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FINITE ELEMENT METHODS FOR A BI-WAVE EQUATION MODELING D-WAVE SUPERCONDUCTORS*

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Abstract

In this paper we develop two conforming finite element methods for a fourth order bi-wave equation arising as a simplified Ginzburg-Landau-type model for *d*-wave superconductors in absence of applied magnetic field. Unlike the biharmonic operator Δ^2, the bi-wave operator $\Box^2$ is not an elliptic operator, so the energy space for the bi-wave equation is much larger than the energy space for the biharmonic equation. This then makes it possible to construct low order conforming finite elements for the bi-wave equation. However, the existence and construction of such finite elements strongly depends on the mesh. In the paper, we first characterize mesh conditions which allow and not allow construction of low order conforming finite elements for approximating the bi-wave equation. We then construct a cubic and a quartic conforming finite element. It is proved that both elements have the desired approximation properties, and give optimal order error estimates in the energy norm, suboptimal (and optimal in some cases) order error estimates in the H^1 and L^2 norm. Finally, numerical experiments are presented to guage the efficiency of the proposed finite element methods and to validate the theoretical error bounds.

Key words: Bi-wave operator, d-wave superconductors, Conforming finite elements, Error estimates.
Mathematics subject classification: 65N30, 65N12, 65N15.

1. Introduction

This paper concerns finite element approximations of the following boundary value problem:

$$\delta\Box^2 u - \Delta u = f \qquad \text{in } \Omega, \tag{1.1}$$

$$u = \partial_{\bar{n}} u = 0 \qquad \text{on } \partial\Omega, \tag{1.2}$$

where $0 < \delta \ll 1$ is a given (small) number,

$$\Box u := \partial_{xx} u - \partial_{yy} u, \qquad \Box^2 u := \Box(\Box u),$$
$$\bar{n} := (n_1, -n_2), \qquad \partial_{\bar{n}} u := \nabla u \cdot \overline{n},$$

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$\Omega \subset \mathbf{R}^2$ is a bounded domain with piecewise smooth boundary $\partial\Omega$, and $n := (n_1, n_2)$ denotes the unit outward normal to $\partial\Omega$. As $\Box$ is the well-known (2-D) wave operator, we shall call $\Box^2$ the bi-wave operator throughout this paper. It is easy to verify that

$$\Box^2 u = \frac{\partial^4 u}{\partial x^4} - 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}.$$

Hence, equation (1.1) is a fourth order PDE, which can be viewed as a singular perturbation of the Poisson equation by the bi-wave operator. As a comparison, we recall that the biharmonic operator Δ^2 is defined as

$$\Delta^2 u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}.$$

Although there is only a sign difference in the mixed derivative term, the difference between Δ^2 and $\Box^2$ is fundamental because Δ^2 is an elliptic operator while $\Box^2$ is a hyperbolic operator.

Superconductors are materials that have no resistance to the flow of electricity when the surrounding temperature is below some critical temperature. At the superconducting state, the electrons are believed to "team up pairwise" despite the fact that the electrons have negative charges and normally repel each other. The Ginzburg-Landau theory [9] has been well accepted as a good mean field theory for low (critical temperature) T_c superconductors [11]. However, a theory to explain high T_c superconductivity still eludes modern physics. In spite of the lack of satisfactory microscopic theories and models, various generalizations of the Ginzburg-Landau-type models to account for high T_c properties such as the anisotropy and the inhomogeneity have been proposed and developed. In low T_c superconductors, electrons are thought to pair in a form in which the electrons travel together in spherical orbits, but in opposite directions. Such a form of pairing is often called s-wave [11]. However, in high T_c superconductors, experiments have produced strong evidence for d-wave pairing symmetry in which the electrons travel together in orbits resembling a four-leaf clover (cf. [4, 6, 10, 12] and the references therein). Recently, the d-wave pairing has gained substantial support over s-wave pairing as the mechanism by which high-temperature superconductivity might be explained. In generalizing the Ginzburg-Landau models to high T_c superconductors, the key idea is to introduce multiple order parameters in the Ginzburg-Landau free energy functional. These models, which can also be derived from the phenomenological Gorkov equations [6], have built a reasonable basis upon which detailed studies of the fine vortex structures in some high T_c materials have become possible. We refer the reader to [4, 6, 10, 12] and the references therein for a detailed exposition on modeling and analysis of d-wave superconductors.

We obtain equation (1.1) from the Ginzburg-Landau-type d-wave model considered in [4] (also see [10, 12]) in absence of applied magnetic field by neglecting the zeroth order nonlinear terms but retaining the leading terms. In the equation, u (notation ψ_d is instead used in the cited references) denotes the d-wave *order parameter*. We note that the original order parameter ψ_d in the Ginzburg-Landau-type model [4, 10] is a complex-valued scalar function whose magnitude represents the density of superconducting charge carriers, however, to reduce the technicalities and to present the ideas, we assume u is a real-valued scalar function in this paper and remark that the finite element methods developed in this paper can be easily extended to the complex case. We also note that the parameter δ appears in the full model as $\delta = -1/\beta$, where β is proportional to the ratio $\ln(T_{s0}/T)/\ln(T_{d0}/T)$ with T_{s0} and T_{d0} being the critical temperatures of the s-wave and d-wave components. Clearly, $\beta < 0$ (or $\delta > 0$) when $T_{s0} < T < T_{d0}$ and $\beta \searrow -\infty$ (or $\delta \searrow 0$) as $T \nearrow T_{d0}$. Hence, δ is expected to be small for d-wave like superconductors.

The primary goal of this paper is to develop conforming finite element methods for the reduced d-wave model (1.1). Since the bi-wave term is the leading term in the full d-wave model, see [4, Section 4], any good numerical method for (1.1) should be applicable to the full d-wave model. It is easy to see that the energy space for the bi-wave equation (1.1) is $V := \{v \in H^1(\Omega); \Box v \in L^2(\Omega)\}$ (see Section 2). Our main task then is to construct finite element subspaces V^h of the energy space V which should be as simple as possible but also rich enough to have good approximation properties. To this end, we note that $H^2(\Omega) \subset V \subset H^1(\Omega)$, and hence, the desired finite element space V^h should satisfy $V^h \subset V \subset H^1(\Omega)$. This immediately implies that $V^h \subset C^0(\overline{\Omega})$ (see [2, 3]). On the other hand, since V is a proper subspace of $H^1(\Omega)$, the condition $V^h \subset C^0(\overline{\Omega})$ does not guarantee that $V^h \subset V$. Hence, C^0 (Lagrange) finite element spaces are in general not subspaces of V. An intriguing question is what extra conditions are required to make a C^0 finite element space to be a subspace of V. To answer this question, on noting that $H^2(\Omega) \subset V$, one may choose V^h such that $V^h \subset H^2(\Omega)$, that is, V^h is a C^1 finite element space such as Argyris finite element space (cf. [3, Chapter 6]). Trivially, $V^h \subset H^2(\Omega) \subset V$. It turns out (see Section 4) such a choice would work since it can be shown that the finite element solution so defined converges with optimal rate in the energy norm of V. However, since C^1 finite elements require either the use of fifth or higher order polynomials with up to second order derivatives as degrees of freedom [13, 14], or the use of exotic elements [3, Chapter 6], it is expensive and less efficient to solve the bi-wave equation (1.1) using C^1 finite elements. This then motivates us to construct low order non-C^1 finite elements which give genuine subspaces of V and to develop other types of finite element methods such as nonconforming and discontinuous Galerkin methods [7].

The remainder of the paper is organized as follows. Section 2 contains some preliminaries and the functional setting for the bi-wave problem. Well-posedness of the problem and regularity estimates of the weak solution are established. Because $\Box^2$ is a hyperbolic operator, the usual regularity shift for fourth order elliptic problems does not hold for the bi-wave problem, instead, a weaker shifting "rule" only holds. Section 3 devotes to construction and analysis of piecewise polynomial subspaces of V. First, we give a characterization of such subspaces. It is proved that a subspace of V is "necessarily" a C^1 finite element space on a general mesh. However, non-C^1 finite elements are possible on restricted meshes. Second, we construct two such finite elements. The first one is a cubic element and the second is a quartic element. Third, we establish the approximation properties for both proposed finite elements. Because both elements are not affine families, a technique of using affine relatives (cf. [2, 3]) is used to carry out the analysis. Finally, optimal order error estimates in the energy norm of V are proved for the finite element approximations of problem (1.1)–(1.2) using the proposed finite elements. Suboptimal (and optimal in some cases) order error estimates in the L^2-norm are also derived using a duality argument. In Section 4 we present some numerical experiment results to gauge the efficiency of the proposed finite element methods and also to validate our theoretical error bounds.

2. Preliminaries and Functional Setting

Standard space notation is adopted in this paper. We refer the reader to [2,3] for their exact definitions. In addition, $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ are used to denote the L^2-inner products on Ω and on $\partial\Omega$, respectively. C denotes a generic h and δ-independent positive constant. We also introduce the following special space notation:

$$V_0 := \{v \in V \cap H_0^1(\Omega);\ \partial_{\bar{n}} v\big|_{\partial\Omega} = 0\}, \qquad (v,w)_V := \delta(\Box v, \Box w) + (\nabla v, \nabla w),$$
$$\|v\|_V := \sqrt{(v,v)_V}.$$

It is easy to verify that $(\cdot,\cdot)_V$ is an inner product on V, hence, $\|\cdot\|_V$ is the induced norm, and V endowed with this inner product is a Hilbert space. We remark that all above claims do not hold in general if the harmonic term Δu is dropped in (1.1) because the kernels of the bi-wave operator $\Box^2$ and the wave operator $\Box$ may contain non-zero functions satisfying the homogeneous Dirichlet boundary condition [1].

