Journal of Computational Mathematics
Vol.28, No.5, 2010, 579–605.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1003-m2957

WEAK APPROXIMATION OF OBLIQUELY REFLECTED DIFFUSIONS IN TIME-DEPENDENT DOMAINS*

Kaj Nyström and Thomas Önskog
Department of Mathematics and Mathematical Statistics, Umeå University, SE-901 87 Umeå, Sweden
Email: kaj.nystrom@math.umu.se thomas.onskog@math.umu.se

Abstract

In an earlier paper, we proved the existence of solutions to the Skorohod problem with oblique reflection in time-dependent domains and, subsequently, applied this result to the problem of constructing solutions, in time-dependent domains, to stochastic differential equations with oblique reflection. In this paper we use these results to construct weak approximations of solutions to stochastic differential equations with oblique reflection, in time-dependent domains in $\mathbb{R}^d$, by means of a projected Euler scheme. We prove that the constructed method has, as is the case for normal reflection and time-independent domains, an order of convergence equal to 1/2 and we evaluate the method empirically by means of two numerical examples. Furthermore, using a well-known extension of the Feynman-Kac formula, to stochastic differential equations with reflection, our method gives, in addition, a Monte Carlo method for solving second order parabolic partial differential equations with Robin boundary conditions in time-dependent domains.

Mathematics subject classification: 65MXX, 35K20, 65CXX, 60J50, 60J60

Key words: Stochastic differential equations, Oblique reflection, Robin boundary conditions, Skorohod problem, Time-dependent domain, Weak approximation, Monte Carlo method, Parabolic partial differential equations, Projected Euler scheme.

1. Introduction

The use of projected Euler schemes, in the construction of weak approximations of solutions to stochastic differential equations with reflection, originates in the work of Saisho [1], who used this approach to prove existence and uniqueness of solutions to stochastic differential equations with normal reflection in time-independent domains. The ideas in [1] were developed further by Costantini, Pacchiarotti and Sartoretto [2], who presented a projected Euler scheme based on the previous work by Costantini [3] concerning the existence of solutions to the Skorohod problem with oblique reflection in time-independent domains. The algorithm proposed in [2] provides, in particular, a Monte Carlo method for solving second order parabolic partial differential equations with mixed Dirichlet and Robin boundary conditions in fairly general time-independent domains. However, in [2] it is proved that the order of convergence of the proposed algorithm is merely 1/2. In recent years several attempts have been made to find more efficient algorithms for stochastic differential equations, with reflection, but the attempts have only been successful for quite limited sets of boundary conditions. In this context we mention, in particular, the projected Euler schemes suggested by Gobet [4] and by Bossy, Gobet and Talay [5]. The order of convergence of the algorithm proposed in [4], which is based on the

* Received November 5, 2008 / Revised version received June 4, 2009 / Accepted October 20, 2009 / Published online May 1, 2010 /

explicit solution to the Skorohod problem in half-spaces, is 1 in the special case of Neumann boundary conditions with reflection in the conormal direction. On the other hand, the order of convergence of the algorithm in [5] is 1 for all possible directions of reflection, but for this algorithm only a very restricted set of boundary data is allowed. An alternative approach to the problem of weak approximation of solutions to stochastic differential equations with oblique reflection, in time-independent domains with smooth boundary, was given by Mil'shtein [6], who presented two numerical algorithms for second order parabolic partial differential equations with Robin boundary conditions. Although the order of convergence of the fastest of these two algorithms is 1, both algorithms have proved to be difficult to implement due to the fact that a change of coordinate system is required at all time steps at which the approximate solution to the stochastic differential equation is close to the boundary.

An important novelty of the article at hand is that we present an algorithm for weak approximation of stochastic differential equations with oblique reflection in the setting of time-dependent domains and to our knowledge this is indeed an area which is less developed compared to the corresponding problem in time-independent domains. Nevertheless, stochastic differential equations with reflection in time-dependent domains emerge in a variety of applications such as singular stochastic control problems and particle dispersion in volumes with fluctuating size and shape. From a purely theoretical point of view, the study of stochastic differential equations with reflection in time-dependent domains was commenced by Costantini, Gobet and El Karoui [7], who proved existence and uniqueness of solutions to the Skorohod problem with normal reflection in smooth time-dependent domains. Concerning the Skorohod problem we also note that existence and uniqueness for deterministic problems of Skorohod type, in time-dependent intervals, have recently been established by Burdzy et al. (see [8, 9]). In [10] we conducted a thorough study of the multi-dimensional Skorohod problem in time-dependent domains and we proved, in particular, the existence of cádlág solutions to the Skorohod problem, with oblique reflection, in fairly general time-dependent domains. Furthermore, in [10] we, subsequently, used our results on the Skorohod problem to construct solutions to stochastic differential equations with oblique reflection in time-dependent domains. Moreover, in the process of proving these results, we established a number of estimates for solutions, with bounded jumps, to the Skorohod problem. In this article we build on the study in [10] and we use the results in [10] regarding the Skorohod problem to develop an algorithm for weak approximation of stochastic differential equations with oblique reflection in the setting of time-dependent domains. Our approximation procedure is, from a numerical point of view, similar to the projected Euler scheme described in [2], but our setting is different and more general compared to [2] as we, in particular, allow for time-dependent domains, more general functionals as well as reflection in oblique directions. By proceeding along the lines of [2] we also prove that the proposed algorithm has an order of convergence equal to $1/2$ and we emphasize that while this convergence may seem slow, the main advantage of the approach is, and this makes it different from the algorithms and results in [4–6], that the method outlined is applicable in very general situations.

To briefly outline the general result, established in [10], concerning stochastic differential equations with oblique reflection in time-dependent domains, and to formulate the results of this article, we next introduce some notation. Given $d \geq 1$, we let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^d$ and we let $|z| = \langle z, z \rangle^{1/2}$ be the Euclidean norm of z. Whenever $z \in \mathbb{R}^d$, $r > 0$, we let

$$B_r(z) = \left\{ y \in \mathbb{R}^d : |z - y| < r \right\} \quad \text{and} \quad S_r(z) = \left\{ y \in \mathbb{R}^d : |z - y| = r \right\}.$$

Moreover, given $D \subset \mathbb{R}^{d+1}$, $E \subset \mathbb{R}^d$, we let $\bar{D}$, $\bar{E}$ be the closure of D and E, respectively, and we let $d(y, E)$ denote the Euclidean distance from $y \in \mathbb{R}^d$ to E. Given $d \geq 1$, $T > 0$ and an open, connected set $D' \subset \mathbb{R}^{d+1}$ we will refer to

$$D = D' \cap ([0, T] \times \mathbb{R}^d) \tag{1.1}$$

as a time-dependent domain. Given D and $s \in [0, T]$, we define the time sections of D as $D_s = \{z : (s, z) \in D\}$, and we assume that

$$D_s \neq \emptyset \text{ and that } D_s \text{ is bounded and connected for every } s \in [0, T]. \tag{1.2}$$

We let ∂D and ∂D_s, for $s \in [0, T]$, denote the boundaries of D and D_s, respectively. A convex cone of vectors in $\mathbb{R}^d$ is a subset $\Gamma \subset \mathbb{R}^d$ such that $\alpha u + \beta v \in \Gamma$ for all positive scalars α, β and all $u, v \in \Gamma$. To give an example of a closed convex cone, we consider the set $C = C_\Omega = \{\lambda\gamma : \ \lambda > 0, \ \gamma \in \Omega\}$, where Ω is a closed, connected subset of $S_1(0)$ satisfying $\gamma_1 \cdot \gamma_2 > -1$ for all $\gamma_1, \gamma_2 \in \Omega$. Given C we define $C^* = \{\alpha u + \beta v : \ \alpha, \beta \in \mathbb{R}_+, \ u, v \in C\}$. Then C^* is an example of a closed convex cone and we note that $C^* = C^*_{\Omega^*}$ where Ω^* can be viewed as the 'convex hull' of Ω on $S_1(0)$. We let $\Gamma = \Gamma_s(z) = \Gamma(s, z)$ be a function defined on $\mathbb{R}^{d+1}$ such that $\Gamma_s(z)$ is a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_s$, $s \in [0, T]$. Given $\Gamma = \Gamma_s(z) = \Gamma(s, z)$ we let $\Gamma_s^1(z) := \Gamma_s(z) \cap S_1(0)$. Following [3], we assume that

$$\gamma_1 \cdot \gamma_2 > -1 \text{ holds whenever } \gamma_1, \gamma_2 \in \Gamma_s^1(z) \text{ and for all } z \in \partial D_s, \ s \in [0, T]. \tag{1.3}$$

This assumption eliminates the possibility of Γ containing vectors in opposite directions. We also assume that the set

$$G^\Gamma = \left\{ (s, z, \gamma) : \gamma \in \Gamma_s(z), \ z \in \partial D_s, s \in [0, T] \right\} \text{ is closed.} \tag{1.4}$$

The interpretation of the condition in (1.4) is discussed in [10]. Furthermore, we assume that the cone of inward normal vectors at $z \in \partial D_s$, denoted $N_s(z)$, is non-empty for all $z \in \partial D_s$, $s \in [0, T]$. Note that we allow for the possibility of several inward normal vectors at the same boundary point. Given $N_s(z)$ we let $N_s^1(z) := N_s(z) \cap S_1(0)$. Then the spatial domain D_s is said to verify the uniform exterior sphere condition if there exists a radius $r_0 > 0$ such that

$$B_{r_0}(z - r_0 n) \subseteq \mathbb{R}^d \setminus D_s \tag{1.5}$$

whenever $z \in \partial D_s$, $n \in N_s^1(z)$. Note that $B_{r_0}(z - r_0 n)$ is the open Euclidean ball with center $z - r_0 n$ and radius r_0. We say that a time-dependent domain D satisfies a uniform exterior sphere condition in time if the uniform exterior sphere condition in (1.5) holds, with the same radius r_0, for all spatial domains D_s, $s \in [0, T]$. Furthermore, following [7], we let

$$l(r) = \sup_{\substack{s,u \in [0,T] \\ |s-u| \leq r}} \ \sup_{z \in \overline{D_u}} d(z, D_s) \tag{1.6}$$

be the modulus of continuity of the variation of D in time. In particular, in our work on the Skorohod problem in [10] we assume that

$$\lim_{r \to 0^+} l(r) = 0. \tag{1.7}$$

While the condition in (1.7) concerns the temporal changes of D, we also need an assumption concerning the variation of the cone $\Gamma_s(z)$. To formulate this assumption we let

$$h(E,F) = \max\Bigg(\sup\{d(z,E) : z \in F\}, \sup\{d(z,F) : z \in E\}\Bigg) \tag{1.8}$$

denote the Hausdorff distance between the sets $E, F \subset \mathbb{R}^d$. Moreover, let $\{(s_n, z_n)\}$ be a sequence of points in $\mathbb{R}^{d+1}$, $s_n \in [0,T]$, $z_n \in \partial D_{s_n}$, such that

$$\lim_{n\to\infty} s_n = s \in [0,T], \lim_{n\to\infty} z_n = z \in \partial D_s.$$

We assume, for any such sequence of points $\{(s_n, z_n)\}$, that

$$\lim_{n\to\infty} h(\Gamma_{s_n}(z_n), \Gamma_s(z)) = 0. \tag{1.9}$$

Given $T > 0$, $t \in [0,T]$, we let $\mathcal{C}([t,T], \mathbb{R}^d)$ denote the class of continuous functions from $[t,T]$ to $\mathbb{R}^d$. We let m be a positive integer and we let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be given functions which are bounded and continuous. Finally, we let $\mathcal{BV}([t,T], \mathbb{R}^d)$ denote the set of functions $\lambda = \lambda_s : [t,T] \to \mathbb{R}^d$ with bounded variation and we let $|\lambda|$ denote the total variation of $\lambda \in \mathcal{BV}([t,T], \mathbb{R}^d)$.

Definition 1.1. *Let $T > 0$ and let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying* (1.2). *Let $\Gamma = \Gamma_s(z)$ be a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_s$, $s \in [0,T]$. Let $t \in [0,T]$ and assume that $x \in \overline{D_t}$. A weak solution to the stochastic differential equation in $\overline{D}$ with coefficients b and σ, reflection along Γ_s on ∂D_s, $s \in [t,T]$, and with initial condition x at t, is a stochastic process $(X^{t,x}, \Lambda^{t,x})$, with paths in $\mathcal{C}([t,T], \mathbb{R}^d) \times \mathcal{BV}([t,T], \mathbb{R}^d)$ and with $\Lambda^{t,x}_t = 0$, which is defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, P)$ and satisfies*

$$X^{t,x}_s = x + \int_t^s b(r, X^{t,x}_r)\,dr + \int_t^s \sigma(r, X^{t,x}_r)\,dW_r + \Lambda^{t,x}_s, \tag{1.10}$$

$$\Lambda^{t,x}_s = \int_t^s \gamma_r\, d|\Lambda^{t,x}|_r, \qquad \gamma_r \in \Gamma_r(X^{t,x}_r) \cap S_1(0), \quad d|\Lambda^{t,x}|\text{-}a.e., \tag{1.11}$$

$$X^{t,x}_s \in \overline{D_s}, \qquad d|\Lambda^{t,x}|(\{s \in [t,T] : X^{t,x}_s \in D_s\}) = 0, \tag{1.12}$$

P-almost surely, whenever $s \in [t,T]$. Here W is a m-dimensional Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}, P)$ and $(X^{t,x}, \Lambda^{t,x})$ is $\{\mathcal{F}_s\}$-adapted.

In Theorem 1.5 in [10] we proved the following theorem which generalizes the corresponding results in [1, 3, 7].

