathbf{z}_h \in \mathbf{U}_h.$$

Proof. Subtracting (4.3) from (4.1). □

Because the scheme is not fully consistent, we have to ensure that the consistency error is small enough. This leads to an assumption of the approximation property of κ_h. This will be discussed in the following subsection.

4.3. Approximation property of the local projection

In order to obtain optimal convergence order, the following assumption must be valid:

Assumption 4.1. *For all patches $M \in \mathcal{M}_h$ and $k \in \mathbb{N}$ it must hold*

$$\|\kappa_{h,k}(v)\|_{0,M} \lesssim \frac{h_M^l}{r^l}|v|_{l,M} \quad \forall v \in H^l(M),\ 0 \le l \le k\,. \tag{4.5}$$

Due to the construction of the vector-valued fluctuation operator, we get a similar estimate for $\kappa_{h,k}(\mathbf{v})$ with $\mathbf{v} \in [H^l(M)]^d$. This assumption ensures that the space $D_{h,k}$ is rich enough.

Lemma 4.4. *Let $\mathbf{u} \in [H^{l_v+1}(\Omega)]^d \times H^{l_p+1}(\Omega)$ and $\mathbf{b} \in (W^{l_v,\infty}(M))^d$ for each $M \in \mathcal{M}_h$. With the approximation Assumption 4.1 the stabilization term can be bounded by:*

$$S_h^{lps}(\mathbf{u},\mathbf{u}) \lesssim \sum_{M\in\mathcal{M}_h} \Big(\frac{h_M^{2l_v}}{r^{2l_v}}((\|\mathbf{b}\|^2_{(W^{l_v,\infty}(M))^d}\tau_M + \gamma_M)|\mathbf{v}|^2_{l_v+1,M} + \frac{h_M^{2l_p}}{r^{2l_p}}\alpha_M|p|^2_{l_p+1,M})\Big)\,,$$

with $0 \le l_v \le r,\ 0 \le l_p \le s$.

Proof. The assertion follows immediately by applying the Cauchy-Schwarz inequality and Assumption 4.1. □

Remark 4.1. Assumption 4.1 is valid if $\pi_{h,k}$ is chosen as the local L^2-projection in $D_{h,k}$ and if $D_{h,k}(M)$ contains the polynomials of degree less or qual to $k-1, k \in \mathbb{N}$, see [18], Remark 1.2.

4.4. Construction of a special interpolant

Assumption 4.2. *The pair $(P_k(\mathcal{T}_h), D_{h,k})$ fulfills this assumption if there exist $\beta_k > 0$ s.t. for every $M \in \mathcal{M}_h$ holds:*

$$\inf_{q_h\in D_{h,k}} \sup_{v_h\in Y_{h,k}(M)} \frac{(v_h,q_h)_M}{\|v_h\|_{0,M}\|q_h\|_{0,M}} \ge \beta_k \tag{4.6}$$

with $Y_{h,k}(M) := P_k(\mathcal{T}_h) \cap H_0^1(M)$.

Following [18], a special interpolant $j_{h,k}$ is constructed, such that the error $v - j_{h,k}v$ is L^2-orthogonal to $D_{h,r}$ for all $v \in H_0^1(\Omega)$. In the following lemma we use the notation of neighborhoods of patches $M \in \mathcal{M}_h$:

$$\omega_M := \bigcup\{K \ :\ K \in \mathcal{T}_h,\ \overline{K} \cap \overline{M} \ne \emptyset\}\,.$$

Lemma 4.5. *Let the Assumption 4.2 be valid for a pair $(P_k(\mathcal{T}_h), D_{h,k})$. Then there is for this k an interpolant $j_{h,k} : H^1(\Omega) \to P_k(\mathcal{M}_h)$ with the properties*

$$j_{h,k}(H_0^1(\Omega)) \subset H_0^1(\Omega)\,, \tag{4.7}$$

$$(v - j_{h,k}v, q_h) = 0 \qquad \forall q_h \in D_{h,k}\ \forall v \in H^1(\Omega), \tag{4.8}$$

and for $1 \le l \le k+1$.fulfilling the approximation property for all $v \in H^l(\Omega)$:

$$\|v - j_{h,k}v\|_{0,M} + \frac{h_M}{k^2}|v - j_{h,k}v|_{1,M} \lesssim \Big(1 + \frac{1}{\beta_k}\Big)\frac{h_M^l}{k^l}\|v\|_{l,\omega_M}\,. \tag{4.9}$$

Proof. Define the linear continuous operator $B_h : Y_{h,k}(M) \to D_{h,k}(M)'$ by

$$\langle B_h v_h, q_h \rangle := (v_h, q_h)_M, \qquad \forall v_h \in Y_{h,k}(M), q_h \in D_{h,k}(M)$$

and $W_{h,k}(M) := \mathrm{Ker}(B_h)$. The Closed-range Theorem yields via Assumption 4.2 that B_h is an isomorphism from $W_{h,k}(M)^\perp$ onto $D_{h,k}(M)'$ with

$$\beta_k \|v_h\|_{0,M} \le \|B_h v_h\|_{D_{h,k}(M)'}, \quad v_h \in W_{h,k}(M)^\perp,$$

where $W_{h,k}(M)^\perp$ is the orthogonal complement of $W_{h,k}(M)$ with respect to $(\cdot,\cdot)_M$.

Let $M \in \mathcal{M}_h$ and $v \in H_0^1(\Omega)$ be arbitrary. Then, there exists a unique $z_h(w, M) \in W_{h,k}(M)^\perp$ with

$$\|z_h(v, M)\|_{0,M} \le \frac{1}{\beta_k} \|v - I_h^k v\|_{0,M}$$

such that

$$\langle B_h z_h(v, M), q_h \rangle = (z_h(v), q_h)_M = (v - I_h^k v, q_h)_M, \quad \forall q_h \in D_{h,k}(M). \tag{4.10}$$

Now, we define local operators $j_{h,k}^M : H^1(M) \to Y_{h,k}(M)$, $M \in \mathcal{M}_h$, by

$$j_{h,k}^M v := (I_h^k v)|_M + z_h(v, M).$$

As $\mathcal{M}_h$ is a partition of Ω, we define a global operator $j_{h,k} : H^1(\Omega) \to Y_{h,k}$ by $(j_{h,k}v)|_M := j_{h,k}^M v$. Due to (2.2) $j_{h,k}$ satisfies for $1 \le l \le k+1$ and all $T \in \mathcal{T}_h$, $v \in H^l(\Omega)$

$$\begin{aligned} \|v - j_{h,k}v\|_{0,M}^2 &\le \left(1 + \frac{1}{\beta_k}\right)^2 \|v - I_h^k v\|_{0,M}^2 \\ &\le C \left(1 + \frac{1}{\beta_k}\right)^2 \sum_{\substack{T \subset M \\ T \in \mathcal{T}_h}} \frac{h_T^{2l}}{k^{2l}} \|v\|_{l,\omega_T}^2. \end{aligned} \tag{4.11}$$

The approximation property in the H^1-seminorm follows from the inverse inequality (2.1) for $z_h(v, M)$ and the approximation property (2.2) for $v - j_{h,k}v$. Finally, the orthogonality property (4.8) is a consequence of (4.10). □

Remark 4.2. The estimate (4.9) is optimal with respect to h_M and seemingly sub-optimal with respect to the polynomial order k due to their arising negative powers.

Remark 4.3. The paper [18] presents different variants for the choice of the discrete spaces $V_{h,r} \times Q_{h,s}$ and $D_{h,r} \times D_{h,s}$ using simplicial and hexahedral elements. There are basically a two-level and a one-level variant (indicated by $\mathcal{M}_h = \mathcal{T}_{2h}$ and $\mathcal{M}_h = \mathcal{T}_h$, respectively).

- Consider the two-level variant first (as shown in Fig. 3.1 for $d = 2$). Given the spaces

$$\mathbf{V}_{h,r} = [P_r(\mathcal{T}_h)]^d \cap V, \qquad Q_{h,s} = P_s(\mathcal{T}_h) \cap Q$$

for simplicial or hexahedral elements, the discontinuous coarse spaces are defined on $\mathcal{M}_h$ with polynomials of reduced polynomial degree as

$$D_{h,r} = [P_{\mathcal{M}_h,r-1}^{dc}]^d, \qquad D_{h,s} = P_{\mathcal{M}_h,s-1}^{dc}. \tag{4.12}$$

The inf-sup constants $\beta_{r/s}$ in Assumption 4.2 are independent of h, see [18], Lemmas 3.1, 3.2. Moreover, the β_k scale like $\mathcal{O}(\sqrt{k})$ for simplicial elements and like $\mathcal{O}(1)$ for quadrilateral elements in the affine linear case, see [19].

• The one-level variant with $M = K$ starts from given discontinuous spaces (4.12) and uses an enrichment of the local spaces $Y_{h,k}(K), k \in \{r, s\}$. For simplicial elements, define the set

$$P_k^{bub}(\hat{K}) = P_k(\hat{K}) + \hat{b} \cdot P_{k-1}(\hat{K}), \quad \hat{b}(\hat{x}) := (d+1)^{d+1} \prod_{i=1}^{d+1} \hat{\lambda}_i(\hat{x})$$

with the barycentric coordinates $\hat{\lambda}_i, i = 1, \cdots, d+1$. The enriched spaces are defined as

$$\begin{aligned} \mathbf{V}_{h,r} &= \{\mathbf{v} \in [H^1(\Omega)]^d \cap \mathbf{V} : \ \mathbf{v}|_K \circ F_K \in [P_r^{bub}(\hat{K})]^d \ \forall K \in \mathcal{T}_h\}, \\ Q_{h,s} &= \{q \in Q \cap H^1(\Omega) : \ q|_K \circ F_K \in P_s^{bub}(\hat{K}) \ \forall K \in \mathcal{T}_h\}. \end{aligned}$$

A similar construction is given in Section 4 of [18] for hexahedral elements. The inf-sup constants $\beta_{r/s}$ in Assumption 4.2 are independent of h, see [18], Lemmas 4.1-4.5.

