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SUPERCONVERGENCE OF A DISCONTINUOUS GALERKIN METHOD FOR FIRST-ORDER LINEAR DELAY DIFFERENTIAL EQUATIONS*

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Abstract

This paper deals with the discontinuous Galerkin (DG) methods for delay differential equations. By an orthogonal analysis in each element, the superconvergence results of the methods are derived at nodal points and eigenpoints. Numerical experiments will be carried our to verify the effectiveness and the theoretical results of the proposed methods.

Mathematics subject classification: 65N12, 65N30.
Key words: Discontinuous Galerkin methods, Delay differential equations, Orthogonal analysis, Superconvergence.

1. Introduction

Delay differential equations frequently arise in a fast variety of scientific problems, such as relativistic dynamics, nuclear reactor, neural network, electric circuit and viscoelasticity mechanics, see, e.g., [11, 14]. The last several decades have witnessed a fast development in computational implementation and numerical analysis for various delay differential equations, see, the monographs e.g., [2, 3] and the references therein. It is noted that most authors have employed finite difference methods, such as linear multistep methods, one-leg methods, Runge-Kutta methods and general linear methods, see, e.g., [19, 20].

Besides the finite difference methods, it is well-known that the finite element methods are also a class of effective numerical methods for solving differential equations, and usually some superconvergence results are available, see, e.g., [1, 4–6, 9, 13, 15–17]. Up to now, however, there have been very few papers on finite elements for solving delay differential equations (DDEs). Generally speaking, solution behaviors of the DDEs are more complicated than those for the standard differential equations since the former depends not only on the present but also on the history. The presence of a delay term could change a system's dynamic properties such as stability, oscillation, bifurcation, chaos and etc. In [10, 18] continuous Galerkin finite element (CGFE) methods are applied to DDEs with one-delay and multi-delay, are respectively, and a number of superconvergence results of the CGFE methods are obtained. In [12], continuous and discrete finite element approximations to a class of parabolic delay differential equations are investigated optimal error estimates in L^2, H^1 and L^∞ norms are obtained.

As an important subclass of finite element methods, the discontinuous Galerkin (DG) methods have been found very useful in scientific engineering. For detail description of the method as well as its development, we refer the readers to the review paper [7] and the special issue of

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Journal of Scientific Computing [8]. Up to now, the DG methods have been proved locally conservative, stable, and high-order accurate. Since the DG methods have many desired properties, it will be interesting to apply such methods to DDEs. In [15], we proved that the DG methods have the ability to preserve stability of the underlying systems. Following our earlier work, the superconvergence results of the methods will be derived at nodal points and eigenpoints in this work.

The rest of the paper is structured as follows. In Section 2, we introduce a class of DG methods for DDEs. In Section 3, we analyze errors of the methods and prove that the DG methods have superconvergence at nodal points and eigenpoints. In Section 4, numerical experiments will be used to confirm the effectiveness and the theoretical results of the methods. Finally, conclusions and discussions for this paper are summarized in Section 5.

2. Delay Differential Equations and Their DG Methods

In this section, we will give a discretization scheme based on the DG methods for a class of linear DDEs. Consider the following DDEs with delay $\tau > 0$:

$$\begin{cases} u'(t) + a(t)u(t) + b(t)u(t-\tau) = f(t), & t_0 \le t \le T, \\ u(t) = \psi(t), & t_0 - \tau \le t \le t_0, \end{cases} \tag{2.1}$$

where the functions $a(t), b(t), f(t), \psi(t)$ are assumed to be continuous on their respective domains so that the above delay system has a unique solution $u \in H^1([t_0, +\infty))$. When $\tau \ge T - t_0$, system (2.1) becomes a linear ordinary differential equation (ODE)

$$\begin{cases} u'(t) + a(t)u(t) = -b(t)\psi(t-\tau) + f(t), & t_0 \le t \le T, \\ u(t_0) = \psi(t_0). \end{cases} \tag{2.2}$$

Such an ODE system has been solved by CGFE methods and DG methods in many references, see, e.g. [1,5,7,9,13,17]. In this work, we always assume $\tau \le T - t_0$ so that the non-degenerate delay systems can be considered.

For the discretization of system (2.1) by a class of DG methods, we divide the interval $[t_0, T]$ with a uniform mesh:

$$\mathcal{J}^h: \quad t_0 < t_1 < \cdots < t_N,$$

where $t_n = t_0 + 2nh, h = \tau/(2k)$, k is a given positive integer and the maximum index N satisfies $t_{N-1} < T \le t_N$. Moreover, we write that the element $J_n = (t_{n-1}, t_n]$, the half-integer node $t_{n-1/2} = (t_n + t_{n-1})/2 (= t_0 + (2n-1)h)$ and the extended interval $J = [t_0, t_N]$, and define m-degree discontinuous finite element space as follows:

$$S^h = \big\{ v : \ v\big|_{J_n} \in P_m(J_n), n = 1, 2, \cdots, N \big\},$$

where $P_m(J_n)$ denotes the set of all polynomials of degree $\le m$ on J_n. Note that when a function $U \in S^h$, it implies that U is allowed to be discontinuous but left-continuous at the nodal points t_n, i.e. $U(t_n) = U(t_n - 0)$. For brevity, we introduce the following notations:

$$U_n^- = U(t_n - 0) = U(t_n) = U_n, \quad U_n^+ = U(t_n + 0), \quad [U_n] = U_n^+ - U_n^-.$$

In general case, the span $[U_n] \ne 0$. As there is no request that $U \in S_h$ is continuous at the nodal points, U has $(m+1)$-degree of freedom on an element J_n.