The variational formulation of (1.1)–(1.2) can be derived easily by testing (1.1) against a test function $v \in V_0$ and using integration by parts formulas. Specifically, it is defined as seeking $u \in V_0$ such that

$$A^\delta(u,v) = \langle f, v\rangle, \tag{2.1}$$

where

$$A^\delta(u,v) := (u,v)_V,$$

and $\langle\cdot,\cdot\rangle$ denotes the pairing between V and its dual, V^*.

We now show that problem (2.1) is well-posed.

Theorem 2.1. *For any $f \in V^*$, there exists a unique solution to* (2.1)*. Furthermore, there holds estimate*

$$\|u\|_V \le \|f\|_{V^*}. \tag{2.2}$$

Proof. We note for $v, w \in V_0$,

$$A^\delta(v,v) \ge \|v\|_V^2, \tag{2.3}$$
$$\big|A^\delta(v,w)\big| \le \|v\|_V \|w\|_V. \tag{2.4}$$

Then, existence and uniqueness follows directly from an application of the Lax-Milgram Theorem (cf. [2,3]) and using the fact that V_0 is a Hilbert space with the inner product $(\cdot,\cdot)_V$. The estimate (2.2) follows from (2.3) and (2.1) after setting $v = u$ and $w = u$. $\Box$

We note $H^2(\Omega)$ is a proper subspace of V, so in general $u \notin H^2(\Omega)$ if $f \in V^*$. However, for smoother function f we have the following regularity results.

Theorem 2.2. *Assume that the boundary $\partial\Omega$ of the domain Ω is sufficiently smooth. Let s_1, s_2 be two nonnegative integers. Then there exist constants $C_{s_1,s_2}, \hat{C}_{s_1,s_2} > 0$ such that the weak solution u of* (2.1) *satisfies*

$$\|\partial_x^{s_1}\partial_y^{s_2} u\|_V \le C_{s_1,s_2}\|\partial_x^{s_1}\partial_y^{s_2} f\|_{V^*} \qquad \text{if } \partial_x^{s_1}\partial_y^{s_2} f \in V^*, \tag{2.5}$$
$$\sqrt{\delta}\|\Box^2\partial_x^{s_1}\partial_y^{s_2} u\|_{L^2} + \sqrt{\delta}\|\nabla\Box\partial_x^{s_1}\partial_y^{s_2} u\|_{L^2} + \|\Delta\partial_x^{s_1}\partial_y^{s_2} u\|_{L^2} \le \hat{C}_{s_1,s_2}\|\partial_x^{s_1}\partial_y^{s_2} f\|_{L^2} \qquad \text{if } \partial_x^{s_1}\partial_y^{s_2} f \in L^2(\Omega). \tag{2.6}$$

Proof. First, we consider the case that u and f have compact support. Let $w := \partial_x^{s_1}\partial_y^{s_2}u$ and $g := \partial_x^{s_1}\partial_y^{s_2}f$. Because equation (1.1) is a linear equation, differentiating the equation immediately verifies that w and g satisfy

$$\delta\Box^2 w - \Delta w = g, \tag{2.7}$$

that is, w is a solution of the bi-wave equation with the source term g. Since u is assumed to have a compact support, then w also satisfies the homogeneous boundary conditions in (1.2). Thus, it follows from Theorem 2.1 that

$$\|w\|_V \le \|g\|_{V^*},$$

which gives (2.5) with $C_{s_1,s_2} = 1$.

To show (2.6), it suffices to prove that

$$\sqrt{\delta}\|\nabla\Box w\|_{L^2} + \|\Delta w\|_{L^2} \le \hat{C}_{s_1,s_2}\|g\|_{L^2}, \tag{2.8}$$

which is equivalent to prove that (2.6) holds for $s_1 = s_2 = 0$. To this end, testing (2.7) with $-\Delta w$ yields

$$-\delta(\Box^2 w, \Delta w) + \|\Delta w\|_{L^2}^2 = -(g, \Delta w).$$

Using the following integral identity

$$(\Box\varphi, \psi) = \langle\partial_{\bar{n}}\Box\varphi, \psi\rangle - \langle\Box\varphi, \partial_n\psi\rangle + (\Box\varphi, \Box\psi)$$

followed by using Green's identity (for Δ) in the first term on the left hand side we get

$$-\delta(\Box^2 w, \Delta w) = \delta\|\nabla\Box w\|_{L^2}^2.$$

Here, we have dropped the boundary integral terms because w has a compact support.

Combining the above two identities for w and using Schwarz inequality yield

$$\delta\|\nabla\Box w\|_{L^2}^2 + \|\Delta w\|_{L^2}^2 \le \frac{1}{2}\|g\|_{L^2}^2 + \frac{1}{2}\|\Delta w\|_{L^2}^2.$$

Hence, the above inequality and (2.7) imply that (2.8) holds with $\hat{C}_{s_1,s_2} = 2\sqrt{2}+1$.

Second, in the case u and f do not have compact support, it is clear that w and g still satisfy (2.7). However, w and its derivatives may not satisfy the homogeneous boundary conditions in (1.2). To get around this difficulty, the well-known tricks are to use the cutoff function technique (see [5, 8]) for interior estimates and to use the flattening boundary technique for boundary estimates. The cutoff function technique involves testing (2.7) by $w\xi$ and $-\Delta w\xi$, instead of w and $-\Delta w$, for a smooth cutoff function ξ. Integrating by parts on the left hand side and using Schwarz inequality and the properties of the cutoff function then yield the desired interior estimate similar to (2.5) and (2.6). The flattening boundary technique involves locally mapping the curved boundary into a flat boundary by a smooth map (this requires the smoothness of the boundary $\partial\Omega$). After the desired boundary estimates are obtained in the new coordinates, they are then transferred to the solution w in the original coordinates. We omit the technical derivations and refer the interested reader to [5, 8] for a detailed exposition of these techniques applying to other linear PDEs. $\Box$

3. Construction and Analysis of Finite Element Methods

3.1. Characterization of finite element subspaces of V

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω with mesh size $h \in (0,1)$, and for a fixed $T \in \mathcal{T}_h$, let $(\lambda_1^T, \lambda_2^T, \lambda_3^T)$ denote the barycentric coordinates, and a_i $(1 \le i \le 3)$ denote the vertices of T. We also let e_i $(1 \le i \le 3)$ denote the edge of T of which a_i is not a vertex, and b_i denote the midpoint of edge e_i. Define the interior and boundary edge sets of $\mathcal{T}_h$

$$\mathcal{E}_h^I := \{e;\ e \cap \partial\Omega = \emptyset\}, \qquad \mathcal{E}_h^B := \{e;\ e \cap \partial\Omega \neq \emptyset\}.$$

We also set

$$\mathcal{E}_h := \mathcal{E}_h^I \cup \mathcal{E}_h^B,$$

and for $T \in \mathcal{T}_h$,

$$\omega(T) := \text{closure}\left(\bigcup_{\partial T' \cap \partial T \neq \emptyset} T'\right).$$

For any $e \in \mathcal{E}_h^I$ such that $e = T_1 \cap T_2$, and $v \in H^1(T_1) \cap H^1(T_2)$, define the jumps of v across e as (assuming the global label of T_1 is bigger than that of T_2)

$$[v]\big|_e := v^{T_1}\big|_e - v^{T_2}\big|_e,$$

where $v^{T_i} = v\big|_{T_i}$, and $[v]\big|_e := v^{T_1}\big|_e$ if $e \in \mathcal{E}_h^B$.

Similarly, for $v \in H^2(T_1) \cap H^2(T_2)$, $\alpha \in \mathbf{R}^2$, we define the jumps of $\partial_\alpha v := \nabla v \cdot \alpha$ as follows:

$$\begin{aligned} [\partial_\alpha v]\big|_e &:= \partial_\alpha v^{T_1}\big|_e - \partial_\alpha v^{T_2}\big|_e \qquad && e = \partial T_1 \cap \partial T_2 \in \mathcal{E}_h^I, \\ [\partial_\alpha v]\big|_e &:= \partial_\alpha v^{T_1}\big|_e && e = \partial T_1 \cap \partial\Omega \in \mathcal{E}_h^B. \end{aligned}$$

We also define the shorthand notation

$$\overline{\nabla} v := (v_x, -v_y), \qquad \overline{|\nabla v|} := \nabla v \cdot \overline{\nabla} v.$$

In the rest of the paper, we shall often encounter the following characterization of the meshes.

