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ON THE FINITE ELEMENT APPROXIMATION OF SYSTEMS OF REACTION-DIFFUSION EQUATIONS BY p/hp METHODS*

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Abstract

We consider the approximation of *systems* of reaction-diffusion equations, with the finite element method. The highest derivative in each equation is multiplied by a parameter $\varepsilon \in (0, 1]$, and as $\varepsilon \to 0$ the solution of the system will contain *boundary layers*. We extend the analysis of the corresponding scalar problem from [Melenk, IMA J. Numer. Anal. 17(1997), pp. 577-601], to construct a finite element scheme which includes elements of size $\mathcal{O}(\varepsilon p)$ near the boundary, where p is the degree of the approximating polynomials. We show that, under the assumption of analytic input data, the method yields *exponential* rates of convergence, independently of ε, when the error is measured in the energy norm associated with the problem. Numerical computations supporting the theory are also presented, which also show that the method yields robust exponential convergence rates when the error in the maximum norm is used.

Mathematics subject classification: 65N30.
Key words: Reaction-diffusion system, Boundary layers, *hp* finite element method.

1. Introduction

The numerical solution of reaction-diffusion problems whose solution contains boundary layers has been studied extensively over the last two decades (see, e.g., the books [5, 6, 8] and the references therein). The presence of boundary layers in the solution cannot be overlooked, and if one wishes to obtain an accurate and robust approximation, special care must be taken when constructing the numerical method. In the context of the Finite Element Method (FEM), the robust approximation of boundary layers requires either the use of the h version on non-uniform meshes (such as the Shishkin [11] or Bakhvalov [1] mesh), or the use of the high order p and hp versions on specially designed (variable) meshes [10]. In both cases, the a-priori knowledge of the position of the layers is taken into account, and mesh-degree combinations can be chosen for which uniform error estimates can be established [2, 4, 10].

In recent years researchers have turned their attention to *systems* of reaction-diffusion problems — see [3] and the references therein for a recent survey. In general, one-dimensional reaction diffusion systems, like the one considered in the present article, have the following

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form: Find $\overrightarrow{u}$ such that

$$L\overrightarrow{u} \equiv \begin{bmatrix} -\varepsilon_1^2 \frac{d^2}{dx^2} & & 0 \\ & \ddots & \\ 0 & & -\varepsilon_m^2 \frac{d^2}{dx^2} \end{bmatrix} \overrightarrow{u} + A\overrightarrow{u} = \overrightarrow{f} \quad \text{in} \quad \Omega = (0,1), \tag{1.1}$$

$$\overrightarrow{u}(0) = \overrightarrow{u}(1) = \overrightarrow{0}, \tag{1.2}$$

where $0 < \varepsilon_1 \leq \varepsilon_2 \leq ... \leq \varepsilon_m \leq 1$,

$$A = \begin{bmatrix} a_{11}(x) & ... & a_{1m}(x) \\ \vdots & & \vdots \\ a_{m1}(x) & ... & a_{mm}(x) \end{bmatrix}, \quad \overrightarrow{f}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}. \tag{1.3}$$

The data $\{\varepsilon_i\}_{i=1}^m$, A and $\overrightarrow{f}$ are given, and the unknown solution is $\overrightarrow{u}(x) = [u_1(x), ..., u_m(x)]^T$. The functions $a_{ij}(x)$ are such that for any $x \in \overline{\Omega} = [0,1]$, the matrix A is invertible (with $\|A^{-1}\|$ bounded) and moreover

$$\overrightarrow{\xi}^T A \overrightarrow{\xi} \geq \alpha^2 \overrightarrow{\xi}^T \overrightarrow{\xi} \quad \forall\, \overrightarrow{\xi} \in \mathbb{R}^m, \tag{1.4}$$

for some constant $\alpha > 0$.

We will restrict ourselves to the case $\varepsilon_i = \varepsilon \ \forall\ i = 1, ..., m$, which allows us to express (1.1)–(1.2) in vector form as: Find $\overrightarrow{u}$ such that

$$L\overrightarrow{u} := -\varepsilon^2 \overrightarrow{u}'' + A\overrightarrow{u} = \overrightarrow{f} \quad \text{in} \quad \Omega = (0,1), \tag{1.5}$$

$$\overrightarrow{u}(0) = \overrightarrow{u}(1) = \overrightarrow{0}. \tag{1.6}$$

The presence of the small parameter ε in the above boundary value problem causes the solution $\overrightarrow{u}$ to contain boundary layers of width $\mathcal{O}(|\varepsilon \ln \varepsilon|)$ near the endpoints of Ω. To illustrate this, we consider the case $m = 2$ with

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \overrightarrow{f}(x) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \varepsilon = 10^{-2}.$$

Figure 1.1 shows the exact solution corresponding to the above data and clearly shows that both components contain a boundary layer.

Our goal in the present article is to extend the analysis of [4] for the analogous scalar problem, to show that under the assumption of analytic input data, the hp version of the FEM on the variable three element mesh $\Delta = \{0, \kappa p\varepsilon, 1 - \kappa p\varepsilon\}, \kappa \in \mathbb{R}^+$ converges at an *exponential* rate (in the energy norm defined in eq. (2.6) below) as the polynomial degree of the approximating basis functions $p \to \infty$. Strictly speaking, the method is not an hp version, since the location and not the number of elements changes as the dimension of the approximating subspace is increased; a more appropriate characterization would be a p version on a variable mesh. In addition to extending the results of [4] to systems, our proof does not use Gauss-Lobatto interpolants (like the one in [4]), but rather we achieve the desired result using the approximation theory from [9] with integrated Legendre polynomials, something that is of interest in its own right. More importantly, the present approach allows us to define the constant κ used in the mesh in a more concrete way.

The rest of the paper is organized as follows: In Section 2 we present the model problem and discuss the properties of its solution. In Section 3 we present the finite element formulation and the design of the p/hp scheme we will be considering, along with our main result of exponential

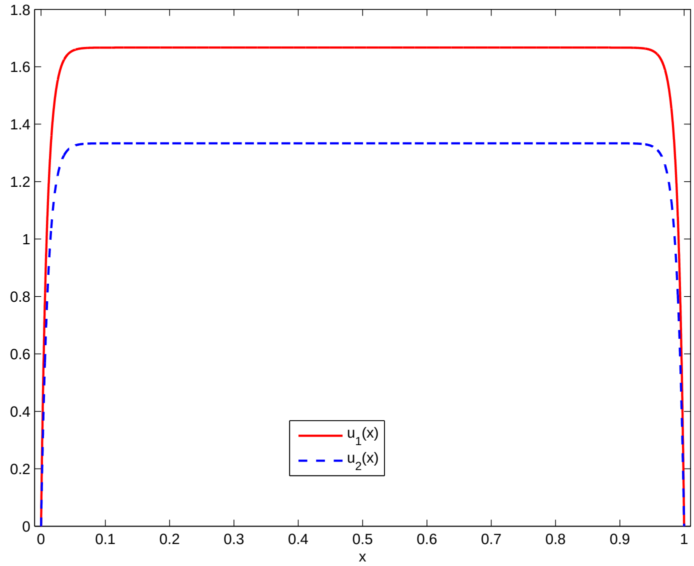

Fig. 1.1. The exact solution with $\varepsilon = 10^{-2}$.

convergence. In Section 4 we present the results of some numerical computations for two model problems, and in Section 5 we summarize our conclusions.