4.5. A priori estimates

Lemma 4.6. *If the Assumption* 4.2 *holds for* $(P_r(\mathcal{T}_h), D_{h,r})$ *then we have a unique discrete solution* $\mathbf{u}_h \in \mathbf{U}_{h,r,s}$ *of* (4.3) *and the upper bound for the discrete pressure holds:*

$$\|p_h\|_0 \le C_1 \|\!|\mathbf{u}_h|\!\|_{lps} + \frac{1}{\beta_r} \|\mathbf{f}\|_{-1},$$

with a constant

$$\begin{aligned} C_1 \sim \sqrt{\nu} + \sqrt{C_P \sigma} + \min\Big(\frac{C_P}{\sqrt{\nu}}; \frac{1}{\sqrt{\sigma}}\Big) \|\mathbf{b}\|_{(L^\infty(\Omega))^d} \\ + \max_M \Big(\sqrt{\tau_M} \|\mathbf{b}\|_{(L^\infty(M))^d} + \sqrt{\mu_M} + \frac{h_M}{r\sqrt{\alpha_M}}\Big) \end{aligned}$$

and the Poincare constant C_P.

Remark 4.4. The design of the constant C_1 implies upper restrictions on the parameters τ_M and μ_M and a lower restriction (away from zero) of α_M. The proof of the estimate in Lemma 4.6 is based on the inf-sup constant of the continuous Stokes problem. The hope in the case of inf-sup stable interpolation pairs $\mathbf{V}_h \times Q_h$ is to omit die pressure stabilization by setting $\alpha_M = 0$. The constraint $\inf_M \alpha_M > 0$ can be removed in this case. See also Remark 4.9.

Theorem 4.1. *Assume for the solution of* (4.1) *the regularity* $\mathbf{u} \in [H^{l_v+1}(\Omega)]^d \times H^{l_p+1}(\Omega)$, *with* $0 \le l_v \le r$, $0 \le l_p \le s$. *Under the condition of Assumptions* 4.1 *and* 4.2 *for the pairs* $(P_k(\mathcal{T}_h), D_{h,k})$, $k \in \{r, s\}$ *we have the a priori estimate for the discrete solution of* (4.3)*:*

$$\|\!|\mathbf{u} - \mathbf{u}_h|\!\|_{lps}^2 \lesssim \sum_{M \in \mathcal{M}_h} \left[C_M^v \left(\frac{h_M}{r}\right)^{2l_v} \|\mathbf{v}\|_{l_v+1,\omega_M}^2 + C_M^p \left(\frac{h_M}{s}\right)^{2l_p} \|p\|_{l_p+1,\omega_M}^2 \right]$$

with the constants

$$\begin{aligned} C_M^v &= \Big(1 + \frac{1}{\beta_r}\Big)^2 r^2 \left(\nu + \frac{h_M^2}{r^4}(\sigma + \frac{1}{\tau_M} + \frac{1}{\alpha_M}) + \mu_M + \|\mathbf{b}\|_{[WL^{l_v,\infty}(M)]^d}^2 \tau_M\right), \\ C_M^p &= \Big(1 + \frac{1}{\beta_s}\Big)^2 s^2 \left(\alpha_M + \frac{h_M^2}{\mu_M s^4}\right). \end{aligned}$$

Proof. The error is split into two parts

$$\mathbf{u} - \mathbf{u}_h = (\mathbf{v} - j_{h,r}\mathbf{v}, p - j_{h,s}p) + (j_{h,r}\mathbf{v} - \mathbf{v}_h, j_{h,s}p - p_h).$$

We start with the approximation error $(\mathbf{v} - j_{h,r}\mathbf{v}, p - j_{h,s}p)$. Lemma 4.5 yields

$$\begin{aligned} |||(\mathbf{v} - j_{h,r}\mathbf{v}, p - j_{h,s}p)|||_{lps}^2 &\lesssim \Big(1 + \frac{1}{\beta_s}\Big)^2 \sum_{M\in\mathcal{M}_h} \frac{h_M^{2l_p}}{s^{2l_p-2}} \alpha_M \|p\|_{l_p+1,\omega_M}^2 \\ &+ \Big(1 + \frac{1}{\beta_r}\Big)^2 \sum_{M\in\mathcal{M}_h} \Big[\nu + \sigma\frac{h_M^2}{r^4} + \mu_M + \tau_M \|\mathbf{b}\|_{(L^\infty(M))^d}^2\Big] \frac{h_M^{2l_v}}{r^{2l_v-2}} \|\mathbf{v}\|_{l_v+1,\omega_M}^2. \end{aligned} \tag{4.13}$$

Now we estimate the remaining part $\mathbf{W}_h := (\mathbf{w}_h, r_h) = (j_{h,r}\mathbf{v} - \mathbf{v}_h, j_{h,s}p - p_h)$. The definition of $|||\cdot|||$ gives

$$\begin{aligned} & |||(j_{h,r}\mathbf{v} - \mathbf{v}_h, j_{h,s}p - p_h)|||_{lps} \\ &= \frac{(A + S_h^{lps})((j_{h,r}\mathbf{v} - \mathbf{v}_h, j_{h,s}p - p_h), \mathbf{W}_h)}{|||\mathbf{W}_h|||_{lps}} \\ &= \frac{(A + S_h^{lps})((\mathbf{v} - \mathbf{v}_h, p - p_h), \mathbf{W}_h)}{|||\mathbf{W}_h|||_{lps}} + \frac{(A + S_h^{lps})((j_{h,r}\mathbf{v} - \mathbf{v}, j_{h,s}p - p), \mathbf{W}_h)}{|||\mathbf{W}_h|||_{lps}} \\ &\equiv I + II. \end{aligned}$$

Applying Lemmas 4.3 and 4.4, the first term is bounded by

$$\begin{aligned} I &= \frac{S_h^{lps}((\mathbf{v}, p), \mathbf{W}_h)}{|||\mathbf{W}_h|||_{lps}} \\ &\lesssim \Big(\sum_{M\in\mathcal{M}_h} \Big(\frac{h_M^{2l_v}}{r^{2l_v}} (\|\mathbf{b}\|_{W^{l_v,\infty}(M)}^2 \tau_M + \gamma_M) |\mathbf{v}|_{l_v+1,M}^2 + \frac{h_M^{2l_p}}{r^{2l_p}} \alpha_M |p|_{l_p+1,M}^2 \Big) \Big)^{\frac{1}{2}}. \end{aligned}$$

Now we consider the terms of II separately. Integration by parts and property (4.8) yield

$$\begin{aligned} & (\nu\nabla(j_{h,r}\mathbf{v} - \mathbf{v}), \nabla\mathbf{w}_h) + (\sigma(j_{h,r}\mathbf{v} - \mathbf{v}), \mathbf{w}_h) + ((\mathbf{b}\cdot\nabla)(j_{h,r}\mathbf{v} - \mathbf{v}), \mathbf{w}_h) \\ &= (\nu\nabla(j_{h,r}\mathbf{v} - \mathbf{v}), \nabla\mathbf{w}_h) + (\sigma j_{h,r}\mathbf{v} - \mathbf{v}), \mathbf{w}_h) - (\kappa_{h,r}((\mathbf{b}\cdot\nabla)\mathbf{w}_h), j_{h,r}\mathbf{v} - \mathbf{v}) \\ &\lesssim \Big(1 + \frac{1}{\beta_r}\Big) \Big(\sum_{M\in\mathcal{M}_h} \frac{h_M^{2l_v}}{r^{2l_v-2}} \Big[\nu + \big(\sigma + \frac{1}{\tau_M}\big)\frac{h_M^2}{r^4}\Big] \|\mathbf{v}\|_{l_v+1,\omega_M}^2 \Big)^{\frac{1}{2}} |||\mathbf{W}_h|||_{lps}. \end{aligned}$$

The orthogonality property (4.8) results in

$$\begin{aligned} |(p - j_{h,s}p, \nabla\cdot\mathbf{w}_h)| &= |(p - j_{h,s}p, \kappa_{h,s}\nabla\cdot\mathbf{w}_h)| \\ &\lesssim \Big(1 + \frac{1}{\beta_s}\Big) \Big(\sum_{M\in\mathcal{M}_h} \frac{h_M^{2l_p+2}}{s^{2l_p+2}} \mu_M^{-1} \|p\|_{l_p+1,\omega_M}^2 \Big)^{\frac{1}{2}} |||\mathbf{W}_h|||_{lps},. \end{aligned}$$

Integration by parts (thanks to continuous discrete pressure) and again (4.8) lead to

$$\begin{aligned} & |(r_h, \nabla\cdot(j_{h,r}\mathbf{v} - \mathbf{v}))| \\ &\le |(\nabla r_h, j_{h,r}\mathbf{v} - \mathbf{v})| \ = \ |\kappa_{h,r}(\nabla r_h), j_{h,r}\mathbf{v} - \mathbf{v})| \\ &\lesssim \Big(1 + \frac{1}{\beta_r}\Big) \Big(\sum_{M\in\mathcal{M}_h} \frac{1}{\alpha_M} \frac{h_M^{2l_v+2}}{r^{2l_v+2}} \|\mathbf{v}\|_{l_v+1,\omega_M}^2 \Big)^{\frac{1}{2}} |||\mathbf{W}_h|||_{lps}. \end{aligned} \tag{4.14}$$

The estimation of the stabilization term is straight forward

$$
\begin{aligned}
&|S_h^{lps}((j_{h,r}\mathbf{v}-\mathbf{v}, j_{h,s}p-p), \mathbf{W}_h)\\
&\le \Big(S_h^{lps}((j_{h,r}\mathbf{v}-\mathbf{v}, j_{h,s}p-p), (j_{h,r}\mathbf{v}-\mathbf{v}, j_{h,s}p-p)\Big)^{\frac{1}{2}} \Big(S_h^{lps}(\mathbf{W}_h, \mathbf{W}_h)\Big)^{\frac{1}{2}}\\
&\lesssim \Big(1+\frac{1}{\beta_r}\Big)\Big(\sum_{M\in\mathcal{M}_h} \frac{h_M^{2l_v}}{r^{2l_v-2}} \Big[\tau_M\|\mathbf{b}\|^2_{(W^{l_v,\infty}(M))^d}+\mu_M\Big]\|\mathbf{v}\|^2_{l_v+1,\omega_M}\Big)^{\frac{1}{2}} |\!|\!|\mathbf{W}_h|\!|\!|_{lps}\\
&\quad + \Big(1+\frac{1}{\beta_s}\Big)\Big(\sum_{M\in\mathcal{M}_h} \alpha_M \frac{h_M^{2l_p}}{s^{2l_p-2}}\|p\|^2_{l_p+1,\omega_M}\Big)^{\frac{1}{2}} |\!|\!|\mathbf{W}_h|\!|\!|_{lps}.
\end{aligned}
$$