Multiplying by $\eta \in S^h$and integrating over the element J_n, (2.1) becomes

$$\int_{J_n} [u'(t) + a(t)u(t) + b(t)u(t-\tau)]\eta(t)dt = \int_{J_n} f(t)\eta(t)dt, \qquad n \geq 1. \tag{2.3}$$

With this, an m-degree DG solution $U \in S^h$ can be defined for $n \geq 1$:

$$\int_{J_n} [U'(t) + a(t)U(t) + b(t)U(t-\tau)]\eta(t)dt + [U_{n-1}]\eta_{n-1}^+ = \int_{J_n} f(t)\eta(t)dt, \tag{2.4}$$

where $\eta_{n-1}^+ = \eta(t_{n-1}+0) \in S^h$. When $t_0-\tau \leq t \leq t_0$, we set $U(t) = u(t) = \psi(t)$. Consequently, we have

$$u'(t) + a(t)u(t) = -b(t)\psi(t-\tau) + f(t) \equiv g(t), \qquad t_0 \leq t \leq t_0+\tau, \tag{2.5}$$

and Eq. (2.3) can be split into

$$\int_{J_n} [U'(t) + a(t)U(t)]\eta(t)dt + [U_{n-1}]\eta_{n-1}^+ = \int_{J_n} g(t)dt, \qquad 1 \leq n \leq k, \tag{2.6}$$

and

$$\int_{J_n} [U'(t) + a(t)U(t) + b(t)U(t-\tau)]\eta(t)dt + [U_{n-1}]\eta_{n-1}^+ = \int_{J_n} f(t)\eta(t)dt, \qquad k < n \leq N. \tag{2.7}$$

Now, write the errors $e(t) = u(t) - U(t)$, $e_n = u(t_n) - U(t_n)$. Then subtracting (2.6) from (2.3) and (2.7) from (2.3) gives

$$\int_{J_n} [e'(t) + a(t)e(t)]\eta(t)dt + [e_{n-1}]\eta_{n-1}^+ = 0, \qquad 1 \leq n \leq k, \tag{2.8}$$

and

$$\int_{J_n} [e'(t) + a(t)e(t) + b(t)e(t-\tau)]\eta(t)dt + [e_{n-1}]\eta_{n-1}^+ = 0, \qquad k < n \leq N. \tag{2.9}$$

On the interval J_n, take the transformation

$$t = t_{n-1/2} + hs, \qquad s \in [-1,1],$$

and introduce the notations

$$\begin{aligned}
&\tilde{u}(s) = u(t_{n-\frac{1}{2}} + hs), && \tilde{U}(s) = U(t_{n-\frac{1}{2}} + hs), \\
&\tilde{a}(s) = a(t_{n-\frac{1}{2}} + hs), && \tilde{b}(s) = b(t_{n-\frac{1}{2}} + hs), \\
&\tilde{e}(s) = \tilde{u}(s) - \tilde{U}(s), && \tilde{\eta}(s) = \eta(t_{n-\frac{1}{2}} + hs), \\
&\tilde{u}_\tau(s) = u(t_{n-\frac{1}{2}} + hs - \tau), && \tilde{U}_\tau(s) = U(t_{n-\frac{1}{2}} + hs - \tau), \qquad \tilde{e}_\tau(s) = \tilde{u}_\tau(s) - \tilde{U}_\tau(s).
\end{aligned}$$

Then (2.8) and (2.9) can be rewritten as

$$\int_{-1}^{1} [\tilde{e}'(s) + h\tilde{a}(s)\tilde{e}(s)]\tilde{\eta}(s)ds + [e_{n-1}]\eta_{n-1}^+ = 0, \quad 1 \leq n \leq k, \tag{2.10}$$

and

$$\int_{-1}^{1} ([\tilde{e}'(s) + h\tilde{a}(s)\tilde{e}(s) + h\tilde{b}(s)\tilde{e}_\tau(s)]\tilde{\eta}(s)ds + [e_{n-1}]\eta_{n-1}^+ = 0, \quad k < n \leq N, \tag{2.11}$$

respectively. For the error analysis of discontinuous finite element, we consider Legendre polynomials on interval $[-1,1]$:

$$l_n(s) = \frac{1}{2^n n!}\frac{d^n}{ds^n}(s^2-1)^n, \quad n = 0, 1 \cdots, \tag{2.12}$$

which satisfy $l_n(\pm 1) = (\pm 1)^n$ and the following orthogonal property (cf. [5])

$$(l_i, l_j) := \int_{-1}^{1} l_i(s) l_j(s) ds = \begin{cases} 0, & \text{for } i \neq j, \\ \dfrac{2}{2j+1}, & \text{for } i = j. \end{cases}$$

Moreover, motivated by the idea from references [1,5], another set of the Radau II polynomials associated with the Legendre polynomials:

$$\varphi_0(s) = 1, \quad \varphi_i(s) = l_i(s) - l_{i-1}(s), \quad i \geq 1$$

will be also used in the subsequent analysis, where each polynomial $\varphi_i(s)$ has i distinct zeros s_α $(\alpha = 1, \cdots, i)$ in $[-1,1]$. These polynomials have some quasiorthogonal properties, which play an important role in the following error analysis.

The lemma below was used to study the convergence of Galerkin methods for different differential equations by Chen [5]. Since the reference is not readily accessible to non-Chinese readers, we collect a concise proof here.

Lemma 2.1. (cf. [5]) *Suppose the sufficiently smooth function $\tilde{u}(s)$ is expanded as*

$$\tilde{u}(s) = b_0(n) + \sum_{i=1}^{\infty} b_i(n)\varphi_i(s), \quad s \in [-1,1]. \tag{2.13}$$

Then its coefficients $\{b_i(n)\}$ satisfy

$$b_i(n) = \mathcal{O}(h^i), \qquad i \geq 0. \tag{2.14}$$

Proof. For sufficiently smooth function $\tilde{u}(s)$ in J_n, expanding $\tilde{u}(s)$ as a Legendre series

$$\tilde{u}(s) = \sum_{i=0}^{\infty} B_i(n) l_i(s), \tag{2.15}$$

where $B_i(n) = (i+\frac{1}{2})\int_{-1}^{1} u(s) l_i(s) ds$. Using integration by parts, we get

$$B_i(n) = \mathcal{O}(h^i). \tag{2.16}$$

Noting that

$$\begin{aligned} \tilde{u}(s) =& b_0(n) + \sum_{i=1}^{\infty} b_i(n)\varphi_i(s) \\ =& b_0(n) - b_1(n) + \sum_{i=1}^{\infty} (b_i(n) - b_{i+1}(n)) l_i(s) \\ =& \sum_{i=0}^{\infty} B_i(n) l_i(s), \end{aligned} \tag{2.17}$$

where $B_i(n) = b_i(n) - b_{i+1}(n)$, we have

$$b_i(n) = \sum_{j=i}^{\infty} B_j(n) = \mathcal{O}(h^i), \qquad i \geq 0. \tag{2.18}$$

This completes the proof of the lemma. $\square$

It is printed out that the relation (2.16), which is used to study the superconvergence of continuous Galerkin methods for DDEs, is also derived in [10] and [18].

