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# A NEW STABILIZED SUBGRID EDDY VISCOSITY METHOD BASED ON PRESSURE PROJECTION AND EXTRAPOLATED TRAPEZOIDAL RULE FOR THE TRANSIENT NAVIER-STOKES EQUATIONS*

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**Abstract**

We consider a new subgrid eddy viscosity method based on pressure projection and extrapolated trapezoidal rule for the transient Navier-Stokes equations by using lowest equal-order pair of finite elements. The scheme stabilizes convection dominated problems and ameliorates the restrictive inf-sup compatibility stability. It has some attractive features including parameter free for the pressure stabilized term and calculations required for higher order derivatives. Moreover, it requires only the solutions of the linear system arising from an Oseen problem per time step and has second order temporal accuracy. The method achieves optimal accuracy with respect to solution regularity.

*Mathematics subject classification:* 65N30, 76D05.
*Key words:* Subgrid eddy viscosity model, Pressure projection method, Extrapolated trapezoidal rule, The transient Navier-Stokes equations.

## 1. Introduction

The flow of an incompressible fluid is governed by the incompressible Navier-Stokes equations

$$\begin{aligned} &\mathbf{u}_t - \nu\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, && \text{in} \quad (0,T]\times\Omega,\\ &\nabla\cdot\mathbf{u} = 0, && \text{in} \quad [0,T]\times\Omega,\\ &\mathbf{u} = 0, && \text{in} \quad (0,T]\times\partial\Omega,\\ &\mathbf{u}(0,x) = \mathbf{u}_0, && \text{in} \quad \Omega,\\ &\int_\Omega p\, dx = 0, && \text{in} \quad (0,T], \end{aligned} \tag{1.1}$$

where $\Omega\subset R^2$ is a bounded domain with boundary $\partial\Omega$, $[0,T]$ is a finite time interval, $\mathbf{u}(t,x)$ is the velocity of the fluid and $p(t,x)$ is the pressure. The viscosity $\nu>0$, which is inverse proportional to the Reynolds number $Re=\mathcal{O}(\nu^{-1})$. The body forces $\mathbf{f}(t,x)$ and the initial velocity

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field $\mathbf{u}_0$ are given. Generally speaking, for the transient Navier-Stokes equations which govern viscous fluid flow, the natural and important Galerkin approximation is a mixed method, however the Galerkin mixed finite element approximation of (1.1) may suffer from three problems: violation of the discrete inf-sup (or Babuska-Brezzi) stability condition, dominating advection, and how to make fully discretization which is a simple, second order temporal accuracy.

The subgrid eddy viscosity model is a numerical stabilization of a convection dominated and underresolved flow. This approach adds an artificial viscosity only on the fine scales, and is referred to artificial viscosity model, which is inspired by earlier work of Guermend [4]. In [4] subgrid scales are augmented by bubble functions. Later, Layton generalized the concept for the stationary convection diffusion problem. In the work of Kaya and Layton [9], this model has been connected with another consistent stabilization technique, also known as variational multiscale method. The model has been analyzed for time-dependent Navier-Stokes equations by John-Kaya [8] and Kaya-*Rivière* [10]. In [11], Kaya-*Rivière* gave algorithm and numerical experiments for variational multiscale method. However, these works require velocity and pressure finite element spaces satisfying the so-called inf-sup condition.

It is well known that the simplest conforming low-order elements like $P_1 - P_1$ triangular element is not stable. This impacts on efficiency, since local mass conservation, the simple logic and regular data structure associate with low-order finite element methods are very attractive and useful on many occasions. To counteract the lack of LBB stability, low-order pairs are usually supplemented by stabilized procedures. Stabilized mixed finite element methods are often developed by using residuals of the momentum equation, e.g., Douglas-Wang method [2], least squares Petrov-Galerkin finite element method [14]. These residual terms must be formulated using mesh-dependent parameters, whose optimal values are usually unknown. Particularly, pressure and velocity derivatives in this residual vanish or are poorly approximated, causing difficulties in the application of consistent stabilization. Other stabilized mixed methods involving non-residual stabilization are also developed, e.g., pressure projection method, it has been applied to the Stokes problem by Bochev [1]; He-Li [6], Li-He-Chen [12] extended this method to the Navier-Stokes problem. Pressure projection method does not require approximation of derivatives, specification of mesh-dependent parameter, or nonstandard data structures. The paper [12] only counteracted the lack of LBB stability condition and made a semi-discrete analysis; the solution has oscillation when the viscosity coefficient is small.

When (1.1) is fully-discretized by accurate and stable methods, well stabilized methods with second-order temporal accuracy are Crank-Nicolson scheme (see Heywood and Rannacher [7]), Crank-Nicolson extrapolation scheme (see Girault and Raviart [3]), and two-level method based on finite element and Crank-Nicolson extrapolation (see He [5]). However, all these discrete forms are nonlinear, and the approximation can still fail for many reasons. One common mode of failure is non-convergence of the iterative nonlinear and linear solvers used to compute the velocity and pressure at the new time levels. We consider herein a simple, second order accurate, and stable method for temporal discretization which addresses the failure cases mentioned above. The method requires the solution of one linear system per time step.

In this paper, we propose a new stabilization finite element method which is combined subgrid eddy viscosity with pressure projection method for the spatial discretization and extrapolated trapezoidal rule for the temporal discretization by using lowest equal-order pair of finite elements. The scheme stabilizes convection domination and ameliorates the restrictive inf-sup compatibility stability, which has second order temporal accuracy of $\mathcal{O}(\triangle t^2 + \nu_T^{\frac{1}{2}} H + h)$, where the constant in the estimate does not depend on the Reynolds number but on the reduced

Reynolds number. When $\nu_T = \mathcal{O}(h),\ H = \mathcal{O}(h^{\frac{1}{2}})$ (or $\nu_T = C,\, H = \mathcal{O}(h)$), the method achieves optimal accuracy with respect to solution regularity. The method can be easily extended to higher equal-order pair of finite elements. Numerical results and the comparison with other stabilized finite element methods will be presented in our forthcoming paper [13].

## 2. Notation and Scheme

We define the spaces $\mathbf{V} = (H_0^1(\Omega))^2,\ Q = L_0^2(\Omega)$ and $\mathbf{L} = \{\mathbb{L} \in (L^2(\Omega))^{2\times 2}\}$ and consider a variational formulation of (1.1): find $\mathbf{u} : [0,T] \to \mathbf{V}, p : (0,T] \to Q$ and $\mathbb{G} : [0,T] \to \mathbf{L}$ such that (see [11])

$$\begin{aligned}
&(\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + b_s(\mathbf{u}, \mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) + \nu_T(\nabla \mathbf{u}, \nabla \mathbf{v}) - \nu_T(\mathbb{G}, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v}), && \forall \mathbf{v} \in \mathbf{V},\\
&b(\mathbf{u}, q) = 0, && \forall q \in Q,\\
&(\mathbb{G} - \nabla \mathbf{u}, \mathbb{L}) = 0, && \forall \mathbb{L} \in \mathbf{L}, \quad (2.1)
\end{aligned}$$

where the parameter $\nu_T > 0$ is the eddy viscosity parameter, $(\cdot,\cdot)$ denotes the $L^2$ inner-product and the bilinear forms are defined below

$$\begin{aligned}
&a(\mathbf{u}, \mathbf{v}) = \nu(\nabla \mathbf{u}, \nabla \mathbf{v}), && \forall \mathbf{u}, \mathbf{v} \in \mathbf{V},\\
&b_s(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}), && \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V},\\
&b(\mathbf{v}, q) = (q, \nabla \cdot \mathbf{v}), && \forall \mathbf{v} \in \mathbf{V}, \quad \forall q \in Q. \quad (2.2)
\end{aligned}$$

In the continuous case, this method reduces to the standard Navier-Stokes equations. However, in the discrete case it leads to different discretizations. In this paper, we consider multiscale finite element approximation of (2.1). Our approach can be understood as LES (Large Eddy Simulation) model.

We now introduce the finite element discretization of (2.1). Let $T_h$ and $T_H$ be two regular triangulations of the domain $\Omega$, such that $h$ (or $H$) denotes the maximum diameter of the elements in $T_h$ (or $T_H$) and satisfies $h \le H$. The boundary $\partial K$ of an element consists of edges $E$. We assume that each edge is oriented by selecting a normal direction $\mathbf{n}_E$. The set of all interior faces will be denoted by $\Gamma_h$. The norm

$$||\mathbf{u}||_{\Gamma_h} = \left( \sum_{E \in \Gamma_h} \int_E \mathbf{u}^2 ds \right)^{\frac{1}{2}}$$

will be used. This paper focuses on the analysis for the unstable velocity-pressure pair of the lowest equal-order finite elements,

$$\begin{aligned}
\mathbf{V}_h &:= \Big\{ \mathbf{v}_h \in (C^0(\Omega))^2 \cap \mathbf{V} : \mathbf{v}_h\big|_K \in (P_1(K))^2, \forall K \in T_h \Big\};\\
Q_h &:= \Big\{ q_h \in C^0(\Omega) \cap Q : q_h\big|_K \in P_1(K), \forall K \in T_h \Big\}. \quad (2.3)
\end{aligned}$$

$\mathbf{L}_H \subset \mathbf{L}$ contains piecewise linear polynomials,

$$\mathbf{L}_H := \Big\{ \mathbb{L}_H \in (C^0(\Omega))^{2\times 2} : \mathbb{L}_H\big|_K \in (P_1(K))^{2\times 2}, \forall K \in T_H \Big\}, \quad (2.4)$$

Let $P_{\mathbf{L}_H} : \mathbf{L} \to \mathbf{L}_H$ be the $L^2$ orthogonal projection onto $\mathbf{L}_H$. Thus, we have

$$(P_{\mathbf{L}_H} \mathbb{L}, \mathbb{G}_H) = (\mathbb{L}, \mathbb{G}_H), \qquad \forall \mathbb{G}_H \in \mathbf{L}_H,\ \forall \mathbb{L} \in \mathbf{L},$$

$$||(I - P_{\mathbf{L}_H})\nabla \mathbf{u}||_0 \le CH||\mathbf{u}||_2, \qquad \forall \mathbf{u} \in (H^2(\Omega))^2. \tag{2.5}$$

We will also use the fact that

$$||I - P_{\mathbf{L}_H}|| \le 1.$$

It is well know that the lowest equal-order finite element pair does not satisfy the so-called inf-sup condition, we define the following stabilized form $G(p^h, q^h)$. Let $\pi_h : Q \to R_0$ be the standard $L^2$ orthogonal projection onto $R_0$ with the following properties:

$$\begin{aligned}
&(p, q^h) = (\pi_h p, q^h), && \forall p \in Q,\ q^h \in R_0,\\
&||\pi_h p||_0 \le C||p||_0, && \forall p \in Q,\\
&||p - \pi_h p||_0 \le Ch||p||_1, && \forall p \in H^1(\Omega) \cap Q,
\end{aligned} \tag{2.6}$$

where $R_0 = \{q^h \in Q : q^h|_K$ is a constant, $\forall K \in T_h\}$. Then we can define the bilinear form $G(\cdot, \cdot)$ by

$$G(p, q) = (p - \pi_h p, q - \pi_h q).$$

Using the above notation, the variational stabilized formulation of problem (2.1) reads: find $\mathbf{u}^h : [0, T] \to \mathbf{V}_h,\ p^h : (0, T] \to Q_h$ satisfying

$$\begin{aligned}
&(\mathbf{u}^h_t, \mathbf{v}^h) + a(\mathbf{u}^h, \mathbf{v}^h) + b_s(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) + S(\mathbf{u}^h, \mathbf{v}^h) - b(\mathbf{v}^h, p^h) = (\mathbf{f}, \mathbf{v}^h), && \forall \mathbf{v}^h \in \mathbf{V}_h,\\
&b(\mathbf{u}^h, q^h) + G(p^h, q^h) = 0, && \forall q^h \in Q_h,
\end{aligned} \tag{2.7}$$

where the bilinear form $S$ is

$$S(\mathbf{u}^h, \mathbf{v}^h) = \nu_T((I - P_{\mathbf{L}_H})\nabla \mathbf{u}^h, (I - P_{\mathbf{L}_H})\nabla \mathbf{v}^h), \quad \forall \mathbf{u}^h,\ \mathbf{v}^h \in \mathbf{V}_h. \tag{2.8}$$

The eddy viscosity parameter $\nu_T > 0$ is to be defined later. Since $P_{\mathbf{L}_H}$ is an $L^2-$ projection, it follows for $\mathbf{v} \in \mathbf{V}$ and $||\nabla \mathbf{v}||_0 > 0$,

$$\begin{aligned}
\nu_T||(I - P_{\mathbf{L}_H})\nabla \mathbf{v}||_0^2 &= \nu_T(||\nabla \mathbf{v}||_0^2 - ||P_{\mathbf{L}_H}\nabla \mathbf{v}||_0^2) = \nu_T(1 - \frac{||P_{\mathbf{L}_H}\nabla \mathbf{v}||_0^2}{||\nabla \mathbf{v}||_0^2})||\nabla \mathbf{v}||_0^2\\
&=: \nu_{add}(\mathbf{v})||\nabla \mathbf{v}||_0^2.
\end{aligned} \tag{2.9}$$

In addition, from

$$0 \le ||P_{\mathbf{L}_H}\nabla \mathbf{v}||_0 \le ||\nabla \mathbf{v}||_0,$$

it follows that

$$0 \le \nu_{add}(\mathbf{v}) \le \nu_T. \tag{2.10}$$

In the following, we consider extrapolated trapezoidal rule on the time.