Theorem 1.1. *Let $T > 0$, let $t \in [0,T]$, and let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain which satisfies* (1.2), (1.7) *and a uniform exterior sphere condition in time with radius r_0 in the sense of* (1.5). *Let $\Gamma = \Gamma_s(z)$ be a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_s$, $s \in [t,T]$, and assume that Γ satisfies* (1.3), (1.4) *and* (1.9). *Assume that* (2.9) *and* (2.10), *stated in Section* 2.2, *hold for some $0 < \rho_0 < r_0$, $\eta_0 > 0$, a and e. Finally, assume that $([0,T] \times \mathbb{R}^d) \setminus \overline{D}$ has the (δ_0, h_0)-property of good projections along Γ, for some $0 < \delta_0 < \rho_0$, $h_0 > 1$, as defined in* (2.11) *and* (2.12) *stated in Section* 2.2. *Let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be given bounded and continuous functions on $\overline{D}$ and let $x \in \overline{D_t}$. Then there exists a weak solution, in the sense of Definition* 1.1, *to the stochastic differential equation in $\overline{D}$ with coefficients b and σ, reflection along Γ_s on ∂D_s, $s \in [t,T]$, and with initial condition x at t.*

Let $T > 0$, $t \in [0,T]$, $D \subset \mathbb{R}^{d+1}$ and let $\Gamma = \Gamma_s(z)$ be as in the statement of Theorem 1.1. Let $x \in \overline{D_t}$ and let $(X^{t,x}, \Lambda^{t,x})$ be a weak solution as in the statement of Theorem 1.1. Let f, g, h, φ and θ be given functions which are bounded on their domains of definition. Given these functions we define, for $t \in [0,T]$ and $x \in \overline{D_t}$, the functional

$$\begin{aligned} F_{t,T}(X^{t,x}, \Lambda^{t,x}) =& f(X_T^{t,x}) \exp(Y_T^{t,x} + Z_T^{t,x}) - \int_t^T g(s, X_s^{t,x}) \exp(Y_s^{t,x} + Z_s^{t,x})\, d\left|\Lambda^{t,x}\right|_s \\ & - \int_t^T h(s, X_s^{t,x}) \exp(Y_s^{t,x} + Z_s^{t,x})\, ds, \end{aligned} \tag{1.13}$$

where the processes $Y_s^{t,x}$ and $Z_s^{t,x}$ are defined as

$$Y_s^{t,x} = -\int_t^s \varphi(r, X_r^{t,x})\, dr, \quad Z_s^{t,x} = -\int_t^s \theta(r, X_r^{t,x})\, d\left|\Lambda^{t,x}\right|_r. \tag{1.14}$$

In this article we develop a numerical algorithm for calculating the expectation

$$u(t,x) = E[F_{t,T}(X^{t,x}, \Lambda^{t,x})]. \tag{1.15}$$

In particular, let N be a large positive integer, let $\Delta^* = (T-t)/N$ and let $\tau_k = t + k\Delta^*$ for $k \in \{0,1,...,N\}$. Then $t = \tau_0 < \tau_1 < .... < \tau_N = T$ and $\{\tau_k\}_{k=0}^N$ defines a partition Δ of the time interval $[t,T]$. Based on Δ we define an approximation of $(X^{t,x}, \Lambda^{t,x}, Y^{t,x}, Z^{t,x})$, denoted $\left(X^\Delta, \Lambda^\Delta, Y^\Delta, Z^\Delta\right)$, by means of a recursive algorithm for $\left(X_{\tau_k}^\Delta, \Lambda_{\tau_k}^\Delta, Y_{\tau_k}^\Delta, Z_{\tau_k}^\Delta\right)$, for $k \in \{0,1,...,N\}$. In particular, we define

$$D_s^\Delta = D_{\tau_k},\ \Gamma_s^\Delta = \Gamma_{\tau_k}, \quad \text{whenever } s \in [\tau_k, \tau_{k+1}),\ k \in \{0,1,...,N-1\}, \tag{1.16}$$

and $D_{\tau_N}^\Delta = D_{\tau_N}$, $\Gamma_{\tau_N}^\Delta = \Gamma_{\tau_N}$. Furthermore, we recursively define the three processes $X^\Delta = X_s^\Delta$, $U^\Delta = U_s^\Delta$ and $\Lambda^\Delta = \Lambda_s^\Delta$, for $s \in [t,T]$ as follows. We let

$$X_{\tau_0}^\Delta = x, \quad U_{\tau_0}^\Delta = x, \quad \Lambda_{\tau_0}^\Delta = 0, \tag{1.17}$$

and, for $k \in \{0,1,...,N-1\}$, we let

$$\begin{aligned} U_{\tau_{k+1}}^\Delta &= U_{\tau_k}^\Delta + b(\tau_{k+1}, X_{\tau_k}^\Delta)\Delta^* + \sigma(\tau_{k+1}, X_{\tau_k}^\Delta)\sqrt{\Delta^*}\Delta_{k+1}\eta, \\ X_{\tau_{k+1}}^\Delta &= \pi_{\partial D_{\tau_{k+1}}^\Delta}^{\Gamma_{\tau_{k+1}}^\Delta}\left(X_{\tau_k}^\Delta + U_{\tau_{k+1}}^\Delta - U_{\tau_k}^\Delta\right), \end{aligned} \tag{1.18}$$

where $\pi_{\partial D_{\tau_{k+1}}^\Delta}^{\Gamma_{\tau_{k+1}}^\Delta}(z)$ is defined as $\pi_{\partial D_{\tau_{k+1}}^\Delta}^{\Gamma_{\tau_{k+1}}^\Delta}(z) = z$ whenever $z \in \overline{D_{\tau_{k+1}}^\Delta}$ and, whenever $z \notin \overline{D_{\tau_{k+1}}^\Delta}$, as a point on $\partial D_{\tau_{k+1}}^\Delta$ which we shall refer to as a projection of z onto $\partial D_{\tau_{k+1}}^\Delta$ along $\Gamma_{\tau_{k+1}}^\Delta$. Furthermore, $\Delta_{k+1}\eta$ is a random variable chosen so that $\sqrt{\Delta^*}\Delta_{k+1}\eta$ mimics the Wiener increment $W_{\tau_{k+1}} - W_{\tau_k}$. The actual definition of $\Delta_{k+1}\eta$ is given in the bulk of the article. Note, however, that as we are assuming that $([0,T] \times \mathbb{R}^d) \setminus \overline{D}$ has the (δ_0, h_0)-property of good projections along Γ, for some $0 < \delta_0 < \rho_0$, $h_0 > 1$, we can only ensure that

$$\pi_{\partial D_{\tau_{k+1}}^\Delta}^{\Gamma_{\tau_{k+1}}^\Delta}(X_{\tau_k}^\Delta + U_{\tau_{k+1}}^\Delta - U_{\tau_k}^\Delta) \text{ is well-defined for } X_{\tau_k}^\Delta + U_{\tau_{k+1}}^\Delta - U_{\tau_k}^\Delta \notin \overline{D_{\tau_{k+1}}^\Delta}$$

if

$$d(X_{\tau_k}^\Delta + U_{\tau_{k+1}}^\Delta - U_{\tau_k}^\Delta, D_{\tau_{k+1}}^\Delta) < \delta_0. \tag{1.19}$$

Now since $W_{\tau_{k+1}} - W_{\tau_k}$ can assume arbitrarily large values we must, in order to make sure that (1.19) holds, somehow truncate $W_{\tau_{k+1}} - W_{\tau_k}$ and this puts a restriction on the possible choices for $\sqrt{\Delta^*}\Delta_{k+1}\eta$. Having ensured that all terms in (1.18) are well-defined, we let, for $k \in \{0, 1, ..., N-1\}$,

$$\Lambda^\Delta_{\tau_{k+1}} = \Lambda^\Delta_{\tau_k} + X^\Delta_{\tau_{k+1}} - X^\Delta_{\tau_k} - U^\Delta_{\tau_{k+1}} + U^\Delta_{\tau_k}. \tag{1.20}$$

At intermediate times $s \in [\tau_k, \tau_{k+1})$, $k \in \{0, 1, ..., N-1\}$, we define

$$X^\Delta_s = X^\Delta_{\tau_k}, \quad U^\Delta_s = U^\Delta_{\tau_k}, \quad \Lambda^\Delta_s = \Lambda^\Delta_{\tau_k}. \tag{1.21}$$

Based on $(X^\Delta, \Lambda^\Delta)$ we define (Y^Δ, Z^Δ) as follows. Let $Y^\Delta_{\tau_0} = Z^\Delta_{\tau_0} = 0$ and introduce, for $k \in \{0, 1, ..., N-1\}$, the notation

$$\Delta_{k+1} \left|\Lambda^\Delta\right| = \left|\Lambda^\Delta\right|_{\tau_{k+1}} - \left|\Lambda^\Delta\right|_{\tau_k} = \left|\Lambda^\Delta_{\tau_{k+1}} - \Lambda^\Delta_{\tau_k}\right|.$$

Then we define

$$Y^\Delta_{\tau_{k+1}} = -\sum_{l=0}^{k} \varphi\left(\tau_{l+1}, X^\Delta_{\tau_l}\right)\Delta^*, \quad Z^\Delta_{\tau_{k+1}} = -\sum_{l=0}^{k} \theta\left(\tau_{l+1}, X^\Delta_{\tau_l}\right)\Delta_{l+1}\left|\Lambda^\Delta\right|. \tag{1.22}$$

The choice to evaluate b, σ, φ and θ at the right end point of each time intervals allows us to require as little regularity as possible, in the time variable, in the arguments outlined in the bulk of the paper (see Lemma 3.1 in [2]). Finally we let

$$Y^\Delta_s = Y^\Delta_{\tau_k}, \quad Z^\Delta_s = Z^\Delta_{\tau_k}, \quad \text{whenever } s \in [\tau_k, \tau_{k+1}), \ k \in \{0, 1, ..., N-1\}. \tag{1.23}$$

This completes the definition of $\left(X^\Delta, \Lambda^\Delta, Y^\Delta, Z^\Delta\right)$ and based on this approximation we introduce the following approximations of $F_{t,T}(X^{t,x}, \Lambda^{t,x})$ and $u\left(t, x\right)$, respectively,

$$\begin{aligned} &F^\Delta_{t,T}(X^\Delta, \Lambda^\Delta) \\ =&f(X^\Delta_{\tau_N})\exp(Y^\Delta_{\tau_N} + Z^\Delta_{\tau_N}) - \sum_{k=0}^{N-1} g(\tau_{k+1}, X^\Delta_{\tau_{k+1}})\exp(Y^\Delta_{\tau_k} + Z^\Delta_{\tau_k})\,\Delta_{k+1}\left|\Lambda^\Delta\right| \\ &- \sum_{k=0}^{N-1} h(\tau_{k+1}, X^\Delta_{\tau_{k+1}})\exp(Y^\Delta_{\tau_k} + Z^\Delta_{\tau_k})\,\Delta^*, \end{aligned} \tag{1.24}$$

and

$$u^\Delta\left(t, x\right) = E[F^\Delta_{t,T}\left(X^\Delta, \Lambda^\Delta\right)]. \tag{1.25}$$

The main result established in this article, see Theorem 3.1 below, is valid under appropriate regularity assumptions on the time-dependent domain D and on the cone of directions of reflection. In particular, we assume, given $T > 0$ and $t \in [0, T]$, that $D \subset \mathbb{R}^{d+1}$ is a time-dependent domain satisfying (1.2), $\Gamma = \Gamma_s(z)$ is a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_s$, $s \in [t, T]$, and that

$$D \text{ and } \Gamma \text{ satisfy (2.14), (2.15), (2.16) and (2.46) stated below.} \tag{1.26}$$

Then, see Lemma 2.1 below, all assumptions in Theorem 1.1 concerning D and Γ are fulfilled. It turns out that in order to be able to actually prove that the suggested algorithm has order of convergence equal to 1/2 we need to impose stronger regularity assumptions on the quantities involved compared what is needed in the proof of Theorem 1.1. However, we note that this is often the case when theoretically verifying the order of convergence of numerical algorithms. In particular, assuming (1.26) we are able to prove that

$$\left|u(t,x) - u^{\Delta}(t,x)\right| \leq C(\Delta^*)^{1/2}, \tag{1.27}$$

for some positive constant C, which is independent of (t,x) and Δ^*, whenever Δ^* is sufficiently small. In particular, $u^{\Delta}(t,x) \to u(t,x)$ as $\Delta^* \to 0$ and the order of convergence equals 1/2. To prove (1.27) a key observation is that $(X^{\Delta}, \Lambda^{\Delta})$ solves the Skorohod problem for $(D^{\Delta}, \Gamma^{\Delta}, U^{\Delta})$ in the sense studied in [10]. Hence, using our results in [10] concerning the Skorohod problem, we are able to proceed similarly to the corresponding proofs in [2]. In particular, at the heart of the matter we prove that

$$\sup_{(t,x)\in D} E\left[\left(\left|\Lambda^{\Delta}\right|_T\right)^q\right] \leq C_q < \infty, \text{ uniformly in } \Delta^* \text{ whenever } \Delta^* \text{ is small enough}, \tag{1.28}$$

for all $q \in \mathbb{N}$ and with C_q independent of Δ. In fact, to establish the order of convergence for the numerical algorithm we will only need (1.28) with $q = 1$. However, as the inequality in (1.28), for arbitrary exponent q, requires no further regularity and is of independent interest, we have included the result as stated in the article. The proof of (1.28) is the particular instance where some of our estimates in [10] concerning the Skorohod problem are used.

Finally, to connect the discussion above to the solvability of second order parabolic equation, we let

$$L = \sum_{i=1}^{d} b_i(t,x)\,\partial_{x_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\,\partial_{x_i x_j}, \tag{1.29}$$

where

$$a_{ij}(t,x) = \sum_{k=1}^{m} \sigma_{ik}(t,x)\,\sigma_{jk}(t,x). \tag{1.30}$$

The following theorem connects stochastic differential equations, with oblique reflection, to second order parabolic partial differential equations with Robin boundary conditions.

Theorem 1.2. *Let $u(t,x)$ be given by* (1.15) *and let $F_{t,T}$ be given by* (1.13). *Let the processes $X^{t,x}$, $\Lambda^{t,x}$, $Y^{t,x}$ and $Z^{t,x}$ be defined as above. Then, under suitable regularity assumptions on the domain D, the operator L and the functions f, g, h, φ and θ, see Theorem* 2.5 *below, $u(t,x)$ is a classical solution to the problem*

$$\begin{cases} u_t(t,x) + Lu(t,x) - \varphi(t,x)\,u(t,x) = h(t,x) & \text{in } D_t,\ t \in [0,T), \\ \langle \nabla_x u(t,x), \gamma(t,x)\rangle - \theta(t,x)\,u(t,x) = g(t,x) & \text{on } \partial D_t,\ t \in [0,T), \\ u(T,x) = f(x) & \text{on } D_T. \end{cases} \tag{1.31}$$

It is clear from Theorem 1.2 that the algorithm developed in this article also yields a numerical method for solving second order parabolic partial differential equations with Robin boundary conditions, in time-dependent domain, with a rate of convergence which is independent of the dimension of the underlying space.