Adding up all inequalities for the estimate of $|\!|\!|\mathbf{W}_h|\!|\!|_{lps} = I + II$ together with the estimate of (4.13) gives the assertion. □

Lemma 4.7. *Under the same conditions as in Lemma* 4.6 *we obtain the following a priori estimate for the pressure:*

$$\|p-p_h\|_0 \le C_1 |\!|\!|\mathbf{u}-\mathbf{u}_h|\!|\!|_{lps}\,,$$

with a constant C_1 with the same parameter dependence as the one in Lemma 4.6*, see* [23]*.*

Now we can calibrate the parameters α_M, τ_M and μ_M with respect to the local mesh size h_M, the polynomial degrees r and s of the discrete ansatz functions and problem data. The parameters are determined by minimizing and balancing the terms of the right hand side of the general a priori error estimation. First, equilibrating the τ_M-dependent terms in C_M^v yields

$$\tau_M \sim \frac{h_M}{\|\mathbf{b}\|_{(W^{l_v,\infty}(M))^d} r^2}. \tag{4.15}$$

Similarly, equilibration of the terms in C_M^v and C_M^p involving μ_M and α_M yields

$$\mu_M \sim \frac{h_M^{l_p-l_v+1}}{k^{l_p-l_v+2}}, \qquad \alpha_M \sim \frac{h_M^{l_v-l_p+1}}{k^{l_v-l_s+2}}, \tag{4.16}$$

where we used $r \sim s$.

Remark 4.5. The design of the stabilization parameters is much simpler than for the original SUPG and PSPG stabilization. Nevertheless, the formula for τ_M is not fully convincing as the $W^{l_v+1,\infty}(M)$-norm of the vector field $\mathbf{b}$ has to be computed. A refined analysis shows that one can replace $\|\mathbf{b}\|_{(W^{l_v,\infty}(M))^d}$ with $\|\mathbf{b}\|_{(L^\infty(M))^d}$, see [21]. This is used in the computations below.

For the following results, we assume that the solution $(\mathbf{v}, p)$ of the continuous Oseen problem is sufficiently smooth.

4.6. Equal-order elements

For equal-order interpolation $r = s \ge 1$, let $l = l_v = l_p \le r$, we obtain from (4.15), (4.16)

$$\tau_M = \frac{\tau_0 h_M}{\|\mathbf{b}\|_{(W^{l,\infty}(M))^d} r^2}, \qquad \mu_M = \frac{\mu_0 h_M}{r^2}, \qquad \alpha_M = \frac{\alpha_0 h_M}{r^2}. \tag{4.17}$$

Then we obtain under the assumptions of Theorem 4.1

$$\|\mathbf{u}-\mathbf{u}_h\|_{lps}^2 \lesssim \Big(1+\frac{1}{\beta_r}\Big)^2 \sum_{M\in\mathcal{M}} \frac{h_M^{2l}}{r^{2l}} \Big[h_M\|p\|^2_{l+1,\omega_M} + r^2\Big(\nu+\frac{\sigma h_M^2}{r^4}+\|\mathbf{b}\|_{(W^{l,\infty}(M))^d}\,\frac{h_M}{r^2}\Big)\|\mathbf{v}\|^2_{l+1,\omega_M}\Big].$$

Remark 4.6. We discuss the optimality of the result:

• For fixed polynomial degree $r = s$, we obtain the optimal convergence rates $\mathcal{O}(h_M^{l+\frac{1}{2}})$ with respect to h_M if $\nu \lesssim h_M$ and $\sigma h_M \lesssim 1$.

• Due to the non-optimal convergence order of the interpolation operators $j_{h,k}$ in the $|\cdot|_1$-norm, these estimates are presumably not optimal with respect to polynomial degree r. Let us assume that in Lemma 4.5 there holds

$$\frac{h_M}{k}|v-j_{h,k}v|_{1,M} \lesssim \Big(1+\frac{1}{\beta_k}\Big)\Big(\frac{h_M}{k}\Big)^l\|v\|_{l,\omega_M}. \tag{4.18}$$

A careful check of the proofs leads to

$$\tau_M \sim \frac{h_M}{\|\mathbf{b}\|_{(W^{l,\infty}(\Omega))^d} r}, \qquad \mu_M \sim \alpha_M \sim \frac{h_M}{r}. \tag{4.19}$$

Then the a-priori estimate in Theorem 4.1 would be optimal with respect to r too with the possible exception of the factors depending on β_r.

Remark 4.7. Please note that the present analysis covers only the case of continuous pressure approximation. For an extension to discontinuous discrete pressure approximation, in particular to the case of $Q_k/P_{-(k-1)}$-elements, we refer to [19] for the two-level case.

4.7. Inf-sup stable elements

For inf-sup stable interpolation with $r = s + 1$, assume $l_v = l_p + 1 = r$ and set

$$\tau_M = \frac{\tau_0 h_M}{\|\mathbf{b}\|_{(W^{l_v,\infty}(M))^d} r^2}, \qquad \alpha_M = \frac{\alpha_0 h_M^2}{r^3}, \qquad \mu_M = \frac{\mu_0}{r} \tag{4.20}$$

according to (4.15) and (4.16). Then we obtain under the assumptions of Theorem 4.1

$$\|\mathbf{u}-\mathbf{u}_h\|_{lps}^2 \lesssim \Big(1+\frac{1}{\beta_r}\Big)^2 \sum_{M\in\mathcal{M}} \frac{h_M^{2l_v}}{r^{2l_v}} \Big(s\|p\|^2_{l_v,\omega_M} + \Big(\nu+\sigma\frac{h_M^2}{r^4}+\|\mathbf{b}\|_{(W^{l_v,\infty}(M))^d}\,\frac{h_M}{r^2}+\frac{1}{r}\Big)r^2\|\mathbf{v}\|^2_{l_v+1,\omega_M}\Big).$$

Remark 4.8. We discuss again the optimality of the result:

• For fixed polynomial degrees of the (inf-sup stable) Taylor-Hood pairs $V_{h,r+1} \times Q_{h,r}$, we obtain the optimal convergence rate $\mathcal{O}(h_M^r)$ with respect to h_M if $\nu \lesssim 1$ and $\sigma h_M^2 \lesssim 1$.

• Assume again that (4.18) is valid in Lemma 4.5. A careful check of the proof leads to

$$\tau_M \sim \frac{h_M}{\|\mathbf{b}\|_{(W^{l_v,\infty}(\Omega))^d} r}, \qquad \alpha_M \sim \frac{h_M^2}{r^2}, \qquad \mu_M \sim 1. \tag{4.21}$$

Then the a-priori estimate in Theorem 4.1 would be optimal with respect to r and s too with the possible exception of the factors depending on β_r and β_s.

Remark 4.9. For inf-sup stable pairs, a natural question is whether one can neglect the PSPG-type stabilization terms, i.e., setting $\alpha_M = 0$ (see also Remark 4.4). This problem is considered in [20] where a special interpolation operator is used which preserves the discrete divergence.

4.8. Some numerical results

Let us validate the design of the LPS parameters and the convergence rates for the Oseen problem (1.4)-(1.6) in $\Omega = (0,1)^2$ with the smooth solution

$$\mathbf{v}(x) = \big(\sin(\pi x_1), -\pi x_2 \cos(\pi x_1)\big), \quad p(x) = \sin(\pi x_1)\cos(\pi x_2)$$

and data $\mathbf{b} = \mathbf{v}$, $\sigma = 1$. A study of the one-level variant for equal-order pairs is given in [24]. The two-level variant is considered in [25] for equal-order and inf-sup stable pairs, see also [19]. Summarizing, all these experiments confirm the calibration of the stabilization parameters w.r.t. h_M and the theoretical a-priori convergence rates.

Here we present some typical results using either Q_2/Q_2 and Q_2/Q_1 pairs for velocity/pressure on unstructured, quasi-uniform meshes for the advection-dominated case $\nu = 10^{-6}$. The coarse spaces of the two-level variant are defined as in Remark 4.3. Table 4.1 shows comparable results for the best variants of the inf-sup stable Q_2/Q_1 and the equal-order Q_2/Q_2 pairs with the exception of the pressure error.

Table 4.1: Comparison of different variants of stabilization for problem (5.4) with $\nu = 10^{-6}, h = 1/64$.

Pair	τ_0	μ_0	α_0	$\lvert\mathbf{v} - \mathbf{v}_h\rvert_1$	$\lVert\mathbf{v} - \mathbf{v}\rVert_0$	$\lVert\nabla(\mathbf{v} - \mathbf{v}_h)\rVert_0$	$\lVert p - p_h\rVert_0$
Q_2/Q_1	0.0000	0.0000	0.0000	2.56E-1	5.42E-4	2.02E-1	2.31E-4
Q_2/Q_1	0.0562	0.5623	0.0000	1.91E-3	6.20E-6	1.66E-4	8.06E-5
Q_2/Q_1	0.0000	0.5623	0.0000	2.61E-3	7.42E-6	1.72E-4	8.05E-5
Q_2/Q_1	3.1623	0.0000	0.0000	1.87E-2	7.50E-5	1.56E-2	1.08E-4
Q_2/Q_2	0.0000	0.0000	0.0000	2.38E+1	5.35E-2	1.45E+1	1.66E+3
Q_2/Q_2	0.0000	0.0000	0.0178	1.65E-2	3.48E-5	9.37E-3	6.96E-6
Q_2/Q_2	0.0562	1.0000	0.0178	9.30E-4	2.85E-6	2.14E-4	4.31E-6
Q_2/Q_2	0.0562	0.0000	0.0178	1.77E-3	4.18E-6	1.46E-3	3.25E-6
Q_2/Q_2	0.0000	5.6234	0.0178	3.26E-3	7.20E-6	2.00E-4	7.56E-6

Remarkably, the importance of the stabilization terms is different. The fine-scale SUPG- and PSPG-type terms are necessary for the equal-order case but not for the inf-sup stable pair. On the other hand, the divergence-stabilization gives clear improvement for the inf-sup stable case and some improvement for the other case. Moreover, the PSPG-type term can be omitted for the inf-sup stable case, see Remark 4.9.

