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ERROR ESTIMATES FOR THE RECURSIVE LINEARIZATION OF INVERSE MEDIUM PROBLEMS*

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Abstract

This paper is devoted to the mathematical analysis of a general recursive linearization algorithm for solving inverse medium problems with multi-frequency measurements. Under some reasonable assumptions, it is shown that the algorithm is convergent with error estimates. The work is motivated by our effort to analyze recent significant numerical results for solving inverse medium problems. Based on the uncertainty principle, the recursive linearization allows the nonlinear inverse problems to be reduced to a set of linear problems and be solved recursively in a proper order according to the measurements. As an application, the convergence of the recursive linearization algorithm [Chen, Inverse Problems 13(1997), pp.253-282] is established for solving the acoustic inverse scattering problem.

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1. Introduction

Motivated by significant scientific and industrial applications, the field of inverse problems has undergone a tremendous growth in the last several decades. A variety of inverse problems, including identification of PDE coefficients, reconstruction of initial data, estimation of source functions, and detection of interfaces or boundary conditions, demand the solution of ill-posed non-linear operator equations see, e.g., [12, 18]. Our focus of this paper is on the inverse medium scattering problem, *i.e.*, the reconstruction of the refractive index of an inhomogeneous medium from measurements of the far field pattern of the scattered fields. The inverse medium scattering problem arises naturally in diverse applications such as radar, sonar, geophysical exploration, medical imaging, and nondestructive testing. There are two major difficulties associated with the nonlinear inverse problem: the ill-posedness and the presence of many local minima. A number of algorithms have been proposed for numerical solutions of this inverse problem. Classical iterative optimization methods offer fast local convergence but often fail to compute the global minimizers because of multiple local minima. Another main difficulty is the ill-posedness, *i.e.*, infinitesimal noise in the measured data may give rise to a large error in the computed solution. It is well known that the ill-posedness of the inverse scattering problem decreases as the frequency increases. However, at high frequencies, the nonlinear equation

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becomes extremely oscillatory and possesses many more local minima. A challenge for solving the inverse problem is to develop solution methods that take advantages of the regularity of the problem for high frequencies without being undermined by local minima.

To overcome the difficulties, stable and efficient regularized recursive linearization methods are developed in [3, 9, 10] for solving the two-dimensional Helmholtz equation and the three-dimensional Maxwell's equations [4] in the case of full aperture data. We refer the reader to [3, 5, 6] for limited aperture data cases. Roughly speaking, these methods use the Born approximation at the lowest frequency k_{min} to obtain the initial guesses which are the low-frequency modes of the medium. Updates are made by using the data at higher frequency sequentially until a sufficiently high frequency k_{max} where the dominant modes of the medium are essentially recovered.

In the case of fixed frequencies, a related continuation approach has been developed on the spatial frequencies [3]. A recursive linearization approach has also been developed in [11] for solving inverse obstacle problems. More recently, direct imaging techniques have been explored to replace the weak scattering for generating the initial guess [2]. Although the numerical results are efficient and robust, the analysis of the computational methods is completely open. Our main goal of this paper is to originate the convergence analysis of the general recursive linearization algorithm for solving the inverse medium problem. Under some reasonable assumptions, we establish the convergence of the algorithm along with an error estimate. Our analysis is inspired by the underlying physics, especially the uncertainty principle.

The outline of the paper is as follows. A formulation of the nonlinear inverse scattering problem is presented in Section 2. Section 3 is devoted to useful properties of the linearized problem. In Section 4, we discuss the significance of the uncertainty principle in the study of inverse problems. Through a singular value decomposition analysis, the uncertainty principle may further be used to characterize the ill-posedness of the inverse problem. A reconstruction method based on the uncertainty principle, recursive linearization, is introduced. We establish the convergence of the recursive linearization approach and derive an error estimate in Section 5. As an example, we apply the convergence result to the algorithm presented in [9] for solving an inverse medium scattering problem in Section 6. Finally, some relevant discussions are provided in the Appendix about the uncertainty principal and its close connection to the inverse medium scattering problem.

2. Inverse Medium Scattering Problem

The scattering of time-harmonic electromagnetic waves by a cylindrical shaped inhomogeneous medium with refractive index $1 + q(x)$ is governed by the following differential equation

$$\Delta\phi(x) + k^2(1 + q(x))\phi(x) = 0, \text{in} \mathbb{R}^2, \tag{2.1}$$

where the real part of the complex valued function ϕ describes the space-dependent part of a velocity potential in the case of acoustic waves or an electric/magnetic field in the case of electromagnetic waves. The real number $k > 0$ is the wave number. Assume that the refractive index $q(x) + 1$ is a positive real function in $\mathbb{R}^2$, the scatterer $q(x)$ is compactly supported in $D(R)$ and belongs to $C_0^2(D)$. Here $D(R)$ denotes a ball in $\mathbb{R}^2$ centered at 0 with radius R. The direct or forward scattering problem in this context is for a given incident wave $\phi_0(x)$ satisfying the Helmholtz equation $\Delta\phi_0 + k^2\phi_0 = 0$ in $\mathbb{R}^2$, to determine the scattered wave $\psi(x) : \mathbb{R}^2 \to \mathbb{C}$

which solves

$$\Delta\psi + k^2(1+q(x))\psi = -k^2 q(x)\phi_0, \qquad \text{in } \mathbb{R}^2, \tag{2.2}$$

along with the Sommerfeld radiation condition:

$$\lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial\psi}{\partial r} - ik\psi\right) = 0, \qquad r = |x|, \tag{2.3}$$

uniformly in all directions. Note that the total field $\phi = \psi + \phi_0$. It is well known that the direct problem has a unique solution.

Let γ be the trace operator to the boundary ∂D of the domain D. By the trace theorem, γ is a linear operator from $H^s(D)$ onto $H^{s-1/2}(\partial D)$ for any $s \geq 1$. We denote $H^s(\Omega)$ and $\tilde{H}^s(\Omega)$ the complex and real Sobolev spaces $W^{s,2}(\Omega)$ on a domain Ω, respectively. For a given incident field ϕ_0 and a scatterer q, we define the map $S(q,k)$ by $S(q,k)\phi_0 = \psi$. The scattering map may be defined as $M(q,k) = \gamma S(q,k)$. Obviously $M(q,k)$ is an operator-valued function in $\mathcal{L}(L^2(D), H^{3/2}(\partial D))$. Its domain of definition is the closure of incident waves ϕ_0 in $L^2(D) = H^0(D)$.

The inverse problem is to determine the medium function q from the given multi-frequency near field measurements $\{M(q,k_j),\ 1 \leq j \leq N\}$ for incoming plane waves with all incident directions (full aperture). The sequence $\{k_j\}$ belongs to the frequency band $[k_{min}, k_{max}]$ produced by some given measurement system.

3. Linearization of the Scattering Map

Since the scattering map $M(q,k)$ is nonlinear, it is natural to study the linearization. By far, most of the progress in solving the inverse problem has been obtained through linearization. Our goal in this section is two folds: we first analyze the regularity, particularly the Fréchet differentiability, of the scattering map and derive some useful properties of its Fréchet derivative $DM(q,k)$. We also analyze the relation between the injectivity property of the linearized scattering map and the uniqueness of the inverse problem. The results obtained here will be employed subsequently to study the connection between the regularity of the scattering map and the uncertainty principle, and to verify the hypotheses for our convergence result.

In order to work on a bounded domain rather than the whole space, we introduce the Dirichlet-to-Neumann map operator $\Lambda(k)$ on the circle ∂D. The map is defined by $\Lambda(k)f = \partial_n u_f$, where u is the unique solution of the exterior Dirichlet problem for the Helmholtz equation outside the domain D, satisfying a Dirichlet boundary condition $u_f = f$ on the circle ∂D and the Sommerfeld radiation condition at the infinity. Here $\partial_n u_f$ denotes the inward normal derivative of u_f on ∂D. It is known that $\Lambda(k)$ is bounded from $H^s(\partial D)$ to $H^{s-1}(\partial D)$ for any $s \geq 1$.