In what follows, the space of squared integrable functions on an interval $\Omega \subset \mathbb{R}$ will be denoted by $L^2(\Omega)$, with associated inner product

$$(u, v)_\Omega := \int_\Omega u(x)v(x)dx.$$

We will also utilize the usual Sobolev space notation $H^k(\Omega)$ to denote the space of functions on Ω with $0, 1, 2, ..., k$ generalized derivatives in $L^2(\Omega)$, equipped with norm and seminorm $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. For vector functions $\overrightarrow{u} = [u_1(x), ..., u_m(x)]^T$, we will write

$$\|\overrightarrow{u}\|^2_{k,\Omega} = \|u_1\|^2_{k,\Omega} + ... + \|u_m\|^2_{k,\Omega}.$$

We will also use the space

$$H^1_0(\Omega) = \left\{ u \in H^1(\Omega) : u|_{\partial\Omega} = 0 \right\},$$

where $\partial\Omega$ denotes the boundary of Ω. Finally, the letter C will be used to denote a generic positive constant, independent of ε or any discretization parameters, and possibly having different values in each occurrence.

2. The Model Problem and its Regularity

We assume that the functions $a_{ij}(x)$ and $f_i(x)$ are analytic on $\overline{\Omega}$ and that there exist constants $C_f, \gamma_f, C_a, \gamma_a > 0$ such that

$$\left\| f_i^{(n)} \right\|_{\infty,\Omega} \le C_f \gamma_f^n n! \quad \forall\, n \in \mathbb{N}_0,\ i = 1, ..., m, \tag{2.1}$$

$$\left\| a_{ij}^{(n)} \right\|_{\infty,\Omega} \le C_a \gamma_a^n n! \quad \forall\, n \in \mathbb{N}_0,\ i, j = 1, ..., m. \tag{2.2}$$

As usual, we cast the problem (1.5)–(1.6) into an equivalent weak formulation, which reads: Find $\overrightarrow{u} \in \left[H_0^1(\Omega)\right]^m$ such that

$$B(\overrightarrow{u}, \overrightarrow{v}) = F(\overrightarrow{v}), \ \forall \overrightarrow{v} \in \left[H_0^1(\Omega)\right]^m, \tag{2.3}$$

where

$$B(\overrightarrow{u}, \overrightarrow{v}) = \varepsilon^2 \sum_{i=1}^{m} (u_i', v_i')_\Omega + \sum_{i=1}^{m}\sum_{j=1}^{m} (a_{ij} u_j, v_i)_\Omega, \tag{2.4}$$

$$F(\overrightarrow{v}) = \sum_{i=1}^{m} (f_i, v_i)_\Omega. \tag{2.5}$$

From (1.4), we get that the bilinear form $B(\cdot,\cdot)$ is coercive with respect to the *energy norm*

$$\|\overrightarrow{u}\|_{E,\Omega}^2 := \varepsilon^2 |\overrightarrow{u}|_{1,\Omega}^2 + \alpha^2 \|\overrightarrow{u}\|_{0,\Omega}^2, \tag{2.6}$$

i.e.,

$$B(\overrightarrow{u}, \overrightarrow{u}) \geq \|\overrightarrow{u}\|_{E,\Omega}^2 \ \ \forall \overrightarrow{u} \in \left[H_0^1(\Omega)\right]^m. \tag{2.7}$$

This, along with the continuity of $B(\cdot,\cdot)$ and $F(\cdot)$, imply the unique solvability of (2.3). We also have the *a priori* estimate

$$\|\overrightarrow{u}\|_{E,\Omega} \leq \frac{1}{\alpha} \left\|\overrightarrow{f}\right\|_{0,\Omega}. \tag{2.8}$$

We now present results on the regularity of the solution to (1.5)–(1.6). Note that by the analyticity of a_{ij} and f_i, we have that u_i are analytic. Moreover, we have the following theorem, whose proof is a straight forward generalization of the proof of Theorem 1 from [4].

Theorem 2.1. *Let* $\overrightarrow{u}$ *be the solution to* (1.5)–(1.6) *with* $0 < \varepsilon \leq 1$. *Then there exist positive constants* C *and* $K \geq 1$, *independent of* ε, *such that*

$$\left\|u_i^{(n)}\right\|_{0,\Omega} \leq CK^n \max\{n, \varepsilon^{-1}\}^n \quad \forall\, n \in \mathbb{N}_0, \ i = 1, ..., m. \tag{2.9}$$

We will now obtain a decomposition for the solution $\overrightarrow{u}$ into a smooth (asymptotic) part, two boundary layer parts and a remainder as follows:

$$\overrightarrow{u} = \overrightarrow{w} + \overrightarrow{u}^- + \overrightarrow{u}^+ + \overrightarrow{r}. \tag{2.10}$$

This decomposition is obtained by inserting the formal ansatz

$$\overrightarrow{u}(x) \sim \sum_{i=0}^{\infty} \varepsilon^i \overrightarrow{u}_i(x), \tag{2.11}$$

into the differential equation (1.5), and equating like powers of ε, so that we can define the smooth part $\overrightarrow{w}$ as

$$\overrightarrow{w}(x) := \sum_{i=0}^{M} \varepsilon^{2i} \overrightarrow{u}_{2i}, \tag{2.12}$$

where the terms $\overrightarrow{u}_{2i}$ are defined recursively by

$$\overrightarrow{u}_0 = A^{-1}\overrightarrow{f}, \tag{2.13}$$

$$\overrightarrow{u}_{2i} = A^{-1}(\overrightarrow{u}_{2i})'', \ i = 0, 2, 4, ... \tag{2.14}$$