The effect of increasing polynomial degree for inf-sup stable Taylor-Hood pairs Q_r/Q_{r-1} with $r \in \{2,3,4,5\}$ is shown in Fig. 4.1 for $\nu = 10^{-6}, \sigma = 1$ and different values of h. Similar results are obtained (but non shown) for equal-order approximation with Q_r/Q_r-elements with $r \in \{1,2,3,4,5\}$.

5. Navier-Stokes Problem

The treatment of the full non-stationary incompressible Navier-Stokes model (1.7)-(1.9) is of major interest in applications. We start with a rather standard variant for the laminar

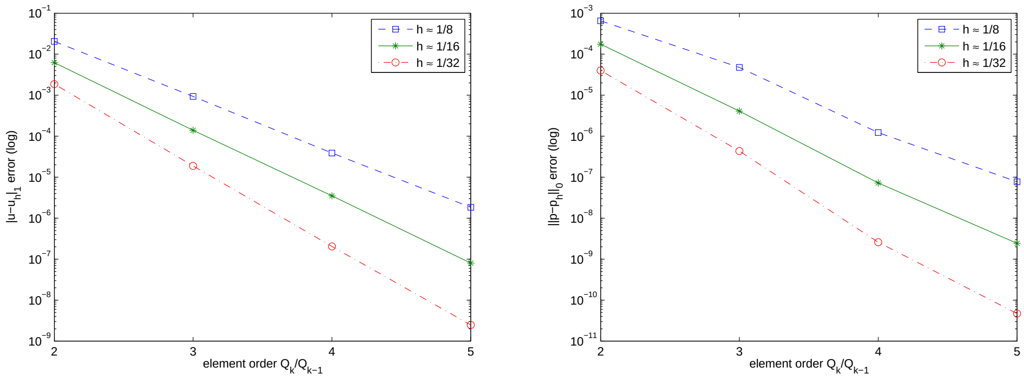

Fig. 4.1. Polynomial convergence for the Oseen problem with $\nu = 10^{-6}, \sigma = 1$ for fixed h. The errors $\|\nabla(u-u_h)\|_0$ (left) and $\|p-p_h\|_0$ (right) are shown.

case where the semi-discretization in time is performed first. Then we show the link to the variational multiscale method which has some potential for the treatment of turbulent flows.

5.1. Horizontal method of lines

We start with the Navier-Stokes model in the following strong form

$$\partial_t \mathbf{v} - \nu \triangle \mathbf{v} + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) + \nabla p = \mathbf{f}, \qquad \nabla \cdot \mathbf{v} = 0 \qquad \text{in } \Omega_T := \Omega \times I \tag{5.1}$$

with homogeneous Dirichlet condition $\mathbf{v} = \mathbf{0}$ on $\partial\Omega$ and time interval $I := [0, T]$.

The velocities and the pressure are sought in the Bochner spaces $\mathcal{V}^v := H^1(I, \mathbf{V})$, and $\mathcal{V}^p := L^2(I, Q)$, respectively. The product space will be denoted by $\mathcal{V} := \mathcal{V}^v \times \mathcal{V}^p$. The test functions are in the space $\mathcal{W} := L^2(I, \mathbf{U})$. The L^2-scalar product over Ω_T will be denoted by $(\cdot, \cdot)_{\Omega_T}$, and the corresponding L^2-norm by $\|\cdot\|_{\Omega_T}$.

The variational formulation of the Navier-Stokes equations (5.1) for $\mathbf{u} = \{\mathbf{v}, p\} \in \mathcal{V}$ reads: Find $\mathbf{u} \in \mathcal{V}$ such that $v(\cdot, 0) = v_0$ and

$$B(\mathbf{u}, \varphi) = \langle \mathbf{f}, \varphi \rangle \quad \forall \varphi \in \mathcal{W}, \tag{5.2}$$

where $B(\mathbf{u}, \varphi)$ is for $\varphi = \{\psi, \xi\}$ defined by

$$B(\mathbf{u}, \varphi) := (\partial_t \mathbf{v}, \psi)_{\Omega_T} - (\mathbf{v} \otimes \mathbf{v}, \nabla \psi)_{\Omega_T} + (\nu \nabla \mathbf{v}, \nabla \psi)_{\Omega_T} - (p, \operatorname{div} \psi)_{\Omega_T} + (\operatorname{div} \mathbf{v}, \xi)_{\Omega_T}\,.$$

A standard numerical approach to (5.2) is to semi-discretize in time first with an A-stable implicit scheme and to apply a fixed-point iteration within each time step. We split the time interval I into subintervals $I_n = (t_{n-1}, t_n]$, $n = 1, \cdots, N$ with $0 = t_0 < t_1 < \cdots < t_N = T$ and $\tau_n := t_n - t_{n-1}$ and introduce the space time slabs $I_n \times \Omega$. Consider, for simplicity only, the implicit Euler scheme. In the n-th time step it holds for $\mathbf{u}^n = \{\mathbf{v}^n, p^n\} := \mathbf{u}|_{I_n}$:

$$A_n(\mathbf{u}^n, \varphi) = g_n(\mathbf{u}^{n-1}, \varphi) \quad \forall \varphi \in \mathcal{W}, \tag{5.3}$$

with

$$A_n(\mathbf{u}, \varphi) := (\nu \nabla \mathbf{v}, \nabla \psi) + (\frac{1}{\tau_n} \mathbf{v}, \psi) - (\mathbf{v} \otimes \mathbf{v}, \nabla \psi) - (p, \operatorname{div} \psi) + (\operatorname{div} \mathbf{v}, \xi),$$

$$g_n(\mathbf{u}, \varphi) := \langle \mathbf{f}, \varphi \rangle + (\frac{1}{\tau_n} \mathbf{v}, \psi)\,.$$

A widely used linearization of (5.3) is the Oseen linearization (1.4)-(1.6) with $\sigma := \tau_n^{-1}$ and $\mathbf{b}$ a suitable divergence-free approximation on $\mathbf{v}^n$ (for instance the last iterate in the nonlinear iteration). Due to $(\mathbf{b} \otimes \mathbf{v}, \nabla\psi) = -((\mathbf{b}\cdot\nabla)\mathbf{v}, \psi)$, it leads to auxiliary Oseen problems

$$(\nu\nabla\mathbf{v}, \nabla\psi) + (\sigma\mathbf{v}, \psi) + ((\mathbf{b}\cdot\nabla))\mathbf{v}, \psi) - (p, \operatorname{div}\psi) + (\operatorname{div}\mathbf{v}, \xi) = \langle\tilde{\mathbf{f}}, \varphi\rangle\,, \tag{5.4}$$

with the linear functional $\tilde{\mathbf{f}}$ given by $\langle\tilde{\mathbf{f}}, \varphi\rangle := \langle\mathbf{f}, \varphi\rangle + (\sigma\mathbf{v}^n, \psi)$. Next, one discretizes the auxiliary problem in space and applies the LPS technique as discussed in Section 4. In particular, the numerical analysis can be applied for each time step.

Remark 5.1. The analysis of Section 4 can be extended to problems resulting from Newton iteration including the term $(\mathbf{v}\cdot\nabla)\mathbf{b}$. Sufficiently small time steps ensure coercivity of the modified bilinear form $A(\cdot,\cdot)$ appearing in (4.1). The analysis of the fully discretized Navier-Stokes problem remains an open problem.

Here, we apply the LPS stabilization to the lid-driven cavity flow as a standard Navier-Stokes benchmark problem (5.1) with $\mathbf{f} = \mathbf{0}$. No-slip data are prescribed with the exception of the upper part of the cavity where $\mathbf{v} = (1, 0)^T$ is given.

A quasi-uniform mesh is used together with the Q_2/Q_1 and Q_2/Q_2 pairs using the two-level LPS variant with scaling parameter τ_0 and μ_0 according to the Oseen case and $\alpha_0 = 0$.

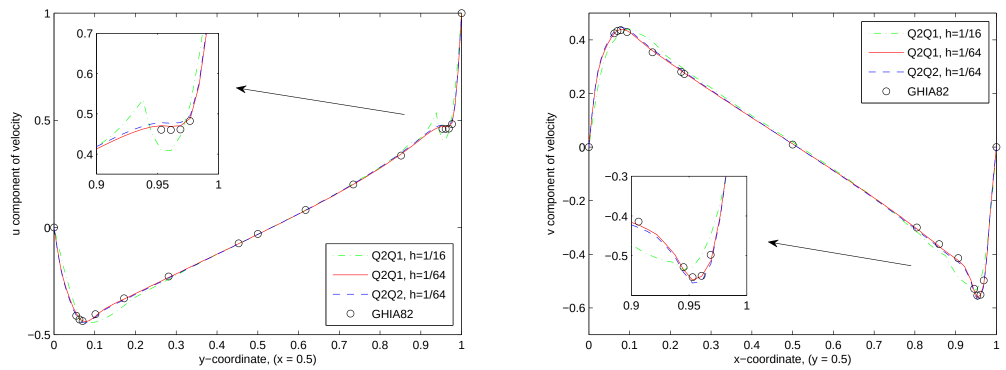

Fig. 5.1. Lid driven-cavity problem with $Re = 5,000$: Cross-sections of the discrete solutions for Q_2/Q_1 pair with $\tau_0 = \alpha_0 = 0$ and $\mu_0 = 1$ and Q_2/Q_2 pair with $\tau_0 = \alpha_0 = \mu_0 = 1$.