Remark 3.1. In the special case that D is a disk of radius R, the following explicit expression holds:

$$\Lambda(k)f = \sum_{j\in\mathbb{Z}} \left(\frac{|j|}{R} - \frac{kH^{(1)}_{|j-1|}(kR)}{H^{(1)}_{|j|}(kR)}\right)\left(f,\ \frac{e^{ij\theta}}{\sqrt{2\pi R}}\right)\frac{e^{ij\theta}}{\sqrt{2\pi R}},$$

where $H^{(1)}_j(r)$ are the Hankel functions of the first kind and (r,θ) are the polar coordinates for the plane.

The restriction to D of the scattered field $S(q,k)\phi_0$ is the unique solution $u \in H^2(D)$ of the equation (2.2) along with the boundary condition

$$\Lambda(k)\gamma u = \frac{\partial u}{\partial n} \quad \text{on} \quad \partial D.$$

The converse is also true, since a solution of the above problem has a unique extension to $\mathbb{R}^2$ satisfying the original scattering problem.

For our analysis, it is crucial to introduce the notion of the singular value decomposition for compact operators. Let $\mathcal{H}$, $\mathcal{H}'$ be Hilbert spaces, T be a compact operator from $\mathcal{H}$ to $\mathcal{H}'$, and T^* denote its adjoint operator. Then there exist a sequence of singular eigenvalues, $\sigma_0 \geq \sigma_1 \geq \cdots$, repeated according to their multiplicity and orthonormal sequences ϕ_j and ψ_j such that $T\phi_j = \sigma_j \psi_j$ and $T^*\psi_j = \sigma_j \phi_j$, for all $j \in \mathbb{N}$. For all ϕ in $\mathcal{H}$, the following singular value decomposition holds

$$\phi = \sum_{j \geq 0} (\phi,\ \phi_j)\phi_j + P_0\phi\ ,$$

where P_0 is the projection onto the kernel of T and $T\phi = \sum_{j\geq 0} \sigma_j(\phi,\ \phi_j)\psi_j$. The system $\big(\sigma_j, \phi_j, \psi_j\big)$ is called the singular system of T.

We next present a useful property of the scattering map.

Theorem 3.1. *Let $\tilde{H}^2(D)$ be the real Sobolev space on the disc D. Assume that k_{min} and k_{max} are positive numbers such that $0 < k_{min} < k_{max}$. Then the scattering map $M(q,k)$ depends analytically on (q,k) in $\tilde{H}^2(D) \times (k_{min}, k_{max})$.*

Proof. We first show that the restriction of the scattered field to the domain D is analytic with respect to (q,k). To do so, we introduce the fundamental solution to the Helmholtz equation:

$$G_k(x,y) = -\frac{i}{4} H_0^{(1)}(k|x-y|),\ x \neq y,\ \ x,y \in \mathbb{R}^2,$$

where $H_0^{(1)}(z)$ is the first kind Hankel function of order 0. For q in $\tilde{H}^2(D)$, define the maps:

$$Kf(x) := k^2 \int_D G_k(x,y) f(y) dy, \tag{3.1}$$

$$Q(q)f(x) := q(x)f(x). \tag{3.2}$$

The following regularity result is well known. We refer the reader to [12] (Thm. 8.2, pp. 208) for a proof.

Proposition 3.1. *Suppose that the function q is in $\tilde{H}^2(D)$. Then the maps Q and K are bounded operators from $L^2(D)$ to $L^2(D)$ and from $L^2(D)$ to $H^2(D)$, respectively.*

The function $H_0^{(1)}(z)$ is analytic in $\mathbb{C}/\mathbb{R}_-$. Since k_{min} is strictly positive, the operator-valued function KQ is analytic on (q,k) in $\tilde{H}^2(D) \times (k_{min}, k_{max})$. Denote I the identity operator from $L^2(D)$ to itself. It follows from Proposition 3.1 that KQ is a compact operator-valued function from $L^2(D)$ to itself. Therefore, by the Fredholm theory, $I + KQ$ is analytic and invertible on $\tilde{H}^2(D) \times (k_{min}, k_{max})$ except possibly at certain isolated points where it is not injective.

We next show that the operator $I + KQ$ is injective for any (q,k) in $\tilde{H}^2(D) \times (k_{min}, k_{max})$. Suppose that for some (q,k) in $\tilde{H}^2(D) \times (k_{min}, k_{max})$ and f in $L^2(D)$, we have

$$(I + KQ)f = 0.$$

Consequently $f(x) = -KQ(x)f$ in D. Proposition 3.1 implies that $f(x)$ belongs to $H^2(D)$. Using the properties of the map K we deduce that $f(x)$ has a unique continuous extension to $\mathbb{R}^2$. For simplicity, the extension is also denoted f. Consequently, we have

$$\Delta f + k^2 f = -k^2 q(x) f(x)$$

in the whole space $\mathbb{R}^2$ along with the radiation condition (2.3). The uniqueness of the forward problem (2.2) implies that f is zero in $\mathbb{R}^2$.

We have proved that the operator $I + KQ$ is injective for any q in $\tilde{H}^2(D)$. Consequently, the operator-valued function $(I + KQ)^{-1}$ is analytic in the whole space $\tilde{H}^2(D) \times (k_{min}, k_{max})$. It is well known that the scattering problem is equivalent to the Lipmann-Schwinger equation:

$$\psi + KQ\psi = -KQ\phi_0 \ .$$

Therefore, the scattered field in the domain D can be rewritten as

$$\psi = -(I + KQ)^{-1} KQ\phi_0.$$

By using the analyticity result proved above, we deduce that $S(q, k)$ is analytic on (q, k) in $\tilde{H}^2(D) \times (k_{min}, k_{max})$. Since the trace operator γ is a bounded linear operator independent of q and k, the scattering map $M(q, k) = \gamma S(q, k)$ maintains the same regularity as the operator $S(q, k)$, which completes the proof of Theorem 3.1.

To introduce the linearization, we consider the following problem

$$\begin{aligned} &\Delta v + k^2(1 + q(x))v = -k^2 \delta q(x)(\psi + \phi_0), \quad &&\text{in } \mathbb{R}^2, \\ &\lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial v}{\partial r} - ikv\right) = 0, &&r = |x|. \end{aligned} \tag{3.3}$$

Define the operator DS by

$$v = DS(q)(\delta q, \phi_0) \ .$$

Then, the linearized scattering map may be formally defined as

$$DM(q, k) = \gamma DS(q) \ .$$

More specifically, it is easily verified that the Fréchet derivative of the scattering map takes the following form

$$DM(q, k)p = -\gamma(I + KQ)^{-1} KQ(p)(I + KQ)^{-1}\phi_0,$$

where $Q(p)$ is the multiplication operator associated with the function p.

In the rest of this section, we study further properties of the linearized scattering map $DM(q, k)$, which plays an important role in the following sections.

We begin with a technical result whose proof is given in [12] for the three-dimensional case. The two-dimensional case could be proved similarly.

Lemma 3.1. *Let B be an open ball such that $\overline{D} \subset B$. Then the set of total fields*

$$\{(I + S(q, k))e^{-ikx.d}, \ \ d \in S^1\}$$

is complete in the closure of

$$Z_k := \left\{ w \in C^2(\overline{B}) : \Delta w + k^2(1 + q(x))w = 0, \ \text{in } B \right\}$$

with respect to the norm $L^2(D)$. Here S^1 denotes the unit circle.

Theorem 3.2. *Let $q \in C_0^2(\bar{D})$. Then there exists an open and dense set $\mathcal{I}_q$ in $\mathbb{R}$ such that $DM(q,k)$ is injective for any $k \in \mathcal{I}_q$. Moreover $\mathbb{R}/\mathcal{I}_q$ is a countable set. In particular $0 \in \mathcal{I}_q$.*

Proof. Multiplying the equation (3.3) by $w \in Z_k$ and integrating by parts, we obtain by Green's theorem and using the fact that $S(q,k)e^{-ikx.d}$ solves the Helmholtz equation outside the disc D that

$$\begin{aligned}&\int_D p(x)w(x)(I + S(q,k))e^{-ikx.d}dx\\ &= -\frac{1}{k^2}\int_{\partial D} w\Lambda(k)(DM(q,k)p)e^{-ikx.d} - \partial_n w(DM(q,k)p)e^{-ikx.d}ds_x\ .\end{aligned}$$

The above equality implies that the formal linearized map $DM(q,k)$ is injective at k if the product of solutions $w \in Z_k$ and $(I + S(q,k))e^{-ikx.d},\ d \in S^1$ is complete in $L^2(D)$. Therefore, from Lemma 3.1, we have $DM(q,k)$ is injective at k if and only if the set $V_k = \{uw : u, w \in Z_k\}$ is complete in $L^2(D)$.

