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OPERATOR SPLITTING SCHEMES FOR THE NON-STATIONARY THERMAL CONVECTION PROBLEMS*

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Abstract

In this work, a new numerical scheme is proposed for thermal/isothermal incompressible viscous flows based on operator splitting. Unique solvability and stability analysis are presented. Some numerical result are given, which show that the proposed scheme is highly efficient for the thermal/isothermal incompressible viscous flows.

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1. Introduction

For the time-dependent thermal and isothermal incompressible viscous flow governed by the Boussinesq and the Navier-Stokes equations, the numerical approximation requires the determination of the fluid's velocity, pressure and temperature. A direct approximation technique requires the solution of a very large nonlinear system of equations at each time step. The fractional step θ-method, developed by Glowinski in [1], is an appealing numerical approximation technique [2–4]. It updates the velocity/pressure and temperature using several sub-steps, which leads to decoupling the difficulties associated with the non-linearities and incompressibility condition, thereby reducing the size of the algebraic systems at each sub-step.

In the last decades, a number of numerical methods have been proposed for the numerical simulation of thermal/isothermal incompressible viscous flows. In [5, 6], the numerical simulation is performed in the stream function-vorticity formulation. Hortmann *et al.* [7] considers the same problem, but solves it with finite volumes in primitive variables for the stationary case. Le Qurin [8] provided accurate transient solutions at high Rayleigh number by using pseudo-spectral discretization with Chebyshev polynomials. In [9], numerical schemes for time-dependent incompressible viscous fluid flow, thermally coupled under the Boussinesq approximation, are presented. The schemes combine an operator splitting in the time discretization and linear finite elements in the space discretization.

In this paper, a new numerical scheme is proposed, which combines an θ scheme in time discretization and linear finite elements in the space discretization. The unique solvability and stability analysis of the proposed scheme are presented. Numerical experiments show that the scheme is efficient for simulating of thermal/isothermal incompressible viscous flows.

The remainder of this paper is organized as follows: in the next section, the mathematical model and some basic notation are introduced. In Section 3, we describe the fractional step θ-time stepping scheme which consists of three steps in each interval of time and a detailed description of the numerical solution of the subproblems is present. In Section 4, the unique

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solvability is presented. In Section 5, the proof of stability of the fractional step θ scheme is given. In Section 6, some numerical result are given to illustrate the theoretical results. Some concluding remarks are given in the final section.

2. The Mathematical Model

Under the well-known Boussinesq approximation, the time-dependent flow is governed by the non-dimensional equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \triangle \mathbf{u} + (\mathbf{u}\cdot\triangledown)\mathbf{u} + \triangledown p = \lambda \mathbf{g} T, \\ \triangledown \cdot \mathbf{u} = 0, \\ \dfrac{\partial T}{\partial t} - \xi \Delta T + (\mathbf{u}\cdot\triangledown)T = 0, \end{cases} \tag{2.1}$$

where $\mathbf{x} \in \Omega \subset R^n$ (n=2, 3), Ω is a bounded region in R^n with a sufficiently regular boundary $\partial\Omega$. The unknowns are the vector function $\mathbf{u}$ (velocity), the scalar function p (pressure) and the scalar function T (temperature). The dimensionless parameters Re, Ra, Pr are the Reynolds, Rayleigh and Prandtl number, respectively. g is the gravity vector $\mathbf{g} = (0, 1)$, $\nu = 1/Re$ is the viscosity, and we also define $\lambda = (Ra)/(PrRe^2)$, $\xi = 1/(RePr)$.

For the sake of completeness, Eqs. (2.1) should be supplemented with appropriate initial and boundary condition:

$$\begin{cases} \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x} \in \Omega\, (\nabla \cdot \mathbf{u}_0 = 0), \\ T(\mathbf{x}, 0) = T_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathbf{u} = 0, & \text{on}\, \partial\Omega, \\ T = T_0, & \text{on}\, \partial\Omega, \end{cases} \tag{2.2}$$

Remark 2.1. It follows from [10] that

(1) The coupling between the first and the third equation in (2.1) involving Re corresponding to mixed convection. For natural convection, $Re = 1$ is taken.

(2) For the Navier-Stokes equations, there is no coupling with the thermal energy equation, and the right hand side of the first formula in (2.1) involves a concentration of external forces $\mathbf{f}$ independent of T. Consequently, it is independent of parameters Ra, Pr and Re.

Next, we will introduce some notations and results which will be frequently used in this paper. Let $(\cdot,\cdot)$, $\|\cdot\|$ denote, the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^n$, respectively. The spaces $H_0^1(\Omega)$ and $H_0^1(\Omega)^n$ are equipped with their usual norm:

$$\|\mathbf{u}\|_1^2 = \int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x}.$$

The norm in $H^s(\Omega)$ will be denoted by $\|\cdot\|_s$. We also use $\langle\cdot,\cdot\rangle$ to denote the duality between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$ for all $s > 0$.

In addition, we also introduce the following Hilbert spaces and notations:

$$\begin{aligned}
&X = H_0^1(\Omega)^2, \qquad W = H^1(\Omega), \qquad W_0 = H_0^1(\Omega), \qquad M = L_0^2(\Omega),\\
&a(\mathbf{u},\mathbf{v}) = \nu(\nabla\mathbf{u},\nabla\mathbf{v}), \qquad b(\varphi,\mathbf{v}) = (\varphi,\mathrm{div}\mathbf{v}), \qquad d(T,\psi) = \xi(\nabla T,\nabla\psi),\\
&V = \big\{v\in X; b(\varphi,v)=0, \quad \forall\varphi\in M\big\},\\
&c(\mathbf{u},\mathbf{w},\mathbf{v}) = \int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}dx,\ \bar{c}(\mathbf{u},T,\psi) = \int_\Omega(\mathbf{u}\cdot\nabla)T\cdot\psi dx.
\end{aligned}$$

For the usual Bochner spaces of the time dependent functions with values in some Banach space X, we use the notations

$$L^p(0,T;X) = \left\{u \;\middle|\; u:(0,T)\to X, \text{measure}; \int_0^T \|u(\tau)\|_X^p d\tau < +\infty\right\},$$

with the standard modification for $p=\infty$. Throughout the paper we use C to denote a generic positive constant whose value may change from place to place.

To simplify our presentation, we will assume

[A1] The vector function $\mathbf{f}$ is sufficiently smooth;

[A2] Assume $\partial\Omega\in C^{k,\alpha}(k\geq 0,\alpha>0)$. Then, there exists an extension of T_0 (denoting by T_0) belonging to $C_0^{k,\alpha}(R^2)$ for $T_0\in C^{k,\alpha}(\partial\Omega)$, satisfying

$$\|T_0\|_{k,q}\leq\varepsilon, \qquad k\geq 0, \quad 1\leq q\leq\infty, \tag{2.3}$$

where ε is a sufficiently small positive constant.

In this notation, the weak form of the problem (2.1) can be defined as follows [11].

Definition 2.1. *Find* $(\mathbf{u},p,T)\in L^2(0,t_1;X)\bigcap L^2(0,t_1,V)\times L^2(0,t^1;M)\times L^2(0,t_1;W)$, *satisfying* $T|_{\partial\Omega}=T_0$, *such that*

$$\begin{cases}
(\mathbf{u}_t,\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v})b(p,v) = \lambda(gT,\mathbf{v}), & \forall\mathbf{v}\in\mathbf{X},\\
b(\varphi,\mathbf{u}) = 0, & \forall\varphi\in M,\\
(T_t,\psi) + d(T,\psi) + \bar{c}(\mathbf{u},T,\psi) = 0, & \forall\psi\in W_0,\\
\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(x) \qquad T(\mathbf{x},0) = T_0(x), & \mathbf{x}\in\Omega.
\end{cases} \tag{2.4}$$

Using properly an operator splitting method for the time discretization, we can decouple those difficulties associated with the nonlinearity and the incompressibility condition.

3. Operator Splitting and Steady Subproblems

Fractional step (or splitting) methods can be the non-stationary thermal convection problems in many different ways. We will use the version advocated in Rannacher [12].

3.1. An operator splitting process

Let $\theta_b = 1-2\theta, m = n+1-\theta$ and $\beta = 1-\alpha$. Assuming $\theta\in(0,1/2)$ and $\alpha\in(0,1)$. We divide the time interval $[t_n,t_{n+1}]$ of the length Δt into three subintervals $[t_n,t_{n+\theta}]$, $[t_{n+\theta},\, t_{n+1-\theta}]$, $[t_{n+1-\theta},t_{n+1}]$ of lengths $\theta\Delta t$, $(1-2\theta)\Delta t$ and $\theta\Delta t$, respectively. Using this partition, the splitting form may be described as follows.

