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ON EQUILIBRIUM PRICING AS CONVEX OPTIMIZATION*

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Abstract

We study competitive economy equilibrium computation. We show that, for the first time, the equilibrium sets of the following two markets: 1. A mixed Fisher and Arrow-Debreu market with homogeneous and log-concave utility functions; 2. The Fisher and Arrow-Debreu markets with several classes of concave non-homogeneous utility functions; are convex or log-convex. Furthermore, an equilibrium can be computed as convex optimization by an interior-point algorithm in polynomial time.

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1. Introduction

The study of competitive economy equilibria occupies a central place in mathematical economics. This study was formally started by Walras [14] over a hundred years ago. In this problem everyone in a population of n agents has an initial endowment of divisible goods and a utility function for consuming all goods—their own and others. Every player sells the entire initial endowment and then uses the revenue to buy a bundle of goods such that his or her utility function is maximized. Walras asked whether prices could be set for everyone's good such that this is possible. An answer was given by Arrow and Debreu in 1954 [1] who showed that such an equilibrium would exist, under very mild conditions, if the utility functions of agents were concave. Their proof was non-constructive and did not offer any algorithm to compute an equilibrium.

Fisher considered a related and different market model where agents were divided into two sets: producers and consumers; see Brainard and Scarf [2,13]. Consumers spend money only to buy goods and maximize their individual utility functions of goods; producers sell their goods only for money. The price equilibrium is an assignment of prices to goods so that when every consumer buys a maximal bundle of goods then the market clears, meaning that all budgets are spent and all goods are sold. Fisher's model is a special case of Walras' model when money is also considered a good so that Arrow and Debreu's result applies.

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In a remarkable piece of work, Eisenberg and Gale [8, 11] give a convex programming (or optimization) formulation whose solution yields equilibrium allocations for the Fisher market with linear utility functions, and Eisenberg [9], published 1961 in *Management Science*, extended this approach to derive a convex program for general concave and homogeneous utility functions of degree 1. Their program consists of maximizing an aggregate utility function of all consumers over a convex polyhedron defined by supply-demand linear constraints. The Lagrange or dual multipliers of these constraints yield equilibrium prices. Thus, finding a Fisher equilibrium becomes solving a convex optimization problem, and it could be computed by the Ellipsoid method or by efficient interior-point methods in polynomial time. Later, Codenotti et al. [5] rediscovered the convex programming formulation, and Jain et al. [12] generalized Eisenberg and Gale's convex model to handling homothetic and quasi-concave utilities introduced by Friedman [10]. Here, polynomial time means that one can compute an ϵ approximate equilibrium in a number of arithmetic operations bounded by polynomial in n and $\log \frac{1}{\epsilon}$; or, if there is a rational equilibrium solution, one can compute an *exact* equilibrium in a number of arithmetic operations bounded by polynomial in n and L, where L is the bit-length of the input data. When the utility functions are linear, the current best arithmetic operations complexity bound is $\mathcal{O}(\sqrt{mn}(m+n)^3 L)$ given by [15].

Little is known on the computational complexity of computing market equilibria for non-homogeneous utility functions or for markets other than the Fisher and Arrow-Debreu settings such as utility functions with externality. This paper is to derive convex programs to solve a couple of more general equilibrium problems. We show that, for the first time, the equilibrium of either of the following two markets:

1. A mixed Fisher and Arrow-Debreu market with homogeneous and log-concave utility functions;
2. The Fisher and Arrow-Debreu markets with several classes of concave non-homogeneous utility functions;

can be computed as convex optimization and by interior-point algorithms in polynomial time. These markets have wide applications in supply chain and communication spectrum management.

First, a few mathematical notations. Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space; $\mathbf{R}^n_+$ denote the subset of $\mathbf{R}^n$ where each coordinate is non-negative. $\mathbf{R}$ and $\mathbf{R}_+$ denote the set of real numbers and the set of non-negative real numbers, respectively.