The rest of this article is organized as follows. In Section 2, which is of preliminary nature, we introduce notation, collect a number of results concerning the geometry of time-dependent domains, recall a few crucial results from [10] concerning the Skorohod problem in time-dependent domain and state a number of results from [11] concerning the problem in (1.31). In Section 3 we outline the details of the numerical algorithm and derive its order of convergence. Finally, in Section 4, we numerically demonstrate the effectiveness of the algorithm by applying it in two specific settings.

2. Preliminaries

In this section we introduce notation, collect a number of results concerning the geometry of time-dependent domains, recall a few crucial results from [10] concerning the Skorohod problem in time-dependent domain and state a number of results from [11] concerning the problem in (1.31).

2.1. Notation

To start with, points in Euclidean $(d+1)$-space $\mathbb{R}^{d+1}$ are denoted by $(s,z)=(s,z_1,\dots,z_d)$. Given a differentiable function $f=f(s,z)$ defined on $\mathbb{R}\times\mathbb{R}^d$, we let $\partial_{z_i}f(s,z)$ denote the partial derivative of f at (s,z) with respect to z_i. Higher order derivatives of f with respect to the space variables will often be denoted by $\partial_{z_iz_j}f(s,z)$, $\partial_{z_iz_jz_k}f(s,z)$ and so on. Furthermore, given a multi-index $\beta=(\beta_1,...,\beta_d)$, $\beta_i\in\mathbb{Z}_+$, we define $|\beta|=\beta_1+...+\beta_d$ and we let $\partial_z^\beta f(s,z)$ denote the associated partial derivative of $f(s,z)$ with respect to the space variables. Time derivatives of f will be denoted by $\partial_s^j f(s,z)$ where $j\in\mathbb{Z}_+$. As in the introduction, we let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{R}^d$ and we let $|z|=\langle z,z\rangle^{1/2}$ be the Euclidean norm of z. Whenever $z\in\mathbb{R}^d$, $r>0$, we let

$$B_r(z)=\Big\{y\in\mathbb{R}^d:|z-y|<r\Big\}\quad\text{and}\quad S_r(z)=\Big\{y\in\mathbb{R}^d:|z-y|=r\Big\}.$$

In addition dz denotes Lebesgue d-measure on $\mathbb{R}^d$. Moreover, given $E\subset\mathbb{R}^d$ we let $\bar{E}$ and ∂E be the closure and boundary of E, respectively, and we let $d(z,E)$ denote the Euclidean distance from $z\in\mathbb{R}^d$ to E. Given $(s,z),(\tilde{s},y)\in\mathbb{R}^{d+1}$, we let

$$d_p((s,z),(\tilde{s},y))=\max\Big\{|z-y|,|s-\tilde{s}|^{1/2}\Big\}$$

denote the parabolic distance between (s,z) and $(\tilde{s},y)$ and for $F\subset\mathbb{R}^{d+1}$, we let $d_p((s,z),F)$ denote the parabolic distance from $(s,z)\in\mathbb{R}^{d+1}$ to F. Moreover, for $(s,z)\in\mathbb{R}^{d+1}$ and $r>0$, we introduce the parabolic cylinder

$$C_r(s,z)\;=\;\Big\{(\tilde{s},y)\in\mathbb{R}^{d+1}:|y-z|<r,|\tilde{s}-s|<r^2\Big\}.$$

Given two real numbers a and b, we let $a\vee b=\max\{a,b\}$ and $a\wedge b=\min\{a,b\}$. Finally, given a Borel set $E\subset\mathbb{R}^{d+1}$, we let χ_E denote the characteristic function associated to E.

Given a time-dependent domain $D'\subset\mathbb{R}^{d+1}$, a function f defined on D', a constant $\alpha\in(0,1]$ and $k\in\mathbb{Z}_+$, we adopt the definition on page 46 in [11] and introduce

$$[f]_{k+\alpha,D'}=\sum_{|\beta|+2j=k}\sup_{(s,z)\in D'}\sup_{(\tilde{s},y)\in D'\setminus\{(s,z)\}}\frac{\left|\partial_z^\beta\partial_s^j f(\tilde{s},z)-\partial_z^\beta\partial_s^j f(\tilde{s},y)\right|}{\left[d_p((s,z),(\tilde{s},y))\right]^\alpha}. \tag{2.1}$$

Furthermore, we let, for $k\geq 1$,

$$\langle f\rangle_{k+\alpha,D'}=\sum_{|\beta|+2j=k-1}\sup_{(s,z)\in D'}\sup_{(\tilde{s},z)\in D'\setminus\{(s,z)\}}\frac{\left|\partial_z^\beta\partial_s^j f(s,z)-\partial_z^\beta\partial_s^j f(\tilde{s},z)\right|}{|s-\tilde{s}|^{(\alpha+1)/2}}, \tag{2.2}$$

with the convention that $\langle f\rangle_{\alpha,D'}=0$. Finally we let

$$|f|_{k+\alpha,D'}=\sum_{|\beta|+2j\le k}\sup_{D'}\left|\partial_z^\beta\partial_s^j f\right|+[f]_{k+\alpha,D'}+\langle f\rangle_{k+\alpha,D'}\,. \tag{2.3}$$

Moreover, using the norm $|\cdot|_{k+\alpha,D'}$, we introduce appropriate function spaces as on page 46 in [11]. In particular, given the index of regularity $k+\alpha$, where $k\in\mathbb{Z}_+$ and $\alpha\in(0,1]$, we let $\mathcal{H}_{k+\alpha}(D')$ represent the Banach space of functions f such that $|f|_{k+\alpha,D'}<\infty$. Furthermore, we let $\mathcal{C}^{\lfloor k/2\rfloor,k}(D')$ denote the space of functions with continuous derivatives in time up to order $\lfloor k/2\rfloor$ in D' and with continuous derivatives in space up to order k in D'.

2.2. Geometry and regularity of time-dependent domains

We here briefly define and discuss the geometric restrictions imposed in Theorem 1.1. However, for a full account of this theory we refer to [10]. Given $T>0$, we let, as defined in the introduction, $D\subset\mathbb{R}^{d+1}$ be a time-dependent domain satisfying (1.2) and $\Gamma=\Gamma_s(z)=\Gamma(s,z)$ be a function defined on $\mathbb{R}^{d+1}$ such that $\Gamma_s(z)$ is a closed convex cone of vectors in $\mathbb{R}^d$ for every $z\in\partial D_s$, $s\in[0,T]$. The spatial domain D_s is said to verify the uniform exterior sphere condition if there exists a radius $r_0>0$ such that (1.5) holds. We note that (1.5) is equivalent to the statement that

$$\langle n,y-z\rangle+\frac{1}{2r_0}|y-z|^2\ge 0, \tag{2.4}$$

for all $y\in\overline{D_s}$, $n\in N_s^1(z)$ and $z\in\partial D_s$. For $z\in\partial D_s$, $s\in[0,T]$, and $\rho,\eta>0$ we define

$$a_{s,z}(\rho,\eta)=\max_{u\in S_1(0)}\min_{s\le\tilde{s}\le s+\eta}\min_{y\in\partial D_{\tilde{s}}\cap\overline{B_\rho(z)}}\min_{\gamma\in\Gamma_{\tilde{s}}^1(y)}\langle\gamma,u\rangle\,. \tag{2.5}$$

The vector u in (2.5) that maximizes the minimum of $\langle\gamma,u\rangle$ over all vectors $\gamma\in\Gamma_{\tilde{s}}^1(y)$ in a time-space neighbourhood of a point (s,z), $z\in\partial D_s$, $s\in[0,T]$, can be regarded as the best approximation of the $\Gamma_{\tilde{s}}^1(y)$-vectors in that neighbourhood. With this interpretation $a_{s,z}(\rho,\eta)$ represents the cosine of the largest angle between the best approximation and a $\Gamma_{\tilde{s}}^1(y)$-vector in the neighbourhood. Hence, in a sense, $a_{s,z}(\rho,\eta)$ quantifies the variation of Γ in a space-time neighbourhood of (s,z). For $z\in\partial D_s$, $s\in[0,T]$ and $\rho,\eta>0$ we define

$$c_{s,z}(\rho,\eta)=\max_{s\le\tilde{s}\le s+\eta}\max_{y\in\partial D_{\tilde{s}}\cap\overline{B_\rho(z)}}\max_{x\in\overline{D_{\tilde{s}}}\cap\overline{B_\rho(z)},\,x\ne y}\max_{\gamma\in\Gamma_{\tilde{s}}^1(y)}\left(\frac{\langle\gamma,y-x\rangle}{|y-x|}\vee 0\right). \tag{2.6}$$

This quantity is close to one if the vectors $\gamma\in\Gamma_{\tilde{s}}^1(y)$, in a time-space neighbourhood, deviate much from the normal vectors and/or the domain is very concave. Hence, in a sense, $c_{s,z}(\rho,\eta)$ quantifies the skewness of Γ and the concavity of D. Note that (2.6) implies

$$\langle\gamma,x-y\rangle+c_{s,z}(\rho,\eta)|y-x|\ge 0, \tag{2.7}$$

for all $y\in\partial D_{\tilde{s}}\cap\overline{B_\rho(z)}$, $x\in\overline{D_{\tilde{s}}}\cap\overline{B_\rho(z)}$, $x\ne y$ and $\gamma\in\Gamma_{\tilde{s}}^1(y)$ with $z\in\partial D_{\tilde{s}}$, $\tilde{s}\in[s,s+\eta]\subset[0,T]$. This condition exhibits some similarity with the uniform exterior sphere property (2.4). For technical reasons we shall also need the quantity

$$e_{s,z}(\rho,\eta)=\frac{c_{s,z}(\rho,\eta)}{(a_{s,z}(\rho,\eta))^2\vee a_{s,z}(\rho,\eta)/2}. \tag{2.8}$$

In the statement of Theorem 1.1 we assume that D satisfies the uniform exterior sphere condition in time with radius r_0, and that there exist $0 < \rho_0 < r_0$ and $\eta_0 > 0$ such that,

$$\inf_{s\in[0,T]} \inf_{z\in\partial D_s} a_{s,z}(\rho_0,\eta_0) = a > 0, \tag{2.9}$$

$$\sup_{s\in[0,T]} \sup_{z\in\partial D_s} e_{s,z}(\rho_0,\eta_0) = e < 1. \tag{2.10}$$

Finally, we say that $([0,T]\times\mathbb{R}^d)\setminus\overline{D}$ has the (δ_0,h_0)-property of good projections along Γ if there exists, for any $z\in\mathbb{R}^d\setminus\overline{D}_s$, $s\in[0,T]$, such that

$$d(z,D_s) < \delta_0, \tag{2.11}$$

at least one projection of z onto ∂D_s along Γ_s, denoted $\pi^{\Gamma_s}_{\partial D_s}(z)$, which satisfies

$$\left| z - \pi^{\Gamma_s}_{\partial D_s}(z) \right| \le h_0 d(z,D_s). \tag{2.12}$$

Note that the assumptions imposed on D and Γ above do not imply any explicit regularity assumptions on D and Γ. However, as stated in the introduction, to actually prove that the order of convergence of the algorithm proposed in this article equals 1/2 we have to impose more rigorous and explicit regularity assumptions on D, Γ and their interaction. The purpose of the rest of this subsection is to formulate such conditions. To start with, given $T>0$ and D as above, we say that D is a $\mathcal{H}_{k+\alpha}$-domain if we can find a $\rho>0$ such that there exists, for all $z_0\in\partial D_{s_0}$, $s_0\in[0,T]$, a function $\psi(s,z)$, $\psi\in\mathcal{H}_{k+\alpha}(C_\rho(s_0,z_0))$, such that

$$D\cap C_\rho(s_0,z_0) = \{\psi(s,z)>0\}\cap C_\rho(s_0,z_0), \tag{2.13a}$$

$$\left((0,T)\times\mathbb{R}^d\right)\cap\partial D\cap C_\rho(s_0,z_0) = \left((0,T)\times\mathbb{R}^d\right)\cap\{\psi(s,z)=0\}\cap C_\rho(s_0,z_0), \tag{2.13b}$$

$$\inf_{(s,z)\in((0,T)\times\mathbb{R}^d)\cap\partial D\cap C_\rho(s_0,z_0)} |\nabla_z\psi(s,z)| > 0. \tag{2.13c}$$

To prove (1.27) we will assume, given $T>0$ and D as above, that

$$D \text{ is a } \mathcal{H}_{k+\alpha}\text{-domain with } k+\alpha\ge 2. \tag{2.14}$$

If (2.14) holds, then there exists a unique unit spatial inward normal, denoted $n_s(x)$, at $x\in\partial D_s$, $s\in[0,T]$. Moreover, assuming (2.14), l is Lipschitz continuous and there exists $r_0>0$ such that D satisfies the uniform exterior sphere condition in time, with radius r_0. Let $\Gamma = \Gamma_s(z) = \Gamma(s,z)$ be a function defined on $\mathbb{R}_+\times\mathbb{R}^d$ such that $\Gamma_s(z)$ is a closed convex cone of unit vectors in $\mathbb{R}^d$ for every $z\in\partial D_s$. To prove (1.27) we will also assume that

$$\begin{aligned}&\Gamma_s(z) = \{\lambda\gamma_s(z):\ \lambda>0\} \text{ for some } S_1(0)\text{-valued function } \gamma_s(z)\\ &\text{which is uniformly continuous whenever } z\in\partial D_s,\ s\in[0,T].\end{aligned} \tag{2.15}$$

In addition we will assume, for Γ as in (2.15), that

$$\beta := \inf_{s\in[0,T]} \inf_{z\in\partial D_s} \langle\gamma_s(z), n_s(z)\rangle > 0. \tag{2.16}$$

Using these assumption, the following lemma is a consequence of Theorem 4.5 in [3], Lemma 2.5 in [10] and the fact that (1.3), (1.4) and (1.9) follows directly from (2.15).