First step. Find $\mathbf{u}^{n+\theta}, p^{n+\theta}, T^{n+\theta}$ such that

$$\begin{cases} \dfrac{\mathbf{u}^{n+\theta}-\mathbf{u}^n}{\theta\Delta t} - \alpha\nu\Delta u^{n+\theta} + \nabla p^{n+\theta} = \beta\nu\Delta u^n - (\mathbf{u}^n\cdot\nabla)\mathbf{u}^n + \lambda\mathbf{g}T^n, \\ \nabla\cdot\mathbf{u}^{n+\theta} = 0, \\ \mathbf{u}^{n+\theta}\big|_{\partial\Omega} = 0, \end{cases} \tag{3.1a}$$

$$\begin{cases} \dfrac{T^{n+\theta}-T^n}{\theta\Delta t} - \alpha\xi\Delta T^{n+\theta} + (\mathbf{u}^{n+\theta}\cdot\nabla)T^{n+\theta} = \beta\xi\triangle T^n, \\ T^{n+\theta}\big|_{\partial\Omega} = T_0^{n+\theta}. \end{cases} \tag{3.1b}$$

Second step. Find $\mathbf{u}^m, T^m$ such that

$$\begin{cases} \dfrac{\mathbf{u}^m-\mathbf{u}^{n+\theta}}{\theta_b\Delta t} - \beta\nu\triangle\mathbf{u}^m + (\mathbf{u}^m\cdot\nabla)\mathbf{u}^m - \lambda\mathbf{g}T^m = \alpha\nu\triangle\mathbf{u}^{n+\theta} - \nabla p^{n+\theta}, \\ \mathbf{u}^m\big|_{\partial\Omega} = 0, \end{cases} \tag{3.2a}$$

$$\begin{cases} \dfrac{T^m-T^{n+\theta}}{\theta_b\Delta t} - \beta\xi\triangle T^m = \alpha\xi\triangle T^{n+\theta} - (\mathbf{u}^{n+\theta}\cdot\nabla)T^{n+\theta}, \\ T^{n+1-\theta}\big|_{\partial\Omega} = T_0^{n+1-\theta}. \end{cases} \tag{3.2b}$$

Third step. Find $\mathbf{u}^{n+1}, p^{n+1}, T^{n+1}$ such that

$$\begin{cases} \dfrac{\mathbf{u}^{n+1}-\mathbf{u}^m}{\theta\Delta t} - \alpha\nu\Delta u^{n+1} + \nabla p^{n+1} = \beta\nu\Delta u^m - (\mathbf{u}^m\cdot\nabla)\mathbf{u}^m + \lambda\mathbf{g}T^m, \\ \nabla\cdot\mathbf{u}^{n+1} = 0, \\ \mathbf{u}^{n+1}\big|_{\partial\Omega} = 0, \end{cases} \tag{3.3a}$$

$$\begin{cases} \dfrac{T^{n+1}-T^m}{\theta\Delta t} - \alpha\xi\Delta T^{n+1} + (\mathbf{u}^{n+1}\cdot\nabla)T^{n+1} = \beta\xi\triangle T^m, \\ T^{n+\theta}\big|_{\partial\Omega} = T_0^{n+\theta}, \end{cases} \tag{3.3b}$$

where $\mathbf{u}^n = \mathbf{u}(t_n,\mathbf{x}), p^n = p(t_n,\mathbf{x}), T^n = T(t_n,\mathbf{x}), \mathbf{f}_n = \mathbf{f}(t_n,\mathbf{x})$. We observe that the nonlinearity and the incompressibility in the original equations have been decoupled by using $\theta-$scheme. The choice of α and β is given by

$$\alpha = \frac{1-2\theta}{1-\theta}, \quad \beta = \frac{\theta}{1-\theta}.$$

With such a choice many computer subprograms are common to both the linear and nonlinear subproblems, saving therefore quite a substantial amount of core memory. In addition, numerical experiment show that $\theta = 1 - 1/\sqrt{2}$ seems to produce the best result, even when the Reynolds number is large [13]. Denoting the corresponding right-hand-sides by $\mathbf{f}$ and f, at each time step. Then, the first and third steps of the $\theta-$scheme consist, of solving the following problem:

$$\alpha_1\mathbf{u} - \beta_1\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0, \quad \text{in}\,\Omega, \qquad \mathbf{u} = \mathbf{g_1} \qquad \text{on}\,\Gamma, \tag{3.4}$$

$$\alpha_1 T - \beta_2\Delta T + (\mathbf{u}\cdot\nabla)T = f, \qquad \text{in}\,\Omega, \qquad T = g_2, \qquad \text{on}\,\Gamma. \tag{3.5}$$

The third step of the $\theta-$scheme consist, of solving the following problem:

$$\alpha_2 T - \beta_3\Delta T = f, \qquad \text{in}\,\Omega, \qquad T = g_2, \quad \text{on}\,\Gamma, \tag{3.6}$$

$$\alpha_2\mathbf{u} - \beta_4\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \lambda\mathbf{g}T = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0, \quad \text{in}\,\Omega, \qquad \mathbf{u} = \mathbf{g_1}, \quad \text{on}\,\Gamma, \tag{3.7}$$

where $\alpha_1 = 1/(\theta\Delta t), \alpha_2 = 1/(\theta_b\Delta t), \beta_1 = \alpha\nu, \beta_2 = \alpha\xi, \beta_3 = \beta\xi, \beta_4 = \beta\nu$.

3.2. The solution of the steady subproblems

To solve the problem (3.4), conjugate gradient methods are used as in the isothermal case [14–16], and the term $-r\nabla(\nabla \cdot u)$ is introduced to accelerate the speed of convergence.

For the problems (3.5)–(3.7), we use the fixed point iterative technique to avoid the non-symmetric part of the elliptic operator as in [9, 14]. The reason is that it is cheaper than the conjugate gradient method used by the least-square technique in the corresponding advective subproblems appearing in Glowinski's $\theta-$scheme [9, 16].

The space discretization is based on finite elements. Therefore, variational formulation have to be given for the steady subproblems and then restrict these formulation to appropriate finite element spaces, i.e. during the process, the LBB condition must be satisfied for pressure and velocity. For parameter r, the bigger value is better on the speed of convergence. However, the value of r is too bigger, the accuracy of convergence will be badly affected, where r is selected from the interval $[10^3, 10^4]\nu$.

4. Unique Solvability of the Scheme

Theorem 4.1. (step 3.1a). *Under the condition* (2.3)*, there exists a unique solution* $\mathbf{u}^{n+\theta}$, $p^{n+\theta}$ *satisfying* (3.1*a*)*.*

Proof. Eq. (3.1a) can be written as

$$\mathcal{A}_1(\mathbf{u}^{n+\theta}, \mathbf{v}) = \mathbf{f}_1,$$

where

$$\mathcal{A}_1(\mathbf{u}^{n+\theta}, \mathbf{v}) = \frac{1}{\theta\Delta t}(\mathbf{u}^{n+\theta}, \mathbf{v}) + \alpha a(\mathbf{u}^{n+\theta}, \mathbf{v}) b(p^{n+\theta}, \mathbf{v}),$$
$$\mathbf{f}_1 = \frac{1}{\theta\Delta t}(\mathbf{u}^n, \mathbf{v}) + \lambda(gT^n, \mathbf{v}) - \beta a(\mathbf{u}^n, \mathbf{v}) - c(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}).$$

Taking $\mathbf{v} = \mathbf{u}^{n+\theta}$ in (4.1), and using the relation $\nabla \cdot \mathbf{u}^{n+\theta}$, we obtain

$$\mathcal{A}_1(\mathbf{u}^{n+\theta}, \mathbf{u}^{n+\theta}) = \frac{1}{\theta\Delta t}\|\mathbf{u}^{n+\theta}\|^2 + \alpha\nu|\mathbf{u}^{n+\theta}|_1^2 \geq \min(\frac{1}{\theta\Delta t}, \alpha\nu)\|\mathbf{u}^{n+\theta}\|_1^2.$$

Combining this and the assumption $[A_1]$, we can derive that problem (3.1a) has a unique solution. □

Theorem 4.2. (step 3.1b). *There exist a unique solution* $T^{n+\theta}$ *satisfying* (3.1b).

Proof. Let $T^{n+\theta} = w + T_0,\ w \in W_0$. Taking the inner product for (3.1b) with $\psi \in W_0$ yields

$$\begin{aligned}&\frac{1}{\theta\Delta t}(w, \psi) + \alpha d(w, \psi) + c(\mathbf{u}^{n+\theta}, w, \psi)\\ =&\frac{1}{\theta\Delta t}(T^n - T_0, \psi) - \beta d(T^n, \psi) - \alpha d(T_0, \psi) - c(\mathbf{u}^{n+\theta}, T_0, \psi).\end{aligned} \tag{4.1}$$

Let $\mathcal{A}_2(w,\psi)=\frac{1}{\theta\Delta t}(w,\psi)+\alpha d(w,\psi)+c(\mathbf{u}^{n+\theta},w,\psi)$. Taking $\psi=w$ gives

$$\mathcal{A}_2(w,w)=\frac{1}{\theta\Delta t}(w,w)+\alpha d(w,w)+c(\mathbf{u}^{n+\theta},w,w)=\frac{1}{\theta\Delta t}\|w\|^2+\alpha\xi|w|_1^2.$$

Using Lax-Milgram Theorem and the assumption [A2] completes the proof of the theorem. □

Theorem 4.3. (step 3.2). *There exist a unique solution u^m, T^m satisfying* (3.2).