**Algorithm 1 (consistent stabilized, extrapolated trapezoidal rule).**
Let $\mathbf{u}_0^h$ be the Stokes projection of $\mathbf{u}_0(x)$ onto $\mathbf{V}_h$. At the first time level, $(\mathbf{u}_1^h, p_1^h) \in (\mathbf{V}_h, Q_h)$ are sought to satisfy

$$
\begin{aligned}
&(\frac{\mathbf{u}_1^h - \mathbf{u}_0^h}{\triangle t}, \mathbf{v}^h) + a(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \mathbf{v}^h) + b_s(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \mathbf{v}^h) \\
&\quad + S(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \mathbf{v}^h) - b(\mathbf{v}^h, \frac{p_1^h + p_0^h}{2}) \\
&=(\mathbf{f}(t_{\frac{1}{2}}), \mathbf{v}^h), \quad \forall \mathbf{v}^h \in \mathbf{V}_h; b(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, q^h) + G(\frac{p_1^h + p_0^h}{2}, q^h) = 0, \qquad \forall q^h \in Q_h. \qquad (2.11)
\end{aligned}
$$

Given a time step $\triangle t > 0$, the method computes $\mathbf{u}_2^h,\ \mathbf{u}_3^h,\ \cdots, p_2^h,\ p_3^h,\ \cdots$, where $t_j = j\triangle t$, $\mathbf{u}_j^h(x) \cong \mathbf{u}(x, t_j)$, $p_j^h(x) \cong p(x, t_j)$. For $n \geq 1$, given $(\mathbf{u}_n^h, p_n^h) \in (\mathbf{V}_h, Q_h)$, find $(\mathbf{u}_{n+1}^h, p_{n+1}^h) \in (\mathbf{V}_h, Q_h)$ satisfying

$$
\begin{aligned}
&(\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{\triangle t}, \mathbf{v}^h) + a(\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}, \mathbf{v}^h) + b_s(A[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h], \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}, \mathbf{v}^h) \\
&\quad + S(\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}, \mathbf{v}^h) - b(\mathbf{v}^h, \frac{p_{n+1}^h + p_n^h}{2}) \\
&=(\mathbf{f}(t_{n+\frac{1}{2}}), \mathbf{v}^h), \forall \mathbf{v}^h \in \mathbf{V}_h; b(\frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2}, q^h) + G(\frac{p_{n+1}^h + p_n^h}{2}, q^h) = 0, \qquad \forall q^h \in Q_h, \qquad (2.12)
\end{aligned}
$$

where, $A[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h] := \frac{3}{2}\mathbf{u}_n^h - \frac{1}{2}\mathbf{u}_{n-1}^h$ is a extrapolation to $t_{n+\frac{1}{2}} := t_n + t_{n+1}/2$.

**Remark 2.1.** At first time level, linear treatment of the nonlinear term can be used: find $(\mathbf{u}_1^h, p_1^h) \in (\mathbf{V}_h, Q_h)$, satisfying

$$
\begin{aligned}
&(\frac{\mathbf{u}_1^h - \mathbf{u}_0^h}{\triangle t}, \mathbf{v}^h) + a(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \mathbf{v}^h) + b_s(A[\mathbf{u}_0^h, \mathbf{u}_{-1}^h], \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \mathbf{v}^h) \\
&\qquad + S(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, \mathbf{v}^h) - b(\mathbf{v}^h, \frac{p_1^h + p_0^h}{2}) = (\mathbf{f}(t_{\frac{1}{2}}), \mathbf{v}^h), \qquad \forall \mathbf{v}^h \in \mathbf{V}_h, \qquad (2.13)\\
&b(\frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}, q^h) + G(\frac{p_1^h + p_0^h}{2}, q^h) = 0, \qquad \forall q^h \in Q_h, \qquad (2.14)
\end{aligned}
$$

where $\mathbf{u}_{-1}^h = 0$.

We can show that this modification affect neither the stability of the method nor the convergence order of the velocity approximation by uniform proofing method.

We shall show that Algorithm 1 is stable and is of second order temporal accuracy $\mathcal{O}(\triangle t^2 +$ *spatial error*$)$ in next sections.

## 3. Stability of the Extrapolated Trapezoidal Rule

We start with the proof of stability, which is the mathematical key to the good properties of the method. The stability of Algorithm 1 is proven in the following proposition.

**Definition 3.1.** $Re_{red} := (\nu + \inf_{t\in(0,T]} \nu_{add}(\phi(t)))^{-1}$, $\forall \phi(t) \in \boldsymbol{V}$.

**Proposition 3.1.** ([1]) *Let $\boldsymbol{V}_h$ and $Q_h$ be the spaces defined in* (2.3). *Then, there exist positive constants $C_2$ and $C_3$, whose values are independent of $h$, $H$ and $\triangle t$, such that*

$$\sup_{\boldsymbol{v}^h \in \boldsymbol{V}_h} \frac{\int_\Omega p^h \nabla \cdot \boldsymbol{v}^h \, dx}{||\nabla \boldsymbol{v}^h||_0} \geq C_2 ||p^h||_0 - C_3 ||(I-\pi_h)p^h||_0, \quad \forall p^h \in Q_h. \tag{3.1}$$

**Proposition 3.2 (stability of extrapolated trapezoidal rule)** *Let $\boldsymbol{f}(t_{i+\frac{1}{2}}) \in (H^{-1}(\Omega))^2$, $i = 1, \cdots, n$ $(n \leq T/\triangle t)$. The stabilized extrapolated trapezoidal rule* (2.11)–(2.12) *is unconditionally stable. For any $h$, $H$, $\triangle t > 0$ and $n \geq 0$, we have*

$$\begin{aligned} &||\boldsymbol{u}_{n+1}^h||_0^2 + Re_{red}^{-1} \sum_{i=0}^{n} \triangle t \left\| \nabla \left( \frac{\boldsymbol{u}_{i+1}^h + \boldsymbol{u}_i^h}{2} \right) \right\|_0^2 + 2 \sum_{i=0}^{n} \triangle t \left\| (I-\pi_h)(\frac{p_{i+1}^h - p_i^h}{2}) \right\|_0^2 \\ \leq & ||\boldsymbol{u}_0^h||_0^2 + C_1^2 Re_{red} \sum_{i=0}^{n} \triangle t \left| \boldsymbol{f}\left(t_{i+\frac{1}{2}}\right) \right\|_{-1}^2. \end{aligned} \tag{3.2}$$

*Proof.* Taking $\mathbf{v}^h = \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2}$, $q^h = \frac{p_1^h + p_0^h}{2}$ in (2.11) gives

$$\begin{aligned} &\left( \frac{\mathbf{u}_1^h - \mathbf{u}_0^h}{\triangle t}, \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \right) + \nu \left\| \nabla \left( \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \right) \right\|_0^2 + \nu_T \left\| (I - P_{\mathbf{L}_H}) \nabla \left( \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \right) \right\|_0^2 \\ &\quad + \left\| (I-\pi_h) \left( \frac{p_1^h + p_0^h}{2} \right) \right\|_0^2 = \left( \mathbf{f}\left(t_{\frac{1}{2}}\right), \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \right). \end{aligned} \tag{3.3}$$

Applying the Cauchy-Schwartz inequality and Young's inequality, this gives

$$\begin{aligned} &\frac{\left\|\mathbf{u}_1^h\right\|_0^2 - \left\|\mathbf{u}_0^h\right\|_0^2}{2\triangle t} + \left( \nu + \nu_{add}\left( \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \right) \right) \left\| \nabla \left( \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \right) \right\|_0^2 + \left\| (I-\pi_h) \left( \frac{p_1^h + p_0^h}{2} \right) \right\|_0^2 \\ \leq & \frac{C_1^2}{2} \left( \nu + \nu_{add}\left( \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \right) \right)^{-1} \left\| \mathbf{f}\left(t_{\frac{1}{2}}\right) \right\|_{-1}^2 + \frac{\nu + \nu_{add}\left( \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \right)}{2} \left\| \nabla \left( \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \right) \right\|_0^2. \end{aligned} \tag{3.4}$$

Thus, on the first time level we obtain

$$\begin{aligned} &\left\|\mathbf{u}_1^h\right\|_0^2 + \left( \nu + \nu_{add}\left( \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \right) \right) \triangle t \left\| \nabla \left( \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \right) \right\|_0^2 + 2\triangle t \left\| (I-\pi_h) \left( \frac{p_1^h + p_0^h}{2} \right) \right\|_0^2 \\ \leq & C_1^2 \left( \nu + \nu_{add}\left( \frac{\mathbf{u}_1^h + \mathbf{u}_0^h}{2} \right) \right)^{-1} \triangle t \left\| \mathbf{f}\left(t_{\frac{1}{2}}\right) \right\|_{-1}^2 + \left\|\mathbf{u}_0^h\right\|_0^2. \end{aligned} \tag{3.5}$$

Now, we consider (2.12) for $n \geq 1$, setting $\mathbf{v}^h = (\mathbf{u}_{n+1}^h + \mathbf{u}_n^h)/2 \in V_h$, this gives

$$\begin{aligned} &\left( \frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{\triangle t}, \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2} \right) + \nu \left\| \nabla \left( \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2} \right) \right\|_0^2 + \nu_T \left\| (I - P_{\mathbf{L}_H}) \nabla \left( \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2} \right) \right\|_0^2 \\ &+ \left\| (I-\pi_h) \left( \frac{p_{n+1}^h + p_n^h}{2} \right) \right\|_0^2 = \left( \mathbf{f}\left(t_{n+\frac{1}{2}}\right), \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2} \right). \end{aligned} \tag{3.6}$$

Applying Cauchy-Schwartz inequality and Young's inequality leads to

$$\left\|\mathbf{u}_{n+1}^h\right\|_0^2 - \left\|\mathbf{u}_n^h\right\|_0^2 + \left( \nu + \nu_{add}\left( \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2} \right) \right) \triangle t \left\| \nabla \left( \frac{\mathbf{u}_{n+1}^h + \mathbf{u}_n^h}{2} \right) \right\|_0^2$$

$$+2\triangle t\left\|(I-\pi_h)\left(\frac{p^h_{n+1}+p^h_n}{2}\right)\right\|^2_0\leq C_1^2\left(\nu+\nu_{add}\left(\frac{\mathbf{u}^h_{n+1}+\mathbf{u}^h_n}{2}\right)\right)^{-1}\triangle t\left\|\mathbf{f}\left(t_{n+\frac{1}{2}}\right)\right\|^2_{-1}. \quad (3.7)$$

Summing (3.7) over the time levels gives

$$||\mathbf{u}^h_{n+1}||^2_0+\sum_{i=1}^{n}\triangle t(\nu+\nu_{add}(\frac{\mathbf{u}^h_{i+1}+\mathbf{u}^h_i}{2}))||\nabla(\frac{\mathbf{u}^h_{i+1}+\mathbf{u}^h_i}{2})||^2_0+2\sum_{i=1}^{n}\triangle t||(I-\pi_h)(\frac{p^h_{i+1}+p^h_i}{2})||^2_0$$
$$\leq||\mathbf{u}^h_1||^2_0+C_1^2\sum_{i=1}^{n}(\nu+\nu_{add}(\frac{\mathbf{u}^h_{i+1}+\mathbf{u}^h_i}{2}))^{-1}\triangle t||\mathbf{f}(t_{i+\frac{1}{2}})||^2_{-1}. \quad (3.8)$$

Finally, using the bound on $||\mathbf{u}^h_1||^h_0$ from (3.5) and Definition 3.1, we obtain that for all $n\geq 1$

$$||\mathbf{u}^h_{n+1}||^2_0+Re^{-1}_{red}\sum_{i=0}^{n}\triangle t||\nabla(\frac{\mathbf{u}^h_{i+1}+\mathbf{u}^h_i}{2})||^2_0+2\sum_{i=0}^{n}\triangle t||(I-\pi_h)(\frac{p^h_{i+1}+p^h_i}{2})||^2_0$$
$$\leq||\mathbf{u}^h_0||^2_0+C_1^2Re_{red}\sum_{i=0}^{n}\triangle t||\mathbf{f}(t_{i+\frac{1}{2}})||^2_{-1}. \quad (3.9)$$

This completes the proof. □

## 4. Convergence of the Extrapolated Trapezoidal Rule

### 4.1. Convergence of velocity

We assume that for the finite element spaces $(\mathbf{V}_h, Q_h)$, the following approximation properties hold: For $(\mathbf{v},p)\in((H^2(\Omega))^2\cap\mathbf{V}, H^1(\Omega)\cap Q)$, there exist approximation $I_h\mathbf{v}\in\mathbf{V}_h$ and $\rho_h q\in Q_h$ such that

$$||\mathbf{v}-I_h\mathbf{v}||_0+h||\nabla(\mathbf{v}-I_h\mathbf{v})||_0\leq Ch^2||\mathbf{v}||_2, \quad (4.1)$$
$$||q-\rho_h q||_0\leq Ch||q||_1, \quad (4.2)$$

where the $L^2-$ projection $\rho_h: Q\to Q_h$ satisfies

$$(q-\rho_h q, q_h)=0, \qquad \forall q\in Q,\quad q_h\in Q_h. \quad (4.3)$$

Throughout the paper we use the following projection.