A calculation shows that

$$L(\overrightarrow{u}-\overrightarrow{w})=\varepsilon^{2M+2}\left(\overrightarrow{u}_{2M}\right)'', \tag{2.15}$$

hence, as $\varepsilon\to 0$, $\overrightarrow{w}(x)$ defined by (2.12) satisfies the differential equation, but not the boundary conditions. To correct this we introduce *boundary layer functions* $\overrightarrow{u}^{+}$ and $\overrightarrow{u}^{-}$ by

$$\begin{aligned} L\overrightarrow{u}^{-} &= \overrightarrow{0} \text{ in } \Omega & \qquad L\overrightarrow{u}^{+} &= \overrightarrow{0} \text{ in } \Omega \\ \overrightarrow{u}^{-}(0) &= -\overrightarrow{w}(0) & \overrightarrow{u}^{+}(0) &= \overrightarrow{0} \\ \overrightarrow{u}^{-}(1) &= \overrightarrow{0}; & \overrightarrow{u}^{+}(1) &= -\overrightarrow{w}(1). \end{aligned} \tag{2.16}$$

Finally, we define $\overrightarrow{r}$ by

$$L\overrightarrow{r}=\varepsilon^{2M+2}\left(\overrightarrow{u}_{2M}\right)'', \tag{2.17a}$$

$$\overrightarrow{r}(0)=\overrightarrow{r}(1)=\overrightarrow{0}. \tag{2.17b}$$

The following results follow from the analogous ones for the scalar problem considered in [4], and their puprose is to provide information on the regularity of each of the components in (2.10).

Lemma 2.1. *Let $\overrightarrow{u}_{2i}$ be defined as in (2.13)–(2.14). Then there exist positive constants C, K_1, K_2, $(K_2>1)$, depending only on A and $\overrightarrow{f}$ such that for any $i,n\in\mathbb{N}_0$*

$$\left\|\left(\overrightarrow{u}_{2i}\right)^{(n)}\right\|_{\infty,\Omega}\le CK_1^{2i}K_2^{n}(2i)!n!.$$

Theorem 2.2. *There exist constants C, $\overline{K}_1$, $\overline{K}_2\in\mathbb{R}^+$ depending only on $\overrightarrow{f}$ and A such that if $0<2M\varepsilon\overline{K}_1\le 1$, then $\overrightarrow{w}(x)$ given by (2.12), satisfies*

$$\left\|\overrightarrow{w}^{(n)}\right\|_{\infty,\Omega}\le C\overline{K}_2^{n}n! \ \ \forall\, n\in\mathbb{N}_0. \tag{2.18}$$

Theorem 2.3. *Let $\overrightarrow{u}^{\pm}$ be the solutions of (2.16). Then there exist constants $\alpha, C, K>0$ independent of ε and n such that for any $x\in\overline{\Omega}$, $n\in\mathbb{N}_0$, and $i=1,...,m$,*

$$\left|\left(u_i^{-}\right)^{(n)}(x)\right|\le CK^{n}e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^{n}, \tag{2.19a}$$

$$\left|\left(u_i^{+}\right)^{(n)}(x)\right|\le CK^{n}e^{-(1-x)\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^{n}. \tag{2.19b}$$

Theorem 2.4. *There are constants C, K_1, $K_2>0$ depending only on the input data such that the remainder $\overrightarrow{r}$ defined by (2.17b) satisfies*

$$\left\|\overrightarrow{r}^{(n)}\right\|_{0,\Omega}\le CK_2^2\varepsilon^{2-n}\left(2M\varepsilon K_1\right)^{2M}, \ \ n=0,1. \tag{2.20}$$

3. The Finite Element Method

For the discretization of (2.3), we choose a finite dimensional subspace S_N of $H_0^1(\Omega)$ and solve the problem: Find $\overrightarrow{u}_N\in[S_N]^m$ such that

$$B\left(\overrightarrow{u}_N,\overrightarrow{v}\right)=F\left(\overrightarrow{v}\right) \ \ \forall\,\overrightarrow{v}\in[S_N]^m. \tag{3.1}$$

The unique solvability of the discrete problem (3.1) follows from (1.4) and (2.7); by the well-known orthogonality relation, we have

$$\|\overrightarrow{u}-\overrightarrow{u}_N\|_E\le\inf_{\overrightarrow{v}\in[S_N]^m}\|\overrightarrow{u}-\overrightarrow{v}\|_E. \tag{3.2}$$

The subspace S_N is chosen as follows: Let $\Delta = \{0 = x_0 < x_1 < ... < x_{\mathcal{M}} = 1\}$ be an arbitrary partition of $\Omega = (0, 1)$ and set

$$I_j = (x_{j-1}, x_j),\ h_j = x_j - x_{j-1},\ j = 1, ..., \mathcal{M}.$$

Also, define the master (or standard) element $I_{ST} = (-1, 1)$, and note that it can be mapped onto the j^{th} element I_j by the linear mapping

$$x = Q_j(t) = \frac{1}{2}(1-t)\,x_{j-1} + \frac{1}{2}(1+t)\,x_j.$$

With $\Pi_p(I_{ST})$ the space of polynomials of degree $\le p$ on I_{ST}, we define our finite dimensional subspaces as

$$S_N \equiv S^{\overrightarrow{p}}(\Delta) = \left\{u \in H_0^1(\Omega) : u(Q_j(t)) \in \Pi_{p_j}(I_{ST}),\, j = 1, ..., \mathcal{M}\right\},$$

and

$$\overrightarrow{S}_0^p(\Delta) := \left[S^{\overrightarrow{p}}(\Delta) \cap H_0^1(\Omega)\right]^m, \tag{3.3}$$

where $\overrightarrow{p} = (p_1, ..., p_{\mathcal{M}})$ is the vector of polynomial degrees assigned to the elements.

The following approximation result from [9] will be the main tool for the analysis of the method. As mentioned earlier, the analysis for the scalar problem in [4] relied on Gauss-Lobatto interpolants (and their approximation properties), which is different from what we present in this work.

Theorem 3.1. *For any $u \in C^\infty\left(\overline{I}_{ST}\right)$ there exists $\mathcal{I}_p u \in \Pi_p(I_{ST})$ such that*

$$u(\pm 1) = \mathcal{I}_p u(\pm 1), \tag{3.4}$$

$$\|u - \mathcal{I}_p u\|_{0,I_{ST}}^2 \le \frac{1}{p^2}\frac{(p-s)!}{(p+s)!}\left\|u^{(s+1)}\right\|_{0,I_{ST}}^2,\ \forall\, s = 0, 1, ..., p, \tag{3.5}$$

$$\left\|(u - \mathcal{I}_p u)'\right\|_{0,I_{ST}}^2 \le \frac{(p-s)!}{(p+s)!}\left\|u^{(s+1)}\right\|_{0,I_{ST}}^2,\ \forall\, s = 0, 1, ..., p. \tag{3.6}$$

The definition below describes the mesh used for the method: If we are in the asymptotic range of p, i.e. $p \ge 1/\varepsilon$, then a single element suffices since p will be sufficiently large to give us exponential convergence without any refinement. If we are in the pre-asymptotic range, i.e. $p < 1/\varepsilon$, then the mesh consists of three elements as described below. We should point out that this is the *minimal* mesh-degree combination for attaining exponential convergence; obviously, refining within each element will retain the convergence rate but would require more degrees of freedom – one such example is the so-called *geometrically graded* mesh discussed in [4] for the scalar problem.