The proof may be completed by using an earlier result of Sun and Uhlmann [19]: there exists an open set $\mathcal{I}_q$ in $\mathbb{R}$ where V_k is complete in $L^2(D)$. The complement of the set $\mathcal{I}_q$ is countable. Furthermore, 0 belongs to the set $\mathcal{I}_q$.

Remark 3.2. The above proof exhibits the important connection between the uniqueness of the inverse problem at fixed frequency and the injectivity of the linearized scattering map. More recently, in [14], the uniqueness of the inverse problem has been established for a fixed frequency outside a countable set.

4. Uncertainty Principle and Recursive Linearization

In an abstract setting, the inverse problem may be formulated as

$$M(q,k_j) = Data(k_j),\ j = 0,\cdots,N,\ \ k_{min} \le k_j \le k_{max}, \tag{4.1}$$

where k_{min} is small enough to be in the regime of the Born approximation and k_{max} is the highest frequency. The operator-valued function $M(q,k)$ represents the totality of information encoded in the full aperture measurements taken outside the medium: the acquisition of each scattered field ψ outside $D(R)$ corresponding to every possible incident field ϕ_0. However, as discussed in the Appendix, information on evanescent waves is often not available in practical situations. Thus, it is essential to reformulate the inverse problem by taking into account the uncertainty principle.

Due to the regularity of the scattered field ψ in the whole space $\mathbb{R}^2$, the operator $M(q,k)$ is compact from its domain of definition in $L^2(D)$ into $L^2(\partial D)$. Hence it has a singular eigenvalue decomposition

$$M(q,k) = \sum_{l=0}^{\infty} \sigma_l \phi_l \otimes \psi_l,$$

where $\{\sigma_l\}$ is a decreasing sequence of positive numbers, $\{\phi_l\}$ and $\{\psi_l\}$ are orthonormal sequences in the closed set of plane waves in $L^2(D)$ and $L^2(\partial D)$, respectively. Recall the standard notation: $(\phi_l \otimes \psi_l)\phi = (\phi,\ \phi_l)\psi_l$.

To illustrate the ill-posedness of the inverse problem (4.1), we examine closely the singular values of the scattering map in the special case where $q(x) = q_0\chi_{D_0}$. Here the number $q_0 > -1$ and χ_{D_0} is the characteristic function of the disc $D(R_0)$. Since in practice the compact support of the medium is unknown and the data is measured far from it, we assume that $R_0 < R$. Denote $\psi = S(q_0\chi_{D_0}, k)\phi_0$ where ϕ_0 is an incoming plane wave. It follows that the total field inside D_0 can be expressed in the polar coordinates as

$$\psi(r,\theta) + \phi_0(r,\theta) = \sum_{m=-\infty}^{+\infty} c_m J_m(kr\sqrt{1+q_0})e^{im\theta}.$$

Since ϕ_0 and ψ are solutions of the Helmholtz equation inside and outside of D_0, respectively, we have

$$\phi_0(r,\theta) = \sum_{m=-\infty}^{\infty} \alpha_m J_m(kr)e^{im\theta}, \text{ for } r \le R_0$$

and

$$\psi(r,\theta) = \sum_{m=-\infty}^{\infty} \xi_m H_m(kr)e^{im\theta}, \text{ for } R_0 \le r,$$

where $\{\xi_m\}$ and $\{\alpha_m\}$ are complex sequences. Here, H_m is the Hankel function of the first kind of order m, J_m is the Bessel function of the first kind of order m.

By using the continuity of the scattered wave and its normal derivative on the boundary ∂D_0, we deduce that the singular values of $M(q_0\chi_{D_0}, k)$ are

$$\sigma_m = \left| \frac{J_m(kR_0\sqrt{1+q_0})J_m'(kR_0) - \sqrt{1+q_0}J_m'(kR_0\sqrt{1+q_0})J_m(kR_0)}{\sqrt{1+q_0}J_m'(kR_0\sqrt{1+q_0})H_m(kR_0) - J_m(kR_0\sqrt{1+q_0})H_m'(kR_0)} \frac{\sqrt{R}H_m(kR)}{(\int_0^{R_0} J_m^2(kr)rdr)^{1/2}} \right|.$$

Therefore, the asymptotic analysis of the Bessel and Hankel functions shows the extremely rapid decay

$$\sigma_m = \mathcal{O}\left(\frac{k^2\sqrt{m+1}}{m(m+1) - \frac{k^2R_0^2}{4}} \left(\frac{R_0}{R}\right)^m \right),$$

as m goes to infinity ($\gg kR$), which indicates the severe ill-posedness of the inverse problem. Note that for a large frequency k, the ill-posedness becomes less severe. In fact, the improvement in the magnitude of the singular eigenvalues by taking larger frequencies is much better than what it appears in the above expression since the asymptotic formula is calculated for large m but fixed kR. By taking into account the far field measurements, for $R \gg R_0$, we obtain

$$\sigma_m = \mathcal{O}\left(\frac{k^{\frac{3}{2}}\sqrt{m(m+1)}}{m(m+1) - \frac{k^2R_0^2}{4}} \left(\frac{kR_0e}{2m}\right)^m \right).$$

The asymptotic result above represents the behavior of singular eigenvalues $\sigma_l(q,k)$, $l \in \mathbb{N}$ with respect to the parameters l and k for more general medium, *i.e.*, they decrease exponentially for large l and increase with respect to k for fixed q and l.

Recall from the previous section that

$$M(q,k) = \gamma KQ(I + KQ)^{-1}\ .$$

It follows from the Courant-Fischer characterization of singular values [17] that

$$\sigma_l(M(q,k)) \le \sigma_l(\gamma KQ(\chi_{D_0}))\|Q\|_{L^\infty}\|(I+KQ)^{-1}\|\ ,$$

where $D(R_0)$ is the smallest disc containing the compact support of $q(x)$. We deduce from Graf's addition formula a series representation of the Green function

$$G_k(x,y) = -\frac{i}{4}\sum_{m=-\infty}^{+\infty} J_m(kr_x)H_m(kr_y)e^{im(\theta_x-\theta_y)},$$

where (r_x,θ_x) and (r_y,θ_y) are the polar coordinates of respectively any vectors x and y satisfying $r_x < r_y$. A direct calculation yields

$$\sigma_l(\gamma KQ(\chi_{D_0})) = \frac{k^2}{4}|H_l(kR)|\left(\int_0^{R_0} J_l^2(kr)rdr\right)^{1/2}.$$

Therefore, for $R \gg R_0$, we have

$$\sigma_l(\gamma KQ(\chi_{D_0})) = \mathcal{O}\left(\frac{R_0\sqrt{k}}{4\sqrt{l(l+1)}}\left(\frac{kR_0 e}{2l}\right)^l\right).$$

Furthermore, since the operator $I+QK$ is a smooth compact perturbation of the identity operator, we conjecture that the variation of the norm $||(I+QK)^{-1}||$ on $[k_{min}, k_{max}]$ has little influence on the exponential behavior of the singular eigenvalues. An on-going project of the authors is to derive this and related frequency stability estimates for the inverse problem. The results will be reported elsewhere.

From a physical point of view, the above singular value decomposition may be understood as follows: a unitary incident wave propagating through the medium in the direction ϕ_l generates a scattered wave propagating in the direction ψ_l with magnitude σ_l. For large l, the scattered wave amplitudes resulting from the diffraction in the directions ϕ_l are infinitesimals hence undetectable by conventional devices.