Definition 3.1. *For $e \in \mathcal{E}_h$, let n and τ denote the outward unit normal and unit tangent vector of e, respectively. We say that e is a* type I *edge if*

$$\overline{n} = \tau \quad \text{or} \quad \overline{n} = -\tau. \tag{3.1}$$

Otherwise, e is called a type II *edge if condition* (3.1) *does not hold.*

Remark 3.1. (a) If e is a type I edge, then $\overline{n} = (n_1, -n_2) = \pm\tau = \pm(\tau_1, \tau_2) = \pm(n_2, -n_1)$. Therefore,

$$\tau = \frac{\sqrt{2}}{2}(\pm 1, \pm 1).$$

That is, the edge e makes an angle of $\frac{\pi}{4}$ in the plane with respect to the x-axis. Examples of meshes such that every triangle in the partition has exactly zero and one type I edges are shown

in Figure 3.1, and examples of meshes such that every triangle has exactly two type I edges are shown in Figure 3.2.
(b) For $T \in \mathcal{T}_h$, $e_i \subset \partial T$, let $n^{(i)}$ and $\tau^{(i)}$ denote the outward (from T) unit normal and unit tangent vector of e_i, respectively. Then using the formula

$$n^{(i)} = -\frac{\nabla \lambda_i^T}{\|\nabla \lambda_i^T\|},$$

we conclude that e_i is a type I edge if and only if

$$\nabla \lambda_i^T \cdot \overline{\nabla} \lambda_i^T = 0.$$

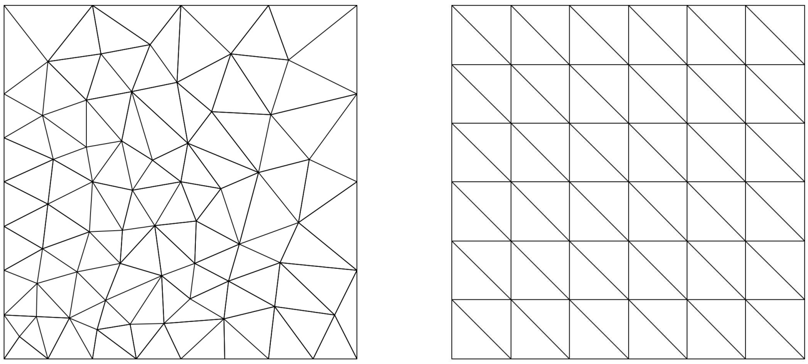

Fig. 3.1. Example of meshes of the domain $\Omega = (0,1)^2$ such that every triangle has no type I edges (left), and one type I edge (right).

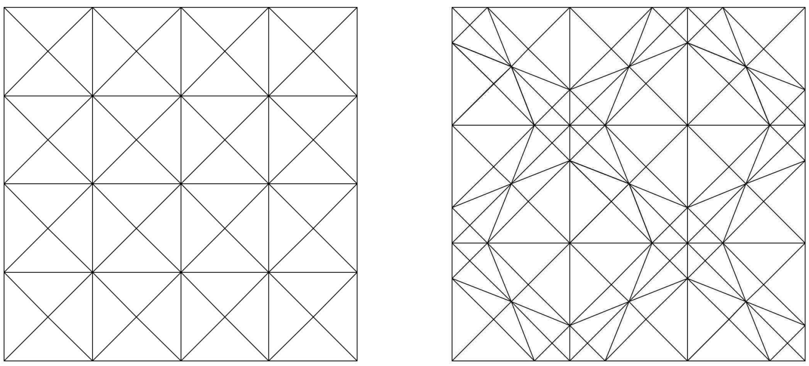

Fig. 3.2. Example of a uniform mesh (left) and a nonuniform mesh (right) of the domain $\Omega = (0,1)^2$ such that every triangle has two type I edges.

To construct finite element subspaces of V, we first provide the following two lemmas, which characterize such spaces.

Lemma 3.1. *Let X^h be a subspace of V consisting of piecewise polynomials, and suppose there exists a type II edge $e \in \mathcal{E}_h^I$ with $e = \partial T_1 \cap \partial T_2$. Then for $v \in X^h$, there holds the inclusion $v \in H^2(T_1 \cup T_2)$.*

Proof. Since X^h is finite-dimensional with $X^h \subset H^1(\Omega)$, we have the inclusion $X^h \subset C^0(\overline{\Omega})$. We also note that it suffices to show $v \in C^1(\overline{T}_1 \cup \overline{T}_2)$ for any $v \in X_h$, which in turn is equivalent to show

$$\left[\partial_\alpha v\right]\big|_e = 0 \qquad \forall \alpha \in \mathbf{R}^2.$$

Let n and τ denote the normal and tangential direction of e, respectively. Rewriting V as

$$V = \{v \in H^1(\Omega);\ \overline{\nabla} v \in H(\mathrm{div};\Omega)\},$$

there holds for $v \in X^h$

$$\left[\partial_{\bar{n}} v\right]\big|_e = 0.$$

Next, using the assumption $\overline{n} \neq \pm\tau$, we can write for any constant vector $\alpha \in \mathbf{R}^2$

$$\left[\partial_\alpha v\right]\big|_e = \frac{1}{1-(\tau\cdot\overline{n})^2}\Big\{\alpha\cdot\big(\tau - \overline{n}(\tau\cdot\overline{n})\big)\left[\partial_\tau v\right]\Big|_e + \alpha\cdot\big(\overline{n} - \tau(\tau\cdot\overline{n})\big)\left[\partial_{\bar{n}} v\right]\Big|_e\Big\}.$$

But $\left[\partial_\tau v\right]\big|_e = 0$ since $v \in C^0(\overline{\Omega})$ and $v\big|_e$ is a polynomial of one variable. Hence, $\left[\partial_{\bar{n}} v\right]\big|_e = 0$ implies that $\left[\partial_\alpha v\right]\big|_e = 0$. The proof is complete. $\square$

Corollary 3.2. *Suppose X^h is a subspace of V consisting of piecewise polynomials, and suppose there exists no type I edges in the set $\mathcal{E}_h^I$. Then $X^h \subset H^2(\Omega)$.*

Lemma 3.3. *Suppose Σ_T is a linearly independent set of parameters uniquely determining a kth-degree polynomial v on an interior triangle $T \in \mathcal{T}_h$ that includes only function and derivative degrees of freedom. Suppose further that v is continuous in $\omega(T)$, $\Box v \in L^2(\omega(T))$, and T has at least two type II edges that are in the set $\mathcal{E}_h^I$. Then $k \geq 5$.*

Proof. If T has three type II edges, then by Lemma 3.1, $v \in H^2(\omega(T))$, and it follows that $k \geq 5$ (cf. [3, p.108], also see [13, 14]).

Suppose T has exactly two type II edges, without loss of generality, assume e_1 is type I. By the proof of Lemma 3.1, v is C^1 across edges e_2 and e_3. Let μ_i denote the order of prescribed derivatives at vertex a_i in the set Σ_T, let m_i denote the number of function value (or equivalent) degrees of freedom in the set Σ_T on edge e_i, and let s_i denote the number of (non-tangential) directional derivative value (or equivalent) degrees of freedom in the set Σ_T on edge e_i. Since v is continuous in $\omega(T)$, we have

$$\begin{aligned} \mu_2 + \mu_3 + m_1 &\geq k-1, \\ \mu_1 + \mu_3 + m_2 &\geq k-1, \\ \mu_1 + \mu_2 + m_3 &\geq k-1, \end{aligned} \tag{3.2}$$

and since ∇v is continuous across e_2 and e_3,

$$\begin{aligned} \mu_1 + \mu_2 + s_3 &\geq k, \\ \mu_1 + \mu_3 + s_2 &\geq k. \end{aligned} \tag{3.3}$$

Adding up the above five inequalities yields

$$4\mu_1 + 3\mu_2 + 3\mu_3 + m_1 + m_2 + m_3 + s_1 + s_2 \geq 5k - 3.$$

Because the set Σ_T is linearly independent, and the dimension of Σ_T equals $(k+1)(k+2)/2$, there holds

$$\begin{aligned}\frac{(k+1)(k+2)}{2} &\geq \sum_{i=1}^{3}\left\{\frac{1}{2}(\mu_i+1)(\mu_i+2)+m_i\right\}+s_2+s_3 \\ &\geq \sum_{i=1}^{3}\frac{1}{2}(\mu_i+1)(\mu_i+2)+5k-3-4\mu_1-3\mu_2-3\mu_3.\end{aligned} \tag{3.4}$$

Thus,

$$(k^2-7k+8) \geq (\mu_1^2-5\mu_1+2)+(\mu_2-2)(\mu_2-1)+(\mu_3-2)(\mu_3-1). \tag{3.5}$$

It is clear that k must be greater than two, therefore, it suffices to show that k cannot equal three or four.

Case $k=3$*:* If $k=3$, by (3.5) we get

$$(\mu_1-3)(\mu_1-2)+(\mu_2-2)(\mu_2-1)+(\mu_3-2)(\mu_3-1)\leq 0,$$

and since μ_i are integer-valued, we have

$$1\leq\mu_3\leq 2,\quad 1\leq\mu_2\leq 2,\quad 2\leq\mu_1\leq 3.$$

But by (3.4), we immediately obtain

$$10=\frac{(k+1)(k+2)}{2}\geq\sum_{i=1}^{3}\frac{1}{2}(\mu_i+1)(\mu_i+2)\geq 12,$$

which is a contradiction.