Let $\psi=T^m$. Then the left hand side of second formula of Eqs. (3.2) can be written as

$$\frac{1}{\theta_b\Delta t}(T^m,\psi)+\beta d(T^m,\psi)=\frac{1}{\theta_b\Delta t}\|T^m\|^2+\beta\xi|T^m|_1^2\ge\min(\frac{1}{\theta_b\Delta t},\beta\xi)\|T^m\|_1^2.$$

Therefore, by virtue of the Lax-Milgram Theorem, existence and uniqueness of the solution T^m has been proven.

For $\mathbf{u}^m$, the first formula of Eq. (3.2) can be written as:

$$\mathcal{A}_3(\mathbf{u}^m,\mathbf{v})=\mathbf{f}_3,$$

where

$$\mathcal{A}_3(\mathbf{u}^m,\mathbf{v})=\frac{1}{\theta_b\Delta t}(\mathbf{u}^m,\mathbf{v})+\beta a(\mathbf{u}^m,\mathbf{v})+c(\mathbf{u}^m,\mathbf{u}^m,\mathbf{v}),$$

$$\mathbf{f}_3=\frac{1}{\theta_b\Delta t}(\mathbf{u}^{n+\theta},\mathbf{v})-\alpha a(\mathbf{u}^{n+\theta},\mathbf{v})+b(p^{n+\theta},\mathbf{v})+\lambda(gT^m,\mathbf{v}).$$

Let $\mathbf{v}$=$\mathbf{u}^m$, combining the fact $\mathbf{u}^m\in X$, we obtain

$$\mathcal{A}_3(\mathbf{u}^m,\mathbf{v})=\frac{1}{\theta_b\Delta t}(\mathbf{u}^m,\mathbf{u}^m)+\beta a(\mathbf{u}^m,\mathbf{u}^m)=\frac{1}{\theta_b\Delta t}\|\mathbf{u}^m\|^2+\beta\nu|\mathbf{u}^m|_1^2.$$

Hence, Ker($\mathcal{A}_3$)={0}, implying that a unique solution exists.

The unique solvability of (3.3a) and (3.3b), representing the third step in the algorithm, follows exactly the same as (3.1a) and (3.1b).

5. Finite Element Approximation and Stability Analysis

In this section, we investigate the numerical approximation method corresponding to (3.1)–(3.3). Firstly, the discrete variational formulation of the θ-method is presented. Then the stability analysis is given.

5.1. Discrete variational approximation

Let $h>0$ be a real positive parameter, T_h be a partitioning of $\bar{\Omega}$ into triangles or quadrilaterals, assumed to be quasi-uniform in the usual sense; i.e., it is regular and satisfies the inverse assumption. Associated with T_h, the finite element subspaces of the approximation of the velocity and pressure and temperature are defined as follows

$$\begin{aligned}
&X_h=\{\mathbf{v}\in X\cap C^0(\bar{\Omega})^d:\mathbf{v}\big|_K\in[P_m(K)]^d\quad\forall K\in T_h\},\\
&M_h=\{\varphi\in M\cap C^0(\bar{\Omega}):\varphi\big|_K\in P_{m-1}(K)\quad\forall K\in T_h\},\\
&W_h=\{\psi\in W\cap C^0(\bar{\Omega}):\psi\big|_K\in P_m(K)\qquad\forall K\in T_h\},\\
&W_{0h}=W_h\cap H_0^1,\\
&V_h=\{\mathbf{v}_h\in X_h\big|\,b(\varphi_h,\mathbf{v}_h)=0,\quad\forall\varphi_h\in M_h\}.
\end{aligned}$$

Analogically to the continuous space we assume that X_h and M_h satisfy the discrete inf-sup condition:

$$\inf_{q\in M_h}\sup_{\mathbf{v}\in X_h}\frac{(q,\nabla\cdot\mathbf{v})}{\|q\|\|\mathbf{v}\|_1}\geq\beta>0. \tag{5.1}$$

Since the norm is equivalent in the finite dimensional spaces, $\forall\mathbf{u}_h\in V_h$, we have

$$D_1|\mathbf{u}_h|\leq\|\mathbf{u}_h\|\leq D_2h^{-1}|\mathbf{u}_h|. \tag{5.2}$$

The constant D_2 depends on the degree of polynomial approximation. In the following we will assume that $D_1=D_2=1$

For each $\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h\in X_h$, we define

$$c(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h)=\frac{1}{2}\Big[((\mathbf{u}_h\cdot\nabla_h)\mathbf{v}_h,\mathbf{w}_h)-((\mathbf{u}_h\cdot\nabla_h)\mathbf{w}_h,v_h)\Big]. \tag{5.3}$$

It holds that

$$c(\mathbf{u}_h,\mathbf{v}_h,\mathbf{v}_h)=0\qquad\forall\mathbf{u},\mathbf{v}\in X_h. \tag{5.4}$$

There exists a function $S(h)$ so that

$$|c(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h)|\leq S(h)|\mathbf{u}_h|\|\mathbf{v}_h\||\mathbf{w}_h|, \tag{5.5}$$

and $S(h)=D_3h^{-1}$ in the conforming case.

From now on we will consider the following fully discrete problem.

Step 1. Find $\mathbf{u_h}^{n+\theta}\in V_h$, $T_h^{n+\theta}\in W_h$, for $\forall\mathbf{v}_h\in V_h,\psi_h\in W_{0h}$, such that

$$\begin{cases}\dfrac{1}{\theta\Delta t}(\mathbf{u}_h^{n+\theta},\mathbf{v}_h)+\alpha a(\mathbf{u}_h^{n+\theta},\mathbf{v}_h)=\\ \qquad\dfrac{1}{\theta\Delta t}(\mathbf{u}_h^{n},\mathbf{v}_h)+\lambda(gT^n,\mathbf{v}_h)-\beta a(\mathbf{u}_h^n,\mathbf{v}_h)-c(\mathbf{u}_h^n,\mathbf{u}_h^n,\mathbf{v}_h),\end{cases} \tag{5.6a}$$

$$\begin{cases}\dfrac{1}{\theta\Delta t}(T_h^{n+\theta},\psi_h)+\alpha d(T_h^{n+\theta},\psi_h)+\bar{c}(\mathbf{u}_h^{n+\theta},T_h^{n+\theta},\psi_h)\\ \qquad\qquad=\dfrac{1}{\theta\Delta t}(T_h^n,\psi_h)-\beta d(T_h^n,\psi_h),\\ T_h^{n+\theta}\big|_{\partial\Omega}=T_0^{n+\theta}.\end{cases} \tag{5.6b}$$

Step 2. Find $\mathbf{u}_h^m\in X_h,T_h^m\in W_h$, for $\forall\mathbf{v}_h\in X_h,\psi_h\in W_{0h}$, such that

$$\begin{cases}\dfrac{1}{\theta_b\Delta t}(\mathbf{u}_h^m,\mathbf{v}_h)+\beta a(\mathbf{u}_h^m,\mathbf{v}_h)+c(\mathbf{u}_h^m,\mathbf{u}_h^m,\mathbf{v}_h)-\lambda(gT_h^m,\mathbf{v}_h)\\ \qquad=\dfrac{1}{\theta_b\Delta t}(\mathbf{u}_h^{n+\theta},\mathbf{v}_h)-\alpha a(\mathbf{u}_h^{n+\theta},\mathbf{v}_h)+b(p_h^{n+\theta},\mathbf{v}_h),\\ \mathbf{u}_h^m\big|_{\partial\Omega}=0,\end{cases} \tag{5.7a}$$

$$\begin{cases}\dfrac{1}{\theta_b\Delta t}(T_h^m,\psi_h)+\beta d(T_h^m,\psi_h)\\ \qquad=\dfrac{1}{\theta_b\Delta t}(T_h^{n+\theta},\psi_h)-\alpha d(T_h^{n+\theta},\psi_h)-\bar{c}(\mathbf{u}_h^{n+\theta},T_h^{n+\theta},\psi_h),\\ T_h^m\big|_{\partial\Omega}=T_0^m.\end{cases} \tag{5.7b}$$

Step 3. Find $\mathbf{u}_h^{n+1} \in V_h, T_h^{n+1} \in W_h$, for $\forall \mathbf{v}_h \in V_h, \psi_h \in W_{0h}$, such that

$$\left\{\begin{array}{l}\frac{1}{\theta\Delta t}(\mathbf{u}_h^{n+1},\mathbf{v}_h)+\alpha a(\mathbf{u}_h^{n+1},\mathbf{v}_h)=\frac{1}{\theta\Delta t}(\mathbf{u}_h^{n+1-\theta},\mathbf{v}_h)+\lambda(gT_h^{n+1-\theta},\mathbf{v}_h)\\ \qquad -\beta a(\mathbf{u}_h^{n+1-\theta},\mathbf{v}_h)-c(\mathbf{u}_h^{n+1-\theta},\mathbf{u}_h^{n+1-\theta},\mathbf{v}_h),\end{array}\right. \tag{5.8a}$$

$$\left\{\begin{array}{l}\frac{1}{\theta\Delta t}(T_h^{n+1},\psi)+\alpha d(T_h^{n+1},\psi)+\bar{c}(\mathbf{u}_h^{n+1},T_h^{n+1},\psi_h)\\ \qquad\qquad =\frac{1}{\theta\Delta t}(T_h^{n+1-\theta},\psi_h)-\beta d(T_h^{n+1-\theta},\psi_h),\\ T_h^{n+1}\big|_{\partial\Omega}=T_0^{n+1}.\end{array}\right. \tag{5.8b}$$

5.2. Stability of the scheme

The stability follows the framework developed in [17, 18]. To simplify our presentation, we assume homogeneous boundary conditions.