We can interpret the uncertainty principle rigorously by assuming a fixed limit (for the observer) $\beta > 0$ of the accuracy of our measurement system, such that the waves propagating in the directions ψ_l with $\sigma_l \le \beta$ can not be observed and hence are lost. Consequently, the available data satisfies the following equations

$$\sum_{\sigma_l\ge\beta} \sigma_l \phi_l \otimes \psi_l = Data(k)\ . \tag{4.2}$$

Since for a fixed q, the singular eigenvalues $k \to \sigma_l$ are increasing, one can detect more scattered waves for large k and hence collect more information about the scatterer.

Recursive linearization approach. Based on the discussion on the uncertainty principle in the Appendix, the observable part of the medium $\tilde{q}(k)$ is a natural solution to the new problem (4.2). For its reconstruction, we present here a recursive linearization algorithm:

- The algorithm requires multi-frequency data and a (systematic) way to generate a good initial guess at the lowest frequency k, k_{min}. For examples, the initial guess is derived from the Born approximation at the lowest frequency in [4, 7, 9] because of the weak scattering.
- The approximation is then used to linearize the inverse problem at the next higher frequency $k_{min} + \delta k$ to produce a better approximation which contains more modes of the scatterer. The process continues recursively until a sufficiently high wavenumber k_{max}, where the dominant modes of the scatterer are essentially recovered.

5. Convergence Analysis

We study the convergence properties of the recursive linearization algorithm. More generally, we consider a general nonlinear ill-posed inverse problems which can be formulated as the abstract equation:

$$F(q,k)=0, \tag{5.1}$$

where $F:\ \mathcal{S}\times[k_{min},k_{max}]\to Y$ is a nonlinear operator indexed by $k\in[k_{min},k_{max}]$. $\mathcal{S}$, Y are Hilbert spaces with inner products $(.,.)_{\mathcal{S}}$, $(.,.)_Y$ and norms $\|.\|_{\mathcal{S}}$, $||.||_Y$, respectively. Assume throughout that Y has a finite dimension (Remark 5.1). We also restrict the attention only to the attainable case, *i.e.*, the nonlinear equation above is assumed to have a (not necessarily unique) solution, $\tilde{q}(k)$ on $[k_{min},k_{max}]$.

The inverse problem is concerned with *the reconstruction of the exact solution $\tilde{q}(k_{max})$ for a given approximation of $\tilde{q}(k_{min})$.*

Remark 5.1. The problem above may be viewed as a generalization of our inverse problem introduced in the previous section. By taking into account the uncertainty principal (the Appendix), the collected multi-frequency data may be considered as a small perturbation of a truncated matrix which contains information on the principal propagating scattered waves. As is shown in the previous section, the inverse problem can be transformed into:

$$\sum_{l=1}^{L}\sigma_l\phi_l\otimes\psi_l-Data(k)=0,\qquad k\in[k_{min},k_{max}],$$

with the integer $L=\max\{l:\sigma_l(q,k)>\beta,\quad k\in[k_{min},k_{max}]\}$. We follow [15] to define the operators

$$r_L: H^{3/2}(\partial D)\to\mathbb{R}^L\quad\text{by}\quad r_L\psi=\big((\psi,\ \psi_l)_{H^{3/2}}\big)_{l=1,\cdots,L}$$

and

$$r_L^*: L^2(\partial D)\to\mathbb{R}^L\quad\text{by}\quad r_L^*\phi=\big((\phi,\phi_l)_{L^2}\big)_{l=1,\cdots,L}.$$

Let $(F_{l,m})_{l,m=1,\cdots,L}$ be the coefficients of the matrix

$$F(q,k)=r_L(M(q,k)-Data(k))r_L^*.$$

Let $F(q,k)=(F_{l,m})_{l,m=1,\cdots,L}$. Because of the Born approximation for sufficiently small k_{min}, $\tilde{q}(k_{min})$ represents the Fourier modes of the real medium less than $2k_{min}$. In this case, the data provides a good approximation. The maps r_L and r_L^* depend obviously on the medium q and the frequency k and could be derived from the singular value decomposition of the data. It follows that these maps have the same regularity as the scattering map. In practice, as we will show in the next section, r_L and r_L^* are linear operators that map the diffracted waves and incident waves respectively into a known and fixed basis. That is, the matrix $(F_{l,m})_{l,m=1,\cdots,L}$ may be substituted by a truncated square matrix including the most important propagating waves. This finite dimension approximation is being used extensively for solving linear ill-posed problems [15]. To get a good approximation of the scattering map, we need to increase the finite rank of the truncated matrix, which may be accomplished by taking a larger band of frequency (to increase k_{max}). The higher the frequency k_{max} is, the more accurate is the approximation.

We next return to the general problem. The following general hypotheses are needed for our convergence analysis

H1 : $F(q,k)$ is twice Fréchet differentiable with respect to (q,k).

H2 : $\tilde{q}(k)$ is twice continuously differentiable.

H3 : There is a positive constant ρ, such that for all k in $[k_{min}, k_{max}]$ and $q \in B_\rho(\tilde{q}(k))$ (a ball of radius ρ centered at $\tilde{q}(k)$)

$$\partial_q F(\tilde{q}(k), k)(q - \tilde{q}(k)) = 0 \Longrightarrow F(q,k) = F(\tilde{q}(k), k) = 0.$$

In the following, we discuss the appropriateness of the hypotheses in the context of the inverse medium scattering problem. The hypothesis **H1** is obviously satisfied for our problem since the scattering map is analytic on (q,k) according to Theorem 3.1. The assumption **H2** is valid for k small (the Born approximation). Due to the uncertainty principle, many scatterers could produce the same data. The regularity of the scattering map as a function of the frequency allows us to choose a smooth function, $\tilde{q}(k)$. Therefore, the hypothesis **H2** is also justified. Finally, the hypothesis **H3** states that for a fixed frequency k, the first order perturbation of the function $F(q,k)$ with respect to the medium q close to the observable part is less sensitive to the high frequency of the medium or conversely a small perturbation of the high frequency of the medium leaves the observable part $\tilde{q}(k)$ unchanged at the first order of the perturbation. The last hypothesis represents the translation of the uncertainty principle into the more general problem proposed in this section.

In order to solve the nonlinear problem (5.1), we use the classical Newton method. Since the Fréchet derivatives are singular (not injective) a regularization method is needed to stabilize the computation procedure. Tikhonov regularization is well-known and easy to implement. The regularized Newton method consists in solving the minimization problem

$$\min\left\{\|A_j\delta q_j + F(q_j, k_{j+1})\|_Y + \alpha\|\delta q\|_{\mathcal{S}}\right\} \tag{5.2}$$

over $\delta q \in \mathcal{S}$ in each iteration to compute $q_{j+1} = q_j + \delta q_j$. Here A_j is the Fréchet derivative $\partial_q F(q_j, k_{j+1})$, A_j^* is its adjoint, α is a small positive regularization parameter and

$$k_j = k_{min} + \frac{j}{N}(k_{max} - k_{min}), \quad j = 0, \dots, N$$

is the discretization grid of $[k_{min}, k_{max}]$. It is known that (5.2) has a unique solution:

$$q_{j+1} = q_j - (\alpha I + A_j^* A_j)^{-1} A_j^* F(q_j, k_{j+1}). \tag{5.3}$$

Essentially, this is the general form of the recursive linearization algorithm in Section 4. The rest of this section is devoted to the convergence study of the algorithm.

We are ready to present the main convergence result.

Theorem 5.1. *Assume that $F(q,k)$ verifies all the hypotheses above and that $\tilde{q}(k_j)$ is a q_j-minimum norm solution,* i.e., *among all solutions of $F(q,k_j) = 0$, $\tilde{q}(k_j)$ minimizes the distance to q_j for $0 \le j \le N$. Then there exist positive constants α, c_0, and N_0, such that, if*

$$\|\tilde{q}(k_{min}) - q_0\|_{\mathcal{S}} \le c_0\alpha,$$

then the estimate

$$\|\tilde{q}(k_{max}) - q_N\|_{\mathcal{S}} \le \frac{C}{N\sqrt{\alpha}} \tag{5.4}$$

holds for all $N \geq N_0$. Here the constant C is independent of N, α.

Before proving the theorem, the following remarks on the usefulness of the theorem are in order.