**Definition 4.1.** *The projection operator* $(R_h,N_h):(\boldsymbol{V},Q)\to(\boldsymbol{V}_h,Q_h)$ *is defined by*

$$B^*((R_h(\boldsymbol{v},q),N_h(\boldsymbol{v},q));(\boldsymbol{v}^h,q^h))$$
$$=B((\boldsymbol{v},q);(\boldsymbol{v}^h,q^h)),\forall(\boldsymbol{v},q)\in(\boldsymbol{V},Q),\qquad\forall(\boldsymbol{v}^h,q^h)\in(\boldsymbol{V}_h,Q_h), \quad (4.4)$$

*where the bilinear forms are defined below*

$$B((\boldsymbol{u},p),(\boldsymbol{v},q))=a(\boldsymbol{u},\boldsymbol{v})-b(\boldsymbol{v},p)+b(\boldsymbol{u},q),$$
$$B^*((\boldsymbol{u}^h,p^h);(\boldsymbol{v}^h,q^h))=B((\boldsymbol{u}^h,p^h);(\boldsymbol{v}^h,q^h))+G(p^h,q^h). \quad (4.5)$$

The projection operator is well defined and satisfy the following approximation properties.

**Lemma 4.1.** *Let* $(\boldsymbol{V}_h, Q_h)$ *be defined above. Then there exists a positive constant* $\beta$*, independent of* $h$*, and* $H$*, such that*

$$\begin{aligned}&|B^*((\boldsymbol{u},p);(\boldsymbol{v},q))|\\ \le &C(\sqrt{\nu}||\nabla \boldsymbol{u}||_0+\frac{1}{\sqrt{\nu}}||p||_0)(\sqrt{\nu}||\nabla \boldsymbol{v}||_0+\frac{1}{\sqrt{\nu}}||q||_0), \forall(\boldsymbol{u},p),(\boldsymbol{v},q)\in(\boldsymbol{V},Q), \end{aligned} \tag{4.6a}$$

$$\begin{aligned}&\beta(\sqrt{\nu}||\nabla \boldsymbol{u}^h||_0+\frac{1}{\sqrt{\nu}}||p^h||_0)\\ \le &\sup_{(\boldsymbol{v}^h,q^h)\in(\boldsymbol{V}_h,Q_h)}\frac{|B^*((\boldsymbol{u}^h,p^h);(\boldsymbol{v}^h,q^h))|}{\sqrt{\nu}||\nabla \boldsymbol{v}^h||_0+\frac{1}{\sqrt{\nu}}||q^h||_0}, \forall(\boldsymbol{u}^h,p^h)\in(\boldsymbol{V}_h,Q_h).\end{aligned} \tag{4.6b}$$

*Under the assumptions of* (4.6)*, the projection operator* $(R_h, N_h)$ *satisfies*

$$\begin{aligned}&\sqrt{\nu}||\nabla(\boldsymbol{v}-R_h(\boldsymbol{v},q))||_0+\frac{1}{\sqrt{\nu}}||q-N_h(\boldsymbol{v},q)||_0\\ \le &C\left(\sqrt{\nu}||\nabla \boldsymbol{v}||_0+\frac{1}{\sqrt{\nu}}||q||_0\right), \qquad \forall(\boldsymbol{v},q)\in(\boldsymbol{V},Q);\end{aligned} \tag{4.7a}$$

$$\begin{aligned}&||\boldsymbol{v}-R_h(\boldsymbol{v},q)||_0+h(\sqrt{\nu}||\nabla(\boldsymbol{v}-R_h(\boldsymbol{v},q))||_0+\frac{1}{\sqrt{\nu}}||q-N_h(\boldsymbol{v},q)||_0)\\ \le &Ch^2\left(\sqrt{\nu}||\boldsymbol{v}||_2+(\frac{1}{\sqrt{\nu}}+\sqrt{\nu})||q||_1\right), \qquad \forall(\boldsymbol{v},q)\in((H^2(\Omega))^2\cap \boldsymbol{V}, H^1(\Omega)\cap Q).\end{aligned} \tag{4.7b}$$

*Proof.* **Step 1.** We will construct a pair $(\hat{\mathbf{v}}^h, \hat{q}^h)$, such that

$$\left|B^*\Big((\mathbf{u}^h,p^h);(\hat{\mathbf{v}}^h,\hat{q}^h)\Big)\right|\ge C\left(\sqrt{\nu}||\nabla \mathbf{u}^h||_0+\frac{1}{\sqrt{\nu}}||p^h||_0\right)\left(\sqrt{\nu}||\nabla(\hat{\mathbf{v}}^h)||_0+\frac{1}{\sqrt{\nu}}||\hat{q}^h||_0\right). \tag{4.8}$$

Setting $(\mathbf{v}^h, q^h) = (\mathbf{u}^h, p^h)$ yields

$$|B^*((\mathbf{u}^h,p^h);(\mathbf{u}^h,p^h))| = \nu||\nabla \mathbf{u}^h||_0^2+||(I-\pi_h)p^h||_0^2. \tag{4.9}$$

For an arbitrary given but fixed pressure $p^h \in Q_h$, let $\mathbf{w}$ and $\mathbf{w}^h$ be the functions that satisfy

$$\begin{aligned}&\int_\Omega p^h\nabla\cdot\mathbf{w}\,dx \ge \tilde{C}_1||p^h||_0||\nabla\mathbf{w}||_0,\\ &||\mathbf{w}-\mathbf{w}^h||_0+h^{\frac{1}{2}}||\mathbf{w}-\mathbf{w}^h||_{\Gamma_h}\le Ch||\nabla\mathbf{w}||_0, \quad \text{(see [3], p.217)}.\end{aligned} \tag{4.10}$$

Assume that $\mathbf{w}^h$ is normalized so that

$$||\nabla\mathbf{w}^h||_0 = \frac{1}{\nu}||p^h||_0. \tag{4.11}$$

From proposition 3.1, we have

$$\int_\Omega p^h\nabla\cdot\mathbf{w}^h\,dx \ge \frac{C_2}{\nu}||p^h||_0^2-\frac{C_3}{\nu}||(I-\pi_h)p^h||_0||p^h||_0. \tag{4.12}$$

Setting $(\mathbf{v}^h, q^h) = (-\alpha\mathbf{w}^h, 0)$, where $\alpha$ is a real, positive parameter, together with (4.12), yields

$$B^*\Big((\mathbf{u}^h,p^h);(-\alpha\mathbf{w}^h,0)\Big) = -\nu\alpha\int_\Omega\nabla\mathbf{u}^h\cdot\nabla\mathbf{w}^h\,dx+\alpha\int_\Omega p^h\nabla\cdot\mathbf{w}^h\,dx$$

$$\geq -\nu\alpha||\nabla \mathbf{u}^h||_0||\nabla \mathbf{w}^h||_0 + \alpha\left(\frac{C_2}{\nu}||p^h||_0^2 - \frac{C_3}{\nu}||(I-\pi_h)p^h||_0||p^h||_0\right)$$
$$\geq -\alpha||\nabla \mathbf{u}^h||_0||p^h||_0 + \alpha\left(\frac{C_2}{\nu}||p^h||_0^2 - \frac{C_3}{\nu}||(I-\pi_h)p^h||_0||p^h||_0\right). \quad (4.13)$$

Using Young's inequality, we have that

$$||\nabla \mathbf{u}^h||_0||p^h||_0 \leq \frac{C_2}{4\nu}||p^h||_0^2 + \frac{\nu}{C_2}||\nabla \mathbf{u}^h||_0^2;$$
$$\frac{C_3}{\nu}||(I-\pi_h)p^h||_0||p^h||_0 \leq \frac{C_2}{4\nu}||p^h||_0^2 + \frac{C_3^2}{\nu C_2}||(I-\pi_h)p^h||_0^2. \quad (4.14)$$

In combination with (4.9), these inequalities lead to

$$B^*((\mathbf{u}^h, p^h); (\mathbf{u}^h - \alpha\mathbf{w}^h, p^h))$$
$$\geq \nu(1-\frac{\alpha}{C_2})||\nabla \mathbf{u}^h||_0^2 + \frac{\alpha C_2}{2\nu}||p^h||_0^2 + (1-\frac{C_3^2\alpha}{\nu C_2})||(I-\pi_h)p^h||_0^2. \quad (4.15)$$

Choosing $\alpha = \min\{\frac{C_2}{2}, \frac{\nu C_2}{2C_3^2}\}$ guarantees that $1-\frac{\alpha}{C_2} \geq \frac{1}{2}$ and $1-\frac{C_3^2\alpha}{\nu C_2} \geq \frac{1}{2}$.

We now set $\hat{\mathbf{v}}^h = \mathbf{u}^h - \alpha\mathbf{w}^h$ and $\hat{q}^h = p^h$. It is easy to see that

$$B^*((\mathbf{u}^h, p^h); (\hat{\mathbf{v}}^h, \hat{q}^h)) \geq \frac{1}{2}(\nu||\nabla \mathbf{u}^h||_0^2 + \frac{\alpha C_2}{2\nu}||p^h||_0^2 + ||(I-\pi_h)p^h||_0^2)$$
$$\geq \frac{1}{6}(\sqrt{\nu}||\nabla \mathbf{u}^h||_0 + \sqrt{\frac{\alpha C_2}{2\nu}}||p^h||_0 + ||(I-\pi_h)p^h||_0)^2$$
$$\geq C(\sqrt{\nu}||\nabla \mathbf{u}^h||_0 + \frac{1}{\sqrt{\nu}}||p^h||_0)^2, \quad (4.16)$$

where the last bound follows from $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$. Finally, we have that

$$\sqrt{\nu}||\nabla(\hat{\mathbf{v}}^h)||_0 + \frac{1}{\sqrt{\nu}}||\hat{q}^h||_0 = \sqrt{\nu}||\nabla(\mathbf{u}^h - \alpha\mathbf{w}^h)||_0 + \frac{1}{\sqrt{\nu}}||p^h||_0$$
$$\leq \sqrt{\nu}||\nabla \mathbf{u}^h||_0 + \sqrt{\nu}\alpha||\nabla \mathbf{w}^h||_0 + \frac{1}{\sqrt{\nu}}||p^h||_0$$
$$\leq C(\sqrt{\nu}||\nabla \mathbf{u}^h||_0 + \frac{1}{\sqrt{\nu}}||p^h||_0). \quad (4.17)$$

This proves (4.6).