Lemma 5.1. *If* $\mathbf{u}_h^{n+\theta}, T_h^{n+\theta}$ *satisfy Eqs.* (5.6a) and (5.6b)*, then the estimates below are derived:*

$$\begin{aligned}&|\mathbf{u}_h^{n+\theta}|^2+\theta\Delta t\alpha\nu\|\mathbf{u}_h^{n+\theta}\|^2+\theta\Delta t\beta\nu\|\mathbf{u}_h^n\|^2+\left(\frac{1}{2}-\frac{\theta\Delta t\beta\nu}{h^2}\right)|\mathbf{u}_h^{n+\theta}-\mathbf{u}_h^n|^2\\ &\quad\le|\mathbf{u}_h^n|^2+(\theta_b\Delta tS(h))^2|\mathbf{u}_h^n|^2\|\mathbf{u}_h^n\|^2+\frac{\theta\Delta t\lambda^2}{\nu}|T_h^n|^2,\end{aligned} \tag{5.9}$$

$$|T_h^{n+\theta}|^2+\xi\theta\Delta t\|T_h^{n+\theta}\|^2+(1-\frac{\beta\xi\theta\Delta t}{h^2})|T_h^{n+\theta}-T_h^n|^2=\left(1-\frac{\beta\xi\theta\Delta t}{h^2}\right)|T_h^n|^2. \tag{5.10}$$

Proof. We take $\mathbf{v}_h=\mathbf{u}_h^{n+\theta}$ in (5.6a) to get

$$\begin{aligned}&\frac{1}{\theta\Delta t}|\mathbf{u}_h^{n+\theta}|^2+\alpha\nu\|\mathbf{u}_h^{n+\theta}\|^2\\ =&\frac{1}{\theta\Delta t}(\mathbf{u}_h^n,\mathbf{u}_h^{n+\theta})+\lambda(\mathbf{g}T_h^n,\mathbf{u}_h^{n+\theta})-\beta a(\mathbf{u}_h^n,\mathbf{u}_h^{n+\theta})-c(\mathbf{u}_h^n,\mathbf{u}_h^n,\mathbf{u}_h^{n+\theta}).\end{aligned} \tag{5.11}$$

Using the identity $(\mathbf{u},\mathbf{v})=\frac{1}{2}[|\mathbf{u}|^2+|\mathbf{v}|^2-|\mathbf{u}-\mathbf{v}|^2]$, the right-hand side of (5.11) becomes

$$\begin{aligned}&\frac{1}{2\theta\Delta t}|\mathbf{u}_h^{n+\theta}|^2+\frac{1}{2\theta\Delta t}|\mathbf{u}_h^n|^2-\frac{1}{2\theta\Delta t}|\mathbf{u}_h^{n+\theta}-\mathbf{u}_h^n|^2+\frac{\beta\nu}{2}\|\mathbf{u}_h^{n+\theta}-\mathbf{u}_h^n\|^2\\ &\quad-\frac{\beta\nu}{2}\|\mathbf{u}_h^{n+\theta}\|^2-\frac{\beta\nu}{2}\|\mathbf{u}_h^n\|^2+\lambda(T_h^n,\mathbf{u}_h^{n+\theta})-c(\mathbf{u}_h^n,\mathbf{u}_h^n,\mathbf{u}_h^{n+\theta}).\end{aligned}$$

It follows from (5.2) that $\|\mathbf{u}_h^{n+\theta}-\mathbf{u}_h^n\|^2\le\frac{1}{h^2}|\mathbf{u}_h^{n+\theta}-\mathbf{u}_h^n|^2$. Consequently,

$$\begin{aligned}&|\mathbf{u}_h^{n+\theta}|^2+(1+\alpha)\theta\Delta t\nu\|\mathbf{u}_h^{n+\theta}\|^2+\theta\Delta t\beta\nu\|\mathbf{u}_h^n\|^2+\left(1-\frac{\theta\Delta t\beta\nu}{h^2}\right)|\mathbf{u}_h^{n+\theta}-\mathbf{u}_h^n|^2\\ \le&|\mathbf{u}_h^n|^2-2\theta\Delta tc(\mathbf{u}_h^n,\mathbf{u}_h^n,\mathbf{u}_h^{n+\theta})+2\theta\Delta t\lambda(\mathbf{g}T_h^n,\mathbf{u}_h^{n+\theta}).\end{aligned} \tag{5.12}$$

We estimate the right-hand side of (5.12) by Young's inequality and (5.5). Because $0<\theta<1-\sqrt{2}/2$ we have $2\theta^2<\theta_b^2$. So, we obtain

$$|\mathbf{u}_h^n|^2+2\theta^2(\Delta tS(h))^2|\mathbf{u}_h^n|^2\|\mathbf{u}_h^n\|^2+\frac{1}{2}|\mathbf{u}_h^{n+\theta}-\mathbf{u}_h^n|^2+\theta\Delta t\nu\|\mathbf{u}_h^{n+\theta}\|^2+\frac{\theta\Delta t\lambda^2}{\nu}|T_h^n|^2. \tag{5.13}$$

Combining (5.13) with (5.12),we obain (5.9).

Taking $\mathbf{T}_h = \mathbf{T}_h^{n+\theta}$ in (5.6b) to get

$$\frac{1}{\theta\Delta t}|T_h^{n+\theta}|^2 + \alpha\xi\|T_h^{n+\theta}\|^2 = \frac{1}{\theta\Delta t}(T_h^n, T_h^{n+\theta}) - \beta\xi(\nabla T_h^n, \nabla T_h^{n+\theta}).$$

Making use of $(\mathbf{u}, \mathbf{v}) = [|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{u}-\mathbf{v}|^2]/2$, then the formula (5.10) is derived. □

Lemma 5.2. *If $\mathbf{u}_h^{n+1-\theta}, T_h^{n+1-\theta}$ satisfy Eq.* (5.7), *the estimates below are derived:*

$$\begin{aligned}
&|\mathbf{u}_h^m|^2 + \theta_b\Delta t\beta\nu\|\mathbf{u}_h^m\|^2 + \theta_b\Delta t\alpha\nu\|\mathbf{u}_h^{n+\theta}\|^2 + (1-3\delta)|\mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}|^2\\
\le&|\mathbf{u}_h^{n+\theta}|^2 + D_3\left(\left(\nu\frac{\Delta t}{h}\right)^2\|\mathbf{u}_h^n\|^2 + (\Delta ts)^2|\mathbf{u}_h^n|^2\|\mathbf{u}_h^n\|^2 + \frac{\Delta t\lambda^2}{\alpha\nu}|T_h^n|^2 + \frac{\Delta t\lambda^2}{\beta\nu}|T_h^m - T_h^n|^2\right), \qquad (5.14)\\
&|T_h^m|^2 + \theta_b\Delta t\beta\xi\|T_h^m\|^2 + \left(\frac{1}{2} - \frac{\theta_b\Delta t\beta\xi}{h^2}\right)|T_h^{n+\theta} - T_h^m|^2 + \theta_b\Delta t\alpha\xi\|T_h^{n+\theta}\|^2\\
=&|T_h^{n+\theta}|^2 + \theta_b\Delta t\beta\xi\|T_h^n\|^2 + c|T_h^{n+\theta} - T_h^n|^2, \qquad (5.15)
\end{aligned}$$

where $D_3 = (\theta_b/\delta)^2(1+(\theta_b/\delta)^2), 2(\theta_b/\theta)^2 < c$.