Remark 5.2. According to the theorem, if the error of the initial guess at low frequency k_{min} is of the order of α (small), the algorithm converges linearly. The constant C depends only on the nonlinear function F. For the inverse medium scattering problem, this may be interpreted as follows: if a good estimate of the first few Fourier modes of the medium at k_{min} is available, then the algorithm will provide a good approximation of the observable part $\tilde{q}(k_{max})$ of the medium at a computational cost $C/\sqrt{\alpha}$. For this convergence result, there is no need to let α go to zero as for most regularized ill-posed problems. In fact, the parameter α represents the computational cost for determining a good approximation of the observable part $\tilde{q}(k_{max})$ of the medium. In addition, it will be shown in the proof of the theorem that α is equivalent to a fraction of the infinimum $\sigma^\star$ of the smallest singular eigenvalue $\sigma_{l_{n_k}}(k)$ over $[k_{min}, k_{max}]$. Based on the uncertainty principle, for larger frequency k_{max}, one can detect more modes of the medium, which implies that the corresponding number L introduced in Remark 5.1 increases. In this case, α acts like $\sigma^\star$ to approach zero and provoke the increase of the computation cost. Without consideration of the uncertainty principle and at a fixed frequency, the unique way to increase the resolution will be to increase the number L (the finite dimension approximation [15]) and hence it is important to find a balance between N and α (the Morozov principle). It should be pointed out that by increasing L, the convergence of our algorithm is slowed considerably due to the severe ill-posedness (the singular eigenvalues are decaying exponentially with respect to L). In fact, it has been proved in dimension three that this convergence is at most logarithmic [16].

Remark 5.3. Note that Equation (5.1) may be solved by other regularization methods, such as the Landweber iteration, the Levenberg-Marquart algorithm, and regularized Gauss-Newton methods. As a minor point, it should be pointed out that the result holds in the trivial case when $F = 0$.

In order to prove the theorem, the following two technical results are useful.

Lemma 5.1. *Assume that A is a compact linear operator from S to Y. Then, the following inequalities hold*

$$
\begin{aligned}
&(a)\ \|(\alpha I + A^*A)^{-1}A^*\|_{\mathcal{L}(Y,\mathcal{S})} \leq \frac{1}{2\sqrt{\alpha}},\\
&(b)\ \|(\alpha I + A^*A)^{-1}A^*A\|_{\mathcal{L}(\mathcal{S},\mathcal{S})} \leq 1,\\
&(c)\ \|(\alpha I + A^*A)^{-1}\|_{\mathcal{L}(\mathcal{S},\mathcal{S})} \leq \frac{1}{\alpha},
\end{aligned}
$$

Proof. Denote by $\{\sigma_l\}_{0\leq l}$ the singular eigenvalues of the operator A. Then, we have

$$
\begin{aligned}
\|(\alpha I + A^*A)^{-1}A^*\|_{\mathcal{L}(Y,\mathcal{S})} &= \max_{\sigma_l} \frac{|\sigma_l|}{\alpha + \sigma_l^2} \leq \frac{1}{2\sqrt{\alpha}},\\
\|(\alpha I + A^*A)^{-1}A^*A\|_{\mathcal{L}(\mathcal{S},\mathcal{S})} &= \max_{\sigma_l} \frac{\sigma_l^2}{\alpha + \sigma_l^2} \leq 1,\\
\|(\alpha I + A^*A)^{-1}\|_{\mathcal{L}(\mathcal{S},\mathcal{S})} &= \max_{\sigma_l} \frac{1}{\alpha + \sigma_l^2} \leq \frac{1}{\alpha}.
\end{aligned}
$$

This completes the proof of the lemma □

Lemma 5.2. *Assume that $F(q,k)$ satisfies the hypotheses* **H1** *and* **H2** *and that $\tilde{q}(k)$ is the unique $q^\star$-minimum-norm solution of $F(q,k)=0$ in $B_\rho(\tilde{q}(k))$,* i.e.,

$$\|\tilde{q}(k)-q^\star\|_{\mathcal{S}} = \min\left\{\|q-q^\star\|_{\mathcal{S}} :\ q\in B_\rho(\tilde{q}(k)),\ \ F(q,k)=0\right\}.$$

Then

$$\tilde{q}(k)-q^\star \in N(\partial_q F(\tilde{q}(k),k))^\perp.$$

Proof. Suppose that $\tilde{q}(k)$ is a $q^\star$-minimum-norm solution and $\tilde{q}(k)-q\in N(\partial_q F(\tilde{q}(k),k))$. There exists $\eta>0$ such that $\tilde{q}(k)+s(q-\tilde{q}(k))\in B_\rho(\tilde{q}(k))$ for all $|s|\le\eta$. The third assumption of the lemma implies that

$$F(\tilde{q}(k)+s(q-\tilde{q}(k)),k)=0.$$

Since $\tilde{q}(k)$ is a $q^\star$-minimum-norm, we have

$$\begin{aligned}&\|\tilde{q}(k)+s(q-\tilde{q}(k))-q^\star\|_{\mathcal{S}}^2\\ =&s^2\|q-\tilde{q}(k)\|_{\mathcal{S}}^2+2s(q-\tilde{q}(k),\tilde{q}(k)-q^\star)+\|\tilde{q}(k)-q^\star\|_{\mathcal{S}}^2\\ \ge&\|\tilde{q}(k)-q^\star\|_{\mathcal{S}}^2,\end{aligned}$$

and therefore

$$s^2\|q-\tilde{q}(k)\|_{\mathcal{S}}^2+2s(q-\tilde{q}(k),\tilde{q}(k)-q^\star)\ge 0, \qquad \text{for all}\quad |s|\le\eta. \tag{5.5}$$

We next show that the inequality (5.5) is true only if $(q-\tilde{q}(k),\tilde{q}(k)-q^\star)=0$. Let

$$a:=\|q-\tilde{q}(k)\|_{\mathcal{S}},\quad b:=(q-\tilde{q}(k),\tilde{q}(k)-q^\star).$$

If $a=0$, the inequality (5.5) directly implies that $b=0$. Now, if $a\neq 0$ the inequality can be rewritten as

$$\left(as+\frac{b}{a}\right)^2-\left(\frac{b}{a}\right)^2\ge 0, \qquad \text{for all}\quad |s|\le\eta.$$

This is possible only when $b=0$. Consequently,

$$(q-\tilde{q}(k),\tilde{q}(k)-q^\star)=0\ ,$$

which is equivalent to $\tilde{q}(k)-q^\star\in N(\partial_q F(\tilde{q}(k),k))^\perp$ since q is arbitrary. □

We are now ready to prove Theorem 5.1.

Proof. (Theorem 5.1.) Denote by $B_\rho(\tilde{q}(k))$ the ball in $\mathcal{S}$ centered at $\tilde{q}(k)$ with radius ρ. From the regularity of the function $F(q,k)$, the estimates

$$\|\partial_{qq}F(q,k)\|_{\mathcal{L}(\mathcal{S}\times\mathcal{S},Y)}\le d_2,\quad \|\partial_{kk}F(q,k)\|_Y\le d_3,\quad \|\partial_{qk}F(q,k)\|_{\mathcal{L}(\mathcal{S},Y)}\le d_6 \tag{5.6}$$

hold for all $k\in[k_{min},k_{max}]$ and $q\in B_\rho(\tilde{q}(k))$ with ρ small enough. Similarly, it follows from the properties of the solution $\tilde{q}(k)$ that for all $k\in[k_{min},k_{max}]$

$$\|\tilde{q}''(k)\|_{\mathcal{S}}\le d_1,\quad \|\partial_q F(\tilde{q}(k),k)\|_{\mathcal{L}(\mathcal{S},Y)}\le d_4,\quad \|\partial_k F(\tilde{q}(k),k)\|_Y\le d_5. \tag{5.7}$$

Here the constants $d_i, i = 1, \cdots, 6$ are positive constants independent of k.