Case $k=4$*:* As in the previous case, if $k=4$ we have

$$(\mu_1-3)(\mu_1-2)+(\mu_2-2)(\mu_2-1)+(\mu_3-2)(\mu_3-1)\leq 0.$$

Since

$$1\leq\mu_3\leq 2,\quad 1\leq\mu_2\leq 2,\quad 2\leq\mu_1\leq 3,$$

and

$$15=\frac{(k+1)(k+2)}{2}\geq\sum_{i=1}^{3}\frac{1}{2}(\mu_i+1)(\mu_i+2),$$

it is not hard to check that there can only be the following three subcases:

$$(\mu_1,\mu_2,\mu_3)=(2,1,2),\quad (\mu_1,\mu_2,\mu_3)=(2,2,1),\quad (\mu_1,\mu_2,\mu_3)=(2,1,1). \tag{3.6}$$

If the first subcase holds, then all degrees of freedom lie on the vertices, therefore, $m_i, s_i=0,\ 1\leq i\leq 3$. However, it follows from (3.3) that

$$3=\mu_1+\mu_2\geq 4,$$

which is a contradiction.

A similar argument can be used to exclude the second subcase in (3.6). Now, suppose $\mu_1 = 2,\ \mu_2 = 1$, and $\mu_3 = 1$. By (3.2) and (3.3), we have

$$m_3 \geq 1, \qquad s_3 \geq 2, \qquad s_2 \geq 1.$$

But this implies that

$$15 \geq \sum_{i=1}^{3} \big\{ \frac{1}{2}(\mu_i + 1)(\mu_i + 2) + m_i \big\} + s_2 + s_3 \geq 16,$$

a contradiction. Thus, the third subcase can not happen, either. Therefore, we must have $k \geq 5$. The proof is complete. □

By Lemmas 3.1 and 3.3, and Corollary 3.2, we conclude that unless certain types of meshes are used, we must resort to either C^1 finite elements such as Argyris, Hsieh-Clough-Tocher, Bogner-Fox-Schmit elements (cf. [2, 3]), or special exotic elements (e.g. macro elements), or nonconforming elements (cf. [7]) to solve problem (1.1)–(1.2), However for special meshes, we now show in the following subsections that it is feasible to construct low order finite element subspaces of V.

3.2. A cubic conforming finite element

To construct a cubic conforming finite element, we assume that $\mathcal{T}_h$ is a triangulation of Ω and every triangle of $\mathcal{T}_h$ has *two* type I edges. Examples of such meshes are shown on a square domain in Figure 3.2. Our cubic finite element $S_3^h := (T, P_T, \Sigma_T)$ is defined as follows:

(i) T is a triangle with two type I edges,

(ii) $P_T = \mathbb{P}_3(T)$, the space of cubic polynomials on T,

(iii) $$\Sigma_T = \begin{cases} v(a_i) & 1 \leq i \leq 3, \\ v(b_i) & 1 \leq i \leq 2, \\ \nabla v(a_i) \cdot (a_j - a_i) & 1 \leq i \leq 2,\ 1 \leq j \leq 3,\ j \neq i, \\ \partial_{\bar{n}} v(b_3), & \end{cases}$$

where e_3 is a type II edge.

Lemma 3.4. *The set Σ_T is unisolvent. That is, any polynomial of degree three is uniquely determined by the degrees of freedom in Σ_T.*

Proof. Suppose $v \in \mathbb{P}_3(T)$ equals zero at all the degrees of freedom in Σ_T. To complete the proof, it suffices to show $v \equiv 0$ since $\dim(\mathbb{P}_3(T)) = \dim(\Sigma_T) = 10$.

Recall that e_3 is a type II edge, e_1 and e_2 are type I edges of T. Let w_i be the restriction of v on $e_i \subset \partial T$ as a function of a single variable, then w_i is a polynomial of degree three which satisfies

$$w_i'(0) = w_i(0) = w_i(\frac{1}{2}) = w_i(1) = 0 \qquad i = 1, 2,$$
$$w_3'(0) = w_3(0) = w_3(1) = w_3'(1) = 0.$$

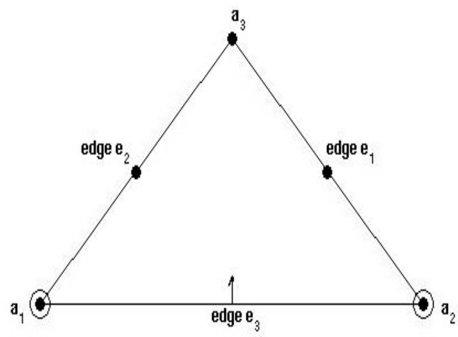

Fig. 3.3. Element S_3^h. Solid dots indicate function evaluation, circles indicate first derivative evaluation, and arrows indicate evaluation of derivatives in the direction $\overline{n}$.

In either case, we conclude $w_i \equiv 0$.

Next, let z_3 be the restriction of $\partial_{\bar{n}} v$ on e_3 as a function of a single variable. Then z_3 is a polynomial of degree two satisfying

$$z_3(0) = z_3(\frac{1}{2}) = z_3(1) = 0,$$

which then infers $z_3 \equiv 0$. From the above calculations, we conclude that $(\lambda_3^T)^2$, λ_1^T, and λ_2^T are factors of v. However, this is not possible (as v is a polynomial of degree three) unless $v \equiv 0$. The proof is complete. $\square$

Let V_3^h be the finite element space associated with S_3^h, that is,

$$V_3^h = \left\{ v|_T \in \mathbb{P}_3(T),\ v \text{ is continuous at every degree of freedom in } \Sigma_T,\ \forall T \in \mathcal{T}_h \right\}.$$

We now show that V_3^h is a subspace of V.

Theorem 3.5. *There holds the inclusion $V_3^h \subset V$.*

Proof. Let $v \in V_3^h$. By the proof of Lemma 3.1, it suffices to show v and $\partial_{\bar{n}} v$ are both continuous across interior edges of $\mathcal{T}_h$. Let T_1 and T_2 be two adjacent triangles with common edge e, and w be the restriction of $v^{T_1} - v^{T_2}$ along e as a function of a single variable. We then have

$$\begin{aligned} &w'(0) = w_1(0) = w(\frac{1}{2}) = w(1) = 0 &&\text{if } e \text{ is type I},\\ &w'(0) = w(0) = w(1) = w'(1) = 0 &&\text{if } e \text{ is type II}. \end{aligned}$$

Thus, $w \equiv 0$ and the inclusion $V_3^h \subset C^0(\Omega) \subset H^1(\Omega)$ holds.

Next, we observe that if e is a type I edge, then

$$\big[\partial_{\bar{n}} v\big]\big|_e = \pm\big[\partial_\tau v\big]\big|_e = 0.$$

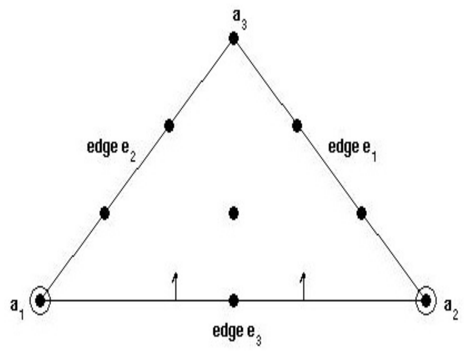

Fig. 3.4. Element S_4^h. Solid dots indicate function evaluation, circles indicate first derivative evaluation, and arrows indicate evaluation of derivatives in the direction $\bar{n}$.

Hence, $\partial_{\bar{n}} v$ is continuous across e. On the other hand, if e is a type II edge, let z be the restriction of $[\partial_{\bar{n}} v]|_e = \partial_{\bar{n}} v^{T_1} - \partial_{\bar{n}} v^{T_2}$ along e as a function of a single variable. Since

$$z(0) = z(\frac{1}{2}) = z(1) = 0,$$

and z is a polynomial of degree two, it follow that $\left[\partial_{\bar{n}} v\right]\big|_e = 0$. So $\partial_{\bar{n}} v$ is also continuous across e. This then concludes the proof. □

Remark 3.2. We note that $V_3^h \not\subset H^2(\Omega)$ because $V_3^h \not\subset C^1(\Omega)$.

3.3. A quartic conforming finite element

In this subsection, we again assume that $\mathcal{T}_h$ is a triangulation of Ω and every triangle of $\mathcal{T}_h$ has *two* type I edges. We then define the following quartic finite element $S_4^h := (T, Q_T, \Xi_T)$:

(i) T is a triangle with two type I edges,

(ii) $Q_T = \mathbb{P}_4(T)$, the space of quartic polynomials on T,

(iii) $$\Xi_T = \begin{cases} v(a_i) & 1 \le i \le 3, \\ v(a_{ii3}),\ v(a_{i33}) & 1 \le i \le 2, \\ v(b_3), & \\ v(a_{123}), & \\ \nabla v(a_i)(a_j - a_i) & 1 \le i \le 2,\ 1 \le j \le 3,\ j \ne i, \\ \partial_{\bar{n}} v(a_{112}),\ \partial_{\bar{n}} v(a_{122}), & \end{cases}$$

where e_3 is a type II edge, and $a_{ij\ell} = \frac{1}{3}\big(a_i + a_j + a_\ell\big)$.