Lemma 2.1. *Let $T>0$, let $D\subset\mathbb{R}^{d+1}$ be a time-dependent domain satisfying* (1.2) *and let $\Gamma=\Gamma_s(z)$ be a closed convex cone of vectors in $\mathbb{R}^d$ for every $z\in\partial D_s$, $s\in[0,T]$. Assume that D and Γ satisfy* (2.14)-(2.16). *Then all assumptions in Theorem* 1.1 *concerning D and Γ are fulfilled. In particular, there exists a constant $0<C_l<\infty$, such that*

$$l(r) \le C_l r, \text{ whenever } r\in[0,T]. \tag{2.17}$$

2.3. Estimates for the Skorohod problem in time-dependent domains

Given $T > 0$, we let $\mathcal{D}\left([0,T],\mathbb{R}^d\right)$ denote the set of cádlág functions $w = w_s : [0,T] \to \mathbb{R}^d$, that is functions which are right continuous with left limits. For $w \in \mathcal{D}\left([0,T],\mathbb{R}^d\right)$ we introduce the norm

$$\|w\|_{s_1,s_2} = \sup_{s_1 \le \tilde{s}_1 \le \tilde{s}_2 \le s_2} |w_{\tilde{s}_2} - w_{\tilde{s}_1}|, \tag{2.18}$$

for $0 \le s_1 \le s_2 \le T$ and, given $\delta > 0$, we let

$$\mathcal{D}^{\delta}\left([0,T],\mathbb{R}^d\right) = \left\{ w \in \mathcal{D}\left([0,T],\mathbb{R}^d\right) : \sup_{s\in[0,T]} |w_s - w_{s^-}| < \delta \right\}, \tag{2.19}$$

denote the set of cádlág functions with jumps bounded by δ. Finally, we let $\mathcal{BV}\left([0,T],\mathbb{R}^d\right)$ denote the set of functions $\lambda = \lambda_s : [0,T] \to \mathbb{R}^d$ with bounded variation and we let $|\lambda|$ denote the total variation of $\lambda \in \mathcal{BV}\left([0,T],\mathbb{R}^d\right)$. In [10] we consider the Skorohod problem in the following form.

Definition 2.1. *Let $d \ge 1$ and $T > 0$. Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (1.2) and let $\Gamma = \Gamma_s(z)$ be, for every $z \in \partial D_s$, $s \in [0,T]$, a closed convex cone of vectors in $\mathbb{R}^d$. Given $w \in \mathcal{D}\left([0,T],\mathbb{R}^d\right)$, with $w_0 \in \overline{D_0}$, we say that the pair (x,λ) is a solution to the Skorohod problem for (D,Γ,w), on $[0,T]$, if $(x,\lambda) \in \mathcal{D}\left([0,T],\mathbb{R}^d\right) \times \mathcal{BV}\left([0,T],\mathbb{R}^d\right)$ and if (w,x,λ) satisfies, for all $s \in [0,T]$,*

$$x_s = w_s + \lambda_s,\ x_s \in \overline{D_s}, \tag{2.20}$$

$$\lambda_s = \int_0^{s^+} \gamma_r \, d|\lambda|_r, \quad \gamma_r \in \Gamma_r^1(x_r)\ \ d|\lambda|\text{-a.e on } \cup_{r\in[0,s]} \partial D_r, \tag{2.21}$$

and

$$d|\lambda|\left(\{s \in [0,T] : (s,x_s) \in D\}\right) = 0. \tag{2.22}$$

In [10] we proved the following theorems.

Theorem 2.1. *Let $T > 0$, $D \subset \mathbb{R}^{d+1}$ and $\Gamma = \Gamma_s(z)$ be as in the statement of Theorem 1.1 and let, in particular, δ_0, ρ_0 and h_0 be as in Theorem 1.1. Then, given*

$$w \in \mathcal{D}^{\left(\frac{\delta_0}{4} \wedge \frac{\rho_0}{4h_0}\right)}\left([0,T],\mathbb{R}^d\right), \quad \text{with } w_0 \in \overline{D_0},$$

there exists a solution (x,λ) to the Skorohod problem for (D,Γ,w), in the sense of Definition 2.1, with $x \in \mathcal{D}^{\rho_0}([0,T],\mathbb{R})$.

Theorem 2.2. *Let $T > 0$, $D \subset \mathbb{R}^{d+1}$ and $\Gamma = \Gamma_s(z)$ be as in the statement of Theorem 1.1 and let, in particular, ρ_0 be as in Theorem 1.1. Let $w : [0,T] \to \mathbb{R}^d$ be a continuous function and let (x,λ) be any solution to the Skorohod problem for (D,Γ,w), in the sense of Definition 2.1. If $x \in \mathcal{D}^{\rho_0}\left([0,T],\mathbb{R}^d\right)$, then x is continuous.*

Remark 2.1. Let $T > 0$, let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (1.2) and let $\Gamma = \Gamma_s(z)$ be a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_s$, $s \in [0,T]$. Then, using Lemma 2.1, it is clear that the conclusions of Theorems 2.1 and 2.2 are valid also for D and Γ satisfying (2.14)-(2.16).

We next outline the estimates derived in [10] that are needed in the proof of (1.27). To do this, recall the definitions of Δ^*, Δ, (D^Δ,Γ^Δ) and $\left(X^\Delta,U^\Delta,\Lambda^\Delta\right)$ stated in the introduction. Then $(X^\Delta,\Lambda^\Delta)$ solves the Skorohod problem for $(D^\Delta,\Gamma^\Delta,U^\Delta)$, in the sense of Definition 2.1, and

$$\begin{aligned} d(X^\Delta_{\tau_k}+U^\Delta_{\tau_{k+1}}-U^\Delta_{\tau_k},D^\Delta_{\tau_{k+1}}) &\leq ||U^\Delta||_{\tau_{k-1},\tau_k}+l(\Delta^*) \\ &\leq (\sup_D|b|)\Delta^*+(\sup_D||\sigma||)\sqrt{\Delta^*}\Delta_{k+1}\eta+l(\Delta^*), \end{aligned} \tag{2.23}$$

whenever $k\in\{0,1,...,N-1\}$. In particular, in the following we assume that $\sqrt{\Delta^*}\Delta_{k+1}\eta$ is generated by some rule which ensures that we always have

$$(\sup_D|b|)\Delta^*+(\sup_D||\sigma||)\sqrt{\Delta^*}\Delta_{k+1}\eta+l(\Delta^*)<\delta_0. \tag{2.24}$$

Hence, if (2.24) holds for all $k\in\{0,1,...,N-1\}$ then the processes $\left(X^\Delta,U^\Delta,\Lambda^\Delta\right)$, introduced in the introduction, are all well-defined. Given $a>0$ and $e\in(0,1)$ we define the positive functions K_1 and K_2 as follows,

$$K_1\left(a,e\right)=\frac{a+2a^2e+2+ae}{a\left(1-e\right)},\quad K_2\left(a,e\right)=\frac{2a^2e+2+ae}{a\left(1-e\right)}. \tag{2.25}$$

The following result is a version of Theorem 3.6 in [10] applied to $\left(X^\Delta,\Lambda^\Delta,U^\Delta\right)$ and (D^Δ,Γ^Δ).

Lemma 2.2. *Let $T>0$, $D\subset\mathbb{R}^{d+1}$, r_0, $\Gamma=\Gamma_s\left(z\right)$, $0<\rho_0<r_0$, η_0, a, e, δ_0 and h_0 be as in the statement of Theorem 1.1. Let $t\in[0,T]$, $x\in\overline{D_t}$, let $\Delta=\{\tau_k\}_{k=0}^N$ be a partition of $[t,T]$ as described in the introduction and assume that*

$$l(\Delta^*)\leq\frac{\rho_0}{4\left(K_2\left(a,e\right)+1\right)}.$$

Let $\sqrt{\Delta^*}\Delta_{k+1}\eta$ be generated by some rule which ensures that (2.24) holds for all $k\in\{0,1,...,N-1\}$. Given Δ, let (D^Δ,Γ^Δ) and $\left(X^\Delta,\Lambda^\Delta,U^\Delta\right)$ be defined as in (1.16)-(1.22) and let, based on a sample of $\{\sqrt{\Delta^*}\Delta_{k+1}\eta\}_{k=0}^{N-1}$,

$$\left(X^\Delta,\Lambda^\Delta,U^\Delta\right)=\left(X^\Delta(\omega),\Lambda^\Delta(\omega),U^\Delta(\omega)\right)$$

be a path of $\left(X^\Delta,\Lambda^\Delta,U^\Delta\right)$. Moreover, assume that $X^\Delta\in\mathcal{D}^{\rho_0}\left([t,T],\mathbb{R}^d\right)$. Then there exist positive constants K and δ^Δ, with K being independent of Δ, such that

$$\left|\Lambda^\Delta\right|_{s_2}-\left|\Lambda^\Delta\right|_{s_1}\leq\frac{K\left(T-t\right)}{\delta^\Delta}\left(\left\|U^\Delta\right\|_{s_1,s_2}+l\left(s_2-s_1\right)+l\left(\Delta^*\right)\right), \tag{2.26}$$

whenever $t\leq s_1\leq s_2\leq T$.

Proof. Let $w\in\mathcal{D}\left([t,T],\mathbb{R}^d\right)$ with $w_t\in\overline{D_t}$ and let w^Δ be defined as

$$w^\Delta_t=w_{\tau_k},\quad \text{whenever } t\in\ [\tau_k,\tau_{k+1})\,,\ k\in\{0,1,...,N-1\}\,, \tag{2.27}$$

and $w^\Delta_T=w_T$. Assume that

$$\left\|w^\Delta\right\|_{\tau_k,\tau_{k+1}}+l\left(\Delta^*\right)<\delta_0, \tag{2.28}$$

holds whenever $k \in \{0, 1, ..., N-1\}$. Given Δ and w^Δ, let x^Δ and λ^Δ be defined based on w^Δ in exactly the same manner as X^Δ and Λ^Δ are defined based on U^Δ in (1.18) and (1.20). Moreover, assume that $x^\Delta \in \mathcal{D}^{\rho_0}\left([t,T], \mathbb{R}^d\right)$. Now let δ' be a fixed positive number satisfying

$$\delta' = \min\{\eta_0, \hat{\delta}'\}, \quad \text{where } \hat{\delta}' \text{ is such that } l(\hat{\delta}') \leq \rho_0/(4(K_2(a,e)+1)). \tag{2.29}$$

Note that δ' is independent of Δ. Now, since w^Δ is a cádlág function, there exists, for any $L > 0$, a constant $\delta_{w^\Delta}(L,t,T)$, satisfying

$$0 < \delta_{w^\Delta}(L,t,T) \leq \delta' \wedge (T-t),$$

and a partition $\tilde{\Delta} = \{\tilde{\tau}_j\}_{j=0}^{\tilde{N}}$, $t = \tilde{\tau}_0 < \tilde{\tau}_1 < .... < \tilde{\tau}_{\tilde{N}} = T$, such that

$$\delta_{w^\Delta}(L,t,T) \leq \tilde{\tau}_{j+1} - \tilde{\tau}_j \leq \delta' \wedge (T-t), \quad \text{for all } j \in \{0, 1, ..., \tilde{N}-1\},$$

and such that

$$\max_{j \in \{0,1,...,\tilde{N}-1\}} \sup_{s_1, s_2 \in [\tilde{\tau}_j, \tilde{\tau}_{j+1})} \left| w^\Delta_{s_1} - w^\Delta_{s_2} \right| < L. \tag{2.30}$$

For $L = \rho_0/2K_1(a,e)$, we obtain, following the proofs of Theorem 3.2 and Theorem 3.6 in [10],

$$\begin{aligned}
\left|\lambda^\Delta\right|_{s_2} - \left|\lambda^\Delta\right|_{s_1} &\leq \frac{1+K_1(a,e)}{a}\left(\frac{T-t}{\delta_{w^\Delta}(L,t,T)} + 2\right)\left\|w^\Delta\right\|_{s_1,s_2} \\
&\quad + \frac{1}{\sqrt{1-e}}\left(\frac{T-t}{\delta_{w^\Delta}(L,t,T)} + 1\right)\left\|w^\Delta\right\|_{s_1,s_2} \\
&\quad + \frac{1+K_1(a,e)}{a}\left(\frac{T-t}{\delta_{w^\Delta}(L,t,T)} + 2\right)\left(l(s_2 - s_1) + l(\Delta^*)\right) \\
&\leq \frac{K(T-t)}{\delta_{w^\Delta}(L,t,T)}\left(\left\|w^\Delta\right\|_{s_1,s_2} + l(s_2-s_1) + l(\Delta^*)\right),
\end{aligned} \tag{2.31}$$

whenever $t \leq s_1 \leq s_2 \leq T$, for some positive constant K, independent of Δ. Note that the last step follows as, by construction, $(T-t)/\delta_{w^\Delta}(L,t,T) \geq 1$. To complete the proof, we observe that U^Δ satisfies all properties required for w^Δ above and, in particular, (2.28) follows by (2.24). Hence, as X^Δ and Λ^Δ are generated, based on U^Δ, in exactly the same manner as x^Δ and λ^Δ are generated based on w^Δ, we can immediately conclude, using (2.31), that

$$\left|\Lambda^\Delta\right|_{s_2} - \left|\Lambda^\Delta\right|_{s_1} \leq \frac{K(T-t)}{\delta_{U^\Delta}(L,t,T)}\left(\left\|U^\Delta\right\|_{s_1,s_2} + l(s_2-s_1) + l(\Delta^*)\right), \tag{2.32}$$

whenever $t \leq s_1 \leq s_2 \leq T$. Hence the lemma holds with $\delta^\Delta = \delta_{U^\Delta}(L,t,T)$. □

The estimate in Lemma 2.2 can be used to deduce the following lemma which we shall use in the proof of (1.27) in Section 3.