Fig. 5.1 shows typical velocity profiles for $Re = 5,000$. The results for $h = \frac{1}{64}$ for both variants are in very good agreement with [26]. The boundary layers are well resolved on this quasi-uniform mesh. Moreover, the solution for a coarse grid with $h = \frac{1}{16}$ is in good agreement with [26] away from the boundary layers. The results for this nonlinear problem confirm the previous remarks for the linearized problem of Oseen type. For the Q_2/Q_1 element, only the divergence stabilization is necessary whereas for the Q_2/Q_2 pair all stabilization terms are important.

In Table 5.1, we compare the positions and values of extrema of the velocity profiles for different values of Re. The results for the two-level LPS scheme with the Q_2/Q_1 pair on the fine mesh with $h \approx 1/256$ are slightly different from the results in [26, 27]. On the other hand, the LPS results on the coarser grid with $h \approx 1/32$ are in good agreement with the case $h \approx 1/256$. We refer to h-convergence studies in [25].

Table 5.1: Lid driven-cavity problem for different values of Re: Maxima and minima on cross-sections $x = 0.5$ and $y = 0.5$; a) LPS Q_2Q_1, $h \approx 1/32$ b) LPS Q_2Q_1, $h \approx 1/256$ c) GHIA82,$h = 1/256$ d) ZHANG90,$h = 1/256$.

Re		$u_{\min}$	$y_{\min}$	$v_{\max}$	$x_{\max}$	$v_{\min}$	$x_{\min}$
1000	a)	-0.38512	0.17578	0.37404	0.16016	-0.52295	0.9062
	b)	-0.38857	0.17188	0.37692	0.15625	-0.52701	0.91016
	c)	-0.38289	0.1719	0.37095	0.1563	-0.51550	0.9063
	d)	-0.39009	0.16992	0.37847	0.15820	-0.52839	0.90820
3200	a)	-0.42407	0.09766	0.42290	0.10156	-0.55544	0.94531
	b)	-0.43588	0.09375	0.43298	0.09766	-0.56791	0.94922
	c)	-0.41933	0.1016	0.42768	0.0938	-0.54053	0.9453
	d)	-0.44006	0.09180	0.43814	0.09570	-0.57228	0.94727
7500	a)	-0.43940	0.07031	0.43749	0.07813	-0.56560	0.96484
	b)	-0.45479	0.06250	0.45837	0.06641	-0.58045	0.96484
	c)	-0.43590	0.0625	0.44030	0.0703	-0.55216	0.9609
	d)	-0.46413	0.06445	0.47129	0.06836	-0.58878	0.96289

5.2. Link to the variational multiscale method

One of the major challenges in computational fluid dynamics is the accurate computation of different quantities of turbulent flows. Several new concepts have been proposed such as the dynamic multilevel methodology (DML) of Dubois et al. [28] or the variational multiscale method (VMS) of Hughes et al. [29]. In VMS, reference is made to residual free bubble techniques, see Brezzi & Russo [30], and subgrid viscosity as introduced by Guermond [31] to motivate an approach to Large-Eddy simulation (LES) where the turbulence model acts only on the fine scales. Here, we will show how the LPS method may be cast in the VMS framework.

In the VMS, see [29], a scale separation is performed. The turbulence model acts only on the finer scales; however, certain model assumptions on the interaction between the scales are made. To fix ideas, we use the three-level partition proposed in [32]. Hence we consider a scale separation into large resolved scales, denoted by $\bar{\mathbf{v}}$, small resolved scales denoted by $\tilde{\mathbf{v}}$ and unresolved scales denoted by $\hat{\mathbf{v}}$. The solution space is partitioned in a corresponding manner

$$\mathcal{V} = \bar{\mathcal{V}} \oplus \tilde{\mathcal{V}} \oplus \hat{\mathcal{V}}.$$

The function space $\mathcal{W}$ is partitioned similarly, $\mathcal{W} = \bar{\mathcal{W}} \oplus \tilde{\mathcal{W}} \oplus \hat{\mathcal{W}}$, with corresponding test functions, for instance, $\bar{\varphi} = \{\bar{\psi}, \bar{\xi}\} \in \bar{\mathcal{W}}$. We now write the exact equations of motions for each scale

$$B(\mathbf{u}, \bar{\varphi}) = \langle \mathbf{f}, \bar{\varphi} \rangle \quad \forall \bar{\varphi} \in \bar{\mathcal{W}}, \tag{5.5}$$

$$B(\mathbf{u}, \tilde{\varphi}) = \langle \mathbf{f}, \tilde{\varphi} \rangle \quad \forall \tilde{\varphi} \in \tilde{\mathcal{W}}, \tag{5.6}$$

$$B(\mathbf{u}, \hat{\varphi}) = \langle \mathbf{f}, \hat{\varphi} \rangle \quad \forall \hat{\varphi} \in \hat{\mathcal{W}}. \tag{5.7}$$

Introducing the linearized Navier-Stokes operator

$$\begin{aligned} B'(\mathbf{u}, \mathbf{u}', \varphi) := (\partial_t \mathbf{v}', \hat{\psi})_{\Omega_T} - (\mathbf{v}' \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{v}', \nabla \psi)_{\Omega_T} \\ - (p', \nabla \cdot \psi)_{\Omega_T} + (\nu \nabla \mathbf{v}', \nabla \psi)_{\Omega_T} + (\nabla \cdot \mathbf{v}', \xi)_{\Omega_T}, \end{aligned}$$

the Reynolds stress projection and the cross stress projection operator

$$R(\mathbf{v}, \psi) := (\mathbf{v} \otimes \mathbf{v}, \nabla \psi)_{\Omega_T}, \qquad C(\mathbf{v}, \hat{\mathbf{v}}, \psi) := (\mathbf{v} \otimes \hat{\mathbf{v}} + \hat{\mathbf{v}} \otimes \mathbf{v}, \nabla \psi)_{\Omega_T},$$

respectively, we may reformulate the exact equations for each scale in a fashion that makes evident the coupling between the scales. Following [32], the exact solution $\bar{\mathbf{v}} \in \bar{\mathcal{V}}$ for the resolved large scales fulfills for all $\bar{\varphi} \in \bar{\mathcal{W}}$ the equation

$$\begin{aligned} &B(\bar{\mathbf{u}}, \bar{\varphi}) + B'(\bar{\mathbf{u}}, \tilde{\mathbf{u}}, \bar{\varphi}) - R(\tilde{\mathbf{v}}, \bar{\psi}) \\ &= \langle \mathbf{f}, \bar{\varphi} \rangle - B'(\bar{\mathbf{u}}, \hat{\mathbf{u}}, \bar{\varphi}) - R(\hat{\mathbf{v}}, \bar{\psi}) + C(\tilde{\mathbf{v}}, \hat{\mathbf{v}}, \bar{\psi}) . \end{aligned} \tag{5.8}$$

The first line in (5.8) describes the influence of the resolved scales on the large scales, the second line includes the influence of the unresolved scales on the large scales. Similarly, the small resolved scales $\tilde{\mathbf{v}} \in \tilde{\mathcal{V}}$ fulfill for all $\tilde{\varphi} \in \tilde{\mathcal{W}}$:

$$\begin{aligned} &B'(\bar{\mathbf{u}}, \tilde{\mathbf{u}}, \tilde{\varphi}) - R(\tilde{\mathbf{v}}, \tilde{\psi}) \\ &= \langle \mathbf{f}, \tilde{\varphi} \rangle - B(\bar{\mathbf{u}}, \tilde{\varphi}) - B'(\bar{\mathbf{u}}, \hat{\mathbf{u}}, \tilde{\varphi}) - R(\hat{\mathbf{v}}, \tilde{\psi}) + C(\tilde{\mathbf{v}}, \hat{\mathbf{v}}, \tilde{\psi}) . \end{aligned} \tag{5.9}$$

The unresolved scales $\hat{\mathbf{v}} \in \hat{\mathcal{V}}$ finally satisfy the following equation for all $\hat{\varphi} \in \hat{\mathcal{W}}$

$$B'(\bar{\mathbf{u}} + \tilde{\mathbf{u}}, \hat{\mathbf{u}}, \hat{\varphi}) + R(\hat{\mathbf{v}}, \hat{\psi}) = \langle \mathbf{f}, \hat{\varphi} \rangle - B(\bar{\mathbf{u}} + \tilde{\mathbf{u}}, \hat{\varphi}) .$$

As a consequence, the equation for the unresolved scales is driven by the residual of the resolved scales. With the given equations we state the modeling assumptions as follows:

Modelling assumption (M1): The unresolved scales $\hat{\mathbf{v}}$ have no "direct" influence on the large scales, i.e.the second line of equation (5.8) is set to zero:

$$-B'(\bar{\mathbf{u}}, \hat{\mathbf{u}}, \bar{\varphi}) - R(\hat{\mathbf{v}}, \bar{\psi}) + C(\tilde{\mathbf{v}}, \hat{\mathbf{v}}, \bar{\psi}) = 0 \quad \forall\ \bar{\varphi} \in \bar{\mathcal{W}}. \tag{5.10}$$

Modelling assumption (M2): The influence of the unresolved scales on the small scales is modeled by an artificial viscosity term $S : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ with $\mathcal{X} := (\bar{\mathcal{V}} \oplus \tilde{\mathcal{V}}) \cup (\bar{\mathcal{W}} \oplus \tilde{\mathcal{W}})$, acting only on the small resolved scales. Hence we assume in (5.9) that for $\tilde{\varphi} \in \tilde{\mathcal{W}}$:

$$S(\tilde{\mathbf{u}}, \tilde{\varphi}) \approx B'(\bar{\mathbf{u}}, \hat{\mathbf{u}}, \tilde{\varphi}) + R(\hat{\mathbf{v}}, \tilde{\psi}) - C(\tilde{\mathbf{v}},\ \hat{\mathbf{v}}, \tilde{\psi}) . \tag{5.11}$$

(M1) may hold true when the main features of the flow are resolved; hence this is the large eddy assumption. (M2) implies that the unresolved scales only has the effect of dissipating energy from the small resolved scales. Roughly speaking, if (M1) is satisfied then the design of the subgrid model is less important as long as it allows for a sufficient rate of energy dissipation from the resolved small scales to the unresolved scales. By the conservation properties of the Galerkin method, insufficient dissipation may cause buildup of energy in high frequency modes leading to spurious oscillations. Excessive dissipation will cause too much damping of the resolved small scales leading to poorer resolution of the large scales through the Reynolds stress coupling.