A function $u : \mathbf{R}^n_+ \to \mathbf{R}_+$ is said to be *concave* if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n_+$ and any $0 \le \alpha \le 1$, we have

$$u(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \ge \alpha u(\boldsymbol{x}) + (1-\alpha)u(\boldsymbol{y}).$$

It is *homothetic* if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n_+$ and any $\alpha > 0$,

$$u(\boldsymbol{x}) \ge u(\boldsymbol{y}) \text{iff} u(\alpha \boldsymbol{x}) \ge u(\alpha \boldsymbol{y}).$$

It is *monotone increasing* if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n_+$, $\boldsymbol{x} \ge \boldsymbol{y}$ implies that $u(\boldsymbol{x}) \ge u(\boldsymbol{y})$. It is *homogeneous* of degree d if for any $\boldsymbol{x} \in \mathbf{R}^n_+$ and any $\alpha > 0$, $u(\alpha \boldsymbol{x}) = \alpha^d u(\boldsymbol{x})$.

2. The Fisher and Arrow-Debreu Markets

Without loss of generality, assume that there is one unit of good for each type of good $j \in P$ with $|P| = n$. Let consumer $i \in C$ (with $|C| = m$) in the Fisher market have an initial money endowment $w_i > 0$ to spend and buy goods to maximize his or her utility function for a given price vector $\boldsymbol{p} \in \mathbf{R}^n_+$:

$$\begin{array}{ll} \text{maximize} & u_i(\boldsymbol{x}_i) \\ \text{subject to} & \boldsymbol{p}^T \boldsymbol{x}_i \le w_i \\ & \boldsymbol{x}_i \ge \mathbf{0}; \end{array} \tag{2.1}$$

where variable $\boldsymbol{x}_i = (x_{i1}; ...; x_{in})$ is a column vector whose jth coordinates x_{ij} represents the amount of good j bought by consumer i, $j = 1, ..., n$. Let $u_i(\boldsymbol{x}_i)$ be concave and monotonically increasing. We also assume that every consumer is interested in buying at least one type of good and every type of good is sought by at least one consumer. Then, a price vector $\boldsymbol{p} \ge \mathbf{0}$, together with good allocations $\boldsymbol{x}_i$, $i = 1, ..., m$, is called an equilibrium if $\boldsymbol{x}_i$ is optimal for (2.1) given $\boldsymbol{p}$, and $\sum_i \boldsymbol{x}_i = \boldsymbol{e}$ (the vector of all ones). The last condition means that all goods in the market are sold; no shortage and no surplass.

In the Arrow-Debreu market, each consumer i has an initial endowment of the same goods

$$\boldsymbol{w}_i = (w_{i1}; ...; w_{ij}),$$

where, without loss of generality, assume

$$\sum_i w_{ij} = 1, \quad \forall j,$$

that is, there is one unit of good for each type of good $j \in P$ in the market. Then, consumer $i \in C$ in the Arrow-Debreu market solves the following problem to maximize his or her utility function for a given price vector $\boldsymbol{p} \in \mathbf{R}^n_+$:

$$\begin{array}{ll} \text{maximize} & u_i(\boldsymbol{x}_i) \\ \text{subject to} & \boldsymbol{p}^T \boldsymbol{x}_i \le \boldsymbol{p}^T \boldsymbol{w}_i \\ & \boldsymbol{x}_i \ge \mathbf{0}. \end{array} \tag{2.2}$$

In comparison to the Fisher market (2.1), budget w_i there is simply replaced by $\boldsymbol{p}^T \boldsymbol{w}_i$ in the Arrow-Debreu market.

2.1. Necessary and sufficient condition for a Fisher equilibrium

Consider the optimality conditions of (2.1). Besides feasibility, they are

$$\begin{array}{r} (\nabla u_i(\boldsymbol{x}_i)^T \boldsymbol{x}_i) \cdot \boldsymbol{p} \ge w_i \cdot \nabla u_i(\boldsymbol{x}_i), \\ \boldsymbol{p}^T x_i = w_i, \\ x_i \ge 0, \end{array} \tag{2.3}$$

where $\nabla u(\boldsymbol{x})$ denotes any sub-gradient vector of $u(\boldsymbol{x})$ at $\boldsymbol{x}$.