Definition 3.1. *For $\kappa > 0$, $p \in \mathbb{N}$ and $0 < \varepsilon \le 1$, define the spaces $\overrightarrow{S}(\kappa, p)$ of piecewise polynomials by*

$$\overrightarrow{S}(\kappa, p) := \begin{cases} \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0, 1\} & \text{if } \kappa p \varepsilon \ge \frac{1}{2}, \\ \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0, \kappa p \varepsilon, 1 - \kappa p \varepsilon, 1\} & \text{if } \kappa p \varepsilon < \frac{1}{2}. \end{cases}$$

In both cases, the polynomial degree is uniformly p on all elements.

Before we state the main theorem of the paper, we cite a useful computation.

Lemma 3.1. *Let* $p \in \mathbb{N}, \lambda \in (0,1]$. *Then*

$$\frac{(p-\lambda p)!}{(p+\lambda p)!} \leq \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^p p^{-2\lambda p} e^{2\lambda p+1}.$$

Proof. Using Stirling's approximation

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}} \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e$$

for the factorial (cf. [7]), we have

$$\begin{aligned}\frac{(p-\lambda p)!}{(p+\lambda p)!} &\leq \frac{\sqrt{2\pi(1-\lambda)p}}{\sqrt{2\pi(1+\lambda)p}} \frac{\left(\frac{(1-\lambda)p}{e}\right)^{(1-\lambda)p}}{\left(\frac{(1+\lambda)p}{e}\right)^{(1+\lambda)p}} \frac{e}{e^{\frac{1}{12(1+\lambda)p+1}}} \\ &\leq \frac{[(1-\lambda)\,p]^{(1-\lambda)p}}{[(1+\lambda)\,p]^{(1+\lambda)p}} e^{2\lambda p} e^{1-\frac{1}{12(1+\lambda)p+1}} \\ &\leq \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^p p^{-2\lambda p} e^{2\lambda p} e.\end{aligned}$$

This completes the proof of the lemma. □

We now present our main result.

Theorem 3.2. *Let* $\overrightarrow{f}$ *and* A *be composed of functions that are analytic on* $\overline{\Omega}$ *and satisfy the conditions in* (2.1)–(2.2). *Let* $\overrightarrow{u} = [u_1, ..., u_m]^T$ *be the solution to* (1.5)–(1.6). *Then there exist constants* $\kappa, C, \beta > 0$ *depending only on* $\overrightarrow{f}$ *and* A *such that there exists* $\mathcal{I}_p\overrightarrow{u} = [\mathcal{I}_p u_1, ..., \mathcal{I}_p u_m]^T \in \overrightarrow{S}(\kappa, p)$ *with* $\mathcal{I}_p\overrightarrow{u} = \overrightarrow{u}$ *on* $\partial\Omega$ *and*

$$\|\overrightarrow{u} - \mathcal{I}_p\overrightarrow{u}\|_{E,\Omega}^2 \leq Cp^3 e^{-\beta p}.$$

Proof. We consider three separate cases.

Case 1: $\kappa p\varepsilon \geq \frac{1}{2}$ (asymptotic case), $\Delta = \{0, 1\}$

From Theorem 2.1 we have

$$\left\|\overrightarrow{u}^{(n)}\right\|_{0,\Omega}^2 \leq CK^{2n} \max\{n, \varepsilon^{-1}\}^{2n},$$

and by Theorem 3.1 there exists $\mathcal{I}_p\overrightarrow{u} \in \overrightarrow{S}(\kappa, p)$ such that $\overrightarrow{u} = \mathcal{I}_p\overrightarrow{u}$ on $\partial\Omega$ and for any $s = 0, 1, ..., p$

$$\begin{aligned}\left\|(\overrightarrow{u} - \mathcal{I}_p\overrightarrow{u})'\right\|_{0,\Omega}^2 &\leq \frac{(p-s)!}{(p+s)!}\left\|\overrightarrow{u}^{(s+1)}\right\|_{0,\Omega}^2 \\ &\leq \frac{(p-s)!}{(p+s)!} CK^{2(s+1)} \max\{s+1, \varepsilon^{-1}\}^{2(s+1)}.\end{aligned}$$

Let $s = \lambda p$ for some $\lambda \in (0, 1]$ to be selected shortly. Then, since $p \geq 1/(2\kappa\varepsilon)$, we have

$$\max\{s+1, \varepsilon^{-1}\}^{2(s+1)} = \max\{\lambda p+1, \varepsilon^{-1}\}^{2(\lambda p+1)} = (\lambda p+1)^{2(\lambda p+1)},$$

provided $\kappa \leq \lambda/2$. This, along with Lemma 3.1, gives

$$\begin{aligned}
\left\|(\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u})'\right\|_{0,\Omega}^2 &\leq \frac{(p-\lambda p)!}{(p+\lambda p)!} C K^{2(\lambda p+1)} (\lambda p+1)^{2(\lambda p+1)} \\
&\leq \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^p p^{-2\lambda p} e^{2\lambda p+1} C K^{2(\lambda p+1)} (\lambda p+1)^{2(\lambda p+1)} \\
&\leq CeK^2 \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} (eK)^{2\lambda}\right]^p (\lambda p+1)^2 \left(\frac{1+\lambda p}{p}\right)^{2\lambda p} \\
&\leq CeK^2 p^2 \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} (eK)^{2\lambda}\right]^p \left(\frac{1}{p} + \lambda\right)^{2\lambda p}.
\end{aligned}$$

Since

$$\left(\frac{1}{p} + \lambda\right)^{2\lambda p} = \lambda^{2\lambda p} \left[\left(1 + \frac{1}{\lambda p}\right)^{\lambda p}\right]^2 \leq e^2 \lambda^{2\lambda p},$$

we further get

$$\left\|(\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u})'\right\|_{0,\Omega}^2 \leq Cp^2 \left[\frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} (eK\lambda)^{2\lambda}\right]^p.$$

Consequently, if we choose $\lambda = (eK)^{-1} \in (0,1)$ we have

$$\left\|(\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u})'\right\|_{0,\Omega}^2 \leq Cp^2 e^{-\beta_1 p}, \tag{3.7}$$

where

$$\beta_1 = |\ln q_1|, \; q_1 = \frac{(1-\lambda)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} < 1,$$

and the constant $C > 0$ is independent of ε. The choice of λ dictates that the constant κ in the definition of the mesh must satisfy

$$\kappa \leq \frac{1}{2eK}. \tag{3.8}$$

Repeating the previous argument for the L^2 norm of $(\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u})$, we get, using (3.6),

$$\|\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u}\|_{0,\Omega}^2 \leq Ce^{-\beta_1 p}. \tag{3.9}$$

Combining (3.7)–(3.9), and using the definition of the energy norm, we get the desired result.