Using these modeling assumptions and the L^2-projection Πv_0 of the initial conditions onto the resolved scales $\bar{\mathcal{V}}^v \oplus \tilde{\mathcal{V}}^v$ we obtain the formulation: $(\bar{\mathbf{v}} + \tilde{\mathbf{v}})(\cdot, 0) = \Pi \mathbf{v}_0$ and

$$\begin{aligned} &B(\bar{\mathbf{u}} + \tilde{\mathbf{u}}, \bar{\varphi}) = \langle \mathbf{f}, \bar{\varphi} \rangle && \forall \bar{\varphi} \in \bar{\mathcal{W}} , \\ &B(\bar{\mathbf{u}} + \tilde{\mathbf{u}}, \tilde{\varphi}) + S(\tilde{\mathbf{v}}, \tilde{\varphi}) = \langle \mathbf{f}, \tilde{\varphi} \rangle && \forall \tilde{\varphi} \in \tilde{\mathcal{W}}. \end{aligned} \tag{5.12}$$

The subgrid model is chosen to be coercive on the small resolved scales, i.e.,

$$S(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) \geq c \|\nabla \tilde{\mathbf{u}}\|_1^2, \quad \forall \tilde{\mathbf{u}} \in \tilde{\mathcal{W}} ,$$

symmetric $S(\mathbf{u},\varphi)=S(\varphi,\mathbf{u}) \quad \forall \mathbf{u},\varphi\in\mathcal{X}$, and such that it vanishes on the large resolved scales

$$S(\cdot,\bar{\varphi})=0, \quad \forall \bar{\varphi}\in \bar{\mathcal{W}}\cup\bar{\mathcal{V}}. \tag{5.13}$$

Assume that some finite element approximation $\mathcal{V}_h$ of $\mathcal{V}$ represent the resolved scales $\mathcal{V}_h = \bar{\mathcal{V}}\oplus\tilde{\mathcal{V}}$. This space is then decomposed in large and small resolved scales by choosing $\bar{\mathcal{V}}=\mathcal{V}_H$ where $\mathcal{V}_H\subset\mathcal{V}_h$. To indicate its dependence on h we equip the subgrid viscosity with a subscript, $S_h(\cdot,\cdot)$. The same discrete space is used for the test space $\mathcal{W}_h=\bar{\mathcal{W}}\oplus\tilde{\mathcal{W}}$. The discrete version of (5.12) becomes: find $\mathbf{u}_h\in\mathcal{V}_h$ such that $\mathbf{v}_h(\cdot,0)=\pi v_0$ and

$$B(\mathbf{u}_h,\varphi)+S_h(\tilde{\mathbf{u}}_h,\tilde{\varphi})=\langle \mathbf{f},\varphi\rangle \quad \forall\varphi\in\mathcal{W}_h\,, \tag{5.14}$$

or using the scale separation property (5.13) of $S_h(\cdot,\cdot)$

$$B(\mathbf{v}_h,\varphi)+S_h(\mathbf{u}_h,\varphi)=\langle \mathbf{f},\varphi\rangle \quad \forall\varphi\in\mathcal{W}_h\,. \tag{5.15}$$

The properties of $S_h(\cdot,\cdot)$ imply Galerkin orthogonality for the error $\mathbf{u}-\mathbf{u}_h$ on the large resolved scales:

$$B(\mathbf{u}-\mathbf{u}_h,\bar{\varphi})=0 \quad \forall\bar{\varphi}\in\mathcal{W}_H\,. \tag{5.16}$$

Consider again the time subintervals $I_n=(t_{n-1},t_n]$, $n=1,\cdots,N$ with $0=t_0<t_1<\cdots<t_N=T$ and $\tau_n:=t_n-t_{n-1}$. As time integration scheme, we apply, e.g., the Crank-Nicholson scheme. It means that we choose piecewise d-linears ansatz functions and piecewise constant (discontinuous) test function, precisely:

$$\mathcal{V}_h=P^1_\tau(I,X_h)\,, \qquad \mathcal{W}_h=P^0_\tau(I,X_h)\,.$$

The spaces $\mathcal{V}_H$ and $\mathcal{W}_H$ are defined similarly by using X_H. With these finite element spaces we now propose the following FEM: Find $\mathbf{u}_h\in\mathbf{u}_0+\mathcal{V}_h$, so that it holds for $\mathbf{u}^n=\{\mathbf{v}^n,p^n\}:=\mathbf{u}_h|_{I_n}$:

$$A_n^{vms}(\mathbf{u}^n,\varphi)+S_h(\mathbf{u}^n,\varphi)=g_n^{vms}(\mathbf{u}^{n-1},\varphi) \quad \forall\varphi\in\mathcal{W}_h\,, \tag{5.17}$$

with

$$\begin{aligned}
A_n^{vms}(\mathbf{u},\varphi) &:= (\tau_n^{-1}\mathbf{v},\psi)-(\mathbf{v}\otimes\mathbf{v},\nabla\psi)-(p,\nabla\cdot\psi)+(\nabla\cdot\mathbf{v},\xi)+(\mu\nabla\mathbf{v},\nabla\psi),\\
g_n^{vms}(\mathbf{u},\varphi) &:= \langle\mathbf{f},\varphi\rangle+(\tau_n^{-1}\mathbf{v},\psi)-(\mu\nabla\mathbf{v},\nabla\psi)+(\mathbf{v}\otimes\mathbf{v},\nabla\psi)-S_h(\mathbf{u},\varphi)\,.
\end{aligned}$$

A widely used linearization of (5.17) is again the Oseen linearization (1.4)-(1.6). With these notations, we take (4.4) as subgrid model. This subgrid operator satisfies (5.13) on simplicial and on quadrilateral/hexaedral meshes exactly.

The numerical analysis of the VMS approach with the subgrid model of modified LPS type

$$S_h^{lps}(\mathbf{u},\mathbf{z}):=(\kappa_{h,r}(\nabla p),\alpha\kappa_{h,r}(\nabla q))+(\kappa_{h,r}(\nabla\mathbf{v}),\delta\kappa_{h,r}(\nabla\mathbf{w})) \tag{5.18}$$

has been analyzed in [33]. The numerical analysis as well as the practical application of the VMS is currently an area of very active research.

6. Adjoint Stabilization in Optimization

The numerical computation of optimal control problems with constraints given by partial differential equations can be divided into two main approaches: One may consider first the discretized problem and then build the optimality condition. The other possibility is to formulate first the optimality condition on the continuous level and then discretize. Both approaches lead to different discrete adjoint equations when discretization and building the adjoint do not commute. This type of inconsistency takes place when conventional residual based stabilized finite elements for flow problems as for instance *streamline upwind / Petrov–Galerkin* (SUPG) are used, because they are non-symmetric. Consequently, the computed control is significantly affected by the way of defining the discrete optimality condition.

An analytical error estimate for convection-diffusion-reaction equations with the SUPG method is given by Collis and Heinkenschloss in [34] where for the two approaches "discretize-optimize" and "optimize-discretize" different a priori estimates are derived. The estimate for "optimize-discretize" has a better asymptotic behaviour in terms of powers of the mesh size. In numerical tests, the largest difference is observed in the adjoint variable. For convection-diffusion problems with a particular least-squares stabilization Dedé and Quarteroni [35] derived an a posteriori estimate and used it for local mesh refinement. Becker and Vexler [36] presented an a priori estimate for optimal control with such a scalar equation for finite elements with local projection stabilization.

Since more inconsistent terms appear in systems of equations, Abraham et al. investigated numerically the *Galerkin Least-Squares* (GLS) stabilization for the Oseen system in [37]. Herein, a significant difference is observed between both approaches. Moreover, the computed control appears to be very sensitive to the evaluation of stabilization parameters. Therefore, the authors conclude that it is questionable whether the GLS approach is suitable for optimal control problems.

Li and Petzold [38] discussed this topic as well in the context of (a) consistent discrete boundary conditions for the adjoint problem and (b) adaptive mesh refinement. They propose a combination of "discretize-optimize" and "optimize-discretize" by splitting the domain into an inner part and a boundary part. The aspect of stabilization due to the presence of convective terms or due to a saddle point structure of the primal equation is not considered.

Obviously, there is a need for symmetric stabilization so that discretization and building the adjoint commute. The local projection stabilization has this feature. In this section we show that LPS leads to a consistent and stable adjoint problem in the context of optimal control.