3. Error Analysis

In this section, by means of the orthogonal analysis method, we will derive a superconvergence result of the DG methods.

Theorem 3.1. *The m-degree finite element $U(t)$ defined by* (2.4) *for the DDEs* (2.1) *with a smooth solution u has the following superconvergence estimate at nodal points $\{t_n\}$:*

$$|(u-U)(t_n)| = \mathcal{O}(h^{2m+1}), \quad n = 1, \cdots, N,$$

and satisfies the superconvergence estimate at eigenpoints $\{t_{n-1/2}^{(\alpha)}\}$:

$$\left|(u-U)\big(t_{n-\frac{1}{2}}^{(\alpha)}\big)\right| = \mathcal{O}(h^{m+2}), \quad n = 1, \cdots, N; \quad \alpha = 1, \cdots, m,$$

where $t_{n-1/2}^{(\alpha)} = t_{n-1/2} + hs_\alpha$ and s_α, $\alpha = 1, \cdots, m$, are the zeros of the m-degree polynomial $\varphi_m(s)$ in $[-1,1]$.

Proof. Following [5], we first construct the m-degree polynomial approximation of u in the element J_n as follows:

$$u_I(t_{n-1/2} + hs) = \sum_{j=0}^{m} b_j(n)\varphi_j(s) - \sum_{j=1}^{m} b_j^*(n)\varphi_j(s), \quad s \in [-1,1], \tag{3.1}$$

where $b_j^*(n)$ are some coefficients to be defined later. Then the remainder is of the form

$$\tilde{\sigma}(s) := (u - u_I)(t_{n-1/2} + hs) = \sum_{j=m+1}^{\infty} b_j(n)\varphi_j(s) + \sum_{j=1}^{m} b_j^*(n)\varphi_j(s). \tag{3.2}$$

It follows from the facts $\varphi_j(1) = 0$, $\varphi_j(-1) = 2(-1)^j$ $(j \geq 1)$ that

$$\tilde{\sigma}(1) = 0, \quad \tilde{\sigma}(-1) = 2\sum_{j=1}^{m}(-1)^j b_j^*(n) + 2\sum_{j=m+1}^{\infty}(-1)^j b_j(n).$$

Introduce the functionals

$$B_n(\sigma,\eta) = \int_{J_n} \big[\sigma'(t) + a(t)\sigma(t)\big]\eta(t)dt + [\sigma_{n-1}]\eta_{n-1}^+, \qquad 1 \leq n \leq k, \tag{3.3}$$

$$B_n(\sigma,\eta) = \int_{J_n} \big[\sigma'(t) + a(t)\sigma(t) + b(t)\sigma(t-\tau)\big]\eta(t)dt + [\sigma_{n-1}]\eta_{n-1}^+, \quad k < n \leq N, \tag{3.4}$$

where $\sigma(t) = \tilde{\sigma}(s)$ $(t = t_{n-1/2} + hs)$. Since $\tilde{\sigma}(1) = 0$ means $\sigma_n^- = 0$, an integration by parts yields that

$$\int_{J_n} \sigma'(t)\eta(t)dt + [\sigma_{n-1}]\eta_{n-1}^+ = -\int_{J_n} \sigma(t)\eta'(t)dt, \qquad \forall n \geq 1.$$

This implies that (3.3) and (3.4) can be simplified as

$$B_n(\sigma,\eta) = -\textstyle\int_{-1}^1 \tilde{\sigma}(s)(\tilde{\eta}'(s) - h\tilde{a}(s)\tilde{\eta}(s))ds, \qquad 1 \leq n \leq k, \quad (3.5)$$

$$B_n(\sigma,\eta) = -\textstyle\int_{-1}^1 \Big[\tilde{\sigma}(s)\tilde{\eta}'(s) - h\tilde{a}(s)\tilde{\sigma}(s)\tilde{\eta}(s) - h\tilde{b}(s)\sigma(t_{n-k-1/2} + hs)\tilde{\eta}(s)\Big]ds, \; k < n \leq N, \quad (3.6)$$

respectively. Inserting the test function $\tilde{\eta}(s) = \sum_{i=0}^m \beta_i\varphi_i$ into (3.5) and (3.6) gives

$$B_n(\sigma,\eta) = -\sum_{i=0}^m \beta_i \left[\sum_{j=1}^m C_{ij}b_j^*(n) + \sum_{j=m+1}^\infty C_{ij}b_j(n)\right], \qquad 1 \leq n \leq k, \qquad (3.7)$$

$$\begin{aligned} B_n(\sigma,\eta) = &-\sum_{i=0}^m \beta_i \left[\sum_{j=1}^m C_{ij}b_j^*(n) + \sum_{j=m+1}^\infty C_{ij}b_j(n)\right] \\ &+\sum_{i=0}^m \beta_i \left[\sum_{j=1}^m C'_{ij}b_j^*(n-k) + \sum_{j=m+1}^\infty C'_{ij}b_j(n-k)\right], \quad k < n \leq N, \end{aligned} \qquad (3.8)$$

where

$$C_{ij} = \int_{-1}^1 \big[\varphi_i'(s) - ha(s)\varphi_i(s)\big]\varphi_j(s)ds, \quad C'_{ij} = h\int_{-1}^1 b(s)\varphi_i(s)\varphi_j(s)ds. \qquad (3.9)$$

In (3.9), the basis functions satisfy

$$\int_{-1}^1 \varphi_i(s)\varphi_j(s)ds = \begin{cases} \dfrac{2}{2i+1} + \dfrac{1}{2i-1}, & \text{for } i = j, \\ \dfrac{-2}{2i+1}, & \text{for } |i-j| = 1, \\ 0, & \text{for } |i-j| \geq 2. \end{cases} \qquad (3.10)$$

$$\int_{-1}^1 \varphi_i'(s)\varphi_j(s)ds = \begin{cases} -2, & \text{for } i = j \geq 1, \\ 4, & \text{for } i = j+1, \\ 0, & \text{for } i = 0, \text{ or } i < j \text{ or } |i-j| \geq 2. \end{cases} \qquad (3.11)$$