Proof. We take $\mathbf{v}_h = \mathbf{u}_h^m$ in (5.7a) to get

$$\frac{1}{\theta_b\Delta t}|\mathbf{u}_h^m|^2 + \beta\nu\|\mathbf{u}_h^m\|^2 - \lambda(\mathbf{g}T_h^m, \mathbf{u}_h^m) = \frac{1}{\theta_b\Delta t}(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^m) - \alpha a(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^m) + (p_h^{n+\theta}, \nabla\cdot\mathbf{u}_h^m),$$

where

$$\begin{aligned}
(p_h^{n+\theta}, \nabla\cdot\mathbf{u}^m) &= (p_h^{n+\theta}, \nabla\cdot\mathbf{u}_h^m - \mathbf{u}_h^{n+\theta})\\
&= \frac{1}{\theta\Delta t}(\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}) + \alpha a(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta})\\
&\quad + \beta a(\mathbf{u}_h^n, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}) + c(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}) - \lambda(\mathbf{g}T_h^n, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}).
\end{aligned}$$

By virtue of the above formula, we obtain

$$\begin{aligned}
&|\mathbf{u}_h^m|^2 + \theta_b\Delta t\beta\nu\|\mathbf{u}_h^m\|^2 - \theta_b\Delta t\lambda(\mathbf{g}T_h^m, \mathbf{u}_h^m)\\
=&(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^m) - \theta_b\Delta t\alpha a(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^m) + \theta_b\Delta t(p_h^{n+\theta}, \nabla_h\cdot\mathbf{u}_h^m)\\
=&(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^m) - \theta_b\Delta t\alpha a(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^m) + \frac{\theta_b}{\theta}(\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta})\\
&+ \theta_b\Delta t\alpha a(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}) + \theta_b\Delta t\beta a(\mathbf{u}_h^n, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}) + \theta_b\Delta tc(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta})\\
&- \theta_b\Delta t\lambda(\mathbf{g}T_h^n, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}).
\end{aligned}$$

Combining this with the expression $(u, v) = (|u|^2 + |v|^2 - |u-v|^2)/2$, we have

$$\begin{aligned}
&\frac{1}{2}|\mathbf{u}_h^m|^2 + \theta_b\Delta t\beta\nu\|\mathbf{u}_h^m\|^2 + \frac{1}{2}|\mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}|^2 - \theta_b\Delta t\lambda(\mathbf{g}(T_h^m - T_h^n), \mathbf{u}_h^m)\\
\le&\frac{1}{2}|\mathbf{u}_h^{n+\theta}|^2 - \theta_b\Delta t\alpha a(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^m) + \frac{\theta_b}{\theta}(\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}) - \theta_b\Delta t\alpha\nu\|\mathbf{u}_h^{n+\theta}\|^2\\
&+ \theta_b\Delta t\alpha a(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^m) + \theta_b\Delta t\beta a(\mathbf{u}_h^n, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}) + \theta_b\Delta tc(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^m - \mathbf{u}_h^{n+\theta})\\
&+ \theta_b\Delta t\lambda(\mathbf{g}T_h^n, \mathbf{u}_h^{n+\theta}). \qquad (5.16)
\end{aligned}$$

Using Young's inequality with $0 < \delta < 1/3$, estimates (5.2) and (5.5), we arrive at

$$\begin{aligned}&|\mathbf{u}_h^m|^2 + \theta_b \Delta t \beta\nu \|\mathbf{u}_h^m\|^2 + \theta_b \Delta t \alpha\nu \|\mathbf{u}_h^{n+\theta}\|^2 + (1-3\delta)|\mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}|^2 - \frac{\theta_b \Delta t \lambda^2}{\beta\nu}|T_h^m - T_h^n|^2 \\ \leq& |\mathbf{u}_h^{n+\theta}|^2 + \frac{1}{2\delta}(\frac{\theta_b}{\theta})^2 |\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n|^2 + \frac{1}{2\delta}(\frac{\theta_b \Delta t \beta\nu}{h^2})^2 \|\mathbf{u}_h^n\|^2 \\ &+ \frac{1}{2\delta}(\theta_b \Delta t s)^2 |\mathbf{u}_h^n|^2 \|\mathbf{u}_h^n\|^2 + \frac{\theta_b \Delta t \lambda^2}{\alpha\nu}|T_n|^2. \end{aligned} \tag{5.17}$$

The equation for the pressure with $\mathbf{v} = \mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n \in V_{0h}$ reads

$$\begin{aligned} 0 =& (p_h^{n+\theta}, \nabla \cdot (\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n)) \\ =& \frac{1}{\theta \Delta t}|\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n|^2 + \alpha a(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n) + \beta a(\mathbf{u}_h^n, \mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n) \\ &+ c(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n) - \lambda(\mathbf{g}T_h^n, \mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n), \end{aligned} \tag{5.18}$$

which leads to

$$\begin{aligned} &|\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n|^2 + \theta \Delta t \alpha\nu \|\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n\|^2 \\ \leq& \left(\frac{\theta_b \Delta t \nu}{h}\right)^2 \|\mathbf{u}_h^n\|^2 + (\theta_b \Delta t s)^2 |\mathbf{u}_h^n|^2 \|\mathbf{u}_h^n\|^2 + \frac{\theta \Delta t \lambda^2}{\alpha\nu}|T_h^n|^2. \end{aligned} \tag{5.19}$$

Making use of the estimate (5.19) and the inequality (5.17), we obtain the estimate (5.14).

For the temperature T_h^m, taking $\psi_h = T_h^m$ in (5.7b) gives

$$\frac{1}{\theta_b \Delta t}|T_h^m|^2 + \beta\xi \|T_h^m\|^2 = \frac{1}{\theta_b \Delta t}(T_h^{n+\theta}, T_h^m) - \alpha\xi(\nabla T_h^{n+\theta}, \nabla T_h^m) - \bar{c}(\mathbf{u}_h^{n+\theta}, T_h^{n+\theta}, T_h^m). \tag{5.20}$$

From (5.6b), we obtain

$$\begin{aligned} &\bar{c}(\mathbf{u}_h^{n+\theta}, T_h^{n+\theta}, T_h^m - T_h^{n+\theta}) \\ =& \frac{1}{\theta \Delta t}(T_h^n - T_h^{n+\theta}, T_h^m - T_h^{n+\theta}) - \beta\xi(\nabla T_h^n, \nabla(T_h^m - T_h^{n+\theta})) - \alpha\xi(\nabla T_h^{n+\theta}, \nabla(T_h^m - T_h^{n+\theta})). \end{aligned}$$

Combing (5.20) with the above formula and also using $(\mathbf{u}, \mathbf{v}) = [|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{u} - \mathbf{v}|^2]/2$, we obtain

$$\begin{aligned} &|T_h^m|^2 + |T_h^{n+\theta} - T_h^m|^2 + 2\theta_b \Delta t \beta\xi \|T_h^m\|^2 \\ =& |T_h^{n+\theta}|^2 - 2\theta_b \Delta t \alpha\xi \|T_h^{n+\theta}\|^2 + 2\theta_b \Delta t \beta\xi \big(\nabla T_h^n, \nabla(T_h^m - T_h^{n+\theta})\big) \\ &+ \frac{2\theta_b}{\theta}(T_h^{n+\theta} - T_h^n, T_h^m - T_h^{n+\theta}). \end{aligned}$$

Using the Young inequality, we get (5.15). □

Lemma 5.3. *If $\mathbf{u}_h^{n+1-\theta}, T_h^{n+1-\theta}$ satisfy Eq.* (5.7)*, then the estimates below hold:*

$$\begin{aligned} &|\mathbf{u}_h^{n+1}|^2 + \theta \Delta t \alpha\nu \|\mathbf{u}_h^{n+1}\|^2 + \theta \Delta t \beta\nu \|\mathbf{u}_h^m\|^2 + \left(\frac{1}{4} - \frac{2\theta \Delta t \beta\nu}{h^2}\right)|\mathbf{u}_h^{n+1} - \mathbf{u}_h^m|^2 \\ \leq& |\mathbf{u}_h^m|^2 + (1-3\delta)|\mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}|^2 + \frac{33}{8}|\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n|^2 + (2\theta_b \Delta t s)^2 |\mathbf{u}_h^n|^2 \|\mathbf{u}_h^n\|^2 \\ &+ (2\theta_b \Delta t \lambda)^2 |T_h^m - T_h^n|^2 + \frac{\theta \Delta t \lambda^2}{\nu}|T_h^m|^2, \end{aligned} \tag{5.21}$$

$$|T_h^{n+1}|^2 + \xi\theta \Delta t \|T_h^{n+1}\|^2 + \left(1 - \frac{\beta\xi\theta\Delta t}{h^2}\right)|T_h^{n+1} - T_h^m|^2 = \left(1 - \frac{\beta\xi\theta\Delta t}{h^2}\right)|T_h^m|^2. \tag{5.22}$$

Proof. We take $\mathbf{v}_h = \mathbf{u}_h^{n+1}$ in (4.8) to get

$$\begin{aligned}&\frac{1}{\theta\Delta t}|\mathbf{u}_h^{n+1}|^2 + \alpha\nu\|\mathbf{u}_h^{n+1}\|^2\\ =&\frac{1}{\theta\Delta t}(\mathbf{u}_h^m, \mathbf{u}_h^{n+1}) - \beta a(\mathbf{u}_h^m, \mathbf{u}_h^{n+1}) - c(\mathbf{u}_h^m, \mathbf{u}_h^m, \mathbf{u}_h^{n+1}) + \lambda(\mathbf{g}T_h^m, \mathbf{v}_h^{n+1}).\end{aligned} \tag{5.23}$$

As in Lemma 5.1, we obtain

$$\begin{aligned}&|\mathbf{u}_h^{n+1}|^2 + (1+\alpha)\theta\Delta t\nu\|\mathbf{u}_h^{n+1}\|^2 + \theta\Delta t\beta\nu\|\mathbf{u}_h^m\|^2 + \left(1 - \frac{\theta\Delta t\beta\nu}{h^2}\right)|\mathbf{u}_h^{n+1} - \mathbf{u}_h^m|^2\\ \le&|\mathbf{u}_h^m|^2 - 2\theta\Delta t c(\mathbf{u}_h^m, \mathbf{u}_h^m, \mathbf{u}_h^{n+1}) + 2\theta\Delta t\lambda(\mathbf{g}T_h^m, \mathbf{u}_h^{n+1}).\end{aligned} \tag{5.24}$$