Next, we use Lemma 5.1 to prove the existence of positives constants d_0 and α_0, such that the estimate

$$\|\tilde{q}'(k_j) + (\alpha I + \tilde{A}_j^* \tilde{A}_j)^{-1} \tilde{A}_j^* \partial_k F(\tilde{q}(k_j), k_j)\|_{\mathcal{S}} < d_0, \quad 0 \le j \le N, \tag{5.8}$$

holds for all $\alpha \in (0, \alpha_0)$, where $\tilde{A}_j = \partial_q F(\tilde{q}(k_j), k_j)$. Since $\tilde{q}(k)$ is one of the solutions of the problem, the equation $F(\tilde{q}(k), k) = 0$ holds over the interval $[k_{min}, k_{max}]$. By taking into account the smoothness assumptions **H1** and **H2**, one can differentiate the latter equation with respect to k, to get:

$$\partial_q F(\tilde{q}(k), k)\tilde{q}'(k) = -\partial_k F(\tilde{q}(k), k), \quad k \in [k_{min}, k_{max}]\,.$$

Therefore, substituting $\partial_k F(\tilde{q}(k_j), k_j)$ by $-\partial_q F(\tilde{q}(k_j), k)\tilde{q}'(k_j)$ in the estimated norm gives:

$$\begin{aligned}&\|\tilde{q}'(k_j) + (\alpha I + \tilde{A}_j^* \tilde{A}_j)^{-1} \tilde{A}_j^* \partial_k F(\tilde{q}(k_j), k_j)\|_{\mathcal{S}} \\ =&\|\alpha(\alpha I + \tilde{A}_j^* \tilde{A}_j)^{-1} \tilde{q}'(k_j)\|_{\mathcal{S}}, \quad 0 \le j \le N.\end{aligned}$$

Using $c)$ in Lemma 5.1, we obtain

$$\|\tilde{q}'(k_j) + (\alpha I + \tilde{A}_j^* \tilde{A}_j)^{-1} \tilde{A}_j^* \partial_k F(\tilde{q}(k_j), k_j)\|_{\mathcal{S}} \le \|\tilde{q}'(k_j)\|_{\mathcal{S}}, \quad 0 \le j \le N.$$

From the regularity of the function $\|\tilde{q}'(k)\|_{\mathcal{S}}$, there exists a constant d_0 such that $\|\tilde{q}'(k)\|_{\mathcal{S}} < d_0$ for all k in $[k_{min}, k_{max}]$, which implies the estimate (5.8).
For convenience, we introduce the following notations:

$$R_{j,\alpha} = (\alpha I + A_j^* A_j)^{-1} A_j^*, \quad \tilde{R}_{j,\alpha} = (\alpha I + \tilde{A}_j^* \tilde{A}_j)^{-1} \tilde{A}_j^*.$$

Denote by $e_j = \tilde{q}(k_j) - q_j$ the j^{th} iteration error in the approximation.

From (5.3), we have for $j = 0, \ldots, N-1$:

$$\begin{aligned} e_{j+1} = e_j &+ \Big(\tilde{q}'(k_j) + \tilde{R}_{j,\alpha} \partial_k F(\tilde{q}(k_j), k_j)\Big)\frac{1}{N} + \tilde{q}(k_{j+1}) - \tilde{q}(k_j) - \tilde{q}'(k_j)\frac{1}{N} \\ &+ R_{j,\alpha} F(q_j, k_j) + R_{j,\alpha}\Big(F(q_j, k_{j+1}) - F(q_j, k_j) - \partial_k F(q_j, k_j)\frac{1}{N}\Big) \\ &+ \Big(R_{j,\alpha} \partial_k F(q_j, k_j) - \tilde{R}_{j,\alpha} \partial_k F(\tilde{q}(k_j), k_j)\Big)\frac{1}{N}, \end{aligned}$$

and consequently

$$\begin{aligned} \|e_{j+1}\|_{\mathcal{S}} \le{}& \|e_j + R_{j,\alpha} F(q_j, k_j)\|_{\mathcal{S}} + \|\tilde{q}'(k_j) + \tilde{R}_{j,\alpha} \partial_k F(\tilde{q}(k_j), k_j)\|_{\mathcal{S}} \frac{1}{N} \\ &+ \|\tilde{q}(k_{j+1}) - \tilde{q}(k_j) - \tilde{q}'(k_j)\frac{1}{N}\|_{\mathcal{S}} + \Big\|R_{j,\alpha}\Big(F(q_j, k_{j+1}) - F(q_j, k_j) - \partial_k F(q_j, k_j)\frac{1}{N}\Big)\Big\|_{\mathcal{S}} \\ &+ \|R_{j,\alpha} \partial_k F(q_j, k_j) - \tilde{R}_{j,\alpha} \partial_k F(\tilde{q}(k_j), k_j)\|_{\mathcal{S}} \frac{1}{N}. \end{aligned}$$

Note that the first and last terms on the right-hand side represent the linearization with respect to q and k, respectively.

For simplicity, assume that $q_j \in B_\rho(\tilde{q}(k_j))$ for $0 \le j \le N$ and small $\rho > 0$. This assumption will be verified later in the proof.

We focus on the last term on the right hand side of the estimate for $||e_{j+1}||_{\mathcal{S}}$.

$$\begin{aligned}&\left\|R_{j,\alpha}\partial_k F(q_j,k_j)-\tilde{R}_{j,\alpha}\partial_k F(\tilde{q}(k_j),k_j)\right\|_{\mathcal{S}}\\ \leq&\|R_{j,\alpha}-\tilde{R}_{j,\alpha}\|_{\mathcal{L}(Y,\mathcal{S})}\|\partial_k F(\tilde{q}(k_j),k_j)\|_{\mathcal{S}}+\|R_{j,\alpha}\|_{\mathcal{L}(Y,\mathcal{S})}\left\|\partial_k F(q_j,k_j)-\partial_k F(\tilde{q}(k_j),k_j)\right\|_{\mathcal{S}}.\end{aligned}$$

From Lemma 5.1 and the estimates (5.6) and (5.7), we have

$$\|R_{j,\alpha}-\tilde{R}_{j,\alpha}\|_{\mathcal{L}(Y,\mathcal{S})}\leq\frac{9}{4\alpha}\|A_j-\tilde{A}_j\|_{\mathcal{L}(\mathcal{S},Y)}\leq\frac{9d_2}{4\alpha}\|e_j\|_{\mathcal{S}}$$

and thus

$$\left\|R_{j,\alpha}\partial_k F(q_j,k_j)-\tilde{R}_{j,\alpha}\partial_k F(\tilde{q}(k_j),k_j)\right\|_{\mathcal{S}}\leq\left(\frac{9d_2d_5}{4\alpha}+\frac{d_6}{2\sqrt{\alpha}}\right)\|e_j\|_{\mathcal{S}}.$$

Once again by using the estimates in Lemma 5.1, the estimates (5.6) – (5.8), and the result above, we obtain

$$\|e_{j+1}\|_{\mathcal{S}}\leq\|e_j+R_{j,\alpha}F(q_j,k_j)\|_{\mathcal{S}}+\frac{d_0}{N}+\frac{d_1}{N^2}+\frac{d_3}{2\sqrt{\alpha}N^2}+\frac{1}{N}\left(\frac{9d_2d_5}{4\alpha}+\frac{d_6}{2\sqrt{\alpha}}\right)\|e_j\|_{\mathcal{S}}. \tag{5.9}$$

Note that the term $\|e_j+R_{j,\alpha}F(q_j,k_j)\|_{\mathcal{S}}$ is in fact the error of the first Newton iteration for solving the nonlinear problem $F(q,k_j)=0$. We next examine this term. Clearly

$$e_j+R_{j,\alpha}F(q_j,k_j)=e_j-\tilde{R}_{j,\alpha}\tilde{A}_je_j+(\tilde{R}_{j,\alpha}-R_{j,\alpha})\tilde{A}_je_j+R_{j,\alpha}(F(q_j,k_j)+\tilde{A}_je_j).$$

Hence,

$$\begin{aligned}&\|e_j+R_{j,\alpha}F(q_j,k_j)\|_{\mathcal{S}}\\ \leq&\|e_j-\tilde{R}_{j,\alpha}\tilde{A}_je_j\|_{\mathcal{S}}+\|(\tilde{R}_{j,\alpha}-R_{j,\alpha})\tilde{A}_je_j\|_{\mathcal{S}}+\|R_{j,\alpha}(F(q_j,k_j)+\tilde{A}_je_j)\|_{\mathcal{S}}.\end{aligned}$$