Lemma 2.3. *Let $T > 0$, $t \in [0,T]$, let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying* (1.2) *and let $\Gamma = \Gamma_s(z)$ be a closed convex cone of vectors in $\mathbb{R}^d$ for every $z \in \partial D_s$, $s \in [t,T]$. Assume* (1.26)*, let $\Delta = \{\tau_k\}_{k=0}^N$ be a partition of $[t,T]$ as described in the introduction and let $(D^\Delta, \Gamma^\Delta)$ and $\left(X^\Delta, \Lambda^\Delta, U^\Delta\right)$ be defined as in* (1.16)-(1.22)*. Assume that $\Delta_{k+1}\eta$ is, for all $k \in \{0, 1, ..., N-1\}$, a bounded random variable defined on Ω. Let*

$H := \sup_{\omega\in\Omega}\sup_{k\in\{0,1,\dots,N-1\}}|\Delta_{k+1}\eta(\omega)|$, *let* C_l *denote the Lipschitz norm of* l *and let the positive constants* ρ_0, δ_0 *and* h_0 *be as defined in Theorem* 1.1. *Let*

$$B=\delta_0\wedge\frac{\rho_0}{(1+h_0)\vee(2(K_2(a,e)+1))},\tag{2.33}$$

and let

$$\hat{\Delta}^*:=\min\left\{\frac{B}{2(C_l+\sup_D|b|)},\left(\frac{B}{2H\sup_D\|\sigma\|}\right)^2\right\}.\tag{2.34}$$

Then there exists, for any $q\in\mathbb{N}$, *a positive constant* C_q *depending only on* q, D, H *and the coefficients of* (1.31), *such that*

$$\sup_{\Delta^*<\hat{\Delta}^*}\sup_{(t,x)\in D}E\left[\left(\left|\Lambda^\Delta\right|_T\right)^q\right]\le C_q<\infty.\tag{2.35}$$

Proof. First note that (2.24) is satisfied for all $\Delta^*\le\hat{\Delta}^*$. Hence the projections in (1.18) are well defined and, accordingly, so are X^Δ and Λ^Δ. As noted earlier, $\left(X^\Delta,\Lambda^\Delta\right)$ is a solution to the Skorohod problem for $\left(D^\Delta,\Gamma^\Delta,U^\Delta\right)$ and, by means of (2.23), (2.34) and the construction of X^Δ, we can conclude that $\sup_{t\le s\le T}\left|X^\Delta_s-X^\Delta_{s-}\right|\le\rho_0$. Furthermore, from the definition of $\hat{\Delta}^*$ it follows that $l(\Delta^*)\le\rho_0/(4(K_2(a,e)+1))$ whenever $\Delta^*\le\hat{\Delta}^*$. Hence all prerequisites of Lemma 2.2 are satisfied and we can conclude that there exist positive constants K and δ^Δ, with K being independent of Δ, such that (2.26) holds. Concerning δ^Δ we note, as described in (2.29)-(2.30) in the proof of Lemma 2.2, that in general $\delta^\Delta=\delta_{U^\Delta}(L,t,T)$ depends on U^Δ and satisfies $0<\delta_{U^\Delta}(L,t,T)\le\delta'\wedge(T-t)$ where δ' was defined in (2.29). Applying the Cauchy-Schwarz inequality to (2.26), using also the Lipschitz continuity of l, we obtain

$$\begin{aligned}&E\left[\left(\left|\Lambda^\Delta\right|_T\right)^q\right]\\&\le L_1\left(E\left[\left(\frac{T-t}{\delta_{U^\Delta}(L,t,T)}\right)^{2q}\right]\right)^{\frac12}\cdot\left(L_2+E\left[\left(\sup_{t\le s_1\le s_2\le T}\left|U^\Delta_{s_2}-U^\Delta_{s_1}\right|\right)^{2q}\right]\right)^{\frac12},\end{aligned}\tag{2.36}$$

for some positive constants L_1 and L_2. Since

$$\sup_{t\le s_1\le s_2\le T}\left|U^\Delta_{s_2}-U^\Delta_{s_1}\right|\le L_3+\max_{0\le l_1\le l_2\le N-1}\left|\sum_{k=l_1}^{l_2}\sigma\left(\tau_{k+1},X^\Delta_{\tau_k}\right)\sqrt{\Delta^*}\Delta_{k+1}\eta\right|,\tag{2.37}$$

for some positive constant L_3, we see that to complete the proof it suffices to prove that

$$\sup_{\Delta^*<\hat{\Delta}^*}\sup_{(t,x)\in D}E\left[\max_{0\le p\le p'\le N-1}\left|\sum_{k=p}^{p'}\sigma\left(\tau_{k+1},X^\Delta_{\tau_k}\right)\sqrt{\Delta^*}\Delta_{k+1}\eta\right|^{2q}\right]<\infty,\tag{2.38}$$

and that

$$\sup_{\Delta^*<\hat{\Delta}^*}\sup_{(t,x)\in D}E\left[\left(\frac{T-t}{\delta_{U^\Delta}(L,t,T)}\right)^{2q}\right]<\infty.\tag{2.39}$$

The estimate in (2.38) follows directly from the proof of (ii) in Lemma 3.3 in [2]. Regarding (2.39), we first observe that $\delta_{U^\Delta}(L,t,T)$ tends to zero either if $t\to T$ or if the variation of U^Δ is large. As far as the first case is concerned we note that there exists a $\epsilon>0$ such that

$$\left\|U^\Delta\right\|_{t,T}<L,\quad\text{whenever}\quad t\in[T-\epsilon,T].$$

Hence, for $t \in [T-\epsilon, T]$ we can set $\delta_{U^\Delta}(L,t,T) = T-t$, whereby (2.39) easily follows. This also implies that we can, without loss of generality, assume that $0 < \delta_{U^\Delta}(L,t,T) \leq \delta'$ is independent of $T-t$ and we note that

$$\left(\frac{T-t}{\delta_{U^\Delta}(L,t,T)}\right)^{2q} \leq \max\left\{\frac{T^{2q}}{(\delta')^{2q}}, \frac{T^{2q}}{(\tilde{\delta}^{\Delta})^{2q}}\right\}, \tag{2.40}$$

where

$$\tilde{\delta}^{\Delta} = \sup\left\{\delta > 0 : \inf_{\tilde{\Delta}} \max_{j\in\{1,\ldots,\tilde{N}\}} \sup_{s_1,s_2\in[\tilde{\tau}_j,\tilde{\tau}_{j-1})} \left|U^{\Delta}_{s_1} - U^{\Delta}_{s_2}\right| < L\right\}, \tag{2.41}$$

and the infimum is taken over all partitions $\tilde{\Delta} = \{\tilde{\tau}_j\}_{j=0}^{\tilde{N}}$, $0 = \tilde{\tau}_0 < \tilde{\tau}_1 < \ldots. < \tilde{\tau}_{\tilde{N}} = T$, such that $\min_{j\in\{1,\ldots,\tilde{N}\}} |\tilde{\tau}_j - \tilde{\tau}_{j-1}| > \delta$. Since δ' is a fixed positive number independent of Δ, it hence remains to show that

$$\sup_{\Delta^* < \hat{\Delta}^*} \sup_{(t,x)\in D} E\left[\frac{1}{(\tilde{\delta}^{\Delta})^{2q}}\right] < \infty. \tag{2.42}$$

However, (2.42) follows readily from the proof of (i) in Lemma 3.3 in [2] by noting that the variable $\tilde{\delta}^{\Delta}$ is equivalent to the variable $\delta_{W^h}(\rho_0/2K_1(a,e), t, T)$ in [2]. □

As a general comment we note that the question of uniqueness of solutions to the Skorohod oblique reflection problem is, in general, still an open problem also for time-independent domains. In fact, concerning uniqueness, the strongest known result is the following theorem proved in Theorem C.2 of [7].

Theorem 2.3. *Let $T > 0$ and let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying* (1.2) *and* (2.14)*. Let $w \in \mathcal{C}([0,T]\times\mathbb{R}^d)$ with $w_0 \in \overline{D_0}$. Then there exists a unique solution to the Skorohod problem for (D,N,w).*

Let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be given functions and assume that there exists a constant $K > 0$ such that b and σ satisfy

$$|b(s,z)| + |\sigma(s,z)| \leq K, \tag{2.43}$$

$$|b(s,z) - b(t,y)| + |\sigma(s,z) - \sigma(t,y)| \leq K\left(|z-y| + |s-t|^{1/2}\right), \tag{2.44}$$

whenever $(s,z), (t,y) \in \overline{D}$. Let $x \in \mathbb{R}^d$. Then, assuming (2.43)-(2.44) one can apply standard results on the existence of solutions to stochastic differential equations (see e.g. [1]) to conclude that there exists a unique strong solution to

$$X_s^{t,x} = x + \int_t^s b\left(r, X_r^{t,x}\right) dr + \int_t^s \sigma\left(r, X_r^{t,x}\right) dW_r. \tag{2.45}$$

Combining this result with Theorem 2.3 one can verify the validity of the following theorem, see Theorem 3.2 in [7].

Theorem 2.4. *Let $T > 0$, let $t \in [0,T]$, and let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying* (1.2) *and* (2.14)*. Let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be given functions on $\overline{D}$ satisfying* (2.43)-(2.44) *and let $x \in \overline{D_t}$. Then there exists a unique weak solution, in the sense of Definition* 1.1*, to the stochastic differential equation in $\overline{D}$ with coefficients b and σ, reflection along N_s on ∂D_s, $s \in [t,T]$, and with initial condition x at t.*

Note that Theorem 2.4 can be proved without the requirement in (2.44) that b and σ are Hölder continuous of order 1/2 with respect to the time variable. Nevertheless, we include this additional regularity condition, as it enables us to establish the connection to second order partial differential equations described below.

2.4. Parabolic partial differential equations with oblique boundary conditions in time-dependent domains

To be able to prove the convergence for the numerical algorithm outlined in the introduction, we need to ensure that there exists a unique $\mathcal{C}^{1,2}(D)$ -solution u to (1.31) satisfying

(i) u_s is Hölder continuous with exponent 1/2 as a function of s

(ii) $u_{x_i x_j}$ is Lipschitz continuous as a function of x for $i, j \in \{1, 2, ..., d\}$,

(iii) u can be extended to a function $\mathcal{C}^{1,2}\left([0,T] \times \mathbb{R}^d\right)$. (2.46)

If D is a time-dependent domain satisfying (1.2) and (2.14), if the closed convex cone Γ of directions of reflection satisfies (2.15) and (2.16), and if (2.46) holds, then we are able to prove that the numerical approximation described in the introduction converges to the unique solution u (see Theorem 3.1 below). Note that concerning a priori knowledge of the regularity of u we shall need (2.46) and naturally the validity of (2.46) imposes implicit conditions on the structure of the operator L. Note, however, that the statement in (2.46) does not, by necessity, require that the operator L in (1.31) is uniformly elliptic.

In this subsection we state some regularity conditions, under which the relation between second order parabolic partial differential equations with Robin boundary conditions and the expectation of a functional of the solution to a stochastic differential equation with reflection, already stated in Theorem 1.2 above, holds. In the following, we assume that $d = m$ and that there exists $\lambda \in [1, \infty)$ such that the following uniform ellipticity condition holds

$$\lambda^{-1} |\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(s,z)\,\xi_i \xi_j \le \lambda |\xi|^2, \quad \text{whenever } (s,z) \in \mathbb{R}^{d+1}, \xi \in \mathbb{R}^d. \tag{2.47}$$

The following two results concerning regularity of the solution to the partial differential equation in (1.31) can be deduced from Theorem 5.18 in [11] (or from [12] in the time-independent case) and the relation to the stochastic differential equations follows by an application of Itô's formula. A similar result is found in Proposition 3.7 of [7].

Theorem 2.5. *Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying* (1.2) *and* (2.14)*, for some $\alpha \in (0,1)$. Assume that b and σ satisfy* (2.43)-(2.44)*, which in turn implies that $b \in \mathcal{H}_\alpha(D)$ and $a_{ij} \in \mathcal{H}_\alpha(D)$, and assume that $\{a_{ij}\}$ satisfies the uniform ellipticity condition in* (2.47)*. Furthermore, let $\varphi \in \mathcal{H}_\alpha(D)$, $\theta \in \mathcal{H}_{1+\alpha}(D)$, $g \in \mathcal{H}_{1+\alpha}(D)$, $h \in \mathcal{H}_\alpha(D)$ and let f be a bounded and continuous function on $\mathbb{R}^d$. Assume that Γ satisfies* (2.15) *and* (2.16) *and that the d components of γ all belong to $\mathcal{H}_{1+\alpha}(D)$. Then there exists a unique solution u to problem* (1.31) *and u is given by* (1.15)*. If, in addition, $f \in \mathcal{H}_{2+\alpha}(D)$ and the compatibility condition*

$$\langle \nabla_x f(x), \gamma(T,x) \rangle - d(T,x)\, f(x) = g(T,x), \tag{2.48}$$

holds whenever $x \in \partial D_T$, then $u \in \mathcal{H}_{2+\alpha}(D)$.

Theorem 2.6. *Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent $\mathcal{H}_3$-domain satisfying* (1.2)*, assume that b and σ satisfy* (2.43)-(2.44) *and that $\{a_{ij}\}$ satisfies the uniform ellipticity condition in* (2.47)*. Furthermore, let $\varphi \in \mathcal{H}_1(D)$, $\theta \in \mathcal{H}_2(D)$, $f \in \mathcal{H}_3(D)$, $g \in \mathcal{H}_2(D)$, $h = 0$. Assume that Γ satisfies* (2.15) *and* (2.16) *and that the d components of γ all belong to $\mathcal{H}_2(D)$. Then there exists, if the compatibility condition* (2.48) *holds whenever $x \in \partial D_T$, a unique solution $u \in \mathcal{H}_3(D)$ to problem* (1.31) *and u is given by* (1.15)*.*

Note, in the context of the last two theorems, that if $u \in \mathcal{H}_{k+\alpha}(D)$, with $\alpha > 0$ and $k \in \mathbb{Z}_+$, and if D is a $\mathcal{H}_{k+\alpha}$-domain, then u can be extended to a function $\widetilde{u} \in \mathcal{H}_{k+\alpha}\left([0,T] \times \mathbb{R}^d\right)$ (see for example [13]) and, as a consequence, it is clear that (2.46) holds if D is a $\mathcal{H}_3$-domain and the solution to (1.31) belongs to $\mathcal{H}_3(D)$. Hence Theorem 2.6 gives sufficient regularity and compatibility conditions for (2.46) in the case of vanishing source term.

3. Proof of the Order of Convergence for the Numerical Algorithm for Weak Approximation

This section is devoted to the proof of (1.27) under the assumption in (1.26). In the introduction we outlined the numerical algorithm considered in this article by defining, for a partition Δ, the processes $\left(X^\Delta, U^\Delta, \Lambda^\Delta, Y^\Delta, Z^\Delta\right)$ on $(D^\Delta, \Gamma^\Delta)$ as approximations of the processes (X, U, Λ, Y, Z) on (D, Γ). Finally, we defined the functional $F^\Delta_{t,T}(X^\Delta, \Lambda^\Delta)$ as an approximation of $F_{t,T}(X, \Lambda)$. In order to fully specify the numerical algorithm, it remains to determine a rule for generating the random variables $\Delta_{k+1}\eta$, for $k = 0, 1, ..., N-1$, which should approximate the increments of the Wiener process. In the following we discuss two alternative choices of such rules.