By the orthogonality of $l_i(s)$, we can obtain the following estimates:

$$C_{ij} = \begin{cases} -2 + \mathcal{O}(h), & \text{for } i = j \geq 1, \\ 4 + \mathcal{O}(h), & \text{for } i = j+1, \\ \mathcal{O}(h^{|i-j|}), & \text{otherwise}, \end{cases} \qquad (3.12)$$

$$C'_{ij} = \begin{cases} \mathcal{O}(h), & \text{for } i = j \text{ or } j = i+1 \\ \mathcal{O}(h^{|i-j|}), & \text{for } i < j \text{ or } |i-j| \geq 2. \end{cases} \qquad (3.13)$$

In order to determine the coefficients b_{j+1}^*, we set that

$$\sum_{j=1}^m C_{ij}b_j^*(n) = -\sum_{j=m+1}^\infty C_{ij}b_j(n), \qquad i = 1,\cdots,m, \quad 1 \leq n \leq k, \qquad (3.14)$$

and

$$\sum_{j=1}^{m} C_{ij} b_j^*(n) = -\sum_{j=m+1}^{\infty} C_{ij} b_j(n) + \sum_{j=1}^{m} C'_{ij} b_j^*(n-k) + \sum_{j=m+1}^{\infty} C'_{ij} b_j(n-k),$$
$$i = 1, \cdots, m, \quad k < n \le N. \tag{3.15}$$

It follows from (2.14) and (3.12) that the right-hand of the Eq. (3.14) satisfies for $i = 1, 2, \cdots, m$:

$$-\sum_{j=m+1}^{\infty} C_{ij} b_j(n) = \mathcal{O}(h^{j-i} \cdot h^j)\Big|_{j=m+1} = \mathcal{O}(h^{2m+2-i}), \quad 1 \le n \le k, \tag{3.16}$$

A combination of (3.16) and (3.14) leads to a vectorial equation

$$\begin{bmatrix} -2+\mathcal{O}(h) & \mathcal{O}(h) & \mathcal{O}(h^2) & \cdots & \mathcal{O}(h^{m-1}) \\ 4+\mathcal{O}(h) & -2+\mathcal{O}(h) & \mathcal{O}(h) & \cdots & \mathcal{O}(h^{m-2}) \\ \mathcal{O}(h^2) & 4+\mathcal{O}(h) & -2+\mathcal{O}(h) & \cdots & \mathcal{O}(h^{m-3}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{O}(h^{m-1}) & \mathcal{O}(h^{m-2}) & \mathcal{O}(h^{m-3}) & \cdots & -2+\mathcal{O}(h) \end{bmatrix} \begin{bmatrix} b_1^*(n) \\ b_2^*(n) \\ b_3^*(n) \\ \vdots \\ b_m^*(n) \end{bmatrix} = \begin{bmatrix} \mathcal{O}(h^{2m+1}) \\ \mathcal{O}(h^{2m}) \\ \mathcal{O}(h^{2m-1}) \\ \vdots \\ \mathcal{O}(h^{m+2}) \end{bmatrix}.$$

When h is sufficiently small, the coefficient matrix are diagonally dominant. Noting the diagonal terms in the matrix equations are $\mathcal{O}(1)$ or observing from the last line, we have

$$b_j^*(n) = \mathcal{O}(h^{2m+2-j}), \quad j = 1, \cdots, m, \quad 1 \le n \le k. \tag{3.17}$$

Next, we use mathematical induction to prove that (3.17) holds for $1 \le n \le N$. Assume that (3.17) holds for $k < n \le \hat{n}$ $(< N)$. Then, by (2.14), (3.12) and (3.13), we have for $n = \hat{n} + 1$:

$$-\sum_{j=m+1}^{\infty} C_{ij} b_j(n) + \sum_{j=1}^{m} C'_{ij} b_j^*(n-k) + \sum_{j=m+1}^{\infty} C'_{ij} b_j(n-k) = \mathcal{O}(h^{2m+2-i}), \qquad i = 1, \cdots, m. \tag{3.18}$$

This, together with (3.15), implies

$$\sum_{j=1}^{m} C_{ij} b_j^*(n) = \mathcal{O}(h^{2m+2-i}), \quad i = 1, \cdots, m; \quad n = \hat{n} + 1. \tag{3.19}$$

Using a similar derivation for (3.17), we find that (3.17) stays valid for $n = \hat{n} + 1$. Therefore, (3.17) holds for all $n: \ 1 \le n \le N$.

The above arguments show that $B_n(\sigma, \eta)$ can be expressed as

$$B_n(\sigma, \eta) = -\beta_0 \left[\sum_{j=1}^{m} C_{0j} b_j^*(n) + \sum_{j=m+1}^{\infty} C_{0j} b_j(n) \right] = \mathcal{O}(h^{2m+2})\beta_0, \quad 1 \le n \le k, \tag{3.20}$$

or

$$\begin{aligned} B_n(\sigma, \eta) = &-\beta_0 \left[\sum_{j=1}^{m} C_{0j} b_j^*(n) + \sum_{j=m+1}^{\infty} C_{0j} b_j(n) \right] + \beta_0 \left[\sum_{j=1}^{m} C'_{0j} b_j^*(n-k) + \sum_{j=m+1}^{\infty} C'_{0j} b_j(n-k) \right] \\ =&\mathcal{O}(h^{2m+2})\beta_0, \qquad k < n \le N. \end{aligned} \tag{3.21}$$

By the inverse estimate $|\beta_0| \le \hat{c}\|\tilde{\eta}\|_{0,E}$ (e.g. [5]), (3.20) and (3.21), we know that there exists a constant $c_1 > 0$ such that

$$
\begin{aligned}
|B_n(\sigma,\eta)| \le & c_1 h^{2m+2}\left(\int_{-1}^{1}\tilde{\eta}^2 ds\right)^{1/2} \le c_1 h^{2m+1}\left(\int_{J_n}\eta^2 dt\right)^{1/2} \\
\le & \frac{1}{2}\left[c_1^2 h^{4m+2} + \int_{J_n}\eta^2 dt\right], \quad 1 \le n \le N,
\end{aligned}
\tag{3.22}
$$

In the following, we estimate the error $\theta(t) := (u_I - U)(t)$. With (2.8) and (2.9), we have by substituting i for n that

$$
B_i(\theta,\eta) = -B_i(\sigma,\eta), \quad 1 \le i \le N. \tag{3.23}
$$

When $t_0 < t \le t_0 + \tau$, taking both $\eta = \theta$ and $1 \le i \le k$ in (3.23) derives that

$$
\frac{1}{2}(\theta_i^-)^2 - \frac{1}{2}(\theta_{i-1}^+)^2 + (\theta_{i-1}^+ - \theta_{i-1}^-)\theta_{i-1}^+ = -B_i(\sigma,\theta) - \int_{J_i} a(t)\theta^2 dt,
$$

which gives

$$
(\theta_i^-)^2 - (\theta_{i-1}^-)^2 + (\theta_{i-1}^+ - \theta_{i-1}^-)^2 = -2B_i(\sigma,\theta) - 2\int_{J_i} a(t)\theta^2 dt. \tag{3.24}
$$