Lemma 3.6. *The set Ξ_T is unisolvent. That is, any polynomial of degree four is uniquely determined by the degrees of freedom in Ξ_T.*

Proof. Suppose $v \in \mathbb{P}_4(T)$ equals zero at all the degrees of freedom in Ξ_T, and let w_i be the restriction of v to e_i as a function of a single variable. Then

$$w_i'(0) = w_i(0) = w_i(\frac{1}{3}) = w_i(\frac{2}{3}) = w_i(1) = 0 \qquad i = 1, 2,$$
$$w_3'(0) = w_3(0) = w_3(\frac{1}{2}) = w_3(1) = w_3'(1) = 0.$$

Thus, $w_i \equiv 0, i = 1, 2, 3$.

Next, letting z_3 be the restriction of $\partial_{\bar{n}} v$ on e_3 as a function of a single variable, we have

$$z_3(0) = z_3(\frac{1}{3}) = z_3(\frac{2}{3}) = z_3(1) = 0.$$

Hence, $z_3 \equiv 0$. From the above calculations, we conclude that $v = a\lambda_1^T \lambda_2^T (\lambda_3^T)^2$ for some $a \in \mathbf{R}$. However, since $0 = v(a_{123}) = a/81$, we have $a = 0$. The proof is complete. $\square$

Theorem 3.7. *Let V_4^h be the finite element space associated with S_4^h, that is,*

$$V_4^h = \Big\{ v|_T \in \mathbb{P}_4(T),\ v \text{ is continuous at every degree of freedom in } \Xi_T,\ \forall T \in \mathcal{T}_h \Big\}.$$

Then there holds the inclusion $V_4^h \subset V$.

Proof. Let $v \in V_4^h$, and suppose $T^1, T^2 \in \mathcal{T}_h$ are two adjacent triangles with common edge e. Let w be the restriction of $[v]|_e = v^{T_1} - v^{T_2}$ along e as a function of a single variable, from

$$w'(0) = w(0) = w(\frac{1}{3}) = w(\frac{2}{3}) = w(1) = 0 \qquad \text{if } e \text{ is type I},$$
$$w'(0) = w(0) = w(\frac{1}{2}) = w(1) = w'(1) = 0 \qquad \text{if } e \text{ is type II},$$

we conclude $w \equiv 0$. Hence, the inclusion $V_4^h \subset C^0(\Omega) \subset H^1(\Omega)$ holds.

If e is a type II edge, we let z denote the restriction of $[\partial_{\bar{n}} v]|_e = \partial_{\bar{n}} v^{T_1} - \partial_{\bar{n}} v^{T_2}$ along e as a function of one variable. It follows from

$$z(0) = z(\frac{1}{3}) = z(\frac{2}{3}) = z(1) = 0,$$

that $z \equiv 0$. Finally, if e is a type I edge, we use the fact that v is continuous to conclude

$$\big[\partial_{\bar{n}} v\big]\big|_e = \pm\big[\partial_\tau v\big]\big|_e = 0.$$

Thus, $V_4^h \subset V$. $\square$

Remark 3.3. We note that $V_4^h \not\subset H^2(\Omega)$ because $V_4^h \not\subset C^1(\Omega)$.

3.4. Approximation properties of the proposed finite elements

Let $\Pi_k^T v \in \mathbb{P}_k(T)$ denote the standard interpolation of v associated with the finite element S_k^h, and define $\Pi_k^h v \in V_k^h$ such that $\Pi_k^h v\big|_T = \Pi_k^T(v\big|_T)$, $\forall T \in \mathcal{T}_h$. Before stating the approximation properties of the interpolation operator Π_k^T, we first establish the following technical lemma concerning the mesh $\mathcal{T}_h$.

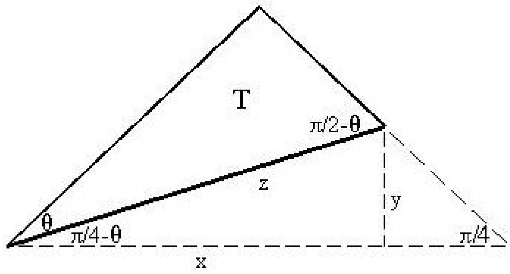

Fig. 3.5. Embedding T into an isosceles triangle

Lemma 3.8. *Suppose $T \in \mathcal{T}_h$ has two type I edges, and without loss of generality, assume $e_3 \subset \partial T$ is a type II edge. Then there exists a constant $C > 0$ that depends only on the minimum angle of T such that*

$$1 - \left(\beta^{(3)}\right)^2 \geq C,$$

where $\beta^{(3)} = \tau^{(3)} \cdot \overline{n}^{(3)}$.

Proof. Since both type I edges of T make an angle of $\frac{\pi}{4}$ with respect to the x-axis (cf. Remark 3.1), there exists $\theta \in (0, \frac{\pi}{4}]$ such that the angles of T are $\frac{\pi}{2}$, θ, and $\frac{\pi}{2} - \theta$.

Next, we embed T into an isosceles triangle as shown in Figure 3.5,
and then obtain

$$\tau^{(3)} = \frac{(x, y)}{z}, \quad \tau^{(3)} \cdot \overline{n}^{(3)} = \frac{-2xy}{z^2}, \quad x = \cos\left(\frac{\pi}{4} - \theta\right) z, \quad y = \sin\left(\frac{\pi}{4} - \theta\right) z.$$

Hence,

$$\tau^{(3)} \cdot \overline{n}^{(3)} = \frac{-2xy}{z^2} = -2\sin\left(\frac{\pi}{4} - \theta\right)\cos\left(\frac{\pi}{4} - \theta\right) = -\cos(2\theta),$$

which implies that

$$1 - (\beta^{(3)})^2 = 1 - (\tau^{(3)} \cdot \overline{n}^{(3)})^2 = 1 - \cos^2(2\theta) = \sin^2(2\theta).$$

The proof is complete. $\square$

Remark 3.4. If $\mathcal{T}_h$ is a uniform criss-cross triangulation of Ω, then $1 - (\beta^{(3)})^2 = 1$ for all type II edges, e_3.

The next theorem establishes the approximation properties of the proposed cubic and quartic finite elements.

Theorem 3.9. *For all $m \geq 0$, $p, q \in [1, \infty]$ which are compatible with the inclusion*

$$W^{k+1,p}(T) \hookrightarrow W^{m,q}(T),$$

there holds

$$\|v - \Pi_k^T v\|_{W^{m,q}(T)} \leq C h_T^{k+1-m+\frac{2}{q}-\frac{2}{p}} \|v\|_{W^{k+1,p}(T)} \qquad \forall v \in W^{k+1,p}(T), \tag{3.7}$$

where $h_T = \mathrm{diam}(T)$.

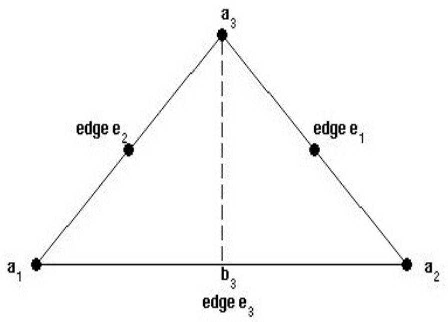

Fig. 3.6. Finite element $\mathcal{S}_3'$.

Proof. The case S_3^h: Since S_3^h is not an affine family in general, the standard scaling technique can not be used directly to prove (3.7). To get around this difficulty, the trick is to introduce an affine "relative" of S_3^h and to estimate the discrepancy between S_3^h and its "relative". To this end, we introduce the following element $\mathcal{S}_3' := (T, P_T, \Sigma_T')$:

(i) T is a triangle with two type I edges,

(ii) $P_T = \mathbb{P}_3(T)$,

(iii) $$\Sigma_T' = \begin{cases} v(a_i) & 1 \le i \le 3, \\ v(b_i) & 1 \le i \le 2, \\ \nabla v(a_i)\cdot(a_j - a_i) & 1 \le i \le 2,\ 1 \le j \le 3,\ j \ne i, \\ \nabla v(b_3)\cdot(a_3 - b_3), & \end{cases}$$

where edge e_3 is of type II.

It is easy to see that Σ_T' is unisolvent in $\mathbb{P}_3(T)$, and that any two triangles are affine equivalent. Therefore for all $p, q \in [1,\infty]$, $0 \le m \le 4$ with $W^{4,p}(T) \hookrightarrow W^{m,q}(T)$, there holds [3]

$$\|v - \Lambda_3^T v\|_{W^{m,q}(T)} \le C h_T^{4-m+\frac{2}{q}-\frac{2}{p}} \|v\|_{W^{4,p}(T)} \qquad \forall v \in W^{4,p}(T), \tag{3.8}$$

where Λ_3^T is the interpolation operator associated with $\mathcal{S}_3'$.

Define $\Theta_3^T := \Pi_3^T - \Lambda_3^T$, and note that for $v \in W^{4,p}(T)$, $\Theta_3^T v\big|_{e_i} = 0$ for $i = 1,2,3$. Consequently,

$$\nabla v(b_3)\cdot(a_3 - b_3) = \frac{1}{1-(\beta^{(3)})^2}\Big\{(a_3 - b_3)\cdot\big(\overline{n}^{(3)} - \tau^{(3)}\beta^{(3)}\big)\partial_{\overline{n}^{(3)}}\big(v - \Lambda_3^T v\big)(b_3)\Big\},$$

where $\overline{n}^{(3)} = (n_1^{(3)}, -n_2^{(3)})$, $\beta^{(3)} := \tau^{(3)}\cdot\overline{n}^{(3)}$, $n^{(3)} = (n_1^{(3)}, n_2^{(3)})$ and $\tau^{(3)}$ denote respectively the unit normal and tangential direction of edge e_3.