To describe the first alternative we note that the term $\sqrt{\Delta^*}\Delta_{k+1}\eta$ in (1.18) and the Wiener increment $W_{k+1} - W_k$ will be identical in law if $\Delta_{k+1}\eta$ is chosen as a Gaussian random variable with zero mean and unit variance. Let $\delta_0 > 0$ be as in the definition of the (δ_0, h_0)-property of good projections along Γ and recall that the existence of a projection onto ∂D_s along Γ_s is only asserted for points $z \in \mathbb{R}^d \setminus \overline{D}_s$, satisfying $d(z, D_s) < \delta_0$. As Gaussian random variables are unbounded we cannot rule out, if choosing $\Delta_{k+1}\eta$ to be Gaussian, the possibility that (1.19) is violated, in which case X^Δ might remain undefined. However, to make this approach operational, we note, see Lemma 4.1 in [14], that

$$P\left(\left|U^\Delta_{\tau_{k+1}} - U^\Delta_{\tau_k}\right| > \delta\right) \leq K(T) \exp\left(-C\frac{\delta^2}{\Delta^*}\right), \tag{3.1}$$

for some positive constants C and $K(T)$. Hence the probability that $\left|U^\Delta_{\tau_{k+1}} - U^\Delta_{\tau_k}\right|$ exceeds a fixed δ decreases exponentially with Δ^*. In particular, as Δ^* tends to zero, the error produced by throwing away those values of $\Delta_{k+1}\eta$ which are too large, decreases much faster compared to the time-discretization error and hence it can be argued that these values of $\Delta_{k+1}\eta$ can be neglected. Recall that assuming (2.14) we know that l is Lipschitz continuous and by C_l we denote the Lipschitz norm of l. Hence the constraint in (2.24), asserting that X^Δ is well defined, is fulfilled if

$$(\sup_D |b|)\Delta^* + (\sup_D \|\sigma\|)\sqrt{\Delta^*}\Delta_{k+1}\eta + C_l\Delta^* < \delta_0. \tag{3.2}$$

Let

$$\tilde{\Delta}^* := \min\left\{\frac{\delta_0}{2\left(C_l + \sup_D |b|\right)}, \left(\frac{\delta_0}{8\sup_D \|\sigma\|}\right)^2\right\}. \tag{3.3}$$

Then, if we consider $\Delta^* < \tilde{\Delta}^*$ we see, by proceeding as above, that only values of $\Delta_{k+1}\eta$ exceeding 4 have to be discarded and we note that if $\Delta_{k+1}\eta$ is generated as a Gaussian variable with unit variance such values occur with very small probability. Furthermore, as Δ^* decreases we can gradually allow for higher values of $\Delta_{k+1}\eta$ and, as Δ^* tends to zero, the law of $\Delta_{k+1}\eta$ converges to that of a Gaussian variable.

To describe the second alternative for generating $\Delta_{k+1}\eta$ we proceed as in [2]. In particular, in this case we let, as in [2], $\Delta_{k+1}\eta$ be defined as a bounded random variable satisfying the constraints

$$E\left[\Delta_{k+1}\eta\right] = 0, \quad E\left[\Delta_{k+1}\eta_i \Delta_{k+1}\eta_j\right] = \delta_{ij}, \quad E\left[\left|\Delta_{k+1}\eta\right|^3\right] < \infty, \tag{3.4}$$

for all $k \in \{0, 1, ..., N-1\}$, where δ_{ij} is the Kronecker delta. To give an example of a choice of $\Delta_{k+1}\eta$ which satisfies these constraints we let $\Delta_{k+1}\eta_i$, $i \in \{1, .., d\}$, be independent random variables which only take on the values ± 1 and satisfy

$$P\left(\Delta_{k+1}\eta_i = \pm 1\right) = \frac{1}{2}. \tag{3.5}$$

Clearly, the advantage of the approach to generating $\Delta_{k+1}\eta$ is that $\Delta_{k+1}\eta$ can be drawn from a simple and easily generated distribution. For $\Delta_{k+1}\eta$ satisfying the constraints in (3.4), we let $H := \sup_{\omega\in\Omega} \sup_{k\in\{0,1,...,N-1\}} \left|\Delta_{k+1}\eta\left(\omega\right)\right|$, and define

$$\tilde{\Delta}^* := \min\left\{\frac{\delta_0}{2\left(C_l + \sup_D |b|\right)}, \left(\frac{\delta_0}{2H \sup_D \|\sigma\|}\right)^2\right\}. \tag{3.6}$$

Then X^Δ is clearly well defined whenever $\Delta^* < \tilde{\Delta}^*$.

Based on the discussion above we conclude that either of the two alternatives can be used to generate $\Delta_{k+1}\eta$. In the following we shall refer to these alternatives as Method 1 and Method 2, respectively.

1. For $\Delta^* < \tilde{\Delta}^*$, with $\tilde{\Delta}^*$ as in (3.3), choose $\Delta_{k+1}\eta$ as a Gaussian random variable with mean zero and unit variance. Redraw the value of $\Delta_{k+1}\eta$ if condition (3.2) is violated.
2. For $\Delta^* < \tilde{\Delta}^*$, with $\tilde{\Delta}^*$ as in (3.6), choose $\Delta_{k+1}\eta$ as a bounded random variable satisfying the conditions in (3.4).

We are now ready to state the main result of this article. Since the derivations, in connection to the numerical algorithm, require less regularity when Method 2 is used, we have chosen to concentrate on this approach in the following. However, in Remark 3.2 we indicate how to obtain a similar result based on Method 1.

Theorem 3.1. *Assume* (1.26) *and assume that $\Delta_{k+1}\eta$ is generated using Method 2. Then*

$$E\left[F_{t,T}\left(X, \Lambda\right) - F^\Delta_{t,T}\left(X^\Delta, \Lambda^\Delta\right)\right] \to 0, \tag{3.7}$$

as $\Delta^* \to 0$. Furthermore, if $\Delta^* < \hat{\Delta}^*$, where $\hat{\Delta}^*$ is defined in Lemma 2.3*, then*

$$E\left[F_{t,T}\left(X, \Lambda\right) - F^\Delta_{t,T}\left(X^\Delta, \Lambda^\Delta\right)\right] \le C\left(\Delta^*\right)^{1/2}, \tag{3.8}$$

where C is a positive constant depending only on T, D, H and the coefficients of (1.31).

Remark 3.1. As in [2] the algorithm stated above can, in the case for normal projection, easily be generalized to the case of piecewise $\mathcal{H}_2$-domains satisfying the consistency and non-degeneracy conditions stated in Theorem 4.6 in [3]. The algorithm can also be generalized to the case of mixed Dirichlet and Robin boundary conditions. For notational clarity, we have chosen not to consider these cases in order to not to divert the focus from the new aspects of the algorithm considered in this article, i.e., the aspects of time-dependency, oblique reflection and second order parabolic partial differential equations with non-zero right hand side.

Below we prove Theorem 3.1 by means of a series of lemmas. First we note that, assuming (1.26), it follows, see Lemma 2.1 that $([0,T]\times\mathbb{R}^d)\setminus\overline{D}$ has the (δ_0,h_0)-property of good projections along Γ. Hence for any $z\in\mathbb{R}^d\setminus\overline{D_s}$, $s\in[0,T]$, such that $d(z,D_s)<\delta_0$, we have

$$\left|\pi_{\partial D_s}^{\Gamma_s}(z)-z\right|\le h_0 d(z,D_s). \tag{3.9}$$

Moreover, using this estimate, the Lipschitz continuity of l, the fact that $\Delta^*<\hat{\Delta}^*$ and (2.23), we obtain

$$\begin{aligned}\Delta_{k+1}\left|\Lambda^\Delta\right| &= \left|\pi_{\partial D_{\tau_{k+1}}}^{\Gamma_{\tau_{k+1}}}(X_{\tau_k}^\Delta+U_{\tau_{k+1}}^\Delta-U_{\tau_k}^\Delta)-(X_{\tau_k}^\Delta+U_{\tau_{k+1}}^\Delta-U_{\tau_k}^\Delta)\right|\\ &\le h_0 d(X_{\tau_k}^\Delta+U_{\tau_{k+1}}^\Delta-U_{\tau_k}^\Delta,D_{\tau_{k+1}}^\Delta)\\ &\le h_0\left|(C_l+\sup_D|b|)\Delta^*+\sup_D\|\sigma\|\sqrt{\Delta^*}\Delta_{k+1}\eta\right|.\end{aligned} \tag{3.10}$$

For $\Delta_{k+1}X^\Delta:=X_{\tau_{k+1}}^\Delta-X_{\tau_k}^\Delta$ we obtain, similarly,

$$\begin{aligned}\left|\Delta_{k+1}X^\Delta\right| &\le \left|U_{\tau_{k+1}}^\Delta-U_{\tau_k}^\Delta\right|+\Delta_{k+1}\left|\Lambda^\Delta\right|\\ &\le(1+h_0)\left|(C_l+\sup_D|b|)\Delta^*+\sup_D\|\sigma\|\sqrt{\Delta^*}\Delta_{k+1}\eta\right|.\end{aligned} \tag{3.11}$$

To formulate the first two lemmas we need to introduce some notation. In particular, we let

$$u^k=u\left(\tau_{k+1},X_{\tau_k}^\Delta\right),\qquad a_i^k=a_i\left(\tau_{k+1},X_{\tau_k}^\Delta\right),\quad \theta^{k+1}=\theta\left(\tau_{k+1},X_{\tau_{k+1}}^\Delta\right), \tag{3.12a}$$

$$u_{x_i}^k=\partial_{x_i}u\left(\tau_{k+1},X_{\tau_k}^\Delta\right),\qquad \varphi^k=\varphi\left(\tau_{k+1},X_{\tau_k}^\Delta\right),\quad \gamma_i^{k+1}=\gamma_i\left(\tau_{k+1},X_{\tau_{k+1}}^\Delta\right). \tag{3.12b}$$

$$u_{x_ix_j}^k=\partial_{x_ix_j}u\left(\tau_{k+1},X_{\tau_k}^\Delta\right),\quad e^k=\exp\left(Y_{\tau_k}^\Delta+Z_{\tau_k}^\Delta\right), \tag{3.12c}$$

and we introduce two error functions $r(t,T,\Delta)$ and $R(t,T,\Delta)$ as follows. We let

$$r(t,T,\Delta)=\sum_{k=0}^{N-1}\int_{\tau_k}^{\tau_{k+1}}e^k\left(\partial_s u\left(s,X_s^\Delta\right)-\partial_s u\left(\tau_{k+1},X_{\tau_k}^\Delta\right)\right)ds, \tag{3.13}$$

and we let

$$\begin{aligned}R(\epsilon,T,\Delta)=\sum_{k=0}^{N-1}\sum_{i,j=1}^{d+2}\Big(&\partial_{v_iv_j}\zeta\left(\tau_{k+1},V_{\tau_k}^\Delta+\nu_k^\Delta\Delta_{k+1}V^\Delta\right)\\ &-\partial_{v_iv_j}\zeta\left(\tau_{k+1},V_{\tau_k}^\Delta\right)\Big)\Delta_{k+1}V_i^\Delta\Delta_{k+1}V_j^\Delta,\end{aligned} \tag{3.14}$$

where $0 < \nu_k^{\Delta} < 1$,

$$v_i = x_i, \quad \left(V_{\tau_k}^{\Delta}\right)_i = \left(X_{\tau_k}^{\Delta}\right)_i, \quad \Delta_{k+1} V_i^{\Delta} = \Delta_{k+1} X_i^{\Delta}, \quad \text{for } i = 1, ..., d, \tag{3.15a}$$

$$v_{d+1} = y, \quad \left(V_{\tau_k}^{\Delta}\right)_{d+1} = Y_{\tau_k}^{\Delta}, \quad \Delta_{k+1} V_{d+1}^{\Delta} = \Delta_{k+1} Y^{\Delta} := Y_{\tau_{k+1}}^{\Delta} - Y_{\tau_k}^{\Delta}, \tag{3.15b}$$

$$v_{d+2} = z, \quad \left(V_{\tau_k}^{\Delta}\right)_{d+2} = Z_{\tau_k}^{\Delta}, \quad \Delta_{k+1} V_{d+1}^{\Delta} = \Delta_{k+1} Z^{\Delta} := Z_{\tau_{k+1}}^{\Delta} - Z_{\tau_k}^{\Delta}, \tag{3.15c}$$

and $\zeta(t, v) = u(t, x) \exp(y + z)$. Following the proof of Lemma 3.1 in [2] we can then deduce the following two technical lemmas. We state the lemmas without proofs and refer the reader to [2] for details. Note that the crucial part in the proof of Lemma 3.2 stated below is that (3.10) and (3.11) are the counterparts of equations (3.13)-(3.15) in [2].