Summing from 1 to n ($1 \le n \le k$) for i in (3.24) gives

$$
(\theta_n^-)^2 - (\theta_0^-)^2 + \sum_{i=1}^{n}[\theta_{i-1}]^2 = -2\sum_{i=1}^{n} B_i(\sigma,\theta) - 2\int_{t_0}^{t_n} a(t)\theta^2 dt. \tag{3.25}
$$

Applying the inequality (3.22) (with $\eta = \theta$) to (3.25) yields

$$
(\theta_n^-)^2 + \sum_{i=1}^{n}[\theta_{i-1}]^2 \le (\theta_0^-)^2 + (2\bar{a}_1 + 1)\int_{t_0}^{t_n}\theta^2 dt + c_1 n h^{4m+2}, \quad 1 \le n \le k, \tag{3.26}
$$

where $\bar{a}_1 = \max_{t_0 \le t \le t_k}|a(t)|$. Because $\theta_0^- = 0$ and $\sum_{i=1}^{n}[\theta_{i-1}]^2 \ge 0$, the inequality (3.26) can be reduced to

$$
(\theta_n^-)^2 \le (2\bar{a}_1 + 1)\int_{t_0}^{t_n}\theta^2 dt + c_1 k h^{4m+2}, \quad 1 \le n \le k. \tag{3.27}
$$

When $t > t_0 + \tau$, taking both $\eta = \theta$ and $k < i \le N$ in (3.23) follows that

$$
(\theta_i^-)^2 - (\theta_{i-1}^-)^2 + (\theta_{i-1}^+ - \theta_{i-1}^-)^2 = -2B_i(\sigma,\eta) - 2\int_{J_i}\left[a\theta^2 + b\theta(t-\tau)\right]dt. \tag{3.28}
$$

Summing from $k+1$ to n ($k < n \le N$) for i in (3.28) yields

$$
(\theta_n^-)^2 - (\theta_k^-)^2 + \sum_{i=k+1}^{n}\left[\theta_{i-1}\right]^2 = -2\sum_{i=k+1}^{n} B_i(\sigma,\eta) - 2\int_{t_k}^{t_n}[a\theta^2 + b\theta(t-\tau)]dt. \tag{3.29}
$$

Applying the inequality (3.22) (with $\eta = \theta$) to (3.29), we obtain for $k < n \le N$ that

$$
(\theta_n^-)^2 + \sum_{i=k+1}^{n}[\theta_{i-1}]^2 \le (\theta_k^-)^2 + (2\bar{a}_2 + 1)\int_{t_k}^{t_n}\theta^2 dt + 2\bar{b}_1\int_{t_k}^{t_n}\theta^2(t-\tau)dt + (n-k)c_1 h^{4m+2}, \tag{3.30}
$$

where $\bar{a}_2 = \max_{t_k \le t \le t_N} |a(t)|$ and $\bar{b}_1 = \max_{t_k \le t \le t_N} |b(t)|$. Also, it holds that

$$\int_{t_k}^{t_n} \theta^2(t-\tau)dt = \int_{t_0}^{t_{n-k}} \theta^2(t)dt \le \int_{t_0}^{t_n} \theta^2 dt, \quad k < n \le N. \tag{3.31}$$

By (3.27), (3.30) and (3.31), it can be concluded that

$$(\theta_n^-)^2 \le 2(\bar{a}_1 + \bar{a}_2 + \bar{b}_1 + 1) \int_{t_0}^{t_n} \theta^2 dt + c_1 N h^{4m+2}, \quad k < n \le N. \tag{3.32}$$

Consequently, we have proved that

$$(\theta_n^-)^2 \le \tilde{c}_1 \int_{t_0}^{t_n} \theta^2 dt + \tilde{c}_2 h^{4m+2}, \quad 1 \le n \le N, \tag{3.33}$$

where $\tilde{c}_1 = 2(\bar{a}_1 + \bar{a}_2 + \bar{b}_1 + 1)$ and $\tilde{c}_2 = c_1 N$. Taking

$$\eta = (t - t_{n-1})\theta'(t) \in P_m(J_n)$$

in the second inequality of (3.22), then, when $1 \le n \le k$, we have

$$\begin{aligned}
\int_{J_n} (t-t_{n-1})(\theta')^2 dt \le & - \int_{J_n} (t-t_{n-1}) a\theta\theta' dt + c_1 h^{2m+1} \left[\int_{J_n} [(t-t_{n-1})\theta']^2 dt \right]^{\frac{1}{2}} \\
\le & \bar{a}_1 \int_{J_n} (t-t_{n-1})|\theta\theta'| dt + c_1 h^{2m+\frac{3}{2}} \left[\int_{J_n} (t-t_{n-1})(\theta')^2 dt \right]^{\frac{1}{2}} \\
\le & \bar{a}_1 \left[\varepsilon \int_{J_n} (t-t_{n-1})(\theta')^2 dt + \frac{1}{4\varepsilon} \int_{J_n} (t-t_{n-1})\theta^2 dt \right] \\
& + c_1 \left[\varepsilon \int_{J_n} (t-t_{n-1})(\theta')^2 dt + \frac{1}{4\varepsilon} h^{4m+3} \right] \\
\le & (\bar{a}_1 + c_1)\varepsilon \int_{J_n} (t-t_{n-1})(\theta')^2 dt + \frac{\bar{a}_1 h}{4\varepsilon} \int_{J_n} \theta^2 dt + \frac{c_1}{4\varepsilon} h^{4m+3},
\end{aligned} \tag{3.34}$$

where we have used the inequality $\alpha\beta \le \varepsilon\alpha^2 + \beta^2/4\varepsilon$ with $\varepsilon > 0$ being an arbitrary constant. When $1 \le n \le k$, similarly, we can deduce that

$$\begin{aligned}
\int_{J_n} (t-t_{n-1})(\theta')^2 dt \le & (\bar{a}_2 + \bar{b}_1 + c_1)\varepsilon \int_{J_n} (t-t_{n-1})(\theta')^2 dt + \frac{\bar{a}_2 h}{4\varepsilon} \int_{J_n} \theta^2 dt \\
& + \frac{\bar{b}_1 h}{4\varepsilon} \int_{J_{n-k}} \theta^2 dt + \frac{c_1}{4\varepsilon} h^{4m+3}.
\end{aligned} \tag{3.35}$$