Thus, the complete necessary and sufficient conditions for a Fisher equilibrium are the following:

$$\begin{array}{l} (\nabla u_i(\boldsymbol{x}_i)^T \boldsymbol{x}_i) \cdot \boldsymbol{p} \ge w_i \cdot \nabla u_i(\boldsymbol{x}_i), \quad \forall i \\ \boldsymbol{p}^T \boldsymbol{x}_i = w_i, \\ \sum_i \boldsymbol{x}_i \le \boldsymbol{e}, \\ \boldsymbol{p}^T \boldsymbol{e} \le \sum_i w_i, \\ \boldsymbol{x}_i, \boldsymbol{p} \ge \mathbf{0}, \ \forall i. \end{array} \tag{2.4}$$

Note here that the condition $\boldsymbol{p}^T\boldsymbol{x}_i = w_i$ should be implied by the rest of conditions in (2.4): Multiplying $\boldsymbol{x}_i \geq \boldsymbol{0}$ to both sides of the first inequality in (2.4), we have $\boldsymbol{p}^T\boldsymbol{x}_i \geq w_i$ for all i, which, together with other inequality conditions in (2.4), imply

$$\sum_i w_i \geq \boldsymbol{p}^T\boldsymbol{e} \geq \boldsymbol{p}^T\left(\sum_i \boldsymbol{x}_i\right) = \sum_i \boldsymbol{p}^T\boldsymbol{x}_i \geq \sum_i w_i,$$

that is, every inequality in the sequence must be tight which implies $\boldsymbol{p}^T\boldsymbol{x}_i = w_i$ for all i. Thus, the reduced necessary and sufficient Fisher equilibrium conditions become

$$\begin{array}{c}(\nabla u_i(\boldsymbol{x}_i)^T\boldsymbol{x}_i)\cdot\boldsymbol{p} \geq w_i\cdot\nabla u_i(\boldsymbol{x}_i), \;\; \forall i\\ \sum_i \boldsymbol{x}_i \leq \boldsymbol{e},\\ \boldsymbol{p}^T\boldsymbol{e} \leq \sum_i w_i,\\ \boldsymbol{x}_i, \boldsymbol{p} \geq \boldsymbol{0}, \;\forall i.\end{array} \tag{2.5}$$

The inequalities and equalities in (2.5) are all linear, except the first

$$(\nabla u_i(\boldsymbol{x}_i)^T\boldsymbol{x}_i)\cdot\boldsymbol{p} \geq w_i\cdot\nabla u_i(\boldsymbol{x}_i).$$

An immediate observation is, if every consumer i is interested in exactly one type of good, that is, $u_i(x_i)$ is a univariate concave function $u_i(x_{i\bar{j}_i})$ for some $\bar{j}_i \in P$, then the above condition becomes a single inequality:

$$(u_i'(x_{i\bar{j}_i})\cdot x_{i\bar{j}_i})\cdot p_{\bar{j}_i} \geq w_i\cdot u_i'(x_{i\bar{j}_i}),$$

or simply

$$x_{i\bar{j}_i}\cdot p_{\bar{j}_i} \geq w_i.$$

One can transfer this non-linear inequality to

$$\log(x_{i\bar{j}_i}) + \log(p_{\bar{j}_i}) \geq \log(w_i),$$

which is a convex inequality (meaning that the set of feasible solutions is convex). Thus, the Fisher equilibrium set is convex and can be found by solving a convex optimization problem. It turns out that this simple trick works for other utilities as well, as we shall present in the next subsection.