Next, let q_3 be the basis function associated with the degree of freedom $\nabla v(b_3)(a_3 - b_3)$ in Σ_T'. We then have

$$\Theta_3^T v = \frac{1}{1-(\beta^{(3)})^2}\Big\{(a_3 - b_3)\cdot\big(\overline{n}^{(3)} - \tau^{(3)}\beta^{(3)}\big)\partial_{\overline{n}^{(3)}}\big(v - \Lambda_3^T v\big)(b_3)\Big\} q_3.$$

Therefore,

$$\|\Theta_3^T v\|_{W^{m,q}(T)} \le \frac{1}{1-(\beta^{(3)})^2}\Big\{|a_3-b_3|\cdot|\overline{n}^{(3)}-\tau^{(3)}\beta^{(3)}|\cdot\|v-\Lambda_3^T v\|_{W^{1,\infty}(T)}\|q_3\|_{W^{m,q}(T)}\Big\}.$$

Finally, by (3.8) and Lemma 3.8 we get

$$\begin{aligned}
&1-(\beta^{(3)})^2 \ge C, && |a_3-b_3| \le Ch_T,\\
&|\overline{n}^{(3)}-\tau^{(3)}\beta^{(3)}| \le 2, && \|v-\Lambda_3^T v\|_{W^{1,\infty}(T)} \le Ch_T^{3-\frac{2}{p}}\|v\|_{W^{4,p}(T)},\\
&\|q_3\|_{W^{m,q}(T)} \le Ch_T^{-m+\frac{2}{q}},
\end{aligned}$$

where C only depends on the minimum angle of T. Hence,

$$\|\Theta_3^T v\|_{W^{m,q}(T)} \le Ch_T^{4-m+\frac{2}{q}-\frac{2}{p}}\|v\|_{W^{4,p}(T)},$$

and consequently,

$$\begin{aligned}
\|v-\Pi_3^T v\|_{W^{m,q}(T)} &\le \|v-\Lambda_3^T v\|_{W^{m,q}(T)} + \|\Theta_3^T v\|_{W^{m,q}(T)}\\
&\le Ch_T^{4-m+\frac{2}{q}-\frac{2}{p}}\|v\|_{W^{4,p}(T)}.
\end{aligned}$$

The case S_4^h: We use a similar argument to show (3.7) for the element S_4^h. First, we introduce the following "relative" $S_4' := (T, Q_T, \Xi_T')$ of S_4^h:

(i) T is a triangle with two type I edges,

(ii) $Q_T = \mathbb{P}_4(T)$,

(iii) $$\Xi_T' = \begin{cases}
v(a_i) & 1\le i\le 3,\\
v(a_{ii3}),\ v(a_{ii3}) & 1\le i\le 2,\\
v(b_3), &\\
v(a_{123}), &\\
\nabla v(a_i)\cdot(a_j-a_i) & 1\le i\le 2,\ 1\le j\le 3,\ j\ne i,\\
\nabla v(a_{112})\cdot(a_3-a_{112}), &\\
\nabla v(a_{122})\cdot(a_3-a_{122}), &
\end{cases}$$

where edge e_3 is of type II.

Next, let Λ_4^T be the interpolation operator associated with $\mathcal{S}_4'$, and set $\Theta_4^T := \Pi_4^T - \Lambda_4^T$. Let r_1 be the basis function of the element $\mathcal{S}_4'$ that is associated with the degree of freedom $\nabla v(a_{112})(a_3-a_{112})$, and let r_2 be the basis function that is associated with the degree of freedom $\nabla v(a_{122})(a_3-a_{122})$. Then for $v\in W^{5,p}(T)$

$$\begin{aligned}
\Theta_4^T v = \frac{1}{1-(\beta^{(3)})^2}\Big\{&(a_3-a_{112})\cdot\big(\overline{n}^{(3)}-\tau^{(3)}\beta^{(3)}\big)\partial_{\overline{n}^{(3)}}\big(v-\Lambda_4^T v\big)(a_{112})r_1\\
&+(a_3-a_{122})\cdot\big(\overline{n}^{(3)}-\tau^{(3)}\beta^{(3)}\big)\partial_{\overline{n}^{(3)}}\big(v-\Lambda_4^T v\big)(a_{122})r_2\Big\}.
\end{aligned}$$

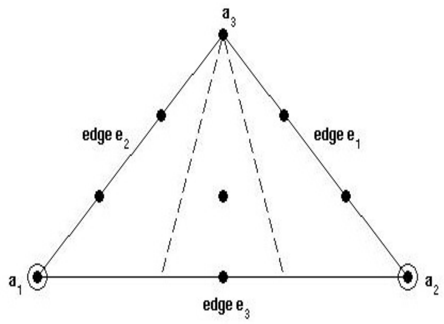

Fig. 3.7. Element $\mathcal{S}_4'$.

Using the fact $\mathcal{S}_4'$ is affine equivalent and applying Lemma 3.8 we get

$$1-(\beta^{(3)})^2 \ge C, \qquad |a_3-a_{112}|,\ |a_3-a_{122}| \le Ch_T,$$
$$|\overline{n}^{(3)}-\tau^{(3)}\beta^{(3)}| \le 2, \qquad \|v-\Lambda_4^T v\|_{W^{1,\infty}(T)} \le Ch_T^{4-\frac{2}{p}}\|v\|_{W^{5,p}(T)},$$
$$\|r_i\|_{W^{m,q}(T)} \le Ch_T^{-m+\frac{2}{q}},\ i=1,2.$$

Therefore,

$$\|\Theta_4^T\|_{W^{m,q}(T)} \le Ch_T^{5-m+\frac{2}{q}-\frac{2}{p}}\|v\|_{W^{5,p}(T)},$$

and consequently,

$$\begin{aligned}\|v-\Pi_4^T\|_{W^{m,q}(T)} &\le \|v-\Lambda_4^T v\|_{W^{m,q}(T)} + \|\Theta_4^T v\|_{W^{m,q}(T)}\\ &\le Ch_T^{5-m+\frac{2}{q}-\frac{2}{p}}\|v\|_{W^{5,p}(T)}.\end{aligned}$$

The proof is complete. □

We note that if a uniform criss-cross mesh is used such that every triangle has two type I edges (see Figure 3.2), then $\nabla v(b_3)(a_3-b_3) = \pm\partial_{\bar{n}} v$ in the definition of Σ_T'. This observation leads to the following corollary.

Corollary 3.10. *Suppose $\mathcal{T}_h$ is the uniform criss-cross triangulation of Ω, then $S_3^h = S_3'$. Hence, S_3^h is an affine family.*

4. Finite Element Formulation and Convergence Analysis

Let V_k^h $(k=3,4)$ be the finite element subspaces of V constructed in the previous section. Define

$$V_{k0}^h := \Big\{ v\in V_k^h;\ v\big|_{\partial\Omega} = \partial_{\bar{n}} v\big|_{\partial\Omega} = 0\Big\}.$$

Based on the weak formulation (2.1), we define our finite element method for problem (1.1)–(1.2) as seeking $u_h \in V_{k0}^h$ such that

$$A^\delta(u_h,v_h) = (f,v_h) \qquad \forall v_h \in V_{k0}^h. \tag{4.1}$$

On noting (2.3)–(2.4), an application of Cea's Lemma [3] yields the following result.

Lemma 4.1. *There exists a unique solution to* (4.1)*. Furthermore, the following error estimate holds:*

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_{k0}^h} \|u - v_h\|_V.$$

Combining Lemma 4.1 and Theorem 3.9 with $p = q = 2,\ m = 1, 2$ we immediately get the following energy norm error estimate.

Theorem 4.1. *If* $u \in H^s(\Omega)$ $(s \geq 3)$ *then*

$$\|u - u_h\|_V \leq Ch^{\ell-2}\big(\sqrt{\delta} + h\big)\|u\|_{H^\ell}, \qquad \ell = \min\{k+1, s\}.$$

Next, using a duality argument, we obtain an error estimate in the L^2-norm.