Synthesizing (3.34) and (3.35) yields

$$\begin{aligned}
\int_{J_n} (t-t_{n-1})(\theta')^2 dt \le & \bar{a}\varepsilon \int_{J_n} (t-t_{n-1})(\theta')^2 dt + \frac{\bar{b} h}{4\varepsilon} \int_{J_n} \theta^2 dt + \frac{\bar{b}_1 h}{4\varepsilon} \int_{J_{n-k}} \theta^2 dt + \frac{c_1}{4\varepsilon} h^{4m+3}, \\
& 1 \le n \le N,
\end{aligned} \tag{3.36}$$

where $\bar{a} = \max\{\bar{a}_1 + c_1, \bar{a}_2 + \bar{b}_1 + c_1\}$ and $\bar{b} = \max\{\bar{a}_1, \bar{a}_2\}$. Let $\varepsilon = 1/(2\bar{a})$, then the inequality (3.36) becomes

$$\int_{J_n} (t-t_{n-1})(\theta')^2 dt \le \bar{a}\bar{b} h \int_{J_n} \theta^2 dt + \bar{a}\bar{b}_1 h \int_{J_{n-k}} \theta^2 dt + \bar{a} c_1 h^{4m+3}, \quad 1 \le n \le N. \tag{3.37}$$

By the local inverse property (cf. [5]), there exists a constant $\bar{b}_2 > 0$ such that

$$\begin{aligned}\int_{J_n} (\theta')^2 dt \leq & \bar{b}_2 h^{-1} \int_{J_n} (t - t_{n-1})(\theta')^2 dt \\ \leq & \bar{a}\bar{b}\bar{b}_2 \int_{J_n} \theta^2 dt + \bar{a}\bar{b}_1\bar{b}_2 \int_{J_{n-k}} \theta^2 dt + \bar{a}\bar{b}_2 c_1 h^{4m+2}, \quad 1 \leq n \leq N. \end{aligned} \tag{3.38}$$

Also, the identity

$$\theta(t) = \theta(t_n - 0) + \int_{t_n}^{t} \theta' dt, \quad t \in J_n$$

implies that

$$\theta^2 \leq 2|\theta_n^-|^2 + 2h \int_{J_n} (\theta')^2 dt, \quad t \in J_n, \tag{3.39}$$

which gives

$$\int_{J_n} \theta^2 dt \leq 2h|\theta_n^-|^2 + 2h^2 \int_{J_n} (\theta')^2 dt. \tag{3.40}$$

A combination of (3.33), (3.38) and (3.40) shows that there exist constants $\gamma_1, \gamma_2 > 0$ such that

$$\int_{J_n} \theta^2 dt \leq \gamma_1 h \int_{t_0}^{t_n} \theta^2 dt + \gamma_2 h^{4m+3}. \tag{3.41}$$

Write $\alpha_n = \int_{J_n} \theta^2 dt$. Then (3.41) can be read as

$$\alpha_n \leq \gamma_1 h \sum_{j=1}^{n} \alpha_j + \gamma_2 h^{4m+3}, \quad 1 \leq n \leq N. \tag{3.42}$$

Assume the stepsize h satisfy $0 < \gamma_1 h \leq \gamma_0 < 1$. Then applying the discrete Gronwall inequality to (3.42) yields

$$\begin{aligned}\alpha_n \leq & \frac{\gamma_2}{1 - \gamma_1 h} \exp\left(\frac{\gamma_1 nh}{1 - \gamma_1 h}\right) h^{4m+3} \\ \leq & \frac{\gamma_2}{1 - \gamma_0} \exp\left[\frac{\gamma_1 (t_N - t_0)}{1 - \gamma_0}\right] h^{4m+3}, \quad 1 \leq n \leq N. \end{aligned} \tag{3.43}$$

Namely, it holds that

$$\int_{J_n} \theta^2 dt \leq \gamma_3 h^{4m+3}, \quad 1 \leq n \leq N, \tag{3.44}$$

where $\gamma_3 = \exp[\gamma_1(t_N - t_0)/(1 - \gamma_0)]$. Substituting (3.33) into (3.44), we know that there exists a constant $\gamma_4 > 0$ such that

$$|\theta_n^-| \leq \gamma_4 h^{2m+1}, \quad 1 \leq n \leq N. \tag{3.45}$$

A combination of (3.38), (3.39), (3.44) and (3.45) follows that

$$|\ \theta\ | \leq \gamma h^{2m+1}, \quad t \in J_n, \tag{3.46}$$

where $\gamma > 0$ is a given constant.

Since $\sigma(t) = (u - u_I)(t)$ and $\theta(t) = (u_I - U)(t)$, the global error of the DG finite element is given by

$$e(t) = (\sigma + \theta)(t) = \sum_{j=m+1}^{\infty} b_j(n)\varphi_j(s) + \sum_{j=1}^{m} b_j^*(n)\varphi_j(s) + \mathcal{O}(h^{2m+1}). \tag{3.47}$$

Using the fact of that $\varphi_j(1) = 0$ for all i, we obtain the following superconvergence result at the nodal points:

$$|(u-U)(t_n)| = \mathcal{O}(h^{2m+1}), \quad 1 \le n \le N.$$

Moreover, when taking $s = s_\alpha$ $(\alpha = 1, \cdots, m)$ in (3.47), we can conclude the superconvergence result at the eigenpoints $t^{(\alpha)}_{n-1/2}$:

$$|(u-U)(t^{(\alpha)}_{n-1/2})| = \mathcal{O}(h^{m+2}), \quad 1 \le n \le N; \quad \alpha = 1, \cdots, m.$$

This completes the proof of the theorem. □

4. A Numerical Example

In this section, we present a numerical example to confirm the superconvergence result obtained in the last section. In the numerical experiment, high-order accurate quadrature approximations are used if needed so that error in integration is negligible compared with DG errors.