2.2. Necessary and sufficient condition for an Arrow-Debreu equilibrium

Similarly, the necessary and sufficient Arrow-Debreu equilibrium conditions become

$$\begin{array}{c}(\nabla u_i(\boldsymbol{x}_i)^T\boldsymbol{x}_i)\cdot\boldsymbol{p} \geq (\boldsymbol{p}^T\boldsymbol{w}_i)\cdot\nabla u_i(\boldsymbol{x}_i), \;\forall i\\ \sum_i \boldsymbol{x}_i \leq \boldsymbol{e},\\ \boldsymbol{x}_i, \boldsymbol{p} \geq \boldsymbol{0}, \;\forall i,\end{array} \tag{2.6}$$

where the third condition, $\boldsymbol{p}^T\boldsymbol{e} \leq \sum_i \boldsymbol{p}^T\boldsymbol{w}_i$, in (2.5) is unnecessary here since $\sum_i \boldsymbol{w}_i = \boldsymbol{e}$. Note that the price $\boldsymbol{p}$ in the conditions can be scaled by any positive number, and we are interested in not-all-zero prices.

3. Convex Optimization for the Fisher Market where Consumers May Retain Money

If $u_i(\boldsymbol{x}_i)$ is homogeneous of degree 1 (this is without loss of generality since any homogeneous function with a positive degree can be monotonically transformed to a homogeneous function with degree 1) and $\log(u_i(\boldsymbol{x}_i))$ is concave in $\boldsymbol{x}_i \in \mathbf{R}^n_+$, the Fisher equilibrium problem can be solved as an aggregate social convex maximization problem (see Eisenberg and Gale [8, 9, 11]):

$$\begin{array}{ll} \text{maximize} & \sum_i w_i \log(u_i(\boldsymbol{x}_i)) \\ \text{subject to} & \sum_i \boldsymbol{x}_i = \boldsymbol{e}, \ \ \forall j, \\ & \boldsymbol{x}_i \geq \mathbf{0}, \qquad \forall i; \end{array} \tag{3.1}$$

where the objective function may be interpreted as a socially aggregated utility.

These homogeneous and log-concave functions include many classical utilities:

- All constant elasticity functions

$$u_i(\boldsymbol{x}) = \left(\sum_{j=1}^{n} (a_j x_j)^{(\sigma-1)/\sigma}\right)^{\sigma/(\sigma-1)}, \ \ a_j \geq 0, \ 0 < \sigma < \infty;$$

- Piece-wise concave linear function

$$u_i(\boldsymbol{x}) = \min_k \{(\boldsymbol{a}^k)^T \boldsymbol{x}\}, \ \ \boldsymbol{a}^k \geq \mathbf{0}, \ k = 1, ..., K;$$

- The Cobb-Douglass utility function

$$u_i(\boldsymbol{x}) = \prod_{j=1}^{n} x_j^{a_j}, \ \ a_j \geq 0.$$

Jain et al. [12] showed how to transform a homothetic utility function into an equivalent homogeneous degree 1 and log-concave function. Thus, the Fisher equilibrium problem with homothetic utilities can be also solved as a convex optimization problem.

3.1. The mixed market Equilibrium

In the classical Fisher market, consumers spend money only to buy goods and maximize their individual utility functions of goods; producers sell their goods only for money. Now consider a market where each consumer has an incentive retain certain amount of money from his or her own budget, that is, his or her utility becomes

$$\begin{array}{ll} \text{maximize} & u_i(\boldsymbol{x}_i, s_i) \\ \text{subject to} & \boldsymbol{p}^T \boldsymbol{x}_i + s_i \leq w_i \\ & \boldsymbol{x}_i, s_i \geq \mathbf{0}, \end{array} \tag{3.2}$$

where again $\boldsymbol{x}_i = (x_{i1}; ...; x_{in})$ and its jth component x_{ij} represents the amount of good j bought by consumer i, and s_i denotes the retained money (e.g., deposited in a bank for a short-time interest gain). We assume that $u_i(\boldsymbol{x}_i, s_i)$ is a monotone increasing and concave function of $(\boldsymbol{x}_i, s_i) \geq \mathbf{0}$.

One such example is

$$u_i(\boldsymbol{x}_i, s_i) = \hat{u}_i(\boldsymbol{x}_i) + s_i = \hat{u}_i(\boldsymbol{x}) - \boldsymbol{p}^T\boldsymbol{x}_i + w_i$$

which is a profit function after subtracting cost $\boldsymbol{p}^T\boldsymbol{x}_i$. This utility contains agent i's own decision variable $\boldsymbol{x}_i$ as well as externality decision variable $\boldsymbol{p}$, and it has a number of applications in managing supply chains and resource allocations.