To estimate $c(\mathbf{u}_h^m, \mathbf{u}_h^m, \mathbf{u}_h^{n+1})$, we use (5.7a) with $\mathbf{v}_h = \mathbf{u}_h^{n+1} - \mathbf{u}_h^m$ to obtain

$$\begin{aligned}&- c(\mathbf{u}_h^m, \mathbf{u}_h^m, \mathbf{u}_h^{n+1}) = -c(\mathbf{u}_h^m, \mathbf{u}_h^m, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m)\\ =&\frac{1}{\theta_b\Delta t}(\mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m) + \beta a(\mathbf{u}_h^m, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m) + \alpha a(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m)\\ &- \lambda(\mathbf{g}T_h^m, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m) b(p_h^{n+\theta}, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m).\end{aligned} \tag{5.25}$$

For pressure, we take $v_h = u_h^{n+1} - u_h^m$

$$\begin{aligned}&b(p_h^{n+\theta}, u_h^{n+1} - u_h^m)\\ =&-\frac{1}{\theta\Delta t}(\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m) - \alpha a(\mathbf{u}_h^{n+\theta}, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m) - \beta a(\mathbf{u}_h^n, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m)\\ &- c(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m) + \lambda(\mathbf{g}T_h^n, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m).\end{aligned} \tag{5.26}$$

Combining (5.25) with (5.26), we obtain

$$\begin{aligned}&- 2\theta\Delta t c(\mathbf{u}_h^m, \mathbf{u}_h^m, \mathbf{u}_h^{n+1})\\ =&\frac{2\theta}{\theta_b}(\mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m) - 2(\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m)\\ &+ 2\theta\Delta t\beta a(\mathbf{u}_h^m - \mathbf{u}_h^n, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m) - 2\theta\Delta t c(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m)\\ &- 2\theta\Delta t\lambda(\mathbf{g}(T_h^m - T_h^n), \mathbf{u}_h^{n+1} - \mathbf{u}_h^m)\end{aligned} \tag{5.27}$$

We therefore obtain the following inequality from (5.24) and (5.27):

$$\begin{aligned}&|\mathbf{u}_h^{n+1}|^2 + (1+\alpha)\theta\Delta t\nu\|\mathbf{u}_h^{n+1}\|^2 + \theta\Delta t\beta\nu\|\mathbf{u}_h^m\|^2 + \left(1 - \frac{\theta\Delta t\beta\nu}{h^2}\right)|\mathbf{u}_h^{n+1} - \mathbf{u}_h^m|^2\\ \le&|\mathbf{u}_h^m|^2 - 2\theta\Delta t c(\mathbf{u}_h^m, \mathbf{u}_h^m, \mathbf{u}_h^{n+1}) + 2\theta\Delta t\lambda(\mathbf{g}T_h^m, \mathbf{u}_h^{n+1})\\ \le&|\mathbf{u}_h^m|^2 + \frac{2\theta}{\theta_b}(\mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m) - 2(\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m)\\ &+ 2\theta\Delta t\beta a(\mathbf{u}_h^m - \mathbf{u}_h^n, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m) - 2\theta\Delta t c(\mathbf{u}_h^n, \mathbf{u}_h^n, \mathbf{u}_h^{n+1} - \mathbf{u}_h^m)\\ &- 2\theta\Delta t\lambda(\mathbf{g}(T_h^m - T_h^n), \mathbf{u}_h^{n+1} - \mathbf{u}_h^m) + 2\theta\Delta t\lambda(\mathbf{g}T_h^m, \mathbf{u}_h^{n+1}).\end{aligned} \tag{5.28}$$

Using Young's inequalities, (5.28) can be written as

$$\begin{aligned}&|\mathbf{u}_h^{n+1}|^2 + \theta\Delta t\alpha\nu\|\mathbf{u}_h^{n+1}\|^2 + \theta\Delta t\beta\nu\|\mathbf{u}_h^m\|^2 + \left(\frac{1}{4} - \frac{2\theta\Delta t\beta\nu}{h^2}\right)|\mathbf{u}_h^{n+1} - \mathbf{u}_h^m|^2\\ \le&|\mathbf{u}_h^m|^2 + \left[2\left(\frac{\theta}{\theta_b}\right)^2 + \frac{1}{8}\right]|\mathbf{u}_h^m - \mathbf{u}_h^{n+\theta}|^2 + \frac{33}{8}|\mathbf{u}_h^{n+\theta} - \mathbf{u}_h^n|^2 + (2\theta_b\Delta ts)^2|\mathbf{u}_h^n|^2\|\mathbf{u}_h^n\|^2\\ &+ (2\theta_b\Delta t\lambda)^2|T_h^m - T_h^n|^2 + \frac{\theta\Delta t\lambda^2}{\nu}|T_h^m|^2.\end{aligned} \tag{5.29}$$

For $\theta \in (0, (-7+4\sqrt{7})/9) \subset (0, 1-\sqrt{2}/2)$, there exist a δ, such that $2(\theta/\theta_b)^2 + 1/8 \leq 1 - 3\delta$. So we obtain (5.21).

For T_h^{n+1}, similar to the proof of Lemma 5.1, we can derive (5.22). □

Theorem 5.1. *For $\theta \in (0, (-7+4\sqrt{7})/9)$ and corresponding δ, if $(\theta_b \Delta t \beta \xi)/h^2 < 1/2$ and $(1-(\beta\xi\theta\Delta t)/h^2)(1+c) < 1$ hold, then we have*

$$T_h^{n+1}, T_h^m, T_h^{n+\theta} \in L^\infty\big(0,T,L^2(\Omega)\big) \bigcap L^2(0,T,W_h).$$

Proof. Combing (5.10),(5.15) and (5.22), the estimate below is derived

$$\begin{aligned} &|T_h^{n+1}|^2 + \xi\theta\Delta t \|T_h^{n+1}\|^2 + \left(1 - \frac{\beta\xi\theta\Delta t}{h^2}\right)|T_h^{n+1} - T_h^m|^2 \\ =&(1 - \frac{\beta\xi\theta\Delta t}{h^2})|T_h^m|^2 \leq \left(1 - \frac{\beta\xi\theta\Delta t}{h^2}\right)(1+c)|T_h^n|^2. \end{aligned} \tag{5.30}$$

If $(1-(\beta\xi\theta\Delta t)/h^2)(1+c) < 1$, then we obtain

$$|T_h^{n+1}|^2 + \xi\theta\Delta t \|T_h^{n+1}\|^2 + \left(1 - \frac{\beta\xi\theta\Delta t}{h^2}\right)|T_h^{n+1} - T_h^m|^2 \leq |T_h^n|^2. \tag{5.31}$$

Summing up (5.31) for $n = 0, 1, 2, \cdots, r$, with $r \in Z^+$, we have

$$|T_h^{r+1}|^2 + \xi\theta\Delta t \sum_{n=0}^{r} \|T_h^{n+1}\|^2 + \left(1 - \frac{\beta\xi\theta\Delta t}{h^2}\right) \sum_{n=0}^{r} |T_h^{n+1} - T_h^m|^2 \leq |T_{0h}|^2 \leq |T_0|^2.$$

Therefore

$$T_h^{n+1} \in L^\infty(0,T,L^2(\Omega)) \bigcap L^2(0,T,W_h).$$

According to (5.10), we see that

$$T_h^{n+\theta} \in L^\infty(0,T,L^2(\Omega)) \bigcap L^2(0,T,W_h).$$

From (5.15), we have

$$T_h^m \in L^\infty(0,T,L^2(\Omega)) \bigcap L^2(0,T,W_h).$$

This completes the proof of the theorem. □

Lemma 5.4. *For any $0 < \delta < 1$ and $r \in Z^+$, if the condition*

$$C\left\{\left(\frac{\nu\Delta t}{h}\right)^2 + (\Delta t s)^2 \Lambda_T\right\} \leq (1-\delta)\alpha\theta\Delta t\nu. \tag{5.32}$$

holds, then we have

$$|\mathbf{u}_h^{r+1}|^2 + \delta\theta\Delta t\alpha\nu \sum_{n=0}^{r} \|\mathbf{u}_h^{n+1}\|^2 + \theta\Delta t\beta\nu \sum_{n=0}^{r} \|\mathbf{u}_h^m\|^2 + \left(\frac{1}{4} - \frac{2\theta\Delta t\beta\nu}{h^2}\right) \sum_{n=0}^{r} |\mathbf{u}_h^{n+1} - \mathbf{u}_h^m|^2 \leq \Lambda_r. \tag{5.33}$$

Proof. Using (5.9), (5.14), (5.19) and (5.21), we have

$$\begin{aligned} &|\mathbf{u}_h^{n+1}|^2 + \theta\Delta t\alpha\nu\|\mathbf{u}_h^{n+1}\|^2 + \theta\Delta t\beta\nu\|\mathbf{u}_h^m\|^2 + \left(\frac{1}{4} - \frac{2\theta\Delta t\beta\nu}{h^2}\right)|\mathbf{u}_h^{n+1} - \mathbf{u}_h^m|^2 \\ \leq& |\mathbf{u}_h^m|^2 + C\left\{\left(\frac{\nu\Delta t}{h}\right)^2 \|\mathbf{u}_h^n\|^2 + (\Delta t s)^2 |\mathbf{u}_h^n|^2 \|\mathbf{u}_h^n\|^2\right\} + C\frac{\Delta t\lambda^2}{\nu}\{|T_h^n|^2 + |T_h^m - T_h^n|^2\}. \end{aligned} \tag{5.34}$$