**Step 2.** Using the triangle inequality, we have that

$$\sqrt{\nu}||\nabla(\mathbf{v} - R_h(\mathbf{v}, q))||_0 + \frac{1}{\sqrt{\nu}}||q - N_h(\mathbf{v}, q)||_0$$
$$\leq \sqrt{\nu}||\nabla \mathbf{v}||_0 + \sqrt{\nu}||\nabla(R_h(\mathbf{v}, q))||_0 + \frac{1}{\sqrt{\nu}}||q||_0 + \frac{1}{\sqrt{\nu}}||N_h(\mathbf{v}, q)||_0$$
$$\leq \sqrt{\nu}||\nabla \mathbf{v}||_0 + \frac{1}{\sqrt{\nu}}||q||_0 + \beta^{-1} \sup_{(\mathbf{v}^h, q^h)\in(\mathbf{V}_h, Q_h)} \frac{B^*((R_h(\mathbf{v}, q), N_h(\mathbf{v}, q)); (\mathbf{v}^h, q^h))}{\sqrt{\nu}||\nabla \mathbf{v}^h||_0 + \frac{1}{\sqrt{\nu}}||q^h||_0}$$
$$\leq \sqrt{\nu}||\nabla \mathbf{v}||_0 + \frac{1}{\sqrt{\nu}}||q||_0 + \beta^{-1} \sup_{(\mathbf{v}^h, q^h)\in(\mathbf{V}_h, Q_h)} \frac{B((\mathbf{v}, q), (\mathbf{v}^h, q^h))}{\sqrt{\nu}||\nabla \mathbf{v}^h||_0 + \frac{1}{\sqrt{\nu}}||q^h||_0}$$
$$\leq C(\sqrt{\nu}||\nabla \mathbf{v}||_0 + \frac{1}{\sqrt{\nu}}||q||_0). \quad (4.18)$$

**Step 3.** It follows from the definition of $(R_h, N_h)$, the triangle inequality, and (4.6) that

$$
\begin{aligned}
&\sqrt{\nu}||\nabla(\mathbf{v}-R_h(\mathbf{v},q))||_0+\frac{1}{\sqrt{\nu}}||q-N_h(\mathbf{v},q)||_0\\
\leq&\sqrt{\nu}||\nabla(\mathbf{v}-I_h\mathbf{v})||_0+\sqrt{\nu}||\nabla(I_h\mathbf{v}-R_h(\mathbf{v},q))||_0+\frac{1}{\sqrt{\nu}}||q-\rho_hq||_0+\frac{1}{\sqrt{\nu}}||\rho_hq-N_h(\mathbf{v},q)||_0\\
\leq&\sqrt{\nu}||\nabla(\mathbf{v}-I_h\mathbf{v})||_0+\frac{1}{\sqrt{\nu}}||q-\rho_hq||_0\\
&+\beta^{-1}\sup_{(\mathbf{v}^h,q^h)\in(\mathbf{V}_h,Q_h)}\frac{|B^*((I_h\mathbf{v}-R_h(\mathbf{v},q),\rho_hq-N_h(\mathbf{v},q));(\mathbf{v}^h,q^h))|}{\sqrt{\nu}||\nabla\mathbf{v}^h||_0+\frac{1}{\sqrt{\nu}}||q^h||_0}\\
\leq&\sqrt{\nu}||\nabla(\mathbf{v}-I_h\mathbf{v})||_0+\frac{1}{\sqrt{\nu}}||q-\rho_hq||_0\\
&+\beta^{-1}\sup_{(\mathbf{v}^h,q^h)\in(\mathbf{V}_h,Q_h)}\frac{|B^*((I_h\mathbf{v}-\mathbf{v},\rho_hq-q);(\mathbf{v}^h,q^h))|+|G(q,q^h)|}{\sqrt{\nu}||\nabla\mathbf{v}^h||_0+\frac{1}{\sqrt{\nu}}||q^h||_0}\\
\leq&Ch(\sqrt{\nu}||\mathbf{v}||_2+(\sqrt{\nu}+\frac{1}{\sqrt{\nu}})||q||_1).
\end{aligned}
\tag{4.19}
$$

To derive the estimate in $L^2$-norm, we consider the dual linearized problem for $(\phi,\psi)\in(\mathbf{V},Q)$ satisfying

$$B((\mathbf{w},r);(\phi,\psi))=(\mathbf{w},\mathbf{v}-R_h(\mathbf{v},q)),\quad\forall(\mathbf{w},r)\in(\mathbf{V},Q),\tag{4.20}$$

which satisfies

$$\sqrt{\nu}||\phi||_2+\frac{1}{\sqrt{\nu}}||\psi||_1\leq C||\mathbf{v}-R_h(\mathbf{v},q)||_0.\tag{4.21}$$

Obvious, setting $(\mathbf{w},r)=(\mathbf{v}-R_h(\mathbf{v},q),q-N_h(\mathbf{v},q))$ in (4.20), and $(\mathbf{v}^h,q^h)=(I_h\phi,\rho_h\psi)$ in (4.4), respectively, we see that

$$
\begin{aligned}
&||\mathbf{v}-R_h(\mathbf{v},q)||_0^2\\
=&B^*((\mathbf{v}-R_h(\mathbf{v},q),q-N_h(\mathbf{v},q));(\phi-I_h\phi,\psi-\rho_h\varphi))+G(q,\rho_h\psi)-G(q-N_h(\mathbf{v},q),\psi)\\
\leq&C(\sqrt{\nu}||\nabla(\mathbf{v}-R_h(\mathbf{v},q))||_0+\frac{1}{\sqrt{\nu}}||q-N_h(\mathbf{v},q)||_0)(\sqrt{\nu}||\nabla(\phi-I_h\phi)||_0+\frac{1}{\sqrt{\nu}}||\psi-\rho_h\psi||_0)\\
&+G(q,\rho_h\psi-\psi)+G(q,\psi)-G(q-N_h(\mathbf{v},q),\psi)\\
\leq&Ch\{(\sqrt{\nu}||\nabla(\mathbf{v}-R_h(\mathbf{v},q))||_0+\frac{1}{\sqrt{\nu}}||q-N_h(\mathbf{v},q)||_0)(\sqrt{\nu}||\phi||_2+\frac{1}{\sqrt{\nu}}||\psi||_1)+h||q||_1||\psi||_1\}\\
\leq&Ch\{(\sqrt{\nu}||\nabla(\mathbf{v}-R_h(\mathbf{v},q))||_0+\frac{1}{\sqrt{\nu}}||q-N_h(\mathbf{v},q)||_0+\sqrt{\nu}h||q||_1)(\sqrt{\nu}||\phi||_2+\frac{1}{\sqrt{\nu}}||\psi||_1)
\end{aligned}
\tag{4.22}
$$

Thus, we deduce

$$||\mathbf{v}-R_h(\mathbf{v},q)||_0\leq Ch^2(\sqrt{\nu}||\mathbf{v}||_2+\left(\frac{1}{\sqrt{\nu}}+\sqrt{\nu})||q||_1\right).\tag{4.23}$$

This proves (4.7). □

For the finite element error analysis, we will require upper bounds on the nonlinear term. The nonlinear form $b_s(\cdot,\cdot,\cdot)$ satisfies the following bound, gived in the following lemma.

**Lemma 4.2.** ([12]) *Let $\Omega \subset R^2$, for all $\boldsymbol{u}, \ \boldsymbol{v}, \ \boldsymbol{w} \in \boldsymbol{V}$,*

$$|b_s(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \le C(\Omega)||\nabla\boldsymbol{u}||_0||\nabla\boldsymbol{v}||_0||\nabla\boldsymbol{w}||_0, \tag{4.24a}$$

$$\begin{aligned}&|b_s(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| + |b_s(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})| + |b_s(\boldsymbol{w},\boldsymbol{v},\boldsymbol{u})| \\ \le& C(\Omega)||\boldsymbol{u}||_0^{\frac12}||\nabla\boldsymbol{u}||_0^{\frac12}(||\nabla\boldsymbol{v}||_0||\boldsymbol{w}||_0^{\frac12}||\nabla\boldsymbol{w}||_0^{\frac12} + ||\nabla\boldsymbol{w}||_0||\boldsymbol{v}||_0^{\frac12}||\nabla\boldsymbol{v}||_0^{\frac12}),\end{aligned} \tag{4.24b}$$

*and*

$$|b_s(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \le C(\Omega)\sqrt{||\boldsymbol{u}||_0||\nabla\boldsymbol{u}||_0}||\nabla\boldsymbol{v}||_0||\nabla\boldsymbol{w}||_0. \tag{4.25}$$

*If $\boldsymbol{v}, \nabla\boldsymbol{v} \in (L^\infty(\Omega))^2$, then*

$$|b_s(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \le C(\Omega)(||\boldsymbol{v}||_{(L^\infty(\Omega))^2} + ||\nabla\boldsymbol{v}||_{(L^\infty(\Omega))^2})||\boldsymbol{u}||_0||\nabla\boldsymbol{w}||_0, \tag{4.26}$$

$$|b_s(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \le C(\Omega)(||\boldsymbol{u}||_0||\nabla\boldsymbol{v}||_{(L^\infty(\Omega))^2} + ||\nabla\boldsymbol{u}||_0||\boldsymbol{v}||_{(L^\infty(\Omega))^2})||\nabla\boldsymbol{w}||_0. \tag{4.27}$$

*If $\boldsymbol{u} \in \boldsymbol{V}, \ \boldsymbol{v} \in (H^2(\Omega))^2 \cap \boldsymbol{V}$ and $\boldsymbol{w} \in (L^2(\Omega))^2$, then*

$$|b_s(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| + |b_s(\boldsymbol{v},\boldsymbol{u},\boldsymbol{w})| + |b_s(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})| \le C||\boldsymbol{u}||_2||\boldsymbol{v}||_2||\boldsymbol{w}||_0. \tag{4.28}$$

*We also recall the following property of $b_s$: $b_s(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0$.*

**Lemma 4.3.** *Let $\triangle t = t_{n+1} - t_n$ for all $n$ and denote $t_{n+1/2} = (t_{n+1} + t_n)/2$. Let $\psi(\cdot,t)$ be a function such that $\psi_t(\cdot,t) \in C^0(0,T;L^2(\Omega))$. Then there exists an $\theta \in (0,1)$ such that*

$$\left\|\frac{\psi(\cdot,t_{n+1}) - \psi(\cdot,t_n)}{\triangle t}\right\| \le C||\psi_t(\cdot,t_{n+\theta})||. \tag{4.29}$$

*If $\psi_{tt}(\cdot,t) \in C^0(0,T;L^2(\Omega))$, then there exist $\theta_1, \theta_2 \in (0,1)$ such that*

$$\left\|\frac{\psi(\cdot,t_{n+1}) + \psi(\cdot,t_n)}{2} - \psi(\cdot,t_{n+\frac12})\right\| \le C\triangle t^2||\psi_{tt}(\cdot,t_{n+\theta_1})||, \tag{4.30}$$

$$\left\|\frac32\psi(\cdot,t_n) - \frac12\psi(\cdot,t_{n-1}) - \psi(\cdot,t_{n+\frac12})\right\| \le C\triangle t^2||\psi_{tt}(\cdot,t_{n+\theta_2})||. \tag{4.31}$$

*If $\psi_{ttt}(\cdot,t) \in C^0(0,T;L^2(\Omega))$, then there exists $\theta_3 \in (0,1)$ such that*

$$\left\|\frac{\psi(\cdot,t_{n+1}) - \psi(\cdot,t_n)}{\triangle t} - \psi_t(\cdot,t_{n+\frac12})\right\| \le C\triangle t^2||\psi_{ttt}(\cdot,t_{n+\theta_3})||. \tag{4.32}$$

*If $\psi_{tttt}(\cdot,t) \in C^0(0,T;L^2(\Omega))$, then there exists $\theta_4 \in (-1,1)$ such that*

$$\begin{aligned}&\left\|\frac{\frac{\psi(\cdot,t_{n+1})-\psi(\cdot,t_n)}{\triangle t} - \psi_t(\cdot,t_{n+\frac12})}{\triangle t} - \frac{\frac{\psi(\cdot,t_n)-\psi(\cdot,t_{n-1})}{\triangle t} - \psi_t(\cdot,t_{n-\frac12})}{\triangle t}\right\| \\ \le& C\triangle t^2||\psi_{tttt}(\cdot,t_{n+\theta_4})||.\end{aligned} \tag{4.33}$$

**Theorem 4.1.** *Let $(\boldsymbol{u}^h, p^h) \in (\boldsymbol{V}_h, Q_h)$, $\boldsymbol{u} \in L^2(0,T;(H^2(\Omega))^2)$, $\boldsymbol{u}_t \in L^2(0,T;(H^2(\Omega))^2)$, $\boldsymbol{u}_{tt} \in L^2(0,T;(H^2(\Omega))^2)$, $p_{tt} \in L^2(0,T;L^2(\Omega))$, $\boldsymbol{u}_{ttt} \in C^0(0,T;(L^2(\Omega))^2)$, $\boldsymbol{u}_{tttt} \in C^0(0,T;(L^2(\Omega))^2)$. Then there is a $C = C(Re_{red},\boldsymbol{u},p,T) < \infty$, such that $\forall n \in 0,1,\cdots,N-1$ ($N = \frac{T}{\triangle t}$), the error of Algorithm 1 satisfies*

$$\left|\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}_{n+1}^h\right\|_0 + \left(Re_{red}^{-1}\sum_{i=0}^n \triangle t\left\|\nabla\left(\frac{\left(\boldsymbol{u}(t_{i+1}) - \boldsymbol{u}_{i+1}^h\right) + \left(\boldsymbol{u}(t_i) - \boldsymbol{u}_i^h\right)}{2}\right)\right\|_0^2\right)^{\frac12}$$

$$+\left(\triangle t\sum_{i=0}^{n}\left\|(I-\pi_h)\left(\frac{\left(p\left(t_{i+1}\right)-p_{i+1}^h\right)+\left(p\left(t_i\right)-p_i^h\right)}{2}\right)\right\|_0^2\right)^{\frac{1}{2}}$$
$$\leq C\left(Re_{red},\boldsymbol{u},p\right)\left(h+\nu_T^{\frac{1}{2}}H+\triangle t^2\right).$$