6.1. Oseen system with control and Karush-Kuhn-Tucker system

The Oseen problem as in (1.4)-(1.6) which additional control $\mathbf{q} \in M := [L^2(\Omega)]^d$ in the momentum equation is given by:

$$-\nu\Delta\mathbf{v} + (\mathbf{b}\cdot\nabla)\mathbf{v} + \sigma\mathbf{v} + \nabla p + B\mathbf{q} = f \quad \text{in } \Omega\,, \tag{6.1a}$$

$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega\,, \tag{6.1b}$$

$$\mathbf{v} = 0 \quad \text{on } \partial\Omega\,, \tag{6.1c}$$

with a continuous linear operator $B : M \to M$. All functions are in the same spaces as previously in Section 4. Note, that the control $\mathbf{q}$ is written in bold face in order to prevent confusion with the pressure test function q. The objective functional under consideration is of the form:

$$J(\mathbf{u},\mathbf{q}) := \frac{1}{2}\|C\mathbf{v} - C\widehat{\mathbf{v}}\|^2 + \frac{\alpha}{2}\|\mathbf{q}\|^2\,, \tag{6.2}$$

with a linear continuous operator $C : M \to M$, a target state $\widehat{\mathbf{v}} \in M$ and a regularization parameter $\alpha > 0$. Hence, the optimal control problem reads

$$\arg\min \Big\{ J(\mathbf{u},\mathbf{q}) : \mathbf{u} \text{ is solution of (6.1) for control } \mathbf{q} \in M \Big\}. \tag{6.3}$$

In optimization problems, $\mathbf{u} = (\mathbf{v}, p)$ is called primal state, whereas the adjoint (or dual) state $\mathbf{u}^* = (\mathbf{v}^*, p^*)$ describes the sensitivity of $J(\mathbf{u}, q)$ with respect to changes in the control. The Karush-Kuhn-Tucker (KKT) system for $\mathbf{u}$, $\mathbf{u}^*$ and $\mathbf{q}$ is an equation system whose solution gives the solution of the optimal control problem (6.3). In the case of the Oseen problem the KKT system is given by:

$$\left.\begin{array}{rcl} -\nu\Delta\mathbf{v} + (\mathbf{b}\cdot\nabla)\mathbf{v} + \sigma\mathbf{v} + \nabla p + B\mathbf{q} &=& f \quad \text{in } \Omega\,, \\ \operatorname{div}\mathbf{v} &=& 0 \quad \text{in } \Omega\,, \\ \mathbf{v} &=& 0 \quad \text{on } \partial\Omega\,, \end{array}\right\} \tag{6.4}$$

$$\left.\begin{array}{rcl} -\nu\Delta\mathbf{v}^* - (\mathbf{b}\cdot\nabla)\mathbf{v}^* + \sigma\mathbf{v}^* - \nabla p^* + C\mathbf{v} &=& C\widehat{\mathbf{v}} \quad \text{in } \Omega\,, \\ -\operatorname{div}\mathbf{v}^* &=& 0 \quad \text{in } \Omega\,, \\ \mathbf{v}^* &=& 0 \quad \text{on } \partial\Omega\,. \end{array}\right\} \tag{6.5}$$

$$\alpha\mathbf{q} + B^*\mathbf{v}^* = 0 \quad \text{in } \Omega\,. \tag{6.6}$$

The equations for the adjoint state (6.5) are also of Oseen type, so that the LPS method would be a possible way to discretize. In the next subsection we will give the corresponding discrete version.

6.2. Stabilized discrete Karush-Kuhn-Tucker system

Starting with the continuous KKT system (6.4)-(6.6) and discretize the primal and dual Oseen system with local projection stabilization, leads to:

$$A(\mathbf{u}_h, \mathbf{z}_h) + (B\mathbf{q}_h, \mathbf{w}_h) + S_h^{lps}(\mathbf{u}_h, \mathbf{z}_h) = (\mathbf{f}, \mathbf{w}_h) \qquad \forall \mathbf{z}_h = (\mathbf{w}_h, q_h) \in X_{r,s}, \tag{6.7}$$

$$A(\mathbf{z}_h, \mathbf{u}_h^*) + S_h^{lps}(\mathbf{u}_h^*, \mathbf{z}_h) = (C(\widehat{\mathbf{v}} - \mathbf{v}_h), \mathbf{w}_h) \qquad \forall \mathbf{z}_h = (\mathbf{w}_h, q_h) \in X_{r,s}, \tag{6.8}$$

$$(\alpha\mathbf{q}_h, \mathbf{y}_h) + (\mathbf{v}_h, B\mathbf{y}_h) = 0 \qquad \forall \mathbf{y}_h \in M_h\,. \tag{6.9}$$

Here, M_h is a finite dimensional subspace of M. Note, that the same discrete system is obtained if we start with the discretized primal problem with LPS stabilization and form the KKT system for the discrete primal equation. Due to the symmetry of LPS, discretization and optimization commute. This is definitely different to residual-based stabilization where different discrete KKT systems are obtained depending whether the KKT system is considered for the discretized primal equation ("discretize-optimize") or the continuous KKT system is discretized ("optimize-discretize").

For the following lemma we need the discrete solution $\mathbf{u}_h(\mathbf{q}) = (\mathbf{v}_h(\mathbf{q}), p_h(\mathbf{q})) \in X_{r,s}$ of the discrete primal equation with the exact (optimal) control $\mathbf{q}$:

$$\begin{aligned} &A(\mathbf{u}_h(\mathbf{q}), \mathbf{z}_h) + (B\mathbf{q}, \mathbf{w}_h) + S_h^{lps}(\mathbf{u}_h(\mathbf{q}), \mathbf{z}_h) \\ &= (\mathbf{f}, \mathbf{w}_h) \qquad \forall \mathbf{z}_h = (\mathbf{w}_h, q_h) \in X_{r,s}\,. \end{aligned} \tag{6.10}$$

Analogously, the discrete adjoint solution for the exact (optimal) velocity is denoted by $\mathbf{u}_h^*(\mathbf{v}) = (\mathbf{v}_h^*(\mathbf{v}), p_h^*(\mathbf{v}))$:

$$A(\mathbf{z}_h, \mathbf{u}_h^*(\mathbf{v})) + S_h^{lps}(\mathbf{u}_h^*(\mathbf{v}), \mathbf{z}_h) = (C(\widehat{\mathbf{v}} - \mathbf{v}), \mathbf{w}_h) \quad \forall \mathbf{z}_h = (\mathbf{w}_h, q_h) \in X_{r,s}\,. \tag{6.11}$$

Lemma 6.1. *We suppose* $\{\mathbf{u},\mathbf{u}^*,\mathbf{q}\}\in[H^{r+1}(\Omega)]^{3d+2}$ *for the continuous solution of the optimal control problem* (6.4)-(6.6)*. Then it holds for the control* q_h *of the discretized system with local projection stabilization* (6.7)-(6.9)*:*

$$\|\mathbf{q}-\mathbf{q}_h\|_0 \lesssim \Big(1+\frac{1}{\alpha\sigma}\Big)\|\mathbf{q}-I_{M_h}\mathbf{q}\|_0+\frac{1}{\alpha\sigma^{1/2}}\|\mathbf{v}_h(\mathbf{q})-\mathbf{v}\|_0+\frac{1}{\alpha}\|\mathbf{v}_h^*(\mathbf{q})-\mathbf{v}^*\|_0\,, \qquad (6.12)$$

for an arbitrary interpolation operator $I_{M_h}:M\to M_h$*.*

Proof. We introduce the interpolant $I_{M_h}\mathbf{q}$ and apply the triangle inequality. Hence, it is sufficient to bound the projection error $\|I_{M_h}\mathbf{q}-\mathbf{q}_h\|_0$ by the right hand side of (6.12). Due to the existence and uniqueness of the continuous and the discrete Oseen problems the solution operators $\mathcal{S}:L^2(\Omega)\to\mathbf{V}$ and $\mathcal{S}_h:L^2(\Omega)\to\mathbf{V}_r$, defined by $\mathcal{S}(\mathbf{q})=\mathbf{u}$ and $\mathcal{S}_h(\mathbf{q})=\mathbf{u}_h$ respectively, we can consider the reduced functionals

$$j(\mathbf{q}):=J(\mathcal{S}(\mathbf{q}),\mathbf{q})\,,\qquad j_h(\mathbf{q}):=J(\mathcal{S}_h(\mathbf{q}),\mathbf{q})\,.$$

The reduced optimization problems are:

$$\min_{\mathbf{q}\in M} j(\mathbf{q})\qquad\text{and}\qquad\min_{\mathbf{q}_h\in M_h} j_h(\mathbf{q}_h)\,.$$

The corresponding (continuous and discrete) optimality conditions are

$$\begin{aligned} j'(\mathbf{q})(\mathbf{y}) &= (\mathbf{v}^*,B\mathbf{y})+(\alpha\mathbf{q},\mathbf{y})=0 \qquad \forall\mathbf{y}\in M\,,\\ j_h'(\mathbf{q}_h)(\mathbf{y}_h) &= (\mathbf{v}_h^*,B\mathbf{y}_h)+(\alpha\mathbf{q}_h,\mathbf{y}_h)=0 \quad \forall\mathbf{y}_h\in M_h\,. \end{aligned}$$

A basic calculus ensures $j''(\cdot)(\mathbf{y},\mathbf{y})\geq\alpha\|\mathbf{y}\|^2$ for all $\mathbf{y}\in M$ and also for the discrete reduced functional:

$$j_h''(\cdot)(\mathbf{y}_h,\mathbf{y}_h)\geq\alpha\|\mathbf{y}_h\|^2\quad\forall\mathbf{y}_h\in M_h\,.$$

Furthermore, since $j_h(\mathbf{q})$ is at most quadratic, it implies for arbitrary $\mathbf{y}_h\in M_h$:

$$j_h''(\mathbf{q}_h)(\cdot,\mathbf{y}_h)=j_h'(\mathbf{q}_h+\mathbf{y}_h)(\cdot)-j_h'(\mathbf{q}_h)(\cdot)\,.$$

We obtain in particular for $\mathbf{y}_h\in M_h$:

$$\alpha\|\mathbf{y}_h\|^2\leq j_h''(\mathbf{q}_h)(\mathbf{y}_h,\mathbf{y}_h)\;=\;j_h'(\mathbf{q}_h+\mathbf{y}_h)(\mathbf{y}_h)-j_h'(\mathbf{q}_h)(\mathbf{y}_h)\,.$$

Let us denote the discrete solution of the adjoint equation for given $\widetilde{\mathbf{u}}_h=(\widetilde{\mathbf{v}}_h,\widetilde{p}_h)=S_h(I_{M_h}\mathbf{q})$ by $\widetilde{\mathbf{u}}_h^*=(\widetilde{\mathbf{v}}_h^*,\widetilde{p}_h^*)$, i.e.:

$$A(\mathbf{z}_h,\widetilde{\mathbf{u}}_h^*)+S_h^{lps}(\widetilde{\mathbf{u}}_h^*,\mathbf{z}_h)=(C(\widehat{\mathbf{v}}-\widetilde{\mathbf{v}}_h),\mathbf{w}_h)\quad\forall\mathbf{z}_h=(\mathbf{w}_h,q_h)\in X_{r,s}\,. \qquad (6.13)$$