Theorem 4.2. *Suppose* $u \in H^s(\Omega)$ $(s \geq 3)$*. Then there holds the following error estimate:*

$$\|u - u_h\|_{L^2} \leq C\hat{C}_{0,0}h^{\ell-1}\big(\sqrt{\delta} + h\big)\|u\|_{H^\ell} \qquad \ell = \min\{k+1, s\}. \tag{4.2}$$

Proof. Denote the error by $e_h := u - u_h$, and let $\varphi \in V_0$ be the solution to the following auxiliary problem:

$$A^\delta(\varphi, v) = \langle e_h, v\rangle \qquad \forall v \in V_0.$$

It follows from Theorems 2.1 and 2.2 that the above problem has a unique solution φ and

$$\sqrt{\delta}\|\nabla\Box\varphi\|_{L^2} + \|\Delta\varphi\|_{L^2} \leq \hat{C}_{0,0}\|e_h\|_{L^2}. \tag{4.3}$$

We then have

$$\|e_h\|_{L^2}^2 = A^\delta(e_h, \varphi) = A^\delta(e_h, \varphi - \Pi_k^h\varphi) \leq \|e_h\|_V\|\varphi - \Pi_k^h\varphi\|_V. \tag{4.4}$$

By the definition of $\|\cdot\|_V$ and (4.3) and employing [2, Proposition 4.1.17], we get

$$\begin{aligned}\|\varphi - \Pi_k^h\varphi\|_V &\leq \sqrt{\delta}\|\Box\varphi - \Pi_k^h\Box\varphi\|_{L^2} + \|\nabla\varphi - \Pi_k^h\nabla\varphi\|_{L^2} \\ &\leq C\sqrt{\delta}h\|\nabla\Box\varphi\|_{L^2} + Ch\|\nabla(\nabla\varphi)\|_{L^2} \\ &\leq Ch\big(\sqrt{\delta}\|\nabla\Box\varphi\|_{L^2} + \|\Delta\varphi\|_{L^2}\big) \\ &\leq C\hat{C}_{0,0}h\|e_h\|_{L^2}.\end{aligned} \tag{4.5}$$

Thus, it follows from Theorem 4.1, (4.4), and (4.5) that

$$\|e_h\|_{L^2} \leq C\hat{C}_{0,0}h^{\ell-1}(\sqrt{\delta} + h)\|u\|_{H^\ell}.$$

The proof is complete. □

We conclude this section with a few remarks.

Remark 4.1. (a) The energy norm error estimate is optimal, on the other hand, the H^1 and L^2 norm estimates are optimal provided that $\sqrt{\delta} \simeq h$.

(b) All above convergence results only hold for the restricted meshes, that is, every triangle of the mesh $\mathcal{T}_h$ needs to have two type I edges. As already mentioned at the end of Section 3.1, for arbitrary mesh $\mathcal{T}_h$, $V^h \subset V$ will implies that V^h (and V_0^h) needs to be a C^1 finite element space on $\mathcal{T}_h$ such as Argyris, Hsieh-Clough-Tocher, Bogner-Fox-Schmit elements (cf. [3]). In such a case, it follows from Lemma 4.1 that

$$\begin{aligned}\|u-u_h\|_V &\le C \inf_{v_h\in V_0^h} \|u-v_h\|_V \\ &\le C \inf_{v_h\in V_0^h} \bigl\{\sqrt{\delta}\|u-v_h\|_{H^2} + \|u-v_h\|_{H^1}\bigr\} \\ &\le Ch^{\ell-2}(\sqrt{\delta}+h)\|u\|_{H^\ell},\end{aligned}$$

where $\ell = \min\{k+1, s\}$ and $k(\ge 5)$ is the order of the C^1 finite element. Thus, we still get optimal order error estimate in the energy norm. Although, as expected, using C^1 finite elements is not efficient to solve the bi-wave problem (cf. [7]).

5. Numerical Experiments and Rates of Convergence

In this section, we provide some numerical experiments to gauge the efficiency and validate the theoretical error bounds for the finite element S_3^h developed in the previous sections.

Test 1. For this test, we calculate the rate of convergence of $\|u-u_h\|$ for fixed δ in various norms and compare each computed rate with its theoretical estimate. All our computations are done on the square domain $\Omega = (0,1)^2$ using the criss-cross mesh. We use the source function

$$\begin{aligned}f(x,y) =& -2048\pi^4\delta\bigl(\cos^2(4\pi x) - \sin^2(4\pi y)\bigr) - 32\pi^2\Bigl\{\sin^2(4\pi y)\bigl(\cos^2(4\pi x)-\sin^2(4\pi x)\bigr) \\ &+ \sin^2(4\pi x)\bigl(\cos^2(4\pi y)-\sin^2(4\pi y)\bigr)\Bigr\},\end{aligned}$$

so that the exact solution is given by $u(x,y) = \sin^2(4\pi x)\sin^2(4\pi y)$.

We list the computed errors in Table 5.1 for δ-values $10, 1, 10^{-2}$ and 10^{-6}, and also plot the results in Figure 5.2. As expected, the rates of convergence depend on both the parameter h and δ. In fact, Corollary 4.1 tells us that for $\sqrt{\delta} >> h$

$$\begin{aligned}\|u-u_h\|_V &\le Ch^2(\sqrt{\delta}+h)\|u\|_{H^4} \le Ch^2\|u\|_{H^4}, \\ \|u-u_h\|_{H^1} &\le Ch^2(\sqrt{\delta}+h)\|u\|_{H^4} \le Ch^2\|u\|_{H^4}, \\ \|u-u_h\|_{L^2} &\le C\hat{C}_{0,0}h^3(\sqrt{\delta}+h)\|u\|_{H^4} \le C\hat{C}_{0,0}h^3\|u\|_{H^4},\end{aligned}$$

while for $\sqrt{\delta} \le h$

$$\begin{aligned}\|u-u_h\|_V &\le Ch^2(\sqrt{\delta}+h)\|u\|_{H^4} \le Ch^3\|u\|_{H^4}, \\ \|u-u_h\|_{H^1} &\le Ch^2(\sqrt{\delta}+h)\|u\|_{H^4} \le Ch^3\|u\|_{H^4}, \\ \|u-u_h\|_{L^2} &\le C\hat{C}_{0,0}h^3(\sqrt{\delta}+h)\|u\|_{H^4} \le C\hat{C}_{0,0}h^4\|u\|_{H^4}.\end{aligned}$$

We find that the computed bounds agree with these theoretical bounds.

In addition, although a theoretical proof of the following convergence rate has yet to be shown, the computed solutions also indicate that

$$\|u-u_h\|_{2,h} \le Ch(\sqrt{\delta}+h)\|u\|_{H^4},$$

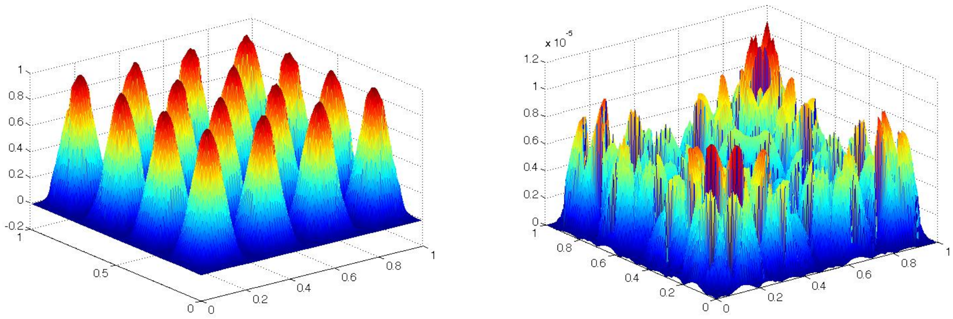

Fig. 5.1. Test 1. Computed solution (left) and error (right) with $\delta = 10^{-2}$ and $h = 0.01$.

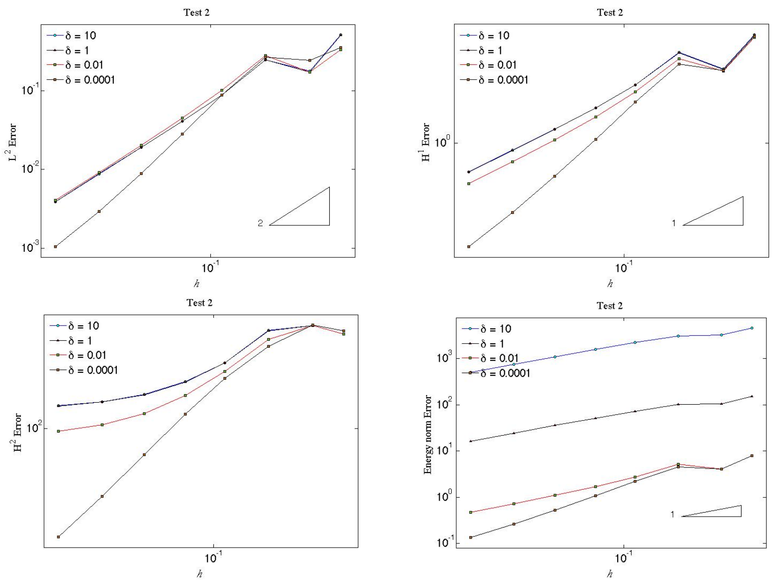

Fig. 5.2. Test 1. L^2 norm, H^1 norm, H^2 norm, and energy norm errors with $\delta = 10, 1, 10^{-2}$ and 10^{-6}.