In view of the properties T_h^n, T_h^m, we suppose

$$\lambda^2\{|T_h^n|^2 + |T_h^m - T_h^n|^2\} \le C|T_h^n|^2.$$

Then (5.34) can be written as

$$\begin{aligned}&|\mathbf{u}_h^{n+1}|^2 + \theta\Delta t\alpha\nu\|\mathbf{u}_h^{n+1}\|^2 + \theta\Delta t\beta\nu\|\mathbf{u}_h^m\|^2 + \left(\frac{1}{4} - \frac{2\theta\Delta t\beta\nu}{h^2}\right)|\mathbf{u}_h^{n+1} - \mathbf{u}_h^m|^2 \\ \le&|\mathbf{u}_h^m|^2 + C\left\{\left(\frac{\nu\Delta t}{h}\right)^2\|\mathbf{u}_h^n\|^2 + (\Delta ts)^2|\mathbf{u}_h^n|^2\|\mathbf{u}_h^n\|^2\right\} + C\frac{\Delta t}{\nu}|T_h^n|^2. \qquad (5.35)\end{aligned}$$

Summing up (5.35) for $n = 0, 1, 2, \cdots, r$, with $r \in Z^+$, gives

$$\begin{aligned}&|\mathbf{u}_h^{r+1}|^2 + \theta\Delta t\alpha\nu\sum_{n=0}^{r}\|\mathbf{u}_h^{n+1}\|^2 + \theta\Delta t\beta\nu\sum_{n=0}^{r}\|\mathbf{u}_h^m\|^2 + \left(\frac{1}{4} - \frac{2\theta\Delta t\beta\nu}{h^2}\right)\sum_{n=0}^{r}|\mathbf{u}_h^{n+1} - \mathbf{u}_h^m|^2 \\ \le&|\mathbf{u}_{0h}|^2 + C\sum_{n=0}^{r}\left\{\left(\frac{\nu\Delta t}{h}\right)^2\|\mathbf{u}_h^n\|^2 + (\Delta ts)^2|\mathbf{u}_h^n|^2\|\mathbf{u}_h^n\|^2\right\} + \frac{C\Delta t}{\nu}\sum_{n=0}^{r}|T_h^n|^2. \qquad (5.36)\end{aligned}$$

Let

$$\begin{aligned}\Lambda_T &= |\mathbf{u}_0|^2 + C\left\{\left(\frac{\nu\Delta t}{h}\right)^2\|\mathbf{u}_0\|^2 + (\Delta ts)^2|\mathbf{u}_0|^2\|\mathbf{u}_0\|^2\right\} + \frac{C}{\nu}|T_0|_{L^2(0,T,L^2)}, \\ \Lambda_r &= |\mathbf{u}_{0h}|^2 + C\left\{\left(\frac{\nu\Delta t}{h}\right)^2\|\mathbf{u}_{0h}\|^2 + (\Delta ts)^2|\mathbf{u}_{0h}|^2\|\mathbf{u}_{0h}\|^2\right\} + \frac{C\Delta t}{\nu}\sum_{n=0}^{r}|T_h^n|^2.\end{aligned}$$

Since the left-hand side of (5.36) is bound for $r = 0$, let us assume that

$$|\mathbf{u}_h^{r+1}|^2 + \delta\theta\Delta t\alpha\nu\sum_{n=0}^{r}\|\mathbf{u}_h^{n+1}\|^2 + \theta\Delta t\beta\nu\sum_{n=0}^{r}\|\mathbf{u}_h^m\|^2 + \left(\frac{1}{4} - \frac{2\theta\Delta t\beta\nu}{h^2}\right)\sum_{n=0}^{r}|\mathbf{u}_h^{n+1} - \mathbf{u}_h^m|^2 \le \Lambda_r.$$

Next, we will show that the inequality holds by induction. Firstly, we suppose the below formula holds for $r = k \in Z^+$ therefore

$$|\mathbf{u}_h^{n+1}|^2 \le \Lambda_k \le \Lambda_T, \qquad n = 0, 1, \cdots, k.$$

Then the right-hand of (5.36) for $k+1$ can be estimated by

$$\begin{aligned}&|\mathbf{u}_h^{k+2}|^2 + \theta\Delta t\alpha\nu\sum_{n=0}^{k+1}\|\mathbf{u}_h^{n+1}\|^2 + \theta\Delta t\beta\nu\sum_{n=0}^{k+1}\|\mathbf{u}_h^m\|^2 + \left(\frac{1}{4} - \frac{2\theta\Delta t\beta\nu}{h^2}\right)\sum_{n=0}^{k+1}|\mathbf{u}_h^{n+1} - \mathbf{u}_h^m|^2 \\ \le&|\mathbf{u}_{0h}|^2 + C\sum_{n=0}^{k+1}\left\{\left(\frac{\nu\Delta t}{h}\right)^2\|\mathbf{u}_h^n\|^2 + (\Delta ts)^2|\mathbf{u}_h^n|^2\|u_h^n\|^2\right\} + \frac{C\Delta t}{\nu}\sum_{n=0}^{k+1}|T_h^n|^2 \\ \le&\Lambda_{k+1} + C\left\{\left(\frac{\nu\Delta t}{h}\right)^2 + (\Delta ts)^2\Lambda_T\right\}\sum_{n=1}^{k+1}\|\mathbf{u}_h^n\|^2.\end{aligned}$$

This completes the proof. □

If the condition (5.32) is satisfied, then we can obtain (5.33). Consequently, we can give the theorem below.

Theorem 5.2. *If $\theta \in (0,(-7+4\sqrt{7})/9)$ and $0<\delta<1/3$ such that $2(\theta/\theta_b)^2+1/8 \leq 1-3\delta$,$(\theta\Delta t\beta\nu)/h^2<1/8$ and if the condition* (5.32) *holds, then*

$$\mathbf{u}_h^n, \mathbf{u}_h^{n+\theta} \in L^\infty(0,T,L^2(\Omega)^2)\cap L^2(0,T,V_h),$$
$$\mathbf{u}_h^m \in L^\infty(0,T,L^2(\Omega)^2)\cap L^2(0,T,\mathbf{X}_h).$$

Proof. According to Lemma 5.4, for $\forall N\in Z^+$

$$|\mathbf{u}_h^{N+1}|^2+\delta\theta\Delta t\alpha\nu\sum_{n=0}^{N}\|\mathbf{u}_h^{n+1}\|^2+\theta\Delta t\beta\nu\sum_{n=0}^{N}\|\mathbf{u}_h^m\|^2+\left(\frac{1}{4}-\frac{2\theta\Delta t\beta\nu}{h^2}\right)\sum_{n=0}^{N}|\mathbf{u}_h^{n+1}-\mathbf{u}_h^m|^2\leq\Lambda_T,$$

which implies that

$$\mathbf{u}_h^{N+1}\in L^\infty(0,T,L^2(\Omega)^2)\cap L^2(0,T,V_h), \mathbf{u}_h^m\in L^2(0,T,\mathbf{X}_h).$$

It follows from (5.9) in Lemma 5.1 that

$$\begin{aligned}|\mathbf{u}_h^{N+\theta}|^2 \leq & |\mathbf{u}_h^N|^2+(\theta_b\Delta tS(h))^2\sum_{n=0}^{N}|\mathbf{u}_h^N|^2\|\mathbf{u}_h^N\|^2+\frac{\theta\Delta t\lambda^2}{\nu}\sum_{n=0}^{N}|T_h^N|^2\\ \leq & |\mathbf{u}_h^N|^2+\big(\theta_b\Delta tS(h)\big)^2\Lambda_T\sum_{n=0}^{N}\|\mathbf{u}_h^N\|^2+\frac{\theta\Delta t\lambda^2}{\nu}\sum_{n=0}^{N}|T_h^N|^2\\ \leq & |\mathbf{u}_h^N|^2+(1-\delta)\alpha\theta\Delta t\nu\sum_{n=0}^{N}\|\mathbf{u}_h^N\|^2+\frac{\theta\Delta t\lambda^2}{\nu}\sum_{n=0}^{N}|T_h^N|^2\\ \leq & \Lambda_T+\alpha\theta\Delta t\nu\sum_{N=0}^{N}\|\mathbf{u}_h^N\|^2+\frac{\theta\Delta t\lambda^2}{\nu}\sum_{n=0}^{N}|T_h^N|^2\\ \leq & \left(1+\frac{1}{\delta}+\theta\lambda^2\right)\Lambda_T.\end{aligned}$$