Lemma 5.1 and the estimates (5.6) and (5.7) provide

$$\|e_j+R_{j,\alpha}F(q_j,k_j)\|_{\mathcal{S}}\leq\|e_j-\tilde{R}_{j,\alpha}\tilde{A}_je_j\|_{\mathcal{S}}+\left(\frac{9d_2d_4}{4\alpha}+\frac{d_2}{2\sqrt{\alpha}}\right)\|e_j\|_{\mathcal{S}}^2. \tag{5.10}$$

To estimate the first term on the right hand side of (5.10), Lemma 5.2 is needed. Let 0, $\sigma_l(k), l=1,\ldots,n_k$ be the singular eigenvalues of the bounded operator $\partial_qF(\tilde{q}(k),k)$. Assume that $\tilde{q}(k_j)$ is the unique q_j-minimum-norm solution of $F(q,k_j)=0$ in $B_\rho(\tilde{q}(k_j))$. It follows from Lemma 5.2 and Hypothesis **H3** that

$$\|e_j-\tilde{R}_{j,\alpha}\tilde{A}_je_j\|_{\mathcal{S}}\leq\frac{\alpha}{\alpha+\sigma^2}\|e_j\|_{\mathcal{S}},$$

where $\sigma=\inf_{[k_{min}k_{max}]}\{\sigma_l(k)\}_{l=1,\ldots,n_k}$. Then the estimate (5.10) becomes

$$\|e_j+R_{j,\alpha}F(q_j,k_j)\|_{\mathcal{S}}\leq\frac{\alpha}{\alpha+\sigma^2}\|e_j\|_{\mathcal{S}}+\left(\frac{9d_2d_4}{4\alpha}+\frac{d_2}{2\sqrt{\alpha}}\right)\|e_j\|_{\mathcal{S}}^2\,. \tag{5.11}$$

Hence

$$\begin{aligned}\|e_{j+1}\|_{\mathcal{S}}\leq&\frac{d_0}{N}+\frac{d_1}{N^2}+\frac{d_3}{2\sqrt{\alpha}N^2}+\left[\frac{\alpha}{\alpha+\sigma^2}+\frac{1}{N}\left(\frac{9d_2d_5}{4\alpha}+\frac{d_6}{2\sqrt{\alpha}}\right)\right]\|e_j\|_{\mathcal{S}}\\ &+\left(\frac{9d_2d_4}{4\alpha}+\frac{d_2}{2\sqrt{\alpha}}\right)\|e_j\|_{\mathcal{S}}^2\,.\end{aligned} \tag{5.12}$$

Now, let δ be a fixed number in $(0,1)$. Then, there exits α in $(0,1)$ satisfying

$$\frac{\alpha}{\alpha+\sigma^2} \leq \frac{\delta}{3}. \tag{5.13}$$

Similarly, there exists $\rho = \rho(\alpha)$, such that

$$\left(\frac{9d_2d_4}{4\alpha} + \frac{d_2}{2\sqrt{\alpha}}\right)\rho \leq \frac{\delta}{3}. \tag{5.14}$$

Therefore, we can choose $N_1 = N_1(\alpha) \in \mathbb{N}$ so that

$$\frac{1}{N}\left(\frac{9d_2d_5}{4\alpha} + \frac{d_6}{2\sqrt{\alpha}}\right) \leq \frac{\delta}{3}, \tag{5.15a}$$

$$\frac{d_0}{N} + \frac{d_1}{N^2} + \frac{d_3}{2\sqrt{\alpha}N^2} \leq (1-\delta)\rho, \tag{5.15b}$$

for all $N \geq N_1$.

Assume that $\|e_0\|_{\mathcal{S}} \leq \rho$. It may be easily shown by recurrence that for all $N \geq N_1$ and $0 \leq j \leq N-1$:

$$\|e_j\|_{\mathcal{S}} \leq \rho, \text{ and } \|e_{j+1}\|_{\mathcal{S}} \leq \frac{d_0}{N} + \frac{d_1}{N^2} + \frac{d_3}{2\sqrt{\alpha}N^2} + \delta\|e_j\|_{\mathcal{S}}.$$

Hence

$$\|e_{j+1}\|_{\mathcal{S}} \leq \left(\frac{d_0}{N} + \frac{d_1}{N^2} + \frac{d_3}{2\sqrt{\alpha}N^2}\right)\frac{1-\delta^{j+1}}{1-\delta} + \delta^{j+1}\|e_0\|_{\mathcal{S}} \tag{5.16}$$

holds for $0 \leq j \leq N-1$. In particular, for $j = N-1$, we have

$$\begin{aligned}
\|e_N\|_{\mathcal{S}} \leq& \left(\frac{d_0}{N} + \frac{d_1}{N^2} + \frac{d_3}{2\sqrt{\alpha}N^2}\right)\frac{1-\delta^N}{1-\delta} + \delta^N\|e_0\|_{\mathcal{S}} \\
=& \frac{1}{N\sqrt{\alpha}}\left[\left((d_0 + \frac{d_1}{N})\sqrt{\alpha} + \frac{d_3}{2N}\right)\frac{1-\delta^N}{1-\delta} + N\delta^N\sqrt{\alpha}||e_0||_{\mathcal{S}}\right].
\end{aligned} \tag{5.17}$$

From (5.13), it follows that

$$\sqrt{\alpha} \leq \frac{\sigma\sqrt{\delta}}{\sqrt{3-\delta}}.$$

Then, there exist $N_0 \geq N_1$ and a positive constant C independent of N and α, such that

$$\left((d_0 + \frac{d_1}{N})\frac{\sigma\sqrt{\delta}}{\sqrt{3-\delta}} + \frac{d_3}{2N}\right)\frac{1-\delta^N}{1-\delta} + \frac{N\delta^{N+\frac{1}{2}}\sigma}{\sqrt{3-\delta}}\|e_0\|_{\mathcal{S}} \leq C.$$

Finally, choosing $\|e_0\|_{\mathcal{S}} = \rho$ and c_0 a constant such that

$$\left(\frac{9d_2d_4}{4} + \frac{d_2\sigma\sqrt{\delta}}{2\sqrt{3-\delta}}\right)c_0 \leq \frac{\delta}{3},$$

the proof of Theorem 5.1 is complete by substituting the above estimates into the estimate (5.17). $\square$

Remark 5.4. It is evident from the inequality (5.13) that α is proportional to σ. Consequently, if we increase the k_{max}, i.e., we take a larger frequency band, then the infinimum σ will naturally decrease. On the other hand, if we fix the frequency and increase L to reconstruct the high frequency modes of the medium, we get the logarithmic convergence found in studying the classical iterative methods. This is provoked by the exponentially decaying of σ.

Remark 5.5. The parameter k does not have to be a frequency. In fact, in [7], for the fixed frequency case, a spatial frequency parameter is used.

Remark 5.6. In our formulation of the problem, it is assumed that the measurement system is efficient and the data is noise-free (without fluctuations). However, in real measurements, there are always various types of errors associated with the data, which represents a major difficulty for solving ill-posed problems. Fortunately, by taking into account the uncertainty principle, the inverse problem becomes regularized and thus a small perturbation in the data, i.e., $\|F_\kappa(q,k) - F(q,k)\|_{\mathcal{L}(\mathcal{S},Y)} < \kappa$ with κ small, essentially has no effect on the convergence analysis.

The derivation of the error between the real medium q and its observable part $\tilde{q}$ is strongly linked to the uniqueness of the inverse problem and the stability of the inverse problem when the frequency is increasing. A future project is to investigate this important connection.

6. Convergence for the Inverse Medium Scattering

In this section, we apply the convergence theorem of Section 5 to the inverse scattering problem introduced in Section 2. In particular, we verify all the hypotheses for our convergence theorem for the recursive linearization algorithm of Chen [9].