*Proof.* Consider the variational formulation corresponding to the Navier-Stokes equations (1.1), for any time $t$,

$$(\mathbf{u}_t,\mathbf{v}^h)+B((\mathbf{u},p),(\mathbf{v}^h,q^h))+b_s(\mathbf{u},\mathbf{u},\mathbf{v}^h)=(f,\mathbf{v}^h),\quad \forall(\mathbf{v}^h,q^h)\in(\mathbf{V}_h,Q_h). \tag{4.34}$$

Then subtract (2.12) from (4.34), taken at $t=t_{n+\frac{1}{2}}$, to get

$$\begin{aligned}&(\mathbf{u}_t(t_{n+\frac{1}{2}})-\frac{\mathbf{u}_{n+1}^h-\mathbf{u}_n^h}{\triangle t},\mathbf{v}^h)+B((\mathbf{u}(t_{n+\frac{1}{2}})-\frac{\mathbf{u}_{n+1}^h+\mathbf{u}_n^h}{2},p(t_{n+\frac{1}{2}})-\frac{p_{n+1}^h+p_n^h}{2}),(\mathbf{v}^h,q^h))\\&-S(\frac{\mathbf{u}_{n+1}^h+\mathbf{u}_n^h}{2},\mathbf{v}^h)-G(\frac{p_{n+1}^h+p_n^h}{2},q^h)+b_s(\mathbf{u}(t_{n+\frac{1}{2}}),\mathbf{u}(t_{n+\frac{1}{2}}),\mathbf{v}^h)\\&-b_s(A[\mathbf{u}_n^h,\mathbf{u}_{n-1}^h],\frac{\mathbf{u}_{n+1}^h+\mathbf{u}_n^h}{2},\mathbf{v}^h)=0.\end{aligned} \tag{4.35}$$

Let the velocity error be decomposed as

$$\zeta_n:=\mathbf{u}(t_n)-\mathbf{u}_n^h=(\mathbf{u}(t_n)-R_h(\mathbf{u}(t_n),p(t_n)))+(R_h(\mathbf{u}(t_n),p(t_n))-\mathbf{u}_n^h)=:\xi_n+\mathbf{e}_n^h. \tag{4.36}$$

Let $\eta_n^h:=N_h(\mathbf{u}(t_n),p(t_n))-p_n^h$. For $a=\xi,\ \mathbf{e}^h,\ \zeta$ or $\eta^h$, we define $a_{n+\frac{1}{2}}=(a_{n+1}+a_n)/2$. By adding and subtracting

$$\begin{aligned}&(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\triangle t},\mathbf{v}^h)+B^*((\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{p(t_{n+1})+p(t_n)}{2});(\mathbf{v}^h,q^h))+b_s(\mathbf{u}(t_{n+\frac{1}{2}})\\&+S(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\mathbf{v}^h)+A[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})]+A[\mathbf{u}_n^h,\mathbf{u}_{n-1}^h],\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\mathbf{v}^h)\end{aligned} \tag{4.37}$$

to (4.35), we obtain the error equation

$$\begin{aligned}&(\frac{\zeta_{n+1}-\zeta_n}{\triangle t},\mathbf{v}^h)+B^*((\zeta_{n+\frac{1}{2}},\frac{p(t_{n+1})+p(t_n)}{2}-\frac{p_{n+1}^h+p_n^h}{2});(\mathbf{v}^h,q^h))+S(\zeta_{n+\frac{1}{2}},\mathbf{v}^h)\\=&-b_s(A[\mathbf{u}_n^h,\mathbf{u}_{n-1}^h],\zeta_{n+\frac{1}{2}},\mathbf{v}^h)-b_s(A[\zeta_n,\zeta_{n-1}],\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\mathbf{v}^h)+T_n(\mathbf{u},p;\mathbf{v}^h,q^h),\end{aligned} \tag{4.38}$$

where

$$\begin{aligned}&T_n(\mathbf{u},p;\mathbf{v}^h,q^h)\\=&(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\triangle t}-\mathbf{u}_t(t_{n+\frac{1}{2}}),\mathbf{v}^h)+b_s(\mathbf{u}(t_{n+\frac{1}{2}}),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}-\mathbf{u}(t_{n+\frac{1}{2}}),\mathbf{v}^h)\\&-b_s(A[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})]-\mathbf{u}(t_{n+\frac{1}{2}}),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\mathbf{v}^h)+S(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\mathbf{v}^h)\\&+a(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}-\mathbf{u}(t_{n+\frac{1}{2}}),\mathbf{v}^h)-b(\frac{p(t_{n+1})+p(t_n)}{2}-p(t_{n+\frac{1}{2}}),\mathbf{v}^h)\\&+G(\frac{p(t_{n+1})+p(t_n)}{2},q^h).\end{aligned} \tag{4.39}$$

Using the velocity error decomposition (4.36) and setting $\mathbf{v}^h = \mathbf{e}^h_{n+\frac{1}{2}}, q^h = \eta^h_{n+\frac{1}{2}}$ in (4.38) yield

$$\begin{aligned}
&\frac{1}{2\triangle t}(||\mathbf{e}^h_{n+1}||_0^2 - ||\mathbf{e}^h_n||_0^2) + (\nu + \nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}}))||\nabla(\mathbf{e}^h_{n+\frac{1}{2}})||_0^2 + ||(I-\pi_h)\eta^h_{n+\frac{1}{2}}||_0^2 \\
=& -(\frac{\xi_{n+1}-\xi_n}{\triangle t}, \mathbf{e}^h_{n+\frac{1}{2}}) - S(\xi_{n+\frac{1}{2}}, \mathbf{e}^h_{n+\frac{1}{2}}) - b_s(A[\mathbf{u}^h_n, \mathbf{u}^h_{n-1}], \xi_{n+\frac{1}{2}}, \mathbf{e}^h_{n+\frac{1}{2}}) \\
&- b_s(A[\zeta_n, \zeta_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{e}^h_{n+\frac{1}{2}}) + T_n(\mathbf{u}, p; \mathbf{e}^h_{n+\frac{1}{2}}, \eta^h_{n+\frac{1}{2}}) \\
&- G(\frac{p(t_{n+1}) + p(t_n)}{2}, \eta^h_{n+\frac{1}{2}}),
\end{aligned} \tag{4.40}$$

where the conclusion follows from

$$B^*\left(\left(\xi_{n+\frac{1}{2}}, \frac{p(t_{n+1})+p(t_n)}{2} - \frac{N_h(\mathbf{u}(t_{n+1}), p(t_{n+1})) + N_h(\mathbf{u}(t_n), p(t_n))}{2}\right); (\mathbf{e}^h_{n+\frac{1}{2}}, \eta^h_{n+\frac{1}{2}})\right) - G\left(\frac{p(t_{n+1})+p(t_n)}{2}, \eta^h_{n+\frac{1}{2}}\right) = 0.$$

We estimate the terms on the right hand side of (4.40). All bilinear terms are estimated essentially in the same way: using the Cauchy-Schwartz inequality (or the estimate for the dual pairing ) and Young's inequality.

$$\begin{aligned}
|(\frac{\xi_{n+1}-\xi_n}{\triangle t}, \mathbf{e}^h_{n+\frac{1}{2}})| &\le C||\frac{\xi_{n+1}-\xi_n}{\triangle t}||_{-1}||\nabla(\mathbf{e}^h_{n+\frac{1}{2}})||_0 \\
&\le \frac{\nu + \nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}{14}||\nabla(\mathbf{e}^h_{n+\frac{1}{2}})||_0^2 + \frac{C}{\nu + \nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}||\frac{\xi_{n+1}-\xi_n}{\triangle t}||_{-1}^2.
\end{aligned} \tag{4.41}$$

Moreover, $S$ is treated as follows:

$$|S(\xi_{n+\frac{1}{2}}, \mathbf{e}^h_{n+\frac{1}{2}})| \le \frac{\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}{28}||\nabla(\mathbf{e}^h_{n+\frac{1}{2}})||_0^2 + 7\nu_{add}(\xi_{n+\frac{1}{2}})||\nabla(\xi_{n+\frac{1}{2}})||_0^2. \tag{4.42}$$

We estimate the first nonlinear term in (4.40) next. Adding and subtracting the quantity $b_s(A[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})], \xi_{n+\frac{1}{2}}, \mathbf{e}^h_{n+\frac{1}{2}})$, and using Lemma 4.2, and the Young's inequality, we get

$$\begin{aligned}
&|b_s(A[\mathbf{u}^h_n, \mathbf{u}^h_{n-1}], \xi_{n+\frac{1}{2}}, \mathbf{e}^h_{n+\frac{1}{2}})| \\
\le& |b_s(A[\mathbf{e}^h_n, \mathbf{e}^h_{n-1}], \xi_{n+\frac{1}{2}}, \mathbf{e}^h_{n+\frac{1}{2}})| + |b_s(A[\xi_n, \xi_{n-1}], \xi_{n+\frac{1}{2}}, \mathbf{e}^h_{n+\frac{1}{2}})| \\
&+ |b_s(A[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})], \xi_{n+\frac{1}{2}}, \mathbf{e}^h_{n+\frac{1}{2}})| \\
\le& \frac{\nu + \nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}{14}||\nabla(\mathbf{e}^h_{n+\frac{1}{2}})||_0^2 + \frac{C}{\nu + \nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}||\nabla(A[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})])||_0^2||\nabla(\xi_{n+\frac{1}{2}})||_0^2 \\
&+ \frac{C}{\nu + \nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}||\nabla(A[\xi_n, \xi_{n-1}])||_0^2||\nabla(\xi_{n+\frac{1}{2}})||_0^2 \\
&+ \frac{C}{\nu + \nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}||A[\mathbf{e}^h_n, \mathbf{e}^h_{n-1}]||_0||\nabla(A[\mathbf{e}^h_n, \mathbf{e}^h_{n-1}])||_0||\nabla(\xi_{n+\frac{1}{2}})||_0^2.
\end{aligned} \tag{4.43}$$

The second and third terms in (4.43) involving the operator $A[\cdot,\cdot]$, which can be controlled by using its definition and regularity assumption on $\mathbf{u}$,

$$||\nabla(A[\mathbf{u}(t_n), \mathbf{u}(t_{n-1})])||_0 \le C$$

$$||\nabla(A[\xi_n, \xi_{n-1}])||_0 \leq \frac{3}{2}||\nabla(\xi_n)||_0 + \frac{1}{2}||\nabla(\xi_{n-1})||_0. \tag{4.44}$$

For the fourth term of (4.43), we apply the inverse estimate, resulting in

$$\begin{aligned}
&||A[\mathbf{e}_n^h, \mathbf{e}_{n-1}^h]||_0||\nabla(A[\mathbf{e}_n^h, \mathbf{e}_{n-1}^h])||_0 \\
\leq& C(||\mathbf{e}_n^h||_0 + ||\mathbf{e}_{n-1}^h||_0)(||\nabla(\mathbf{e}_n^h)||_0 + ||\nabla(\mathbf{e}_{n-1}^h)||_0) \\
\leq& Ch^{-1}(||\mathbf{e}_n^h||_0 + ||\mathbf{e}_{n-1}^h||_0)^2,
\end{aligned} \tag{4.45}$$

so that

$$\begin{aligned}
&||A[\mathbf{e}_n^h, \mathbf{e}_{n-1}^h]||_0||\nabla(A[\mathbf{e}_n^h, \mathbf{e}_{n-1}^h])||_0||\nabla(\xi_{n+\frac{1}{2}})||_0^2 \\
\leq& Ch^{-1}||\nabla(\xi_{n+\frac{1}{2}})||_0^2(||\mathbf{e}_n^h||_0^2 + ||\mathbf{e}_{n-1}^h||_0^2).
\end{aligned} \tag{4.46}$$