Lemma 3.1. *Assume* (1.26) *and assume that* $\Delta_{k+1}\eta$ *is generated using Method 2. If* $\Delta^* < \tilde{\Delta}^*$*, where* $\tilde{\Delta}^*$ *is defined as in* (3.6)*, then for all* $(t, x) \in D$*, the following expansion of* $E[F_{t,T}(X, \Lambda)] - E\left[F_{t,T}^{\Delta}\left(X^{\Delta}, \Lambda^{\Delta}\right)\right]$ *holds, for some constants* $0 < \hat{\nu}_k^{\Delta} < 1$ *and* $0 < \tilde{\nu}_{i,k}^{\Delta} < 1$.

$$\begin{aligned} &\left| E[F_{t,T}(X, \Lambda)] - E\left[F_{t,T}^{\Delta}\left(X^{\Delta}, \Lambda^{\Delta}\right)\right] \right| \\ &= - E\left[\sum_{k=0}^{N-1} e^k G_{t,T}^{\Delta,k}\left(X^{\Delta}, \Lambda^{\Delta}\right)\right] - E[r(t, T, \Delta)] - E[R(t, T, \Delta)], \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} &G_{t,T}^{\Delta,k}\left(X^{\Delta}, \Lambda^{\Delta}\right) \\ =&\theta^{k+1} \sum_{i=1}^{d} \left[u_{x_i}\left(\tau_{k+1}, X_{\tau_k}^{\Delta} + \hat{\nu}_k^{\Delta} \Delta_{k+1} X^{\Delta}\right) - u_{x_i}\left(\tau_{k+1}, X_{\tau_k}^{\Delta}\right)\right] \Delta_{k+1} X_i^{\Delta} \Delta_{k+1} \left|\Lambda^{\Delta}\right| \\ &+ \sum_{i=1}^{d} \gamma_i^{k+1} \sum_{j=1}^{d} (u_{x_i x_j}\left(\tau_{k+1}, X_{\tau_k}^{\Delta} + \tilde{\nu}_{i,k}^{\Delta} \Delta_{k+1} X^{\Delta}\right) \\ &- u_{x_i x_j}\left(\tau_{k+1}, X_{\tau_k}^{\Delta}\right)) \Delta_{k+1} X_j^{\Delta} \Delta_{k+1} \left|\Lambda^{\Delta}\right| \\ &+ \frac{1}{2}\left(u^k \left(\theta^{k+1}\right)^2 - \sum_{i,j=1}^{d} u_{x_i x_j}^k \gamma_i^{k+1} \gamma_j^{k+1}\right) \left(\Delta_{k+1} \left|\Lambda^{\Delta}\right|\right)^2 \\ &+ \frac{1}{2}\left(2\varphi^k \sum_{i=1}^{d} u_{x_i}^k a_i^k + \sum_{i,j=1}^{d} u_{x_i x_j}^k a_i^k a_j^k + u^k \left(\varphi^k\right)^2\right) \left(\Delta^*\right)^2 \\ &+ \left(\sum_{i=1}^{d} u_{x_i}^k \gamma_i^{k+1} - u^k \theta^{k+1}\right) \varphi^k \Delta^* \Delta_{k+1} \left|\Lambda^{\Delta}\right|. \end{aligned} \tag{3.17}$$

Lemma 3.2. *Assume* (1.26) *and assume that* $\Delta_{k+1}\eta$ *is generated using Method* 2*. Let*

$$H_N = \max_{k \in \{0,1,...,N-1\}} \left|\Delta_{k+1} \eta\right|.$$

If $\Delta^* < \tilde{\Delta}^*$*, where* $\tilde{\Delta}^*$ *is defined as in* (3.6)*, then the upper bound*

$$\begin{aligned}
&\left| E\left[F_{t,T}\left(X,\Lambda\right)\right] - E\left[F^{\Delta}_{t,T}\left(X^{\Delta},\Lambda^{\Delta}\right)\right]\right| \\
\leq & CE\left[H_N|\Lambda^{\Delta}|_T\right]\sqrt{\Delta^*} + C\bigg(1 + E\left[|\Lambda^{\Delta}|_T\right] + E\left[H_N^2|\Lambda^{\Delta}|_T\right]\bigg)\Delta^* \\
&+ \bigg| E\left[r\left(t,T,\Delta\right)\right]| + |E\left[R\left(t,T,\Delta\right)\right]\bigg| + \mathcal{O}_{pol}\left(\Delta^*\right),
\end{aligned} \tag{3.18}$$

holds uniformly with respect to $(t,x) \in D$ *for some positive constant* C *depending only on* T*,* D*,* H *and the coefficients of* (1.31)*. By* $\mathcal{O}_{pol}\left(\Delta^*\right)$ *we mean a quantity that is bounded by a positive constant times* $\left(\Delta^*\right)^{-k}$ *for any* $k \geq 0$*.*

Furthermore, the estimates of the error terms $|E\left[r\left(t,T,\Delta\right)\right]|$ and $|E\left[R\left(t,T,\Delta\right)\right]|$ derived in [2], in the time-independent case, can also be carried over to the time-dependent case considered in this article. Hence we can state the following result, which is essentially Lemma 3.2 in [2], without proof.

Lemma 3.3. *Assume* (1.26) *and assume that* $\Delta_{k+1}\eta$ *is generated using either Method 1 or Method 2. Then*

$$\lim_{\Delta^*\to 0}\sup_{(t,x)\in D}|E\left[r\left(t,T,\Delta\right)\right]| \to 0, \qquad \lim_{\Delta^*\to 0}\sup_{(t,x)\in D}|E\left[R\left(t,T,\Delta\right)\right]| \to 0. \tag{3.19}$$

Furthermore, if $\Delta^* < \tilde{\Delta}^*$*, where* $\tilde{\Delta}^*$ *is defined as in* (3.3) *or* (3.6)*, then*

$$\sup_{(t,x)\in D}|E\left[r\left(t,T,\Delta\right)\right]| \leq C\sqrt{\Delta^*}, \qquad \sup_{(t,x)\in D}|E\left[R\left(t,T,\Delta\right)\right]| \leq C\sqrt{\Delta^*}, \tag{3.20}$$

for some positive constant C *which is independent of* Δ^**.*

Now, as H_N and H_N^2 are bounded in the case of Method 2, Theorem 3.1 follows immediately by combining Lemma 2.3 for $q = 1$ with Lemmas 3.2 and 3.3.

Remark 3.2. For $\Delta_{k+1}\eta$ chosen according to Method 1 it was shown in [15] that if the domain D is convex and time-independent, and if the coefficients b and σ are time-independent, then the result in Lemma 2.3 regarding Λ^{Δ} still holds. However, as H_N is expected to increase without bound as N increases, Lemma 3.2 cannot be applied in this case. Nevertheless, under much stronger regularity conditions compared to those stated in (1.26), for example assuming that the domain D and the coefficients of (1.31) are smooth and time-independent, it was shown in Lemma 4.1 in [14] that the probability of $\Delta_{k+1}\left|\Lambda^{\Delta}\right|$ being non-zero is of order $\sqrt{\Delta^*}$. Hence, applying the Hölder inequality to the third term in (3.17), we obtain, for arbitrary $1 < q < \infty$ and for some positive constant L,

$$\begin{aligned}
E\left[\sum_{k=0}^{N-1}\left(\Delta_{k+1}\left|\Lambda^{\Delta}\right|\right)^2\right] &= \sum_{k=0}^{N-1} E\left[\left(\Delta_{k+1}\left|\Lambda^{\Delta}\right|\right)^2 \mathbf{1}_{\Delta_{k+1}|\Lambda^{\Delta}|>0}\right] \\
&\leq \sum_{k=0}^{N-1}\left(E\left[\left(\Delta_{k+1}\left|\Lambda^{\Delta}\right|\right)^{2p}\right]\right)^{\frac{1}{p}}\left(E\left[\left(\mathbf{1}_{\Delta_{k+1}|\Lambda^{\Delta}|>0}\right)^q\right]\right)^{\frac{1}{q}} \\
&\leq L\sum_{k=0}^{N-1}\left(E\left[\left(\Delta_{k+1}\left|\Lambda^{\Delta}\right|\right)^{2p}\right]\right)^{\frac{1}{p}}\left(\Delta^*\right)^{\frac{1}{2q}}.
\end{aligned} \tag{3.21}$$

Now by (3.10), which applies also when $\Delta_{k+1}\eta$ is generated according to Method 1, we have

$$\begin{aligned}\left(E\left[\left(\Delta_{k+1}\left|\Lambda^{\Delta}\right|\right)^{2p}\right]\right)^{\frac{1}{p}} &\leq \left(E\left[pL^{2p}\left(\Delta_{k+1}\eta\right)^{2p}\left(\Delta^{*}\right)^{p}\right]\right)^{\frac{1}{p}} \\ &= p^{\frac{1}{p}}L^{2}\left(E\left[\left(\Delta_{k+1}\eta\right)^{2p}\right]\right)^{\frac{1}{p}}\Delta^{*}, \qquad (3.22)\end{aligned}$$

for some positive constant L, independent of p and Δ. Furthermore, using basic properties of the normal distribution we have that,

$$\begin{aligned}\left(E\left[\left(\Delta_{k+1}\eta\right)^{2p}\right]\right)^{\frac{1}{p}} &\leq \left(\frac{(2p)!}{2^{p}p!}\right)^{\frac{1}{p}} = \frac{1}{2}\Big((2p)\,(2p-1)\cdot\ldots\cdot(p+1)\Big)^{\frac{1}{p}} \\ &\leq \frac{1}{2}\Big((2p)^{p}\Big)^{\frac{1}{p}} = p. \qquad (3.23)\end{aligned}$$

Hence, we conclude that there exist positive constants L and L', independent of q and Δ, such that

$$E\left[\sum_{k=0}^{N-1}\left(\Delta_{k+1}\left|\Lambda^{\Delta}\right|\right)^{2}\right] \leq L'\sum_{k=0}^{N-1}(p)^{\frac{1}{p}}\,p\left(\Delta^{*}\right)^{1+\frac{1}{2q}} \leq \frac{Lq}{q-1}\left(\Delta^{*}\right)^{\frac{1}{2q}}. \qquad (3.24)$$

Similar results can also be found for the other terms in (3.17) and, as $1 < q < \infty$ is arbitrary, this enables us to obtain an order of convergence equal to $1/2-\varepsilon$ whenever $\Delta_{k+1}\eta$ is chosen according to Method 1. The problem of generalizing these results, valid only for time-independent domains, to the time-dependent setting considered in this article is left as a subject for future research.

Remark 3.3. Although the solution to (1.31) is unique we cannot be certain that the solution to the corresponding stochastic differential equation with reflection is pathwise unique. This also explains why there is no need to require that the oblique projections of x onto ∂D along Γ are unique. If we, however, use Theorem 2.6 to assert that (2.46) holds, the regularity assumptions are sufficient to ensure that the oblique projections are indeed unique. This is a consequence of Theorem 4.6 in [3].

4. Numerical Examples

In this section we empirically evaluate the performance of the algorithm described in this article based on two examples. In both examples we consider $D \subset \mathbb{R}^3$ and the examples are constructed in such a way that the solution u to (1.31) can be explicitly found. As we shall see, the numerical results suggest that the order of convergence of the suggested algorithm does not fall below the theoretical bound of $1/2$.

4.1. A reflected Langevin type equation on a square

Let $T > 0$ be given and let $\rho : [0,T] \to \mathbb{R}_+$ be some function. Based on ρ we let $D \subset \mathbb{R}^3$ be a time-dependent domain with time sections

$$D_t = \left\{x = (x_1, x_2) \in \mathbb{R}^2,\ (x_1, x_2) \in [-\rho(t), \rho(t)] \times [-\rho(t), \rho(t)]\right\}, \qquad (4.1)$$

whenever $t \in [0,T]$. Note that the time sections are squares with side lengths specified by the function ρ. Inspired by the numerical example in Section 4 of [2], we let the drift and diffusion coefficients be

$$b(t,x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad \text{and} \quad \sigma(t,x) = \begin{pmatrix} \dfrac{\sin(x_1+x_2)}{\kappa(t)} & 0 \\ 0 & \dfrac{\cos(x_1+x_2)}{\kappa(t)} \end{pmatrix}, \tag{4.2}$$

for some function $\kappa : [0,T] \to \mathbb{R}_+$. Furthermore, we let, in (1.31), $f(x) = x_1^2 + x_2^2$ and we set $\gamma = n$, $\varphi \equiv 0$, $\theta \equiv 0$, $g(t,x) = -2\rho(t)$ and $h(t,x) = \mu(t)$, for some function $\mu : [0,T] \to \mathbb{R}$. Note that $n = n_t$ denotes the inward normal on ∂D_t. Then the solution u to (1.31) is given by

$$u(t,x) = x_1^2 + x_2^2 + \xi(t), \tag{4.3}$$

for some function $\xi : [0,T] \to \mathbb{R}_+$ which we specify below based on κ and μ. Indeed, applying (4.2) and (4.3) to the differential operator L, we obtain

$$Lu = \sum_{i=1}^{2} b_i(t,x)\,\partial_{x_i}u + \frac{1}{2}\sum_{i,j=1}^{2}\sum_{k=1}^{2}\sigma_{ik}(t,x)\,\sigma_{jk}(t,x)\,\partial_{x_ix_j}u = \frac{1}{(\kappa(t))^2}. \tag{4.4}$$

Accordingly, the partial differential equation and the terminal condition for u reduce to

$$\partial_t u(t,x) + Lu(t,x) - \varphi(t,x)\,u(t,x) - h(t,x) = \xi'(t) + \frac{1}{(\kappa(t))^2} - \mu(t) = 0, \tag{4.5}$$

and

$$u(T,x) - f(x) = \xi(T) = 0, \tag{4.6}$$

respectively. Hence, given κ and μ, the unknown function ξ is easily obtained as the solution to the ordinary differential equation

$$\xi'(t) = \mu(t) - \frac{1}{(\kappa(t))^2}, \quad \text{with terminal condition } \xi(T) = 0. \tag{4.7}$$

Finally, considering the boundary condition, we note that on the subset of the boundary given by $\{(t,\rho(t),x_2) \in \mathbb{R}^3 : x_2 \in [-\rho(t),\rho(t)], t \in [0,T]\} \subset \partial D$, the boundary condition is satisfied since

$$\langle \nabla_x u(t,x), \gamma(t,x)\rangle = \sum_{i=1}^{2} (\partial_{x_i}u)\,n_i = 2x_1(-1) = -2\rho(t) = g(t,x). \tag{4.8}$$

Analogous relations hold on the other parts of the boundary. Observe that the time-dependence of the domain enters in (1.31) through the right hand side in the Neumann condition. To conclude we can, under fairly weak assumptions on κ, μ and ρ, always find the explicit form of the solution u to (1.31). Note that the functional $F_{t,T}^{\Delta}$ corresponding to this problem equals

$$F_{t,T}^{\Delta}\left(X^{\Delta},\Lambda^{\Delta}\right) = \left|X_{\tau_N}^{\Delta}\right|^2 + 2\sum_{k=0}^{N-1}\rho\left(\tau_{k+1}, X_{\tau_{k+1}}^{\Delta}\right)\Delta_{k+1}\left|\Lambda^{\Delta}\right| - \sum_{k=0}^{N-1}\mu(\tau_{k+1})\,\Delta^*. \tag{4.9}$$

To numerically evaluate the algorithm described in this article, for the problem outlined above, we set $T = 1$ and we estimate $u(0,0.2,0.2)$ for the following three specifications of the functions κ, μ and ρ.

$$\begin{aligned} &(i) && \kappa(t)=\sqrt{t+1}, \quad \mu\equiv 0, \quad \rho\equiv 1, \\ &(ii) && \kappa(t)=\sqrt{t+1}, \quad \mu\equiv 0, \quad \rho(t)=1-\frac{2t(T-t)}{T^2}, \\ &(iii) && \kappa(t)=\sqrt{t+1}, \quad \mu(t)=\frac{4t(T-t)}{T^2}, \quad \rho(t)=1-\frac{2t(T-t)}{T^2}. \end{aligned} \tag{4.10}$$

In the first case in (4.10) the domain is time-independent and the setup is quite similar to the one studied in Section 4 of [2]. In the second case the complexity is increased by allowing for genuinely time-dependent domains and in the third case we complicate the situation further by introducing an inhomogeneity in the partial differential equation. Numerical simulations based on $M=10^8$ trajectories, corresponding to a statistical error of approximately $1.4\cdot 10^{-4}$ in the first case and $1.1\cdot 10^{-4}$ in the other two cases, are found in Figure 4.1. Evidently the orders of convergence are asymptotically close to 1/2 in cases (ii) and (iii), i.e. in the cases where the domain is time-dependent. In the first case, the results are significantly more instable and harder to interpret and hence, somewhat surprisingly, the results are more stable in the time-dependent case. The instability arises as the approximated value oscillates around the correct value. This property of the Euler approximation is well known and has been described in the literature, see for example [16].