Consider the following delay differential equation with variable coefficients:

$$\begin{cases} u'(t) = e^{-t}\cos(\pi t)u(t) + t\sin(\pi t)u(t-1) + f(t), & 0 < t \le 6, \\ u(t) = \sin(\pi t), & -1 \le t \le 0, \end{cases} \tag{4.1}$$

where function $f(t)$ is defined such that the system has an exact solution $u(t) = \sin(\pi t)$.

Taking stepsizes $h = 0.1,\ 0.1/2,\ 0.1/4,\ 0.1/8,\ 1/16$, and then applying the one-degree, two-degree DGFE methods to the above system, respectively, we can obtain several sets of finite element solutions. The errors $e(t)$ are plotted in Figs. 4.1 and 4.2. The figures shows that the DG methods are effective for solving the DDEs and superconvergence is observed at both nodal points and eigenpoints. In order to give a further observation for the superconvergence of the methods at the nodal points, we introduce a quantity

$$p = \frac{\ln\left(\max\limits_{1\le n\le N} |e(t_n)|\right)}{\ln(h)}$$

to characterize the convergence order of a DG method on the interval $(0, 6]$. The convergence orders of one-degree and two-degree elements at nodal points are computed in Table 4.1, where it is found that the one-degree element has approximately convergence order 3 and the two-degree element has approximately convergence order 5. This confirms the superconvergence result stated in Theorem 3.1.

Table 4.1: Convergence orders of DG methods at nodal points

stepsize	0.1	0.1/2	0.1/4	0.1/8	0.1/16
1-degree element	3.09	3.09	3.09	3.08	3.08
2-degree element	5.93	5.70	5.56	5.23	5.10

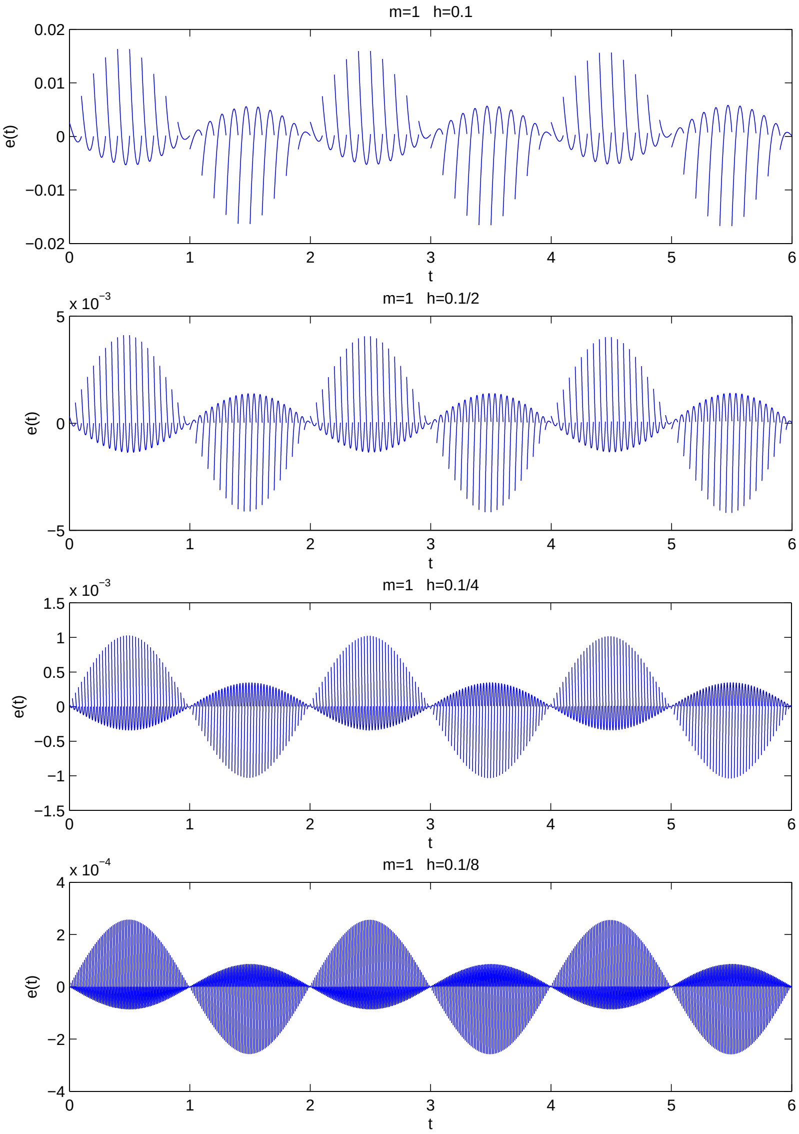

Fig. 4.1. The errors $e(t)$ of one-degree element.

5. Conclusions and Discussions

In the present paper, the DG methods are investigated for solving first-order linear delay differential equations. For the error analysis, we assume that Eq. (2.1) has a smooth solution. The main superconvergence results are obtained with the help of an m-degree polynomial approximation of u in the element J_n. The approximation is constructed by adding some lower order terms in the remainder of expansion so that the remainder satisfies some orthogonal condition in the element. Then a desired superclose function to DG solution is derived. The superconvergence results of the methods are derived by an orthogonal analysis in each element.