Case 2: $\kappa p \varepsilon < \frac{1}{2}$ (pre-asymptotic case), $\Delta = \{0, \kappa p \varepsilon, 1 - \kappa p \varepsilon, 1\}$

The mesh consists of three elements I_i, $i = 1,2,3$ and we decompose $\overrightarrow{u}$ as in (2.10):

$$\overrightarrow{u} = \overrightarrow{w} + \overrightarrow{u}^- + \overrightarrow{u}^+ + \overrightarrow{r}.$$

The expansion order M is chosen as the integer part of $\eta \kappa p/2$, where $\eta > 0$ is a fixed parameter satisfying

$$\frac{1}{2}\eta \overline{K}_1 \leq 1, \qquad \frac{1}{2}\eta K_1 =: \delta < \frac{1}{2},$$

with $\overline{K}_1$ and K_1 the constants from Theorems 2.2 and 2.4, respectively. The choice of η guarantees that as $\kappa p \varepsilon < \frac{1}{2}$, we have

$$2M\varepsilon \overline{K}_1 = \eta \kappa p \varepsilon \overline{K}_1 < \frac{1}{2}\eta \overline{K}_1 \leq 1$$

and

$$2M\varepsilon K_1 = \eta\kappa p\varepsilon K_1 < \frac{1}{2}\eta K_1 = \delta < \frac{1}{2}.$$

Thus the assumptions of Theorem 2.2 are satisfied and the remainder $\overrightarrow{r}$ is small by Theorem 2.4 — in particular, we have for $n = 1, 2$:

$$\left\| (\overrightarrow{r})^{(n)} \right\|_{0,\Omega} \leq C\varepsilon^{2-n} K_2^2 \left(2M\varepsilon K_1\right)^{2M} \leq C\varepsilon^{2-n}\delta^{\eta\kappa p} \leq C\varepsilon^{2-n} e^{-\beta_2 p}, \tag{3.10}$$

where $\beta_2 = |\ln q_2|$, $q_2 = \delta^{\eta\kappa} < 1$.

We next analyze the approximation of each of the remaining three terms in the decomposition (2.10).

For the approximation of $\overrightarrow{w}$, we have, by Theorem 3.1, that there exists $\mathcal{I}_p\overrightarrow{w} \in \overrightarrow{S}(\kappa, p)$ such that $\overrightarrow{w} = \mathcal{I}_p\overrightarrow{w}$ on $\partial\Omega$ and for any $s = 0, 1, ..., p$

$$\left\| (\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w})' \right\|_{0,\Omega}^2 \leq \frac{(p-s)!}{(p+s)!} \left\| \overrightarrow{w}^{(s+1)} \right\|_{0,\Omega}^2 \leq \frac{(p-s)!}{(p+s)!} C K^{2(s+1)} \left((s+1)!\right)^2,$$

where we used Theorem 2.2. Letting $s = \overline{\lambda}p$, for some $\overline{\lambda} \in (0, 1]$ to be selected shortly, and using Lemma 3.1, we get

$$\begin{aligned}
&\left\| (\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w})' \right\|_{0,\Omega}^2 \\
&\leq \left[\frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}} \right]^p p^{-2\overline{\lambda}p} e^{2\overline{\lambda}p+1} C K_2^{2\overline{\lambda}p+2} \left[\left(\overline{\lambda}p+1\right)^{\overline{\lambda}p+1+1/2} e^{-\overline{\lambda}p-1} \right]^2 \\
&\leq C\left(\overline{\lambda}p+1\right)^3 \left[\frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}} \right]^p K_2^{2\overline{\lambda}p} \left(\frac{1+\overline{\lambda}p}{p} \right)^{2\lambda p} \\
&\leq C\left(\overline{\lambda}p+1\right)^3 \left[\frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}} \right]^p K_2^{2\overline{\lambda}p}\overline{\lambda}^{2\overline{\lambda}p} \left[\left(1+\frac{1}{\overline{\lambda}p}\right)^{\overline{\lambda}p} \right]^2 \\
&\leq Cp^3 \left[\frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}} \left(K_2\overline{\lambda}\right)^{2\overline{\lambda}} \right]^p.
\end{aligned}$$

Thus, we choose $\overline{\lambda} = 1/K_2 \in (0, 1)$ and we have

$$\left\| (\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w})' \right\|_{0,\Omega}^2 \leq Cp^3 e^{-\beta_3 p}, \tag{3.11}$$

where

$$\beta_3 = |\ln q_3| \ , \ q_3 = \frac{\left(1-\overline{\lambda}\right)^{(1-\overline{\lambda})}}{\left(1+\overline{\lambda}\right)^{(1+\overline{\lambda})}} < 1.$$

Repeating the previous argument for the L^2 norm of $(\overrightarrow{w} - \mathcal{I}_p\overrightarrow{w})$, we get, using (3.6),

$$\| \overrightarrow{w} - \mathcal{I}_p\overrightarrow{w} \|_{0,\Omega}^2 \leq Cpe^{-\beta_3 p}. \tag{3.12}$$

We now approximate the boundary layers. We will only consider $\overrightarrow{u}^-$, since $\overrightarrow{u}^+$ is completely analogous. In view of Theorem 2.3, we will construct separate approximations for $\overrightarrow{u}^-$ on the intervals $\widetilde{I}_1 := I_1 = [0, \kappa p\varepsilon]$, and $\widetilde{I}_2 := [\kappa p\varepsilon, 1]$. Let $i \in \{1, ..., m\}$ be arbitrary. Then, by Theorem 3.1 there exists $\mathcal{I}_p u_i^- \in S(\kappa, p)$ such that $\mathcal{I}_p u_i^- = u_i^-$ on $\partial\widetilde{I}_1$ and for any $s = 0, 1, ..., p$

$$\left\| \left(u_i^- - \mathcal{I}_p u_i^-\right)' \right\|_{0,\widetilde{I}_1}^2 \le (\kappa p\varepsilon)^{2s} \frac{(p-s)!}{(p+s)!} \left\| \left(u_i^-\right)^{(s+1)} \right\|_{0,\widetilde{I}_1}^2 . \tag{3.13}$$