An equilibrium is defined as a non-negative price vector $\boldsymbol{p} \in \mathbf{R}^n_+$ at which there exist a bundle of goods ($\boldsymbol{x}_i \in \mathbf{R}^n_+, s_i \geq 0$) for each consumer $i \in C$ such that the following conditions hold:

1. The vector $(\boldsymbol{x}_i; s_i)$ optimizes retailer i's utility (3.2) given her money budget w_i.
2. For each good j, the total amount available equals the total amount consumed by the consumers, that is,
$$\sum_{i \in C} x_{ij} = 1.$$
3. The sum of the spending and retaining money equals the sum of the money possessed by all consumers, that is,
$$\sum_{j \in P} p_j + \sum_{i \in C} s_i = \sum_{i \in C} w_i.$$

The existence of such an equilibrium is immediately implied by the existence of an Arrow-Debreu equilibrium by treating money as an additional "good". One may attempt to prove the existence using the Fisher equilibrium model. However, in such a Fisher equilibrium model the price for the money "good" (s_i) has to be fixed to 1 (as the same as w_i), which is difficult to enforce. Thus, we need to invoke the Arrow-Debreu model by assigning price p_{n+1} to a unit of money. Then, each consumer's problem becomes

$$\begin{array}{ll} \text{maximize} & u_i(\boldsymbol{x}_i, s_i) \\ \text{subject to} & \boldsymbol{p}^T\boldsymbol{x}_i + p_{n+1}s_i \leq p_{n+1}w_i, \\ & \boldsymbol{x}_i, s_i \geq \mathbf{0}, \end{array}$$

where the total supply of money is $\sum_i w_i$. Therefore, the Arrow-Debreu theorem implies that an equilibrium price vector $(\boldsymbol{p}; p_{n+1}) \in \mathbf{R}^{n+1}_+$ exists. In particular, $p_{n+1} > 0$ at every Arrow-Debreu equilibrium since money has a value at least to every producer. By dividing $(\boldsymbol{p}; p_{n+1})$ by p_{n+1}, we have an equilibrium price for all goods, and the price for the money "good" equals 1:

Corollary 3.1. *An equilibrium always exists for the mixed market of* (3.2).

However, it was unknown if the mixed market admits a convex program for computing its equilibrium, or it has to use the more difficult Arrow-Debreu equilibrium framework to compute it, even the utility is homogeneous and log-concave. The computational complexity issue of the mixed market equilibrium problem is important, since there is a fundamental difference between the Fisher and Arrow-Debreu models with respect to computational complexity. For example, when the utility is Leontief

$$u_i(\boldsymbol{x}) = \min_{j \in P} \left\{ \frac{x_j}{a_j} : \ a_j > 0 \right\},$$

a homogeneous of degree one and log-concave function, the Fisher market equilibrium can be computed as a convex program in polynomial time while the existence of an Arrow-Debreu market equilibrium is NP-hard to decide; see Ye [16] and Codenotti et al. [6].

3.2. Convex optimization for computing an equilibrium

We settle the computational complexity issue of the mixed market equilibrium problem (3.2) in this subsection by showing that any optimal solution to a convex program yields an equilibrium if the utility functions are log-concave and homogeneous of degree one.

From (2.4), the necessary and sufficient conditions for the mixed market equilibrium are

$$\begin{array}{l} (\boldsymbol{p};\ 1) \geq \frac{w_i}{\nabla u_i(\boldsymbol{x}_i,s_i)^T(x_i;s_i)} \cdot \nabla u_i(\boldsymbol{x}_i, s_i),\ \forall i \\ \sum_i \boldsymbol{x}_i \leq \boldsymbol{e}, \\ \sum_j p_j + \sum_i s_i \leq \sum_i w_i, \\ \boldsymbol{x}_i, \boldsymbol{p} \geq \mathbf{0},\ \forall i; \end{array} \tag{3.3}$$

where one can see that the price for the money good is set to 1.