Putting (4.44) and (4.46) back into (4.43), we have

$$\begin{aligned}
&|b_s(A[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h], \zeta_{n+\frac{1}{2}}, \mathbf{e}_{n+\frac{1}{2}}^h)| \\
\leq& \frac{\nu + \nu_{add}(\mathbf{e}_{n+\frac{1}{2}}^h)}{14}||\nabla(\mathbf{e}_{n+\frac{1}{2}})||_0^2 + \frac{C}{\nu + \nu_{add}(\mathbf{e}_{n+\frac{1}{2}}^h)}||\nabla(\xi_{n+\frac{1}{2}})||_0^2 \\
&+ \frac{C}{\nu + \nu_{add}(\mathbf{e}_{n+\frac{1}{2}}^h)}(||\nabla(\xi_n)||_0^2 + ||\nabla(\xi_{n-1})||_0^2)||\nabla(\xi_{n+\frac{1}{2}})||_0^2 \\
&+ \frac{C}{\nu + \nu_{add}(\mathbf{e}_{n+\frac{1}{2}}^h)}h^{-1}||\nabla(\xi_{n+\frac{1}{2}})||_0^2(||\mathbf{e}_{n-1}^h||_0^2 + ||\mathbf{e}_n^h||_0^2).
\end{aligned} \tag{4.47}$$

For the second nonlinear term of (4.40), using Lemma 4.2, the assumption that $\mathbf{u}(t), \nabla\mathbf{u}(t) \in (L^\infty(\Omega))^2$ and $||\nabla\mathbf{u}(t)||_0$ are bounded for any $t \in [0, T]$, together with the Young's inequality, we have

$$\begin{aligned}
&|b_s(A[\zeta_n, \zeta_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{e}_{n+\frac{1}{2}}^h)| \\
\leq& |b_s(A[\xi_n, \xi_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{e}_{n+\frac{1}{2}}^h)| + |b_s(A[e_n^h, \mathbf{e}_{n-1}^h], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \mathbf{e}_{n+\frac{1}{2}}^h)| \\
\leq& \frac{\nu + \nu_{add}(\mathbf{e}_{n+\frac{1}{2}}^h)}{14}||\nabla(\mathbf{e}_{n+\frac{1}{2}}^h)||_0^2 + \frac{C}{\nu + \nu_{add}(\mathbf{e}_{n+\frac{1}{2}}^h)}(||\nabla(\xi_n)||_0^2 + ||\nabla(\xi_{n-1})||_0^2) \\
&+ \frac{C}{\nu + \nu_{add}(\mathbf{e}_{n+\frac{1}{2}}^h)}(||\mathbf{e}_n^h||_0^2 + ||\mathbf{e}_{n-1}^h||_0^2).
\end{aligned} \tag{4.48}$$

What is left is to $|T_n(\mathbf{u}, p; \mathbf{e}_{n+\frac{1}{2}}^h, \eta_{n+\frac{1}{2}}^h) - G(\frac{p(t_{n+1})+p(t_n)}{2}, \eta_{n+\frac{1}{2}}^h)|$. Each of four bilinear terms can be controlled by the Cauchy-Schwartz inequality and Young's inequality, together with the estimates in Lemma 4.3. We estimate one at the same time

$$\begin{aligned}
&(\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{\triangle t} - \mathbf{u}_t(t_{n+\frac{1}{2}}), \mathbf{e}_{n+\frac{1}{2}}^h) \\
\leq& \frac{\nu + \nu_{add}(\mathbf{e}_{n+\frac{1}{2}}^h)}{14}||\nabla(\mathbf{e}_{n+\frac{1}{2}}^h)||_0^2 + \frac{C}{\nu + \nu_{add}(\mathbf{e}_{n+\frac{1}{2}}^h)}\triangle t^4||\mathbf{u}_{ttt}(t_{n+\theta_1})||_{-1}^2,
\end{aligned} \tag{4.49}$$

$$\nu(\nabla(\frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2} - \mathbf{u}(t_{n+\frac{1}{2}}), \nabla(\mathbf{e}_{n+\frac{1}{2}}^h))$$

$$\leq \frac{\nu}{14}||\nabla(\mathbf{e}^h_{n+\frac{1}{2}})||_0^2 + \nu C\triangle t^4||\nabla(\mathbf{u}_{tt}(t_{n+\theta_2}))||_0^2, \tag{4.50}$$

$$|(\frac{p(t_{n+1})+p(t_n)}{2} - p(t_{n+\frac{1}{2}}), \nabla\cdot\mathbf{e}^h_{n+\frac{1}{2}})|$$
$$\leq \frac{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}{14}||\nabla(\mathbf{e}^h_{n+\frac{1}{2}})||_0^2 + \frac{C}{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}\triangle t^4||p_{tt}(t_{n+\theta_3})||_0^2, \tag{4.51}$$

for some $\theta_1,\ \theta_2,\ \theta_3 \in (0,1)$. Using (2.5), $S$ is treated as follows:

$$S(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}, \mathbf{e}^h_{n+\frac{1}{2}}) \leq \nu_T||(I-P_{\mathbf{L}_H})\nabla(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2})||_0||(I-P_{\mathbf{L}_H})\nabla(\mathbf{e}_{n+\frac{1}{2}})||_0$$
$$\leq \frac{\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}{28}||\nabla(\mathbf{e}^h_{n+\frac{1}{2}})||_0^2 + C\nu_T H^2||\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}||_2^2. \tag{4.52}$$

For the nonlinear terms of $|T_n(\mathbf{u},p;\mathbf{e}^h_{n+\frac{1}{2}},\eta^h_{n+\frac{1}{2}}) - G(\frac{p(t_{n+1})+p(t_n)}{2},\eta^h_{n+\frac{1}{2}})|$, using Lemmas 4.2, 4.3 and Young's inequality, together with $||\nabla(\mathbf{u}(t))||_0 \leq C$ for any $t\in[0,T]$, we obtain

$$|b_s(\mathbf{u}(t_{n+\frac{1}{2}}), \frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2} - \mathbf{u}(t_{n+\frac{1}{2}}), \mathbf{e}^h_{n+\frac{1}{2}})|$$
$$+ |b_s(A[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})] - \mathbf{u}(t_{n+\frac{1}{2}}), \frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}, \mathbf{e}^h_{n+\frac{1}{2}})|$$
$$\leq C(\Omega)||\nabla(\mathbf{u}(t_{n+\frac{1}{2}}))||_0||\nabla(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2} - \mathbf{u}(t_{n+\frac{1}{2}}))||_0||\nabla(\mathbf{e}^h_{n+\frac{1}{2}})||_0$$
$$+ C(\Omega)||\nabla(\frac{3}{2}\mathbf{u}(t_n) - \frac{1}{2}\mathbf{u}(t_{n-1}) - \mathbf{u}(t_{n+\frac{1}{2}}))||_0||\nabla(\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2})||_0||\nabla(\mathbf{e}^h_{n+\frac{1}{2}})||_0$$
$$\leq \frac{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}{14}||\nabla(\mathbf{e}^h_{n+\frac{1}{2}})||_0^2 + \frac{C}{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}\triangle t^4||\nabla(\mathbf{u}_{tt}(t_{n+\theta_4}))||_0^2 \tag{4.53}$$

for some $\theta_4 \in (0,1)$. Combining (4.49)–(4.53) gives

$$|T_n(\mathbf{u},p;\mathbf{e}^h_{n+\frac{1}{2}},\eta^h_{n+\frac{1}{2}}) - G(\frac{p(t_{n+1})+p(t_n)}{2},\eta^h_{n+\frac{1}{2}})|$$
$$\leq(\frac{\nu}{14} + \frac{3(\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}}))}{14} + \frac{\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}{28})||\nabla(\mathbf{e}^h_{n+\frac{1}{2}})||_0^2$$
$$+ \frac{C}{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}\triangle t^4(||\mathbf{u}_{ttt}(t_{n+\theta_1})||_{-1}^2 + ||p_{tt}(t_{n+\theta_3})||_0^2 + ||\nabla(\mathbf{u}_{tt}(t_{n+\theta_4}))||_0^2)$$
$$+ \nu C\triangle t^4||\nabla(\mathbf{u}_{tt}(t_{n+\theta_2}))||_0^2 + C\nu_T H^2||\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}||_2^2. \tag{4.54}$$

Now, putting (4.41)-(4.42), (4.47)-(4.48) and (4.54) backing into (4.40), we have

$$\frac{1}{2\triangle t}(||\mathbf{e}^h_{n+1}||_0^2 - ||\mathbf{e}^h_n||_0^h) + \frac{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}{2}||\nabla(\mathbf{e}_{n+\frac{1}{2}})||_0^2 + ||(I-\pi_h)\eta^h_{n+\frac{1}{2}}||_0^2$$
$$\leq \frac{C}{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}||\frac{\xi_{n+1}-\xi_n}{\triangle t}||_{-1}^2 + 7\nu_{add}(\xi_{n+\frac{1}{2}})||\nabla(\xi_{n+\frac{1}{2}})||_0^2$$
$$+ \frac{C}{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}||\nabla(\xi_{n+\frac{1}{2}})||_0^2 + \frac{C}{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}(||\nabla(\xi_{n-1})||_0^2 + ||\nabla(\xi_n)||_0^2)||\nabla(\xi_{n+\frac{1}{2}})||_0^2$$
$$+ \frac{C}{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}h^{-1}||\nabla(\xi_{n+\frac{1}{2}})||_0^2(||\mathbf{e}^h_{n-1}||_0^2 + ||\mathbf{e}^h_n||_0^2)$$

$$+\frac{C}{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}(||\nabla(\xi_{n-1})||_0^2+||\nabla(\xi_n)||_0^2)+\frac{C}{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}(||\mathbf{e}^h_{n-1}||_0^2+||\mathbf{e}^h_n||_0^2)$$
$$+\frac{C}{\nu+\nu_{add}(\mathbf{e}^h_{n+\frac{1}{2}})}\triangle t^4(||\mathbf{u}_{ttt}(t_{n+\theta_1})||_{-1}^2+||p_{tt}(t_{n+\theta_3})||_0^2+||\nabla(\mathbf{u}_{tt}(t_{n+\theta_4}))||_0^2)$$
$$+\nu C\triangle t^4||\nabla(\mathbf{u}_{tt}(t_{n+\theta_2}))||_0^2+C\nu_T H^2||\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}||_2^2. \quad (4.55)$$

Multiple both sides of (4.55) by $2\triangle t$ and use Lemma 4.1, then sum over the time levels from 1 to $n$, choosing $R_h(\mathbf{u}(t_0),p(t_0))=\mathbf{u}_0^h$, which gives $\mathbf{e}_0^h=0$, to get

$$||\mathbf{e}^h_{n+1}||_0^2+Re_{red}^{-1}\sum_{i=1}^n\triangle t||\nabla(\mathbf{e}^h_{i+\frac{1}{2}})||_0^2+2\triangle t\sum_{i=1}^n||(I-\pi_h)\eta^h_{i+\frac{1}{2}}||_0^2$$
$$\leq||\mathbf{e}_1^h||_0^2+CRe_{red}h^4||\mathbf{u}_t||^2_{L^2(0,T;(H^2(\Omega))^2)}+C\sup_{t\in[0,T]}\nu_{add}(\xi(t))h^2||\mathbf{u}||^2_{L^2(0,T;(H^2(\Omega))^2)}$$
$$+CRe_{red}h^4||\mathbf{u}||^2_{L^2(0,T;(H^2(\Omega))^2)}+CRe_{red}h^2||\mathbf{u}||^2_{L^2(0,T;(H^2(\Omega))^2)}+C\nu_TH^2||\mathbf{u}||^2_{L^2(0,T;(H^2(\Omega))^2)}$$
$$+CRe_{red}\triangle t^4(||\mathbf{u}_{ttt}||^2_{L^2(0,T;(L^2(\Omega))^2)}+||p_{tt}||^2_{L^2(0,T;H^2(\Omega))}+||\nabla(\mathbf{u}_{tt})||^2_{L^2(0,T;(L^2(\Omega))^2)})$$
$$+\nu C\triangle t^4||\nabla(\mathbf{u}_{tt})||^2_{L^2(0,T;(L^2(\Omega))^2)}+CRe_{red}h\sum_{i=1}^n\triangle t||\mathbf{u}(t_{i+\frac{1}{2}})||_2^2(||\mathbf{e}^h_{i-1}||_0^2+||\mathbf{e}^h_i||_0^2)$$
$$+CRe_{red}\sum_{i=1}^n\triangle t(||\mathbf{e}^h_{i-1}||_0^2+||\mathbf{e}^h_i||_0^2). \quad (4.56)$$

Since $\mathbf{u}\in L^\infty(0,T;(H^2(\Omega))^2)$, the last two terms in (4.56) can be combined as $CRe_{red}(h+1)\triangle t\sum_{i=1}^n||\mathbf{e}_i^h||_0^2$. Using the regularity of $\mathbf{u}$ and $p$, the error equation finally gets

$$||\mathbf{e}^h_{n+1}||_0^2+Re_{red}^{-1}\sum_{i=1}^n\triangle t||\nabla(\mathbf{e}^h_{i+\frac{1}{2}})||_0^2+2\triangle t\sum_{i=1}^n||(I-\pi_h)\eta^h_{i+\frac{1}{2}}||_0^2$$
$$\leq||\mathbf{e}_1^h||_0^2+CRe_{red}h^2+C\sup_{t\in[0,T]}\nu_{add}(\xi(t))h^2+C(\nu+Re_{red})\triangle t^4+C\nu_TH^2$$
$$+C\triangle t\sum_{i=1}^n(Re_{red}+Re_{red}h)||\mathbf{e}_i^h||_0^2. \quad (4.57)$$

To complete the proof, the bound of $||\mathbf{e}_1^h||$ is needed in the above estimate. The bound depend upon the way the first time step is taken. The error equation for $\mathbf{e}_1^h$ is the same as for $\mathbf{e}_n^h$ except for the nonlinear terms, and is treated in the same way except for the nonlinear terms. Therefore, we go directly to the treatment of the nonlinear terms.