Now, by Lemma 2.3, we have

$$\begin{aligned} \left\| \left(u_i^-\right)^{(s+1)} \right\|_{0,\widetilde{I}_1}^2 &= \int_0^{\kappa p\varepsilon} \left| \left(u_i^-\right)^{(s+1)}(x) \right|^2 dx \\ &\le C\kappa p\varepsilon K^{2(s+1)} \max\{s+1, \varepsilon^{-1}\}^{2(s+1)} \max_{x\in[0,\kappa p\varepsilon]} \{e^{-2x\alpha/\varepsilon}\} \\ &\le C\kappa p\varepsilon K^{2(s+1)} \max\{s+1, \varepsilon^{-1}\}^{2(s+1)}. \end{aligned} \tag{3.14}$$

Choose $s = \widetilde{\lambda} p$ for some $\widetilde{\lambda} \in (0, 2\kappa]$ to be selected shortly, with κ satisfying (3.8). Then, since $\kappa p\varepsilon < 1/2$, we get $s + 1 = \widetilde{\lambda} p + 1 \le 2\kappa p + 1 < 1/\varepsilon + 1 < 2/\varepsilon$ and by Lemma 3.1, (3.13) and (3.14), we have

$$\left\| \left(u_i^- - \mathcal{I}_p u_i^-\right)' \right\|_{0,\widetilde{I}_1}^2 \tag{3.15}$$

$$\begin{aligned} &\le C K^{2(\widetilde{\lambda} p+1)} \kappa^{2\widetilde{\lambda} p+1} p^{2\widetilde{\lambda} p+1} 2^{2\widetilde{\lambda} p+1} \varepsilon^{-1} \frac{(p-\widetilde{\lambda} p)!}{(p+\widetilde{\lambda} p)!} \\ &\le C 2^{2\widetilde{\lambda} p+1} K^{2(\widetilde{\lambda} p+1)} \kappa^{2\widetilde{\lambda} p+1} p^{2\widetilde{\lambda} p+1} \varepsilon^{-1} \left[\frac{\left(1-\widetilde{\lambda}\right)^{(1-\widetilde{\lambda})}}{\left(1+\widetilde{\lambda}\right)^{(1+\widetilde{\lambda})}} \right]^p p^{-2\widetilde{\lambda} p} e^{2\widetilde{\lambda} p+1} \\ &\le C e K^2 \kappa p \varepsilon^{-1} \left[\frac{\left(1-\widetilde{\lambda}\right)^{(1-\widetilde{\lambda})}}{\left(1+\widetilde{\lambda}\right)^{(1+\widetilde{\lambda})}} \right]^p (2Ke\kappa)^{2\widetilde{\lambda} p} \\ &\le C p \varepsilon^{-1} e^{-\beta_4 p}, \end{aligned} \tag{3.16}$$

where

$$\beta_4 = |\ln q_4| \ , \ q_4 = \frac{\left(1-\widetilde{\lambda}\right)^{(1-\widetilde{\lambda})}}{\left(1+\widetilde{\lambda}\right)^{(1+\widetilde{\lambda})}} < 1.$$

Now, on the interval $\widetilde{I}_2 = [\kappa p\varepsilon, 1]$, u_i^- is already exponentially small, and by Lemma 2.3

$$\left\| \left(u_i^-\right)' \right\|_{0,\widetilde{I}_2}^2 = \int_{\kappa p\varepsilon}^1 \left| \left(u_i^-\right)' \right|^2 dx \le C\varepsilon^{-2} (1-\kappa p\varepsilon) \max_{x\in\widetilde{I}_2} \left\{ e^{-2x\alpha/\varepsilon} \right\} \le C\varepsilon^{-2} e^{-2\kappa p\alpha}.$$

Thus, we approximate u_i^- by its linear interpolant $\mathcal{I}_1 u_i^-$, and we have

$$\left\| \left(u_i^- - \mathcal{I}_1 u_i^-\right)' \right\|_{0,\widetilde{I}_2}^2 \le \left\| \left(u_i^-\right)' \right\|_{0,\widetilde{I}_2}^2 + \left\| \left(\mathcal{I}_1 u_i^-\right)' \right\|_{0,\widetilde{I}_2}^2 \le C\varepsilon^{-2} e^{-2\kappa p\alpha},$$

which along with (3.16) give

$$\left\| \left(u_i^- - \mathcal{I}_p u_i^-\right)' \right\|_{0,\Omega}^2 \le C p \varepsilon^{-2} e^{-\beta_5 p}, \tag{3.17}$$

for some $\beta_5 > 0$ independent of ε. Repeating the previous arguments for the L^2 norm of $\left(u_i^- - \mathcal{I}_p u_i^-\right)$, we get

$$\left\|u_i^- - \mathcal{I}_p u_i^-\right\|_{0,\Omega}^2 \leq Ce^{-\beta_5 p}. \tag{3.18}$$

Using the same techniques, similar bounds can be obtained for $\overrightarrow{u}^+$.

Combining (3.10), (3.12), (3.17), (3.18) and the analogous bounds for $\overrightarrow{u}^+$, we have

$$\begin{aligned}
&\|\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u}\|_{0,\Omega}^2 \\
=&\left\|\left(\overrightarrow{w} + \overrightarrow{u}^- + \overrightarrow{u}^+\right) - \left(\mathcal{I}_p \overrightarrow{w} + \mathcal{I}_p \overrightarrow{u}^- + \mathcal{I}_p \overrightarrow{u}^+ + \overrightarrow{r}\right)\right\|_{0,\Omega}^2 \\
\leq& \|\overrightarrow{w} - \mathcal{I}_p \overrightarrow{w}\|_{0,\Omega}^2 + \left\|\overrightarrow{u}^- - \mathcal{I}_p \overrightarrow{u}^-\right\|_{0,\Omega}^2 + \left\|\overrightarrow{u}^+ - \mathcal{I}_p \overrightarrow{u}^+\right\|_{0,\Omega}^2 + \|\overrightarrow{r}\|_{0,\Omega}^2 \\
\leq& Cpe^{-\beta p},
\end{aligned}$$

for some $\beta > 0$, independent of ε. Similarly,

$$\begin{aligned}
&|u_i - \mathcal{I}_p u_i|_{1,\Omega}^2 \\
\leq& |w_i - \mathcal{I}_p w_i|_{1,\Omega}^2 + \left|u_i^- - \mathcal{I}_p u_i^-\right|_{1,\Omega}^2 + \left|u_i^+ - \mathcal{I}_p u_i^+\right|_{1,\Omega}^2 + |r_i|_{1,\Omega}^2 \\
\leq& C\varepsilon^{-2} p^3 e^{-\beta p},
\end{aligned}$$

so that

$$\begin{aligned}
&\|\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u}\|_{E,\Omega}^2 \\
=&\varepsilon^2 \sum_{i=1}^m |u_i - \mathcal{I}_p u_i|_{1,\Omega}^2 + \alpha^2 \|\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u}\|_{0,\Omega}^2 \leq Cp^3 e^{-\beta p}
\end{aligned}$$

as desired. □

Remark 3.1. In contrast to the analysis for the scalar problem carried out in [4], our approach allows for the choice of κ in the definition of the mesh, to be made more specific, even when the data of the problem is not constant. As was shown in the proof of the above theorem, κ can be chosen based on the constant of analyticity of the input data.