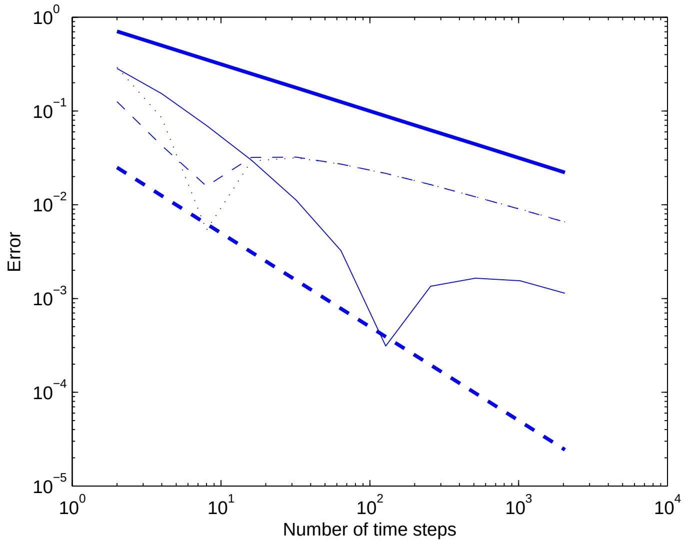

Fig. 4.1. Plot of the error in the estimate of the solution to (4.6) as a function of the number of time steps. Legend: case (i) (solid); case (ii) (dash); case (iii) (dot); reference line with slope 1/2 (solid thick); reference line with slope 1 (dash thick).

4.2. A reflected geometric Brownian motion on a disc

Let $T>0$ be given and let $\rho:[0,T]\to\mathbb{R}_+$ be some function. Based on ρ we let $D\subset\mathbb{R}^3$ be a time-dependent domain with time sections

$$D_t=\left\{x\in\mathbb{R}^2:|x|\leq\rho(t)\right\}, \tag{4.11}$$

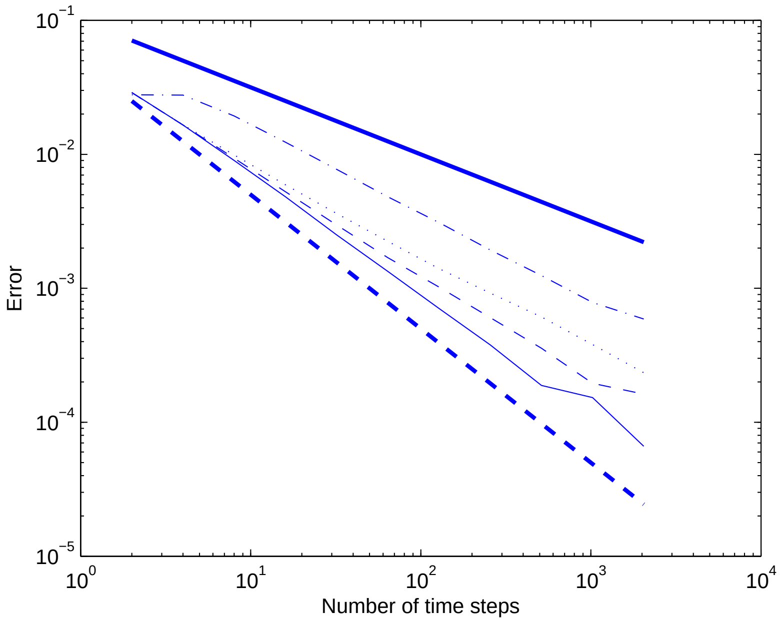

Fig. 4.2. Plot of the error in the estimate of the solution to (4.13) as a function of the number of time steps. Legend: case (i) (solid); case (ii) (dash); case (iii) (dot); case (iv) (dash dot); reference line with slope 1/2 (solid thick); reference line with slope 1 (dash thick).

whenever $t \in [0, T]$. Note that the time sections are discs with radii specified by the function ρ. We let the drift and diffusion coefficients be

$$b(t,x) = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} \quad \text{and} \quad \sigma(t,x) = \begin{pmatrix} \beta x_1 & 0 \\ 0 & \beta x_2 \end{pmatrix}, \tag{4.12}$$

for some choice of constants $\alpha, \beta \in \mathbb{R}$. Furthermore, we let, in (1.31), $f(x) = x_1^2 + x_2^2$ and we set $\gamma = n$, $\varphi(t,x) = \hat{\varphi}(t)$, $\theta(t,x) = \hat{\theta}(t)$ and $h(t,x) = \mu(t)\left(x_1^2 + x_2^2\right)$, for some functions $\hat{\varphi}, \hat{\theta}, \mu : [0,T] \to \mathbb{R}$. Note that $n = n_t$ denotes the inward normal on ∂D_t. Then the solution u to (1.31) is given by

$$u(t,x) = \xi(t)\left(x_1^2 + x_2^2\right), \tag{4.13}$$

for some function $\xi : [0,T] \to \mathbb{R}_+$ which we specify below based on $\hat{\varphi}$, $\hat{\theta}$ and μ. Note that $g(t,x)$ in (1.31) will also be specified below based on $\hat{\theta}, \rho$ and ξ. Indeed, applying (4.12) and (4.13) to the differential operator L, we obtain

$$\begin{aligned} Lu &= \sum_{i=1}^{2} b_i(t,x)\,\partial_{x_i} u + \frac{1}{2}\sum_{i,j=1}^{2}\sum_{k=1}^{2} \sigma_{ik}(t,x)\,\sigma_{jk}(t,x)\,\partial_{x_i x_j} u \\ &= \xi(t)\left(x_1^2 + x_2^2\right)\left(2\alpha + \beta^2\right), \end{aligned} \tag{4.14}$$

and, hence, the partial differential equation and terminal condition for u reduce to

$$\begin{aligned} &\partial_t u(t,x) + Lu(t,x) - \varphi(t,x)\,u(t,x) - h(t,x) \\ &= \left(x_1^2 + x_2^2\right)\left(\xi'(t) + \xi(t)\left(2\alpha + \beta^2\right) - \xi(t)\,\hat{\varphi}(t) - \mu(t)\right) = 0, \end{aligned} \tag{4.15}$$

and

$$u(T,x) - f(x) = \left(x_1^2 + x_2^2\right)(\xi(T) - 1) = 0, \tag{4.16}$$

respectively. Hence, given α, β, $\hat{\varphi}$ and μ, the function ξ is easily obtained as the solution to the ordinary differential equation

$$\xi'(t) + \Big(\left(2\alpha + \beta^2\right) - \hat{\varphi}(t)\Big)\xi(t) = \mu(t), \quad \text{with terminal condition } \xi(T) = 1. \tag{4.17}$$

Finally considering the boundary condition, we obtain

$$\begin{aligned}
&\langle \nabla_x u(t,x), \gamma(t,x)\rangle - \theta(t,x)\, u(t,x) \\
&= \left\langle \xi(t)(2x_1, 2x_2), -\frac{(x_1,x_2)}{\sqrt{x_1^2+x_2^2}} \right\rangle - \hat{\theta}(t)\,\xi(t)\left(x_1^2+x_2^2\right) \\
&= -2\xi(t)\,\rho(t) - \xi(t)\,(\rho(t))^2\,\hat{\theta}(t) := g(t),
\end{aligned} \tag{4.18}$$

and, as mentioned above, we specify g based on the specific form of $\xi(t)$. As in the previous example, the time-dependence of the domain enters (1.31) through the right hand side of the Neumann condition. To conclude we can, under fairly weak assumptions on α, β, $\hat{\varphi}$, $\hat{\theta}$, μ and ρ, always find the explicit form of the solution u to (1.31). Note that the functional $F_{t,T}^{\Delta}$ corresponding to this problem is

$$\begin{aligned}
&F_{t,T}^{\Delta}\left(X^{\Delta}, \Lambda^{\Delta}\right) \\
&= \left|X_{\tau_N}^{\Delta}\right|^2 \exp\left(Y_{\tau_N}^{\Delta} + Z_{\tau_N}^{\Delta}\right) - \sum_{k=0}^{N-1} g(\tau_{k+1}) \exp\left(Y_{\tau_k}^{\Delta} + Z_{\tau_k}^{\Delta}\right) \Delta_{k+1} \left|\Lambda^{\Delta}\right| \\
&\quad - \sum_{k=0}^{N-1} \mu(\tau_{k+1}) \left|X_{\tau_{k+1}}^{\Delta}\right|^2 \exp\left(Y_{\tau_k}^{\Delta} + Z_{\tau_k}^{\Delta}\right) \Delta^*,
\end{aligned} \tag{4.19}$$

where g is defined in (4.18) and the processes $Y_{\tau_k}^{\Delta}$ and $Z_{\tau_k}^{\Delta}$ are given as

$$Y_{\tau_k}^{\Delta} = -\sum_{k=0}^{k-1} \hat{\varphi}(\tau_{k+1})\,\Delta^* \quad \text{and} \quad Z_{\tau_k}^{\Delta} = -\sum_{k=0}^{k-1} \hat{\theta}(\tau_{k+1})\,\Delta_{k+1}\left|\Lambda^{\Delta}\right|. \tag{4.20}$$

Note that the processes $Y_{\tau_k}^{\Delta}$ and $Z_{\tau_k}^{\Delta}$ are zero for the setup considered in Section 4.1.

To numerically evaluate the algorithm described in this article, for the problem outlined above, we set $T = 1$, $\alpha = 3/8$, $\beta = 1/2$ (so that $2\alpha + \beta^2 = 1$) and we estimate $u(0, 0.2, 0.2)$, for the following four specifications of the functions $\hat{\varphi}$, $\hat{\theta}$, μ and ρ.

$$\begin{aligned}
&(i) && \hat{\varphi} \equiv 0, \quad \hat{\theta} \equiv 0, \quad \mu \equiv 0, \quad \rho \equiv 1, \\
&(ii) && \hat{\varphi} \equiv 0, \quad \hat{\theta} \equiv 0, \quad \mu \equiv 0, \quad \rho(t) = 1 - \frac{2t(T-t)}{T^2}, \\
&(iii) && \hat{\varphi} \equiv 0, \quad \hat{\theta}(t) = \sin\left(\frac{\pi t}{T}\right), \quad \mu \equiv 0, \quad \rho(t) = 1 - \frac{2t(T-t)}{T^2}, \\
&(iv) && \hat{\varphi} \equiv -1, \quad \hat{\theta}(t) = \sin\left(\frac{\pi t}{T}\right), \quad \mu(t) = \sin\left(\frac{\pi t}{T}\right), \quad \rho(t) = 1 - \frac{2t(T-t)}{T^2}.
\end{aligned} \tag{4.21}$$

The first case in (4.21) represents a homogenous partial differential equation in a time-independent domain and the second case represents the same equation but in a time-dependent domain. In the last two cases we add, respectively, a u-term in the boundary condition and a non-zero right hand side in the partial differential equation. Numerical simulations based on $M = 10^8$ trajectories, corresponding to a statistical error of approximately $3.4 \cdot 10^{-5}$ in the first three cases and $7.8 \cdot 10^{-5}$ in the last case, are found in Figure 4.2. We observe that the asymptotic orders of convergence decrease, from just under 1 in case (i) to around 0.6 in case (iv), as the complexity of the problem is increased. Note also that, in general, the convergence is more stable for this problem compared to the problem considered in Section 4.1.

References

[1] Y. Saisho, Stochastic differential equations for multi-dimensional domain with reflecting boundary, *Probab. Theory Rel.*, **74** (1987), 455–477.

[2] C. Costantini, B. Pacchiarotti and F. Sartoretto, Numerical approximation for functionals of reflecting diffusion processes, *SIAM J. Appl. Math.*, **58** (1998), 73–102.

[3] C. Costantini, The Skorohod oblique reflection problem in domains with corners and application to stochastic differential equations, *Probab. Theory Rel.*, **91** (1992), 43–70.

[4] E. Gobet, Euler schemes and half-space approximation for the simulation of diffusion in a domain, *ESAIM: Probability and Statistics*, **5** (2001), 261–297.

[5] M. Bossy, E. Gobet and D. Talay, A symmetrized Euler scheme for an efficient approximation of reflected diffusions, *J. Appl. Probab.*, **41** (2004), 877–889.

[6] G.N. Mil'shtein, Application of the numerical integration of stochastic equations for the solution of boundary value problems with Neumann boundary conditions, *Theor. Probab. Appl.*, **41** (1996), 170–177.

[7] C. Costantini, E. Gobet and N. El Karoui, Boundary sensitivities for diffusion processes in time dependent domains, *Appl. Math. Opt.*, **54** (2006), 159–187.

[8] K. Burdzy, Z. Chen and J. Sylvester, The heat equation and reflected Brownian motion in time dependent domains. II. Singularities of solutions, *J. Funct. Anal.*, **204**:1 (2003), 1–34.

[9] K. Burdzy, W. Kang and K. Ramanan, The Skorokhod problem in a time dependent interval, *Stoch. Proc. Appl.*, **119** (2009), 428–452.

[10] K. Nyström and T. Önskog, The Skorohod oblique reflection problem in time-dependent domains, *To appear in Ann. Prob.*, (2009).

[11] G.M. Lieberman, Second Order Parabolic Differential Equations, World Scientific, River Edge, NJ, 1996.

[12] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'ceva, Linear and Quasi-linear Equations of Parabolic Type, Translations of Mathematical Monographs 23, AMS, Providence, RI, 1968.

[13] G.J. Minty, On the extensions of Lipschitz, Lipschitz-Hölder continuous, and monotone functions, *Bulletin of the American Mathematical Society*, **76** (1970), 334–339.

[14] E. Gobet, Weak approximation of killed diffusion using Euler schemes, *Stoch. Proc. Appl.*, **87** (2000), 167–197.

[15] L. Słomiński, On approximation of solutions of multidimensional stochastic differential equations with reflecting boundary conditions, *Stoch. Proc. Appl.*, **50** (1994), 197–220.

[16] K. Nyström and T. Önskog, On Monte Carlo algorithms applied to Dirichlet problems for parabolic operators in the setting of time-dependent domains, *Monte Carlo Methods and Applications*, **15**:1 (2009), 11–47.