Adding and subtracting

$$b_s(\mathbf{u}(t_{\frac{1}{2}})+\frac{\mathbf{u}_0^h+\mathbf{u}_1^h}{2}+\frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2},\frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2},\mathbf{v}^h)$$

to the nonlinear terms, we obtain

$$b_s(\mathbf{u}(t_{\frac{1}{2}}),\mathbf{u}(t_{\frac{1}{2}}),\mathbf{v}^h)-b_s(\frac{\mathbf{u}_0^h+\mathbf{u}_1^h}{2},\frac{\mathbf{u}_0^h+\mathbf{u}_1^h}{2},\mathbf{v}^h)$$
$$=b_s(\frac{\mathbf{u}_0^h+\mathbf{u}_1^h}{2},\zeta_{\frac{1}{2}},\mathbf{v}^h)+b_s(\zeta_{\frac{1}{2}},\frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2},\mathbf{v}^h)+b_s(\mathbf{u}(t_{\frac{1}{2}}),\mathbf{u}(t_{\frac{1}{2}})-\frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2},\mathbf{v}^h)$$

$$+ b_s(\mathbf{u}(t_{\frac{1}{2}}) - \frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2}, \frac{\mathbf{u}(t_0)+\mathbf{u}(t_1)}{2}, \mathbf{v}^h) \tag{4.58}$$

Taking $\mathbf{v}^h = \mathbf{e}^h_{\frac{1}{2}}$, the first term in (4.58) can be treated as (4.43)–(4.47); the second term can be treated as (4.48); the third and fourth can be treated as (4.53). This leads to upper bound

$$\begin{aligned}&||\mathbf{e}^h_1||_0^2 + \triangle t Re_{red}^{-1}||\nabla(\mathbf{e}^h_1)||_0^2 + \triangle t||(I-\pi_h)\eta^h_1||_0^2\\ \le& CRe_{red}h^2\triangle t + C\sup_{t\in[0,T]}\nu_{add}(\xi(t))h^2\triangle t + C(\nu+Re_{red})\triangle t^5 + C\nu_T H^2\triangle t.\end{aligned} \tag{4.59}$$

We thus substitute the bound of $||\mathbf{e}^h_1||_0^2$ into (4.57) to get

$$\begin{aligned}&||\mathbf{e}^h_{n+1}||_0^2 + Re_{red}^{-1}\sum_{i=0}^n \triangle t||\nabla(\mathbf{e}^h_{i+\frac{1}{2}})||_0^h + 2\triangle t\sum_{i=0}^n ||(I-\pi_h)\eta^h_{i+\frac{1}{2}}||_0^2\\ \le& C\{Re_{red}h^2 + C\sup_{t\in[0,T]}\nu_{add}(\xi(t))h^2 + (\nu+Re_{red})\triangle t^4 + \nu_T H^2\\ &+ \triangle t\sum_{i=0}^n (Re_{red}+Re_{red}h)||\mathbf{e}^h_i||_0^2\}.\end{aligned} \tag{4.60}$$

Hence, it follows from the discrete Gronwall inequality that there exists $C = C(Re_{red}, \Omega, T, \mathbf{u}, p)$, such that for any $n \ge 0$,

$$\begin{aligned}&||\mathbf{e}^h_{n+1}||_0^2 + Re_{red}^{-1}\sum_{i=0}^n \triangle t||\nabla(\mathbf{e}^h_{i+\frac{1}{2}})||_0^h + 2\triangle t\sum_{i=0}^n ||(I-\pi_h)\eta^h_{i+\frac{1}{2}}||_0^2\\ \le& C\{h^2 + \triangle t^4 + \nu_T H^2\}.\end{aligned} \tag{4.61}$$

Finally, the statement of the theorem follows from the triangle inequality.

**Corollary 4.1.** *If one choose $(\nu_T, H) = (h, h^{\frac{1}{2}})$, or $(\nu_T, H) = (C, h)$, we obtain the optimal error estimate:*

$$\begin{aligned}&||\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}^h_{n+1}||_0 + (\triangle t\sum_{i=0}^n Re_{red}^{-1}||\nabla(\frac{(\boldsymbol{u}(t_{i+1})-\boldsymbol{u}^h_{i+1}) + (\boldsymbol{u}(t_i)-\boldsymbol{u}^h_i)}{2})||_0^2)^{\frac{1}{2}}\\ &+ (\triangle t\sum_{i=0}^n ||(I-\pi_h)(\frac{(p(t_{i+1})-p^h_{i+1}) + (p(t_i)-p^h_i)}{2})||_0^2)^{\frac{1}{2}}\\ \le& C(Re_{red}, \Omega, T, \boldsymbol{u}, p)(h+\triangle t^2).\end{aligned} \tag{4.62}$$

**Remark 4.1.** If one choose $(\nu_T, H) = (h, h^{\frac{1}{2}})$, or $(\nu_T, H) = (C, h)$, we obtain the error in the following:

$$\begin{aligned}&||\mathbf{u}(t_{n+1}) - \mathbf{u}^h_{n+1}||_0 + (\triangle t\sum_{i=0}^n Re_{red}^{-1}h^2||\nabla(\frac{(\mathbf{u}(t_{i+1})-\mathbf{u}^h_{i+1}) + (\mathbf{u}(t_i)-\mathbf{u}^h_i)}{2})||_0^2)^{\frac{1}{2}}\\ \le& C(Re_{red}, \Omega, T, \mathbf{u}, p)(h^2+\triangle t^2).\end{aligned} \tag{4.63}$$

## 4.2. Error estimate for the pressure

In order to prove pressure convergence, we need to derive a bound on time difference of velocity error

$$||\frac{(\mathbf{u}(t_{n+1}) - \mathbf{u}^h_{n+1}) - (\mathbf{u}(t_n) - \mathbf{u}^h_n)}{\triangle t}||_0.$$

Let the pressure error be decomposed as

$$\Phi_n := p(t_n) - p_n^h = (p(t_n) - N_h(\mathbf{u}(t_n), p(t_n))) + (N_h(\mathbf{u}(t_n), p(t_n)) - p_n^h) =: \phi_n + \eta_n^h. \quad (4.64)$$

**Lemma 4.4.** *Let $(\boldsymbol{u}^h, p^h) \in (\boldsymbol{V}_h, Q_h)$, $\boldsymbol{u} \in L^2(0,T;(H^2(\Omega))^2)$, $\boldsymbol{u}_t \in L^2(0,T;(H^2(\Omega))^2)$, $\boldsymbol{u}_{tt} \in L^2(0,T;(H^2(\Omega))^2)$, $p_{ttt} \in L^2(0,T;L^2(\Omega))$, $\boldsymbol{u}_{ttt} \in C^0(0,T;(L^2(\Omega))^2)$, $\boldsymbol{u}_{tttt} \in C^0(0,T;(L^2(\Omega))^2)$ and $h \sim \triangle t$. Then there is a $C = C(Re_{red}, \Omega, u, p, T) < \infty$, such that $\forall n \in 0, 1, \cdots, N-1$, the error in Algorithm 1 satisfied*

$$\begin{aligned}
&||\frac{\boldsymbol{e}_{n+1}^h - \boldsymbol{e}_n^h}{\triangle t}||_0^2 + \frac{1}{4} Re_{red}^{-1} \sum_{i=1}^n \triangle t ||\nabla(\frac{\boldsymbol{e}_{i+1}^h - \boldsymbol{e}_{i-1}^h}{\triangle t})||_0^2 + \frac{1}{2}\sum_{i=1}^n \triangle t ||(I-\pi_h)(\frac{\eta_{i+1}^h - \eta_{i-1}^h}{\triangle t})||_0^2 \\
\le & C\{h^2 + \nu_T H^2 + \triangle t^4\}
\end{aligned}$$

*Proof.* Subtracting $n$ time level of (4.38) from $(n-1)$ time level of (4.38) and setting $\mathbf{v}^h = (\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h)/\triangle t \in \mathbf{V}_h$, $q^h = (\eta_{n+1}^h - \eta_{n-1}^h)/\triangle t \in Q_h$, we obtain

$$\begin{aligned}
&||\frac{\mathbf{e}_{n+1}^h - \mathbf{e}_n^h}{\triangle t}||_0^2 - ||\frac{\mathbf{e}_n^h - \mathbf{e}_{n-1}^h}{\triangle t}||_0^2 + \frac{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t})\triangle t}{2} ||\nabla(\frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t})||_0^2 \\
&+ \frac{\triangle t}{2}||(I-\pi_h)(\frac{\eta_{n+1}^h - \eta_{n-1}^h}{\triangle t})||_0^2 \\
= &-(\frac{\xi_{n+1} - 2\xi_n + \xi_{n-1}}{\triangle t}, \frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t}) - S(\frac{\xi_{n+1} - \xi_{n-1}}{2}, \frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t}) \\
&- b_s(A[\mathbf{u}_n^h, \mathbf{u}_{n-1}^h], \zeta_{n+\frac{1}{2}}, \frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t}) + b_s(A[\mathbf{u}_{n-1}^h, \mathbf{u}_{n-2}^h], \zeta_{n-\frac{1}{2}}, \frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t}) \\
&- b_s(A[\zeta_n, \zeta_{n-1}], \frac{\mathbf{u}(t_{n+1}) + \mathbf{u}(t_n)}{2}, \frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t}) \\
&+ b_s(A[\zeta_{n-1}, \zeta_{n-2}], \frac{\mathbf{u}(t_n) + \mathbf{u}(t_{n-1})}{2}, \frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t}) \\
&+ T_n(\mathbf{u}, p, \frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t}, \frac{\eta_{n+1}^h - \eta_{n-1}^h}{\triangle t}) - T_{n-1}(\mathbf{u}, p, \frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t}, \frac{\eta_{n+1}^h - \eta_{n-1}^h}{\triangle t}) \\
&- G(\frac{p(t_{n+1}) + p(t_n)}{2}, \frac{\eta_{n+1}^h - \eta_{n-1}^h}{\triangle t}) + G(\frac{p(t_n) + p(t_{n-1})}{2}, \frac{\eta_{n+1}^h - \eta_{n-1}^h}{\triangle t}). \quad (4.65)
\end{aligned}$$

We estimate the two bilinear terms on the right-hand side of (4.65), using the Cauchy-Schwartz inequality and Young's inequality to get

$$\begin{aligned}
&|(\frac{\xi_{n+1} - 2\xi_n + \xi_{n-1}}{\triangle t}, \frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t})| \\
\le & C_1 ||\frac{\xi_{n+1} - 2\xi_n + \xi_{n-1}}{\triangle t}||_{-1} ||\nabla(\frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t})||_0 \\
\le & \frac{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h - \mathbf{e}_n^h}{\triangle t})}{24} \triangle t ||\nabla(\frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t})||_0 + \frac{C}{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h - \mathbf{e}_n^h}{\triangle t})} \triangle t ||\frac{\xi_{n+1} - 2\xi_n + \xi_{n-1}}{\triangle t^2}||_{-1}^2. \quad (4.66)
\end{aligned}$$

$$\begin{aligned}
&|S(\frac{\xi_{n+1} - \xi_{n-1}}{2}, \frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t})| \\
\le & \frac{\nu_{add}(\frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t})}{48} \triangle t ||\nabla(\frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t})||_0^2 + 3\nu_{add}(\frac{\xi_{n+1} - \xi_{n-1}}{\triangle t}) \triangle t ||\nabla(\frac{\xi_{n+1} - \xi_{n-1}}{\triangle t})||_0^2. \quad (4.67)
\end{aligned}$$