Due to the optimality conditions,

$$j_h'(\mathbf{q}_h)(\mathbf{y}_h)=0=j'(\mathbf{q})(\mathbf{y}_h),$$

it follows especially for $\mathbf{y}_h:=I_{M_h}\mathbf{q}-\mathbf{q}_h$:

$$\begin{aligned} \alpha\|I_{M_h}\mathbf{q}-\mathbf{q}_h\|^2 &\leq j_h'(I_{M_h}\mathbf{q})(I_{M_h}\mathbf{q}-\mathbf{q}_h)-j'(\mathbf{q})(I_{M_h}\mathbf{q}-\mathbf{q}_h)\\ &=(\widetilde{\mathbf{v}}_h^*-\mathbf{v}^*,B(I_{M_h}\mathbf{q}-\mathbf{q}_h))+(\alpha(I_{M_h}\mathbf{q}-\mathbf{q}),I_{M_h}\mathbf{q}-\mathbf{q}_h)\\ &\lesssim\|\widetilde{\mathbf{v}}_h^*-\mathbf{v}^*\|_0\cdot\|I_{M_h}\mathbf{q}-\mathbf{q}_h\|_0+\alpha\|I_{M_h}\mathbf{q}-\mathbf{q}\|_0\cdot\|I_{M_h}\mathbf{q}-\mathbf{q}_h\|_0\,. \end{aligned}$$

In the last step we used the continuity of B. Now we apply Young's inequality and get

$$\frac{\alpha}{2}\|I_{M_h}\mathbf{q}-\mathbf{q}_h\|_0^2 \lesssim \frac{1}{\alpha}\|\widetilde{\mathbf{v}}_h^*-\mathbf{v}^*\|^2+\alpha\|I_{M_h}\mathbf{q}-\mathbf{q}\|_0^2\,.$$

By taking the square root on both sides we obtain

$$\|I_{M_h}\mathbf{q}-\mathbf{q}_h\|_0 \lesssim \frac{1}{\alpha}\|\widetilde{\mathbf{v}}_h^*-\mathbf{v}^*\|_0+\|I_{M_h}\mathbf{q}-\mathbf{q}\|_0\,.$$

Hence, it remains to bound $\|\widetilde{\mathbf{v}}_h^*-\mathbf{v}^*\|_0$ by the right-hand side of (6.12). This can be done by introducing the discrete adjoint velocities $\mathbf{v}_h^*(\mathbf{q})$ which is the velocity component of the solution of the discrete adjoint equation (6.11):

$$\|\widetilde{\mathbf{v}}_h^*-\mathbf{v}^*\|_0 \le \|\widetilde{\mathbf{v}}_h^*-\mathbf{v}_h^*(\mathbf{q})\|_0+\|\mathbf{v}_h^*(\mathbf{q})-\mathbf{v}^*\|_0\,. \tag{6.14}$$

Subtracting (6.8) and (6.13) leads to the equation

$$\begin{aligned} &A(\mathbf{z}_h,\mathbf{u}_h^*-\widetilde{\mathbf{u}}_h^*)+S_h^{lps}(\mathbf{u}_h^*-\widetilde{\mathbf{u}}_h^*,\mathbf{z}_h)\\ &=(C(\widetilde{\mathbf{v}}_h-\mathbf{v}_h),\mathbf{w}_h)\quad \forall \mathbf{z}_h=(\mathbf{w}_h,q_h)\in X_{r,s}\,. \end{aligned}\tag{6.15}$$

Since this discrete adjoint Oseen problem is stable, $\widetilde{\mathbf{v}}_h^*-\mathbf{v}_h^*(\mathbf{q})$ depends continuously on the right hand side $C(\widetilde{\mathbf{v}}_h-\mathbf{v}_h)$. As a consequence, the first term of the right hand side of (6.14) can be bounded by

$$\|\widetilde{\mathbf{v}}_h^*-\mathbf{v}_h^*(\mathbf{q})\|_0 \lesssim \sigma^{-1/2}\|\widetilde{\mathbf{v}}_h-\mathbf{v}\|_0\,.$$

The discrete primal equation (6.7) is stable, too, so that it makes sense to introduce the discrete primal velocity $\mathbf{v}_h(\mathbf{q})$ for the exact control $\mathbf{q}$ according to (6.10). Then the difference $\|\widetilde{\mathbf{v}}_h-\mathbf{v}_h(\mathbf{q})\|_0$ can be bounded by the difference of the corresponding control.

$$\begin{aligned} \|\widetilde{\mathbf{v}}_h-\mathbf{v}\|_0 &\le \|\widetilde{\mathbf{v}}_h-\mathbf{v}_h(\mathbf{q})\|_0+\|\mathbf{v}_h(\mathbf{q})-\mathbf{v}\|_0\\ &\lesssim \sigma^{-1/2}\|I_{M_h}\mathbf{q}-\mathbf{q}\|_0+\|\mathbf{v}_h(\mathbf{q})-\mathbf{v}\|_0\,. \end{aligned}$$

Summing up, we obtain

$$\|\mathbf{q}-\mathbf{q}_h\|_0 \lesssim \Big(1+\frac{1}{\alpha\sigma}\Big)\|I_{M_h}\mathbf{q}-\mathbf{q}\|_0+\frac{1}{\alpha\sigma^{1/2}}\|\mathbf{v}_h(\mathbf{q})-\mathbf{v}\|_0+\frac{1}{\alpha}\|\mathbf{v}_h^*(\mathbf{q})-\mathbf{v}^*\|_0\,.$$

Hence, the L^2-error in the control is bounded by the sum of interpolation error of the control plus the discretization error of the primal and dual velocity, both for the exact control. □

Remark 6.1. The dependence of σ^{-1} seems to be problematic for small σ. However, this dependence can be avoided by a standard duality argument, so that the a priori results with respect to the H^1-seminorm of the previous sections are shifted to the L^2-norm.

Now, we are able to derive an a priori estimate for $\mathbf{u}-\mathbf{u}_h$ in terms of the discretization error for given exact control, $\mathbf{u}-\mathbf{u}_h(\mathbf{q})$, the discretization error for the adjoint velocities with exact control, $\mathbf{v}^*-\mathbf{v}_h^*(\mathbf{q})$ and the interpolation error of the control, $\mathbf{q}-I_{M_h}\mathbf{q}$:

Theorem 6.1. *For the optimal control discretized with LPS we have the following a priori estimate:*

$$\begin{aligned} |\!|\!|\mathbf{u}-\mathbf{u}_h|\!|\!|_{lps} &\lesssim |\!|\!|\mathbf{u}-\mathbf{u}_h(\mathbf{q})|\!|\!|_{lps}+\sigma^{-1/2}(1+\frac{1}{\alpha\sigma})\|\mathbf{q}-I_{M_h}\mathbf{q}\|_0\\ &\quad+\frac{1}{\alpha\sigma}\|\mathbf{v}_h(\mathbf{q})-\mathbf{v}\|_0+\frac{1}{\alpha\sigma^{1/2}}\|\mathbf{v}^*-\mathbf{v}_h^*(\mathbf{q})\|_0\,. \end{aligned}\tag{6.16}$$

Proof. It is sufficient to show:

$$\|\mathbf{u}_h(\mathbf{q}) - \mathbf{u}_h\|_{lps} \lesssim \sigma^{-1/2}\|\mathbf{q} - \mathbf{q}_h\|_0 \,, \tag{6.17}$$

because then the assertion follows with the previous lemma. Let us first observe that due to the coercivity property of the discrete operator it holds:

$$\begin{aligned}\|\mathbf{u}_h(\mathbf{q}) - \mathbf{u}_h\|_{lps}^2 &= A(\mathbf{u}_h(\mathbf{q}) - \mathbf{u}_h, \mathbf{u}_h(\mathbf{q}) - \mathbf{u}_h) + S_h^{lps}(\mathbf{u}_h(\mathbf{q}) - \mathbf{u}_h, \mathbf{u}_h(\mathbf{q}) - \mathbf{u}_h)\\ &= (B(\mathbf{q}_h - \mathbf{q}), \mathbf{u}_h(\mathbf{q}) - \mathbf{u}_h)\\ &\le \sigma^{-1/2}\|B(\mathbf{q}_h - \mathbf{q})\|_0 \|\mathbf{u}_h(\mathbf{q}) - \mathbf{u}_h\|_{lps}.\end{aligned}$$

Now, we obtain with the continuity of B the bound (6.17). □

Remark 6.2. The quantities $\|\mathbf{u} - \mathbf{u}_h(\mathbf{q})\|_{lps}$, $\|\mathbf{v}_h(\mathbf{q}) - \mathbf{v}\|_0$ and $\|\mathbf{v}_h^*(\mathbf{v}) - \mathbf{v}^*\|_0$ on the right-hand side of the estimate (6.16) are bounded by the a priori estimates of Section 4, because they describe the error of a simple primal problem with the same control $\mathbf{q}$. This completes the error estimates for the optimal control problem.

7. Summary

In this paper we considered the state of the art in the analysis of finite element methods for incompressible flow problems where spurious oscillations of the discrete solution are drastically reduced by using local projection stabilization (LPS) techniques. These methods preserve the favourable stability and approximation properties of classical residual-based stabilization (RBS) techniques but avoid the strong coupling of velocity and pressure in the stabilization terms.

After explaining basic ideas of this approach to the Stokes problem, we presented a unified framework for the analysis of different variants of the LPS method applied to the generalized Oseen problem. The latter problem appears as auxiliary problem within each time step of the calculation of the time-dependent Navier-Stokes problem if an A-stable implicit time semi-discretization is applied first. Moreover, the LPS approach can be identified as a variational multiscale method which provides a potential framework for the treatment of turbulent flows.

One major advantage of LPS-methods is the symmetry of the stabilization terms. This ensures that the operations "discretize" and "optimize" commute within optimal control problems for linear(ized) flow problems.

The extension of local projection stabilization methods to optimal control problems for the incompressible Navier-Stokes problem is a task for future research.

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