Let $u_i(\boldsymbol{x}_i, s_i)$ be homogeneous of degree one and $\log(u_i(\boldsymbol{x}_i, s_i))$ be concave in $(\boldsymbol{x}_i; s)i) \in \mathbf{R}^{n+1}_+$, which includes all constant elasticity, piece-wise concave linear, the Cobb-Douglass utility, and the Leontief utility functions. Now consider the convex optimization problem

$$\begin{array}{ll} \text{maximize} & \sum_i w_i \log(u_i(\boldsymbol{x}_i, s_i)) - s \\ \text{subject to} & \sum_i x_{ij} \leq 1,\ \forall j, \\ & \sum_i s_i - s = 0, \\ & (\boldsymbol{x}_i, s_i) \geq \mathbf{0},\ \forall i. \end{array} \tag{3.4}$$

The first set of constraint inequalities indicates that the demand does not exceed the supply; the second simply denotes the total amount of money retained by all consumers by s. Then, the total amount s is subtracted linearly from the aggregate social utility function. This makes economical sense since this amount has been withdrawn from the exchange market by the consumers so that one should extract them from the aggregated social utility.

We have

Theorem 3.1. *Let $(\bar{\boldsymbol{x}}_i, \bar{s}_i)$, $i = 1, ..., m$, be an optimal solution for convex program* (3.4)*, and let p_j be an optimal Lagrange multiplier for each good j in the first constraint set of* (3.4)*. Then, they form an equilibrium for the mixed market* (3.2)*, and it can be computed in polynomial time.*

Proof. First, the feasible set of the optimization problem (3.4) is linear, compact and convex, the maximal solution exists and the maximum value is finite. Moreover, the objective function to be maximized is concave. Thus, the first-order optimality conditions are necessary and sufficient for an optimal solution $(\bar{\boldsymbol{x}}_i, \bar{s}_i)$. These optimality conditions can be written (using the fact that the optimal Lagrange multiplier for the second constraint automatically equals 1) as:

$$\begin{array}{l} \dfrac{w_i}{u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)} \nabla_{\boldsymbol{x}_i} u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i) \leq \boldsymbol{p}, \qquad \forall i \\ \dfrac{w_i}{u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)} \partial_{s_i} u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i) \leq 1, \qquad \forall i \\ \dfrac{w_i}{u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)} \left(\nabla_{\boldsymbol{x}_i} u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)^T \bar{\boldsymbol{x}}_i + \partial_{s_i} u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)\bar{s}_i\right) = \boldsymbol{p}^T \bar{\boldsymbol{x}}_i + \bar{s}_i, \end{array} \tag{3.5}$$

where $\boldsymbol{p} = (p_1, ..., p_n)$ and p_j is the optimal Lagrange multiplier for each j in the first constraint set of (3.4). The third equality of condition (3.5) is called the complementarity condition, which, together with the fact that $u_i(\boldsymbol{x}_i, s_i)$ is homogeneous of degree one, namely $u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i) =$

$\nabla u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)^T(\bar{\boldsymbol{x}}_i; \bar{s}_i)$, imply

$$\begin{aligned}\boldsymbol{p}^T\bar{\boldsymbol{x}}_i + \bar{s}_i &= \frac{w_i}{u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)}\left(\nabla_{\boldsymbol{x}_i} u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)^T\bar{\boldsymbol{x}}_i + \partial_{s_i} u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)\bar{s}_i\right)\\ &= \frac{w_i}{u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)}(\nabla u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)^T(\bar{\boldsymbol{x}}_i; \bar{s}_i))\\ &= \frac{w_i}{u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)} \cdot u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i) = w_i,\end{aligned}$$

so that

$$\sum_i(\boldsymbol{p}^T\bar{\boldsymbol{x}}_i + \bar{s}_i) = \sum_i w_i.$$

Thus, $(\bar{\boldsymbol{x}}_i, \bar{s}_i)$, $i = 1, ..., m$, and $\boldsymbol{p}$ satisfy the equilibrium conditions of (3.3).