Using Theorem 3.2 and the quasioptimality result (3.2) we have the following.

Corollary 3.1. *Let $\overrightarrow{u}$ be the solution to* (1.5)–(1.6) *and let $\overrightarrow{u}_{FE} \in \overrightarrow{S}_0^p(\Delta)$ be the solution to* (3.1). *Then exist constants $\kappa, C, \sigma > 0$ depending* only *on the input data $\overrightarrow{f}$ and A such that*

$$\|\overrightarrow{u} - \overrightarrow{u}_{FE}\|_E \leq Cp^{3/2} e^{-\sigma p}.$$

4. Numerical Experiments

In this section we present the results of numerical computations for systems of 2 equations (i.e., $m = 2$), having as our goal the illustration our theoretical findings; we refer the interested reader to [12] for a detailed numerical study in which several other cases are considered.

4.1. The constant coefficient case

First we consider the constant coefficient case, in which

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \overrightarrow{f}(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \overrightarrow{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

An exact solution is available, hence the computations we report are reliable. We will be plotting the percentage relative error in the energy norm, given by

$$100 \times \frac{\|\overrightarrow{u}_{EXACT} - \overrightarrow{u}_{FEM}\|_{E,\Omega}}{\|\overrightarrow{u}_{EXACT}\|_{E,\Omega}}, \tag{4.1}$$

versus the number of degrees of freedom N, on a semilog scale.

To illustrate the necessity to resolve the boundary layers, we first consider the p version FEM on the uniform mesh $\{0, 1/2, 1\}$. Figure 4.1 shows the performance of this method, which for relatively large values of ε (i.e. $\varepsilon \geq 1/p$), exponential rates are achieved. As ε decreases, the convergence rate deteriorates to an algebraic one due to the fact that the layers are not resolved.

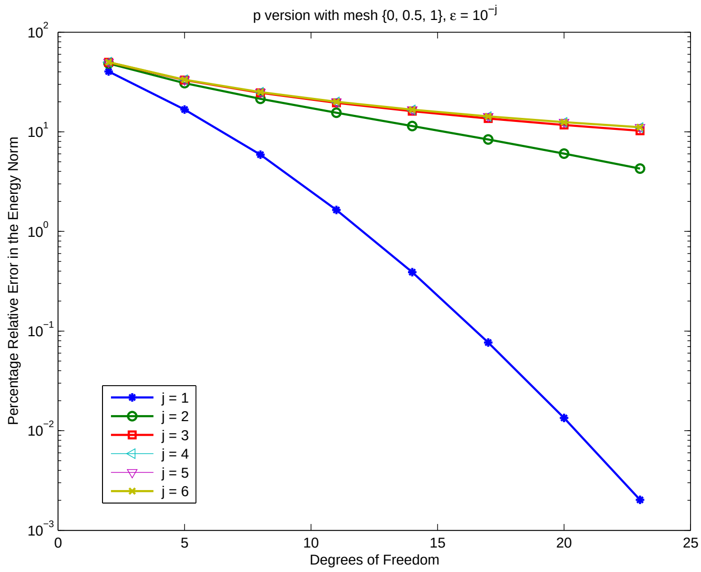

Fig. 4.1. Energy norm convergence for the p version.

Figure 4.2 shows the performance of the p/hp version on the 3 element mesh $\{0, \kappa p\varepsilon, 1 - \kappa p\varepsilon, 1\}$ for different values of ε and we observe that the method not only does not deteriorate as $\varepsilon \to 0$, but it actually performs better, when the error is measured in the energy norm. This suggests that there is a positive power of ε in the error estimate of Corollary 3.1. In fact, for the corresponding scalar problem with constant coefficients and polynomial right hand side, this was shown to be true in [10], where the estimate

$$\|\overrightarrow{u}_{EXACT} - \overrightarrow{u}_{FEM}\|_{E,\Omega} \leq C\varepsilon^{1/2}\beta^{p}, \quad C \in \mathbb{R}^{+}, \ \beta \in (0, 1)$$

was proven. This allows for the derivation of an analogous estimate in the maximum norm which, although will not contain any positive powers of ε, will show that the method converges at an exponential rate independently of ε. (See [13] for a proof of this fact for high order h version FEM on Shishkin meshes.)

To illustrate the above claim, we show a final computational result, in which the error is measured in the maximum norm

$$\|\overrightarrow{u}_{EXACT} - \overrightarrow{u}_{FEM}\|_{\infty,\Omega} = \max_{k=1,2} \left\{ \max_{[0,1]} |\overrightarrow{u}_{EXACT} - \overrightarrow{u}_{FEM}| \right\}.$$

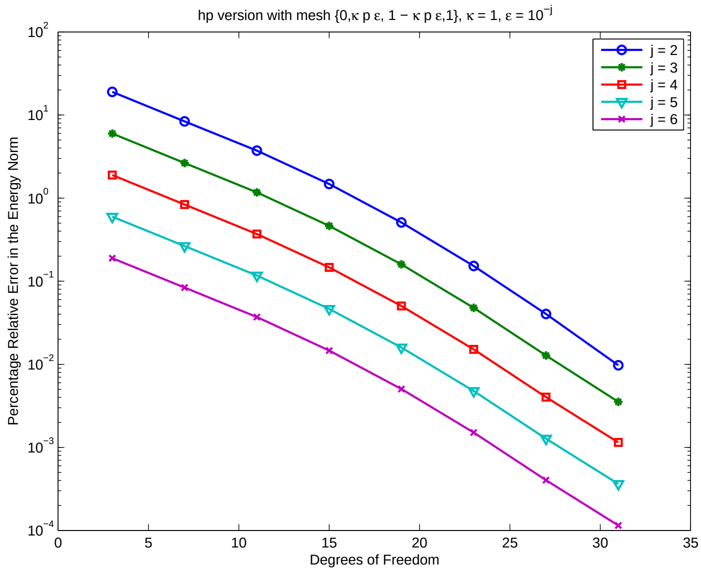

Fig. 4.2. Energy norm convergence for the p/hp version.