where

$$\|u - u_h\|_{2,h}^2 := \sum_{T \in \mathcal{T}_h} \|u - u_h\|_{H^2(T)}^2.$$

Table 5.1: Test 1. Errors with estimated rates of convergence

δ	h	$\|\cdot\|_{L^2}$ err. (cnv. rate)	$\|\cdot\|_{H^1}$ err.(cnv. rate)	$\|\cdot\|_{h,2}$ err. (cnv. rate)	$\|\cdot\|_V$ err.(cnv. rate)
10	0.5000	4.17(−)	26.4(−)	311.62(−)	2191.62(−)
	0.3333	2.76E-01(6.694)	9.54(2.514)	284.05(0.228)	1147.49(1.596)
	0.2000	1.59E-01(1.079)	2.99(2.273)	211.50(0.577)	535.69(1.491)
	0.1000	4.41E-03(5.176)	3.75E-01(2.995)	52.54(2.009)	153.70(1.801)
	0.0500	4.64E-04(3.248)	9.11E-02(2.041)	21.61(1.282)	39.84(1.948)
	0.0400	2.31E-04(3.117)	5.82E-02(2.010)	16.79 (1.130)	25.61(1.980)
	0.0200	2.79E-05(3.054)	1.45E-02(2.004)	8.05(1.060)	6.44(1.992)
	0.0100	3.45E-06(3.014)	3.62E-03(2.001)	3.98(1.016)	1.61(1.998)
	0.0083	1.99E-06(3.006)	2.52E-03(2.000)	3.31(1.006)	1.12(1.999)
	0.0067	1.02E-06(3.004)	1.61E-03(2.000)	2.65(1.004)	0.72(1.999)
1	0.5000	3.93(−)	25.3(−)	306.88(−)	238.43(−)
	0.2500	2.75E-01(3.837)	9.52(1.413)	283.58(0.114)	123.22(0.952)
	0.2000	1.57E-01(2.523)	2.98(5.210)	210.95(1.326)	56.24(3.515)
	0.1000	4.40E-03(5.152)	3.75E-01(2.989)	52.53(2.006)	15.71(1.840)
	0.0500	4.63E-04(3.249)	9.09E-02(2.043)	21.58(1.284)	4.07(1.950)
	0.0400	2.31E-04(3.118)	5.81E-02(2.012)	16.76(1.132)	2.61(1.981)
	0.0200	2.78E-05(3.055)	1.45E-02(2.005)	8.03(1.061)	0.66(1.992)
	0.0100	3.44E-06(3.015)	3.61E-03(2.001)	3.97(1.016)	0.16(1.998)
	0.0083	1.99E-06(3.006)	2.51E-03(2.000)	3.31(1.006)	0.11(1.999)
	0.0067	1.02E-06(3.004)	1.61E-03(2.000)	2.64(1.004)	0.07(1.999)
	0.0056	5.89E-07(2.999)	1.12E-03(2.000)	2.20(1.003)	0.05(1.998)
10^{-2}	0.5000	2.15(−)	15.4(−)	276.56(−)	17.4(−)
	0.3333	2.25E-01(3.259)	8.38(0.879)	260.32(0.087)	9.48(0.877)
	0.2000	1.02E-01(3.556)	2.53(5.365)	183.36(1.571)	3.07(5.047)
	0.1000	4.21E-03(4.597)	3.69E-01(2.780)	52.08(1.816)	5.22E-01(2.558)
	0.0500	4.36E-04(3.269)	8.55E-02(2.107)	20.31(1.358)	1.25E-01(2.058)
	0.0400	2.15E-04(3.175)	5.38E-02(2.082)	15.52(1.207)	7.93E-02(2.049)
	0.0200	2.50E-05(3.101)	1.30E-02(2.049)	7.21(1.106)	1.94E-02(2.030)
	0.0100	3.06E-06(3.033)	3.21E-03(2.016)	3.53(1.030)	4.82E-03(2.010)
	0.0083	1.77E-06(3.013)	2.23E-03(2.006)	2.94(1.012)	3.35E-03(2.004)
	0.0067	9.02E-07(3.009)	1.42E-03(2.005)	2.34(1.008)	2.14E-03(2.003)
10^{-6}	0.5000	3.93(−)	23.3(−)	374.18(−)	23.3(−)
	0.2500	2.28E-01(4.108)	7.25(1.686)	233.21(0.682)	7.25(1.686)
	0.2000	9.68E-02(3.831)	1.93(5.929)	149.75(1.985)	1.93(5.929)
	0.1000	3.70E-03(4.708)	2.81E-01(2.782)	45.13(1.731)	2.81E-01(2.782)
	0.0500	4.92E-04(2.914)	5.21E-02(2.429)	13.09(1.786)	5.21E-02(2.429)
	0.0400	2.35E-04(3.298)	2.89E-02(2.647)	8.54(1.915)	2.89E-02(2.647)
	0.0200	1.91E-05(3.626)	4.11E-03(2.813)	2.15(1.989)	4.11E-03(2.813)
	0.0100	1.28E-06(3.898)	5.39E-04(2.931)	0.53(2.024)	5.39E-04(2.931)
	0.0083	6.19E-07(3.981)	3.15E-04(2.943)	0.36(2.035)	3.15E-04(2.943)
	0.0067	2.53E-07(4.005)	1.64E-04(2.934)	0.23(2.039)	1.64E-04(2.933)

Test 2. This test is the same as the first, but we now use the following source function: f=1. We note that the exact solution is unknown. We plot the solution with $h = 0.01$ and δ-values $10, 1, 10^{-2}$, and 10^{-6} in Figure 5.3. As expected, the solution is more and more like the solution of the corresponding Poisson problem as δ gets smaller and smaller.

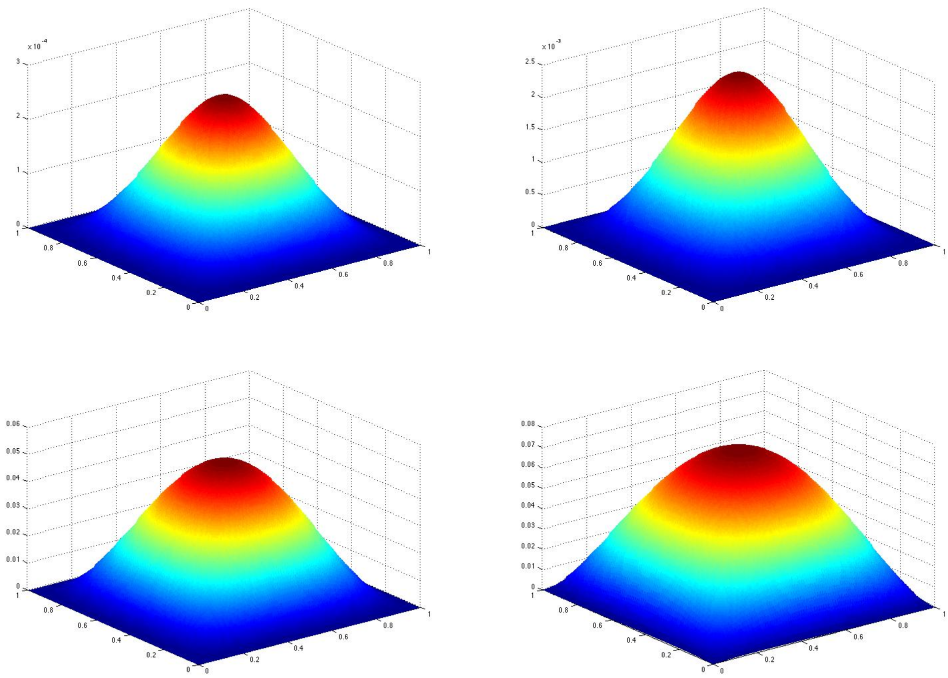

Fig. 5.3. Test 2. Computed solution with source function $f = 1$ and $h = 0.01$ with $\delta = 10$(top left), $\delta = 1$(top right), $\delta = 10^{-2}$(bottom left), and $\delta = 10^{-6}$(bottom right).

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References

[1] D.G. Bourgin and R. Duffin, The Dirichlet problem for the vibrating string equation, *Bull. Amer. Math. Soc.*, **45** (1939), 851-858.
[2] S.C. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods, 3rd edition, Springer (2008).
[3] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[4] Q. Du, Studies of Ginzburg-Landau model for d-wave superconductors, *SIAM J. Appl. Math.*, **59**:4 (1999), 1225-1250.
[5] L.C. Evans, Partial Differential Equations, AMS, 1998.

[6] D.L. Feder and C. Kallin, Microscopic derivation of the Ginzburg-Landau equations for a d-wave superconductor, *Phys. Rev. B*, **55** (1997), 559-574.

[7] X. Feng and M. Neilan, Discontinuous finite element methods for a bi-wave equation modeling d-wave superconductors, *Math. Comput.* (submitted)

[8] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer, Berlin, 2001. Reprint of the 1998 edition.

[9] V.L. Ginzburg and L.D. Landau, On the theory of superconductivity, Zh. ksper. Teoret. Fiz. 20:1064-1082, 1950 (in Russian), in: L.D. Landau, and I.D. ter Haar (Eds.), Man of Physics, Pergamon, Oxford, 1965, 138-167 (in English).

[10] Y. Ren, J.-H. Xu, and C.S. Ting, Ginzburg-Landau equations for mixed $s+d$ symmetry superconductors, *Phys. Rev. B*, **53** (1996), 2249-2252.

[11] M. Tinkham, Introduction to Superconductivity, 2rd Edition, Dover Publications, 2004.

[12] J.-H. Xu, Y. Ren, and C.S. Ting, Ginzburg-Landau equations for a d-wave superconductor with nonmagnetic impurities, *Phys. Rev. B*, **53**: (1996), 12481-12495.

[13] A. Ženíšek, Polynomial approximation on tetrahedrons in the finite element method, *J. Approx. Theory*, **7** (1973), 334-351.

[14] A. Ženíšek, A general theorem on triangular finite $C^{(m)}$-elements, *Rev. Franaise Automat. Informat. Recherche Oprationnelle Sr. Rouge*, **8** (1974), 119-127.