The method developed in Section 5 is based on the singular eigenvalue analysis of the scattering map. These singular values provide the limit in observation of details of the medium by a measurements system. Such an analysis provides a good explanation about the uncertainty principle. Unfortunately, in practice, because of extremely small magnitudes and the noise associate with the data, the computation of all of the singular values becomes a very difficult task. To overcome this difficulty, by following [9], we propose a useful limit to the observation.

It is known that inside the disc D, the solution of the Helmholtz equation can be expressed in the polar coordinates as

$$\phi_0(r,\theta) = \sum_{m=-\infty}^{\infty} \alpha_m J_m(kr) e^{im\theta},$$

where the sequence $\alpha = \{\alpha_m\}$ is complex.

Similarly, outside of the disc D, the solution ψ of the homogeneous Helmholtz equation can be expressed as

$$\psi(r,\theta) = \sum_{m=-\infty}^{\infty} \xi_m H_m(kr) e^{im\theta},$$

where the sequence $\xi = \{\xi_m\}$ is also complex. Here, H_m is the Hankel function of the first kind of order m, J_m is the Bessel function of the first kind of order m.

The scattering matrix $S_{R,k}(q)$ is a linear mapping defined by

$$\xi = S_{R,k}\alpha, \tag{6.1}$$

for every α of an incident field and ξ of the corresponding scattered field. Then the nonlinear inverse scattering is to determine the medium function q from knowledge of the following set of matrices

$$\Big\{S_{R,k_j}(q),\ \ j=1,\dots,N\Big\}.$$

The data represents all the information measured on the boundary of the medium at frequencies $k_j,\ \ j=1,\dots,N$. Furthermore, it has been shown in [9] that the coefficients of the matrix $S_{R,k}$ satisfy

$$(S_{R,k})_{m,n}=\mathcal{O}(H_m(kR)^{-1}),(S_{R,k})_{n,m}=\mathcal{O}(J_m(kR)),$$

for an arbitrary integer n and large integer $m\ge N_0(kR)$ (the Bessel function of order 0). From the asymptotic behavior of the Bessel and Hankel functions for large order m, it is clear that an entry of $S_{R,k}$ whose row or column index is greater in absolute value than $N_0(kR)$ is infinitesimal. Hence, the m-angular expansion coefficient of the scattered field with large m is very small. It follows from the uncertainty principle that the propagating waves associated with those coefficients are not detectable by the measurement system.

Therefore, the information about the medium present in the collected data is essentially contained in the square matrix: $(S_{R,k})_{m,n},\ \ |m|,|n|\le\beta(k)$, where β is the observation limit which is larger than $N_0(kR)$ and depends on the regularity of the medium.

By taking

$$F(q,k)=\begin{cases}(S_{R,k})_{m,n}, & |m|,|n|\le L,\\ 0,\ \text{outside},\end{cases}$$

with $L=\max_{k_{min}\le k\le k_{max}}\beta(k)$, the problem may be reduced to the general one studied in Section 5.

We next apply our convergence theorem to the present problem. By Theorems 3.1 and 3.2, the scattering matrix $S_{R,k}(q)$ depends analytically on the medium and its linearization $S'_{R,k}$ is injective for k outside a countable set in $\mathbb{R}_+$. It is easily verified that the function $F(q,k)$ satisfies all of the hypotheses **H1**, **H2**, and **H3** of the last section. Therefore, Theorem 5.1 presents the first convergence result along with the error estimate for the recursive linearization method of Chen [9]. We believe that the convergence analysis may be applied to investigate convergence properties of other methods based on the general recursive linearization.

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Appendix: Uncertainty Principle

We present here a heuristic justification of our approach from a physical point of view. For convenience, the discussion is focused on the scattering of electromagnetic waves rather than acoustic waves. In this setting, the inverse scattering problem is concerned with the reconstruction of a scatterer (object) from its light field (image). The relation between the object structure and light field is described by the Maxwell equations. The measurements of the fields are usually detected far away from the object, *i.e.*, at least a few wavelengths away, which encode the object information.

An important question arises: what is the highest resolution one can expect in analyzing the light field for an optical system?

At the end of the nineteenth century, Abbe and Rayleigh showed that there is a limit to the sharpness of details that could be observed with an optical microscope. This limit known as the diffraction limit is about half a wavelength ($\lambda/2$). Heisenberg later used the example of an optical experiment to illustrate the uncertainty principle. In [20], interesting examples of the Heisenberg relation were presented based on light diffraction and analyzed in terms of the Fourier integral analysis. The analysis indicates that when the scatterer is larger than $\lambda/2$, almost the whole Fourier modes could be detected in the far field and consequently may be recovered. However, when its dimensions becomes smaller than $\lambda/2$, a large part of its Fourier modes becomes evanescent and thus is lost. The scatterer can no longer be reconstructed. Moreover, in this case, it becomes impossible to distinguish the scatterer's features from those of a smaller scatterer.

The loss of the details is related to the existence of non-radiative components of the field known as evanescent waves which contain small details of the scatterer. In general, a light beam incident on a scatterer $q(x)$ may be converted into propagating components that propagate toward the detector and evanescent components confined on the surface. The propagating waves transmit the low spatial frequencies of the object, which contain the most significant characteristics of the object. The evanescent waves, on the other hand, are related to the high spatial frequency information of the object, which contain information about fine (sub-Rayleigh) features of the object.

Consequently, from the measurements far away from the scatterer, the smallest details that can be detected are always larger than half a wavelength. Denote by $\tilde{q}(k)$ the part of the medium $q(x)$ that can by observed through the detected propagating waves. The uncertainty principle may be easily understood in the case of weak scattering. If the wave number is small compare to the magnitude of the scatterer, it is well known that the knowledge of the scattered fields for all incident waves is equivalent to the knowledge of the Fourier modes of the scatterer in the aperture $D(2k)$, where k is the wave number, *i.e.*,

$$\tilde{q}(k) \sim \mathcal{F}^{-1}(\chi_{D(2k)}\mathcal{F}(q))(x),$$

with $\mathcal{F}$ the Fourier transform. Hence the scatterer can be determined with a resolution $\frac{2\pi}{2k} = \lambda/2$ under the weak scattering assumption.

Note that the uncertainty principle does not limit the resolution but merely provides a limit on the accuracy of the reconstruction of the scatterer for a given wave number. Consequently, *it is impossible to specify precisely the complete features of a scatterer from measurements only at low wave numbers.* In practice, the uncertainty on the reconstructed scatterer can be reduced by increasing the number and the magnitude of wave numbers. An important challenge is to use the uncertainty principle to derive an accurate reconstruction method of the scatterer, which is the focus of the present paper.

Mathematically, the reconstruction of the entire features of the scatterer is assured once the uniqueness of the inverse problem is proved. In fact, it is well known that the scatterer is uniquely determined even when $k = 0$. The uniqueness argument may be easily understood in the case of weak scattering, k small. In weak scattering, the measurement of the far field reconstructs only the Fourier modes $\{\mathcal{F}(q)(\xi), |\xi| \leq 2k\}$. Assume that there is another function $p(x)$ which has the same Fourier modes. Then $\mathcal{F}(p-q)(\xi) = 0$ in $D(2k)$. Since $p-q$ is compactly supported in a bounded domain D, the function $\mathcal{F}(p-q)(\xi)$ is an entire function of $\xi \in \mathbb{C}^2$ which vanishes in a ball. It follows that the function vanishes identically and consequently $p = q$. Furthermore, the rest of Fourier modes of the scatterer can be reconstructed by a spectral extrapolation. Therefore, the scatterer can be completely reconstructed. Moreover, the approximation is exact. In other words, one can reconstruct arbitrarily accurate small details of the scatterer from the knowledge of its Fourier modes on a fixed bounded domain and a guess on its regularity. This conclusion clearly contradicts to the uncertainty principle: one cannot resolve details smaller than one half of the wavelength. The contradiction results from the overlooked stability issue. In fact, for any realistic situation, the measured data is never exact hence only noisy data about those finite Fourier modes may be assumed. As the result, the techniques that reconstruct the scatterer from less information amplify the error in the derivation of the rest of the information. Consequently, the infinitesimal details of the scatterer are lost due to the amplified error, i.e., the ill-posedness of the inverse problem.

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