It is well-known that one can use interior point methods to solve the linearly constrained convex program (3.4) to yield both primal and dual optimal solutions in polynomial time; see [15]. Therefore, an equilibrium for the mixed market (3.2) can be found in polynomial time. □

4. The Markets with Concave and Non-Homogeneous Utilities

As one can see, convex optimization helps to compute equilibrum for most homogeneous utility functions. A natural question arises: Does this approach apply to general *non-homogeneous* utility functions?

Now consider $u_i(\boldsymbol{x}_i)$ in the following additive or separable form:

$$u_i(\boldsymbol{x}_i) = \sum_{j=1}^n a_{ij}(x_{ij} + b_{ij})^{d_{ij}}, \quad \text{or} \quad u_i(\boldsymbol{x}_i) = \sum_{j=1}^n a_{ij}\log(x_{ij} + b_{ij}), \tag{4.1}$$

where $a_{ij}, b_{ij} \geq 0$, and $0 < d_{ij} \leq 1$, for all i and j, are given, and variable x_{ij} represents the amount of goods bought from good j by consumer i, $j = 1, ..., n$. One can see that $u_i(\boldsymbol{x}_i)$ is a concave and monotone increasing function in $\boldsymbol{x}_i = (x_{i1}; ...; x_{in}) \geq \mathbf{0}$.

These utility functions include the Shannon capacity in wireless and Digital Subscribe Lines (DSL) communications under a Frequency Division Multiple Access (FDMA) policy; see, e.g., Cover and Thomas [7]. They include as special case several popular homogeneous utilities:

- linear utility functions: $d_{ij} = 1$ for all j in the first form;
- certain constant elasticity functions: $b_{ij} = 0$ and $d_{ij} = d$, $0 \leq d \leq 1$, for all j in the first form;
- the Cobb-Douglass utility function: $b_{ij} = 0$ in the second form;
- a non-homogeneous Cobb-Douglass utility functions given in [4] in the second form.

Note that $u_i(\boldsymbol{x}_i)$ of (4.1), can be non-homothetic; see, for example, $u(x, y) = \sqrt{x} + y$. Chen at al. [4] developed approximation algorithm with running time polynomial in n and $1/\epsilon$ for the utility function in the second form of (4.1).

Lemma 4.1. *Given* $u_i(\boldsymbol{x}_i)$ *in the forms of* (4.1), $(\nabla u_i(\boldsymbol{x}_i)^T \boldsymbol{x}_i)$ *is concave and* $\log(\nabla_j u_i(\boldsymbol{x}_i))$ *is convex in* $\boldsymbol{x}_i \in \mathbf{R}^n_+$ *for every* j.

Proof. For simplicity, let us omit index i, so that

$$u(\boldsymbol{x}) = \sum_{j=1}^{n} a_j (x_j + b_j)^{d_j}$$

or

$$u(\boldsymbol{x}) = \sum_{j=1}^{n} a_j \log(x_j + b_j).$$

Thus, for the first form

$$\nabla u(\boldsymbol{x}) = (..., a_j d_j (x_j + b_j)^{d_j - 1}, ...),$$

so that

$$\nabla u(\boldsymbol{x})^T \boldsymbol{x} = \sum_j a_j d_j (x_j + b_j)^{d_j - 1} x_j.$$

It is easily see that each $(x_j + b_j)^{d_j-1} x_j$ is concave in $x_j \geq 0$ since $0 \leq d_j \leq 1$ (therefore, so is the log-sum: $\log\left(\sum_j a_j d_j (x_j + b_j)^{d_j-1} x_j\right)$.

Furthermore,

$$\log(\nabla_j u(\boldsymbol{x})) = (d_j - 1)\log(x_j + b_j) + \log(a_j d_j)$$

which is convex in $x_j > 0$ for every j.