As Fig. 4.3 shows, the method is robust and converges at an exponential rate.

4.2. The variable coefficient case

Next, we consider the variable coefficient case, in which

$$A = \begin{bmatrix} 2\,(x+1)^2 & -\left(1+x^2\right) \\ -2\cos(\pi x/4) & 2.2e^{1-x} \end{bmatrix}, \quad \vec{f}\,(x) = \begin{bmatrix} 2e^x \\ 10x+1 \end{bmatrix}, \quad \vec{u}\,(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

An exact solution is not available, and for our computations we use a reference solution obtained with a large number of degrees of freedom on a very fine mesh which includes exponential refinement near the endpoints of the domain (see [12] for more details). We are again interested in the (now estimated) percentage relative error in the energy norm, as given by (4.1).

In Figure 4.4 we show the p/hp version on the 3 element mesh $\{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\}$, for different values of ε, and we observe that the method performs better as $\varepsilon \to 0$, in the variable coefficient case as well. This, does not, strictly speaking, agree with the theory and it could very well be due to the fact that we used a reference solution instead of an exact solution for the computations. Nevertheless, the exponential convergence is visible. Finally, in Figure 4.5 we show the performance of the method when the error is measured in the maximum norm, which again shows its robustness and exponential convergence rate.

5. Conclusions

We have studied the finite element approximation of systems of reaction-diffusion equations whose solution contains boundary layers. We showed that under the assumption of analytic input data, the p/hp version on the variable three element mesh $\{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\}$ yields exponential convergence as $p \to \infty$, independently of ε, when the error is measured in the

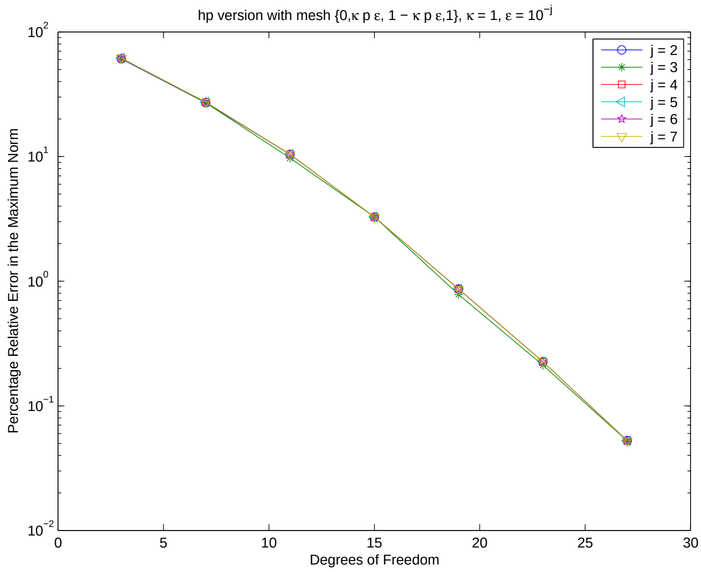

Fig. 4.3. Maximum norm convergence for the p/hp version.

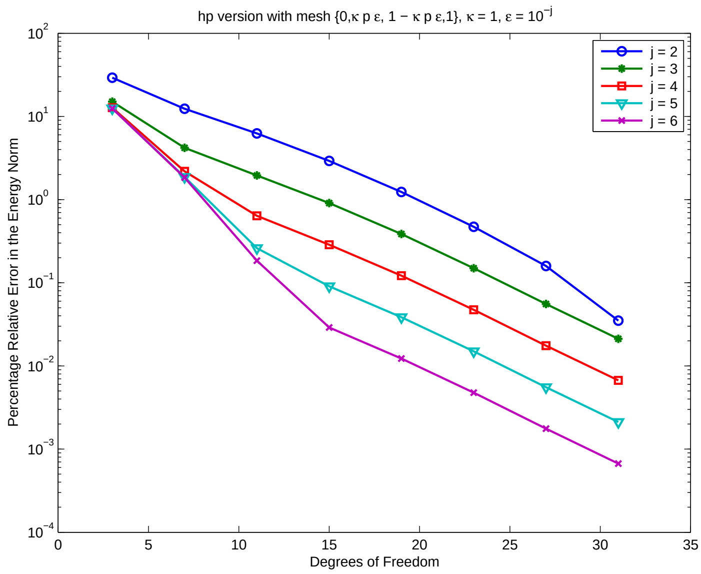

Fig. 4.4. Energy norm convergence for the p/hp version.

energy norm. The constant κ in the mesh was shown to depend on the constant of analyticity of the input data.

Through two numerical experiments, we verified the established exponential rate and observed that, for the problems under consideration, the method performs better as $\varepsilon \to 0$, as was the case for the corresponding scalar problem with constant coefficients and polynomial right hand side studied in [10]. Finally, we illustrated that when the error in the maximum norm is used, the method retains its robustness and exponential convergence rate.

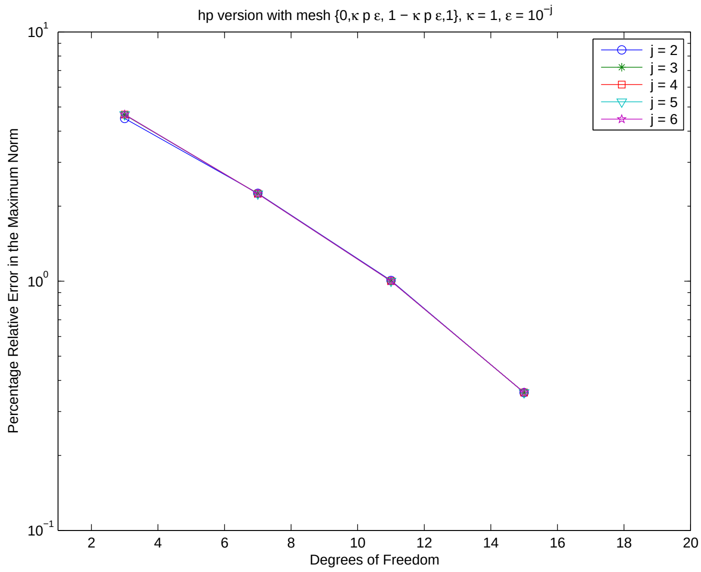

Fig. 4.5. Maximum norm convergence for the p/hp version.

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