Consequently,

$$\mathbf{u}_h^{n+\theta}\in L^\infty(0,T,L^2(\Omega)^2).$$

It follows from (5.14) in Lemma 5.2 that

$$\begin{aligned}|\mathbf{u}_h^m|^2\leq & |\mathbf{u}_h^{N+\theta}|^2+D_3((\nu\frac{\Delta t}{h})^2+(\Delta ts)^2\Lambda_T)\sum_{n=0}^{N}\|\mathbf{u}_h^n\|^2\\ & +D_3\left(\frac{\Delta t\lambda^2}{\alpha\nu}\sum_{n=0}^{N}|T_h^n|^2+\frac{\Delta t\lambda^2}{\beta\nu}\sum_{n=0}^{N}|T_h^m-T_h^n|^2\right)\\ \leq & |\mathbf{u}_h^{N+\theta}|^2+\alpha\theta\Delta t\nu\sum_{n=0}^{N}\|\mathbf{u}_h^n\|^2+D_3\frac{\Delta t\lambda^2}{\nu}\left(\frac{1}{\alpha}\sum_{n=0}^{N}|T_h^n|^2+\frac{1}{\beta}\sum_{n=0}^{N}|T_h^m-T_h^n|^2\right)\\ \leq & \left(1+\frac{2}{\delta}+\lambda^2\left(\theta+\frac{1}{\alpha}+\frac{2}{\beta}\right)\right)\Lambda_T,\end{aligned}$$

where $m=N+1-\theta$. Therefore

$$\mathbf{u}_h^m\in L^\infty(0,T,L^2(\Omega)^2).$$

Using (5.18) in Lemma 5.2 gives

$$\begin{aligned}&\sum_{n=0}^{N}|\mathbf{u}_h^{n+\theta}-\mathbf{u}_h^n|^2+\theta\Delta t\alpha\nu\sum_{n=0}^{N}\|\mathbf{u}_h^{n+\theta}-\mathbf{u}_h^n\|^2\\ \le&\Big(\frac{\theta_b\Delta t\nu}{h}\Big)^2\sum_{n=0}^{N}\|\mathbf{u}_h^n\|^2+(\theta_b\Delta ts)^2\sum_{n=0}^{N}|\mathbf{u}_h^n|^2\|\mathbf{u}_h^n\|^2+\frac{\theta\Delta t\lambda^2}{\alpha\nu}\sum_{n=0}^{N}|T_h^n|^2\\ \le&\theta\Delta t\alpha\nu\sum_{n=0}^{N}\|\mathbf{u}_h^n\|^2+\frac{\theta\Delta t\lambda^2}{\alpha\nu}\sum_{n=0}^{N}|T_h^n|^2\le\Big(\frac{1}{\delta}+\frac{\theta\lambda^2}{\alpha}\Big)\Lambda_T,\end{aligned}$$

which leads to

$$\theta\Delta t\alpha\nu\sum_{n=0}^{N}\|\mathbf{u}_h^{n+\theta}\|^2\le\theta\Delta t\alpha\nu\sum_{n=0}^{N}\|\mathbf{u}_h^{n+\theta}-\mathbf{u}_h^n\|^2+\theta\Delta t\alpha\nu\sum_{n=0}^{N}\|\mathbf{u}_h^n\|^2\le\Big(\frac{2}{\delta}+\frac{\theta\lambda^2}{\alpha}\Big)\Lambda_T.$$

Consequently,

$$u_h^{N+\theta}\in L^2(0,T,V_h).$$

The proof of the theorem is complete. □

6. Numerical Results

In this section, we give results of the simulation performed with our proposed algorithm for the thermally driven cavity flow. The result is obtained in some fixed time instead of steady state flow (from the non-steady problem). Computations are made on a fixed mesh size with fixed time steps. The space steps are denoted by h_x, h_y and the time step by Δt.

Example. We consider 2D domain $\Omega = (0,1)\times(0,1)$. The motion boundary condition is defined by $\mathbf{u} = (1,0)$ at the moving boundary (the top one $y = 1$) and $\mathbf{u} = (0,0)$ elsewhere. Initial condition for $\mathbf{u}$ and T are identically 0. Boundary condition for the temperature is given by

$$\frac{\partial T}{\partial n}=0,\quad \text{on }\ \partial\Omega\big|_{x=0,a};\qquad T=0,\quad \text{on }\ \Omega\big|_{y=0};\qquad T=1,\quad \text{on }\ \Omega\big|_{y=b},$$

which means that the fluid motion is caused by buoyancy from the vertical temperature gradient and by the velocity-driven cavity boundary condition on the top horizontal boundary wall.

Isotherms and streamline for $Re = 2000$ are shown in Figs. 6.1, $Re = 4000$ in Fig.6.2, $Re = 6000$ in Fig. 6.3. Both the results agree very well with those in [5, 9] and shows good stability for different Rayleigh and Reynolsds numbers . In all the computations, we used the $P2 - P1$ (Taylor-Hood) finite element approximations and used splitting parameter $\theta = 0.2$, Prandtl number 0.72, $T = 200$, $\Delta t = 0.02$, $Gr = 100000$, the size of mesh $M \times N = 32 \times 32$.

7. Conclusions

In this paper, we have proposed a numerical scheme based on operator splitting, and provided a theoretical analysis about the unique solvability and stability. The numerical result shows good stability to support the theoretical analysis. The result obtained with such coarse meshes makes the scheme suitable for more complicated memory demanding flows, as long as they preserve the incompressible structure.

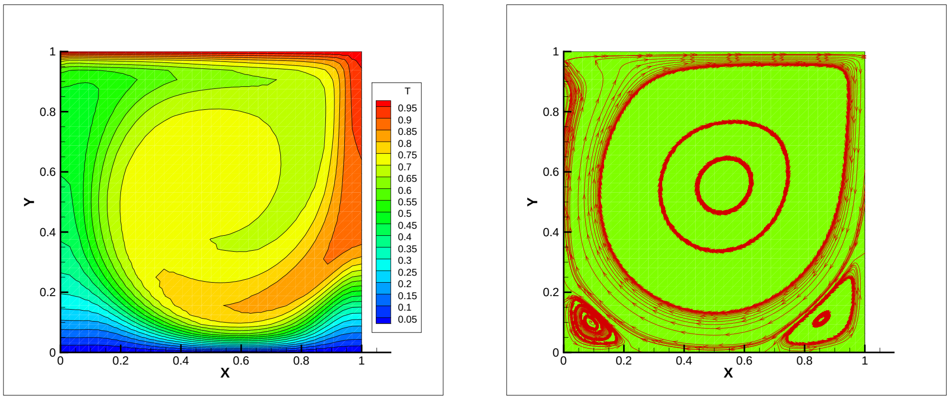

Fig. 6.1. Isotherms (left) and streamline (right) for $Re = 2000$.

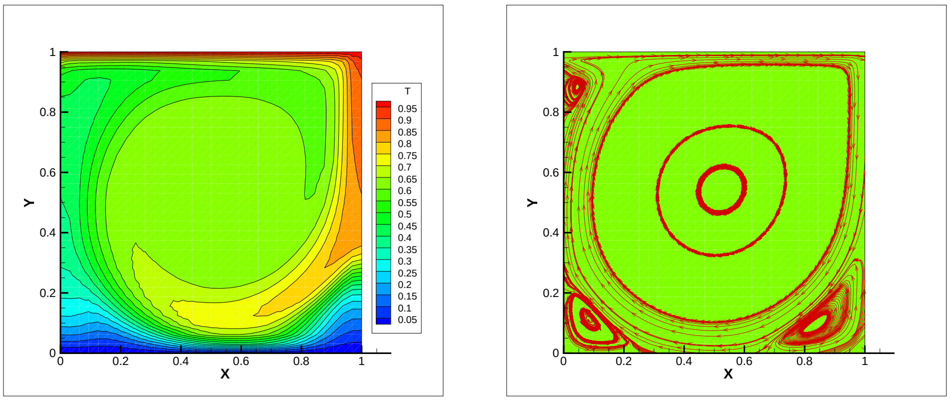

Fig. 6.2. Isotherms (left) and streamline (right) for $Re = 4000$.

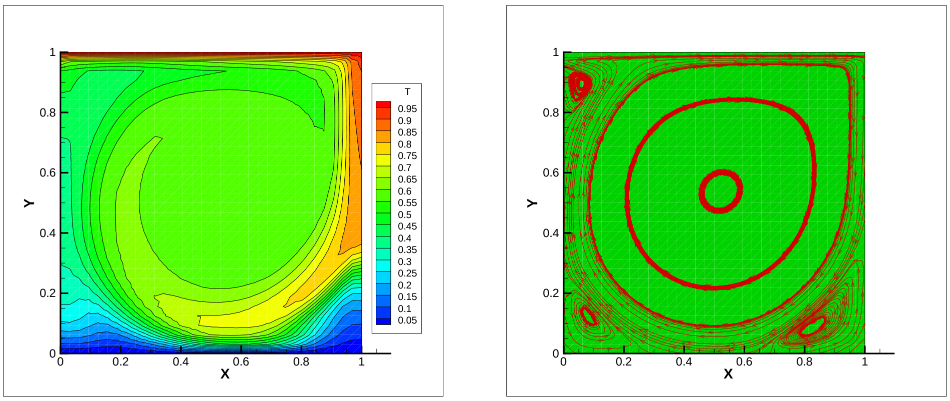

Fig. 6.3. Isotherms (left) and streamline (right) for $Re = 6000$.

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