Similarly, one can prove the lemma for the the second form. □

Then, one can rewrite the nonlinear inequality in (2.5) as

$$\log(\nabla u_i(\boldsymbol{x}_i)^T \boldsymbol{x}_i) + \log(p_j) \geq \log(w_i) + \log(\nabla_j u_i(\boldsymbol{x}_i)), \ \forall j,$$

which is a convex inequality (the set of feasible solutions is convex) by Lemma 4.1. Thus,

Theorem 4.1. *If utilities* $u_i(\boldsymbol{x}_i)$ *are given in the forms of* (4.1), *then the Fisher equilibrium set of* (2.5) *is convex in* $\boldsymbol{x}$ *and* $\boldsymbol{p}$, *and an equilibrium can be computed as a convex optimization problem in polynomial time.*

As for the Arrow-Debreu market, let $y_j = \log(p_j)$. Then,

$$w_i = \boldsymbol{p}^T \boldsymbol{w}_i = \sum_j w_{ij} e^{y_j}.$$

Thus, one can rewrite the nonlinear inequality of (2.6) as

$$\log(\nabla u_i(\boldsymbol{x}_i)^T \boldsymbol{x}_i) + y_j \geq \log\big(\sum_j w_{ij} e^{y_j}\big) + \log(\nabla_j u_i(\boldsymbol{x}_i)), \ \forall j,$$

which is a convex inequality in $\boldsymbol{x}$ and $\boldsymbol{y}$. It is shown in [15] that this inequality admits an efficient barrier function for interior-point algorithms. Thus,

Theorem 4.2. *If utilities* $u_i(\boldsymbol{x}_i)$ *are given in the forms of* (4.1), *then the Arrow-Debreu equilibrium set of* (2.5) *is convex in* $\boldsymbol{x}$ *and* $\log(\boldsymbol{p})$, *and an equilibrium can be computed as a convex optimization problem in polynomial time.*

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References

[1] K.J. Arrow and G. Debreu, Existence of an Equilibrium for a Competitive Economy, *Econometrica*, **22**:3 (1954), 265-290.

[2] W.C. Brainard and H. Scarf, How to Compute Equilibrium Prices in 1891, Cowles Foundation Discussion Paper, 1270, (2000).

[3] L. Chen, Y. Ye, and J. Zhang, On Equilibrium Pricing as Convex Optimization, Proc. WINE'07.

[4] N. Chen, X. Deng, X. Sun, and A.C. Yao, Fisher equilibrium price with a class of concave utility functions, In *the Proceedings of ESA 2004*, 2004, 169-179.

[5] B. Codenotti, S. Pemmaraju, and K. Varadarajan, On the polynomial time computation of equilibria for certain exchange economies, Proc. SODA'05.

[6] B. Codenotti, A. Saberi, K. Varadarajan, and Y. Ye, Leontief Economies Encode Nonzero Sum Two-Player Games, Proc. SODA'06.

[7] T.M. Cover and J.A. Thomas, Elements of Information Theory, John Wiley & Sons, Inc., (1991).

[8] E. Eisenberg, and D. Gale, Consensus of Subjective Probabilities: The Pari-Mutuel Method, *Annals of Mathematical Statistics*, **30** (1959), 165-168.

[9] E. Eisenberg, Aggregation of Utility Functions, *Manage. Sci.*, **7**:4 (1961), 337-350.

[10] J.W. Friedman, Concavity of Production Functions and Non-Increasing Returns to Scale, *Econometrica*, **41**:5 (1973), 981-984.

[11] D. Gale, The Theory of Linear Economic Models, McGraw Hill, N.Y. (1960).

[12] K. Jain, V. Vazirani, and Y. Ye, Market equilibria for homothetic, quasi-concave utilities and economies of scale in production, Proc. SODA'05.

[13] H.E. Scarf, The Computation of Economic Equilibria, With collaboration of T. Hansen, Cowles Foundation Monograph No. 24. New Haven: Yale University Press (1973).

[14] L. Walras, Elements d'economie politique pure; ou, Theorie de la richesse sociale, *Elements of Pure Economics; Or the Theory of Social Wealth*, Lausanne, Paris (1874).

[15] Y. Ye, A Path to the Arrow-Debreu Competitive Market Equilibrium, *Math Program.*, 2006.

[16] Y. Ye, Exchange Market Equilibria with Leontief's Utility: Freedom of Pricing Leads to Rationality, Proc. WINE'05.