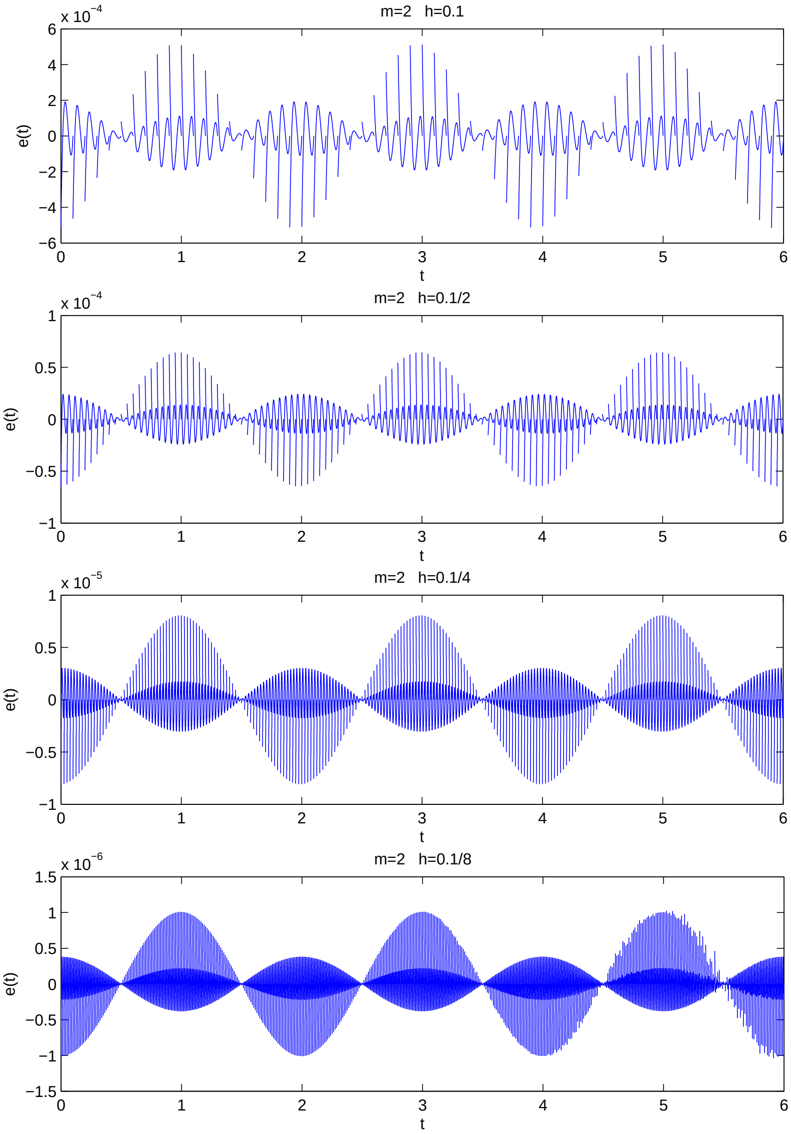

Fig. 4.2. The errors $e(t)$ of two-degree element.

For the case with multiple delays:

$$\begin{cases} u'(t) + a(t)u(t) + \displaystyle\sum_{i=1}^{m} b(t)u(t-\tau_i) = f(t), & t_0 \le t \le T, \\ u(t) = \psi(t), & t \le t_0. \end{cases} \tag{5.1}$$

and the general DDEs with variable delays:

$$\begin{cases} u'(t) + a(t)u(t) + b(t)u(t-\tau(t)) = f(t), & t_0 \le t \le T, \tau(t) \ge 0, \\ u(t) = \psi(t), & t \le t_0, \end{cases} \tag{5.2}$$

the value $u(t-\tau_i)$ or $u(t-\tau(t))$, which is denoted by an m-degree polynomials, may be known when computing in the interval J_n. The inner product $\int_{J_n} b(t)u(t-\tau_i)\eta(t)\ dt$ or

$\int_{J_n} b(t)u(t-\tau(t))\eta(t)\ dt$ can be computed analytically. Therefore, the DG methods may be also a good candidate to solve the above problems. However, it seems difficult to extend the superconvergence result to these cases. This will be investigated in future works.

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References

[1] S. Adjerid, K. Devine, J. Flaherty, L. Krivodonova, A posteriori error estimation for discontinuous Galerkin solutions of hyperbolic problems, *Comput. Method. Appl. M.*, **191** (2002), 1097-1112.
[2] A. Bellen, M. Zennaro, Numerical Methods for Delay Differential Equations, Oxford University Press, Oxford, 2003.
[3] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Differential Equations, Cambridge University Press, Cambridge, 2004.
[4] H. Brunner, Q. Huang, H. Xie, Discontinuous Galerkin methods for delay differential equations of pantograph type, *SIAM J. Numer. Anal.* **48** (2011), 1944-1967.
[5] C. Chen, Structure Theory of superconvergence of Finite Elements, Hunan Press of Science and Technology, Changsha, 2001.
[6] C. Chen, Q. Tang, S. Hu, Finite Element Method with Superconvergence for Nonlinear Hamiltonian Systems, *J. Comput. Math.*, **29** (2011), 167-184.
[7] B. Cockburn, G.E. Karniadakis and C.-W. Shu, The development of the discontinuous Galerkin method, In Discontinuous Galerkin Methods, Theory, Computation and Applications, Lecture Notes in Computational Science and Engineering, Springer-Verlag, **11** (2000), 3-50.
[8] B. Cockburn, C.-W. Shu, Foreword for the special issue on discontinuous Galerkin method, *J. Sci. Comput.*, **22-23** (2005), 1-3.
[9] M. Delfour, W. Hager, F. Trochu, Discontinuous Galerkin methods for ordinary differential equations, *Math. Comput.*, **36** (1981), 455-473.
[10] K. Deng, Z. Xiong, Y. Huang, The Gelerkin continuous finite element method for delay-differential equation with a variable term, *Appl. Math. Comput.*, **186** (2006), 1488-1496.
[11] J.K. Hale, L.S.M. Verduyn, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.
[12] Z. Jiang, The finite element methods for an initial-boundary value problem of the neutral time delay parabolic equations, *Numer. Math. (A J. Chinese Univ.)*, **22** (2000), 305-310.
[13] C. Johnson, Eorror estimates and adaptive time-step control for a class of one-step methods for stiff differential equations, *SIAM J. Numer. Anal.*, **25** (1988), 908-926.
[14] V. Kolmanovskii, A. Myshkis, Introduction to the Theory and Applications of Functional Differential Equations, Kluwer Academic Publishers, Dordrecht, 1999.
[15] D. Li, C. Zhang, Nonlinear stability of discontinuous Galerkin methods for delay differential equation, *Appl. Math. Lett.*, **23**, (2010), 457-461.
[16] D. Li, C. Zhang, H. Qin, LDG method for reaction-diffusion dynamical systems with time delay, *Appl. Math. Comput.*, **217** (2011), 9173-9181.
[17] T. Li, C. Chen, Discontinuous finite elements in solving initial value problems of nonlinear ODE, *Acta Math. Sci. (Ser. A)*, **26** (2006), 63-68.
[18] Z. Xiong, Superconvergence of the continuous Galerkin finite element method for delay-differential equation with several terms, *J. Comput. Appl. Math.*, **198** (2007), 160-166.

[19] C. Zhang, S. Zhou, The asymptotic stability of theoretical and numerical solutions for systems of neutral multidelay-differential equations, *Sci. China (Ser. A)*, **41** (1998), 1151-1157.

[20] C. Zhang, Y. He, The extended one-leg methods for nonlinear neutral delay-integro-differential equations, *Appl. Numer. Math.*, **59** (2009), 1409-1418.