For clarity, we analyze each of the nonlinear terms of (4.65). Using Lemma 4.2 and Young's inequality gives

$$\begin{aligned}
&|-b_s(A[\mathbf{u}_n^h,\mathbf{u}_{n-1}^h],\zeta_{n+\frac{1}{2}},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})+b_s(A[\mathbf{u}_{n-1}^h,\mathbf{u}_{n-2}^h],\zeta_{n-\frac{1}{2}},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\\
\le&|b_s(A[\mathbf{u}_n^h,\mathbf{u}_{n-1}^h],\frac{\zeta_{n+1}-\zeta_{n-1}}{2},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\\
&+|b_s(A[\mathbf{u}_{n-1}^h,\mathbf{u}_{n-2}^h]-A[\mathbf{u}_n^h,\mathbf{u}_{n-1}^h],\zeta_{n-\frac{1}{2}},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|. \qquad (4.68)
\end{aligned}$$

We estimate the first nonlinear term of (4.68),

$$\begin{aligned}
&|b_s(A[\mathbf{u}_n^h,\mathbf{u}_{n-1}^h],\frac{\zeta_{n+1}-\zeta_{n-1}}{2},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\\
\le&|b_s(A[\mathbf{u}(t_n),\mathbf{u}(t_{n-1})],\frac{\xi_{n+1}-\xi_{n-1}}{2},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\\
&+|b_s(A[\mathbf{e}_n^h,\mathbf{e}_{n-1}^h],\frac{\xi_{n+1}-\xi_{n-1}}{2},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|+|b_s(A[\xi_n,\xi_{n-1}],\frac{\xi_{n+1}-\xi_{n-1}}{2},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\\
\le&\frac{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}{24}\triangle t||\nabla(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})||_0^2+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t||\nabla(\frac{\xi_{n+1}-\xi_{n-1}}{\triangle t})||_0^2\\
&+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t(||\nabla(\mathbf{e}_n^h)||_0^2+||\nabla(\mathbf{e}_{n-1}^h)||_0^2)||\nabla(\frac{\xi_{n+1}-\xi_{n-1}}{\triangle t})||_0^2\\
&+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t(||\nabla(\xi_n)||_0^2+||\nabla(\xi_{n-1})||_0^2)||\nabla(\frac{\xi_{n+1}-\xi_{n-1}}{\triangle t})||_0^2. \qquad (4.69)
\end{aligned}$$

For the second nonlinear term of (4.68), using Lemma 4.2, Tayler expansion and the assumption that $||\mathbf{u}_t||_0\le C$ and $||\nabla\mathbf{u}_t||_0\le C$ for any $t\in[0,T]$. Then we apply Young's inequality, resulting in

$$\begin{aligned}
&|b_s(A[\mathbf{u}_{n-1}^h,\mathbf{u}_{n-2}^h]-A[\mathbf{u}_n^h,\mathbf{u}_{n-1}^h],\zeta_{n-\frac{1}{2}},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\\
\le&|b_s(\frac{3}{2}(\mathbf{u}(t_n)-\mathbf{u}(t_{n-1}))-\frac{1}{2}(\mathbf{u}(t_{n-1})-\mathbf{u}(t_{n-2})),\zeta_{n-\frac{1}{2}},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\\
&+|b_s(\frac{3}{2}(\mathbf{e}_n^h-\mathbf{e}_{n-1}^h)-\frac{1}{2}(\mathbf{e}_{n-1}^h-\mathbf{e}_{n-2}^h),\zeta_{n-\frac{1}{2}},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\\
&+|b_s(\frac{3}{2}(\xi_n-\xi_{n-1})-\frac{1}{2}(\xi_{n-1}-\xi_{n-2}),\zeta_{n-\frac{1}{2}},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\\
\le&\frac{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}{24}\triangle t||\nabla(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})||_0^2+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t||\nabla(\zeta_{n-\frac{1}{2}})||_0^2\\
&+\frac{Ch^{-1}\triangle t}{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}(||\frac{\mathbf{e}_n^h-\mathbf{e}_{n-1}^h}{\triangle t}||_0^2+||\frac{\mathbf{e}_{n-1}^h-\mathbf{e}_{n-2}^h}{\triangle t}||_0^2)||\nabla(\zeta_{n-\frac{1}{2}})||_0^2
\end{aligned}$$

$$+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t(||\nabla(\frac{\xi_n-\xi_{n-1}}{\triangle t})||_0^2+||\nabla(\frac{\xi_{n-1}-\xi_{n-2}}{\triangle t})||_0^2)||\nabla(\zeta_{n-\frac{1}{2}})||_0^2. \quad (4.70)$$

For the third and fourth nonlinear terms in (4.65), we also use Lemma 4.2, and the assumption that $||\mathbf{u}(t)||_0$ to obtain $||\nabla(\mathbf{u}(t))||_0$, $||\nabla_t(\mathbf{u}(t))||_0$ are bounded for any $t\in[0,T]$,

$$\begin{aligned}
&|-b_s(A[\zeta_n,\zeta_{n-1}],\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})\\
&\quad+b_s(A[\zeta_{n-1},\zeta_{n-2}],\frac{\mathbf{u}(t_n)+\mathbf{u}(t_{n-1})}{2},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\\
\le&|b_s(A[\zeta_n,\zeta_{n-1}]-A[\zeta_{n-1},\zeta_{n-2}],\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\\
&\quad+|b_s(A[\zeta_{n-1},\zeta_{n-2}],\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2}-\frac{\mathbf{u}(t_n)+\mathbf{u}(t_{n-1})}{2},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\\
\le&\frac{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}{24}\triangle t||\nabla(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})||_0^2+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t(||\nabla(\frac{\xi_n-\xi_{n-1}}{\triangle t})||_0^2\\
&\quad+||\nabla(\frac{\xi_{n-1}-\xi_{n-2}}{\triangle t})||_0^2)+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t(||\frac{\mathbf{e}_n^h-\mathbf{e}_{n-1}^h}{\triangle t}||_0^2+||\frac{\mathbf{e}_{n-1}^h-\mathbf{e}_{n-2}^h}{\triangle t}||_0^2)\\
&\quad+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t(||\nabla(\zeta_{n-1})||_0^2+||\nabla(\zeta_{n-2})||_0^2). \qquad (4.71)
\end{aligned}$$

What is left is to estimate

$$\begin{aligned}
J_n:=&|T_n(\mathbf{u},p,\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t},\frac{\eta_{n+1}^h-\eta_{n-1}^h}{\triangle t})-T_{n-1}(\mathbf{u},p,\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t},\frac{\eta_{n+1}^h-\eta_{n-1}^h}{\triangle t})\\
&-G(\frac{p(t_{n+1})+p(t_n)}{2},\frac{\eta_{n+1}^h-\eta_{n-1}^h}{\triangle t})+G(\frac{p(t_n)+p(t_{n-1})}{2},\frac{\eta_{n+1}^h-\eta_{n-1}^h}{\triangle t})|.
\end{aligned}$$

Using Taylor expansion, the bound of $J_n$ is obtained as the proof of Theorem 4.1.

$$\begin{aligned}
J_n=&C\{|\triangle t^3(\mathbf{u}_{tttt}(t_{n+\theta}),\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})+\triangle t^2b_s(\mathbf{u}(t_{n+\frac{1}{2}}),\mathbf{u}_{tt}(t_{n+\theta}),\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})\\
&-\triangle t^2b_s(\mathbf{u}(t_{n-\frac{1}{2}}),\mathbf{u}_{tt}(t_{n-1+\theta}),\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})\\
&-\triangle t^2b_s(\mathbf{u}_{tt}(t_{n+\theta}),\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})\\
&+\triangle t^2b_s(\mathbf{u}_{tt}(t_{n-1+\theta}),\frac{\mathbf{u}(t_n)+\mathbf{u}(t_{n-1})}{2},\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})+\triangle t^3a(\mathbf{u}_{ttt}(t_{n+\theta}),\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})\\
&-\triangle t^3b(p_{ttt}(t_{n+\theta}),\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})|\}+\triangle tS(\mathbf{u}_t,\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})\\
\le&\frac{\triangle t^5}{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}C(||\mathbf{u}_{tttt}(t_{n+\theta})||_0^2+||\nabla(\mathbf{u}_{ttt}(t_{n+\theta}))||_0^2+||p_{ttt}(t_{n+\theta})||_0^2)\\
&+C\nu_TH^2\triangle t||\nabla(\mathbf{u}_t(t_{n+\theta}))||_0^2+(\frac{\nu}{24}+\frac{\nu+\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}{24})\triangle t||\nabla(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})||_0^2
\end{aligned}$$

$$+ C\nu\triangle t^5||\nabla(\mathbf{u}_{ttt}(t_{n+\theta}))||_0^2 + \frac{\nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}{48}\triangle t||\nabla(\frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t})||_0^2. \tag{4.72}$$

Putting (4.66)-(4.72) back into (4.65), we have

$$\begin{aligned}
&||\frac{\mathbf{e}_{n+1}^h - \mathbf{e}_n^h}{\triangle t}||_0^2 - ||\frac{\mathbf{e}_n^h - \mathbf{e}_{n-1}^h}{\triangle t}||_0^2 + \frac{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})\triangle t}{4}||\nabla(\frac{\mathbf{e}_{n+1}^h - \mathbf{e}_{n-1}^h}{\triangle t})||_0^2\\
&\quad + \frac{\triangle t}{2}||(I-\pi_h)(\frac{\eta_{n+1}^h - \eta_{n-1}^h}{\triangle t})||_0^2\\
\leq& \frac{C}{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_n^h}{\triangle t})}\triangle t||\frac{\xi_{n+1} - 2\xi_n + \xi_{n-1}}{\triangle t^2}||_{-1}^2 + 3\nu_{add}(\frac{\xi_{n+1} - \xi_{n-1}^h}{\triangle t})\triangle t||\nabla(\frac{\xi_{n+1} - \xi_{n-1}}{\triangle t})||_0^2\\
&\quad + \frac{C}{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t||\nabla(\frac{\xi_{n+1} - \xi_{n-1}}{\triangle t})||_0^2\\
&\quad + \frac{C}{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t(||\nabla(\mathbf{e}_n^h)||_0^2 + ||\nabla(\mathbf{e}_{n-1}^h)||_0^2)||\nabla(\frac{\xi_{n+1} - \xi_{n-1}}{\triangle t})||_0^2\\
&\quad + \frac{C}{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t(||\nabla(\xi_n)||_0^2 + ||\nabla(\xi_{n-1})||_0^2)||\nabla(\frac{\xi_{n+1} - \xi_{n-1}}{\triangle t})||_0^2\\
&\quad + \frac{C}{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t||\nabla(\zeta_{n-\frac{1}{2}})||_0^2\\
&\quad + \frac{Ch^{-1}\triangle t}{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}(||\frac{\mathbf{e}_n^h - \mathbf{e}_{n-1}^h}{\triangle t}||_0^2 + ||\frac{\mathbf{e}_{n-1}^h - \mathbf{e}_{n-2}^h}{\triangle t}||_0^2)||\nabla(\zeta_{n-\frac{1}{2}})||_0^2\\
&\quad + \frac{C}{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t(||\nabla(\frac{\xi_n - \xi_{n-1}}{\triangle t})||_0^2 + ||\nabla(\frac{\xi_{n-1} - \xi_{n-2}}{\triangle t})||_0^2)||\nabla(\zeta_{n-\frac{1}{2}})||_0^2\\
&\quad + \frac{C}{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t(||\nabla(\frac{\xi_n - \xi_{n-1}}{\triangle t})||_0^2 + ||\nabla(\frac{\xi_{n-1} - \xi_{n-2}}{\triangle t})||_0^2)\\
&\quad + \frac{C}{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t(||\nabla(\zeta_{n-1})||_0^2 + ||\nabla(\zeta_{n-2})||_0^2)\\
&\quad + \frac{C}{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}\triangle t(||\frac{\mathbf{e}_n^h - \mathbf{e}_{n-1}^h}{\triangle t}||_0^2 + ||\frac{\mathbf{e}_{n-1}^h - \mathbf{e}_{n-2}^h}{\triangle t}||_0^2)\\
&\quad + \frac{\triangle t^5}{\nu + \nu_{add}(\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_{n-1}^h}{\triangle t})}C(||\mathbf{u}_{tttt}(t_{n+\theta})||_0^2 + ||\nabla(\mathbf{u}_{ttt}(t_{n+\theta}))||_0^2 + ||p_{ttt}(t_{n+\theta})||_0^2)\\
&\quad + C\nu_T H^2\triangle t||\nabla(\mathbf{u}_t(t_{n+\theta}))||_0^2 + C\nu\triangle t^5||\nabla(\mathbf{u}_{ttt}(t_{n+\theta})||_0^2
\end{aligned} \tag{4.73}$$

Summing over the time levels from 2 to $n$, we obtain

$$\begin{aligned}
&||\frac{\mathbf{e}_n^h - \mathbf{e}_{n-1}^h}{\triangle t}||_0^2 - ||\frac{\mathbf{e}_2^h - \mathbf{e}_1^h}{\triangle t}||_0^2 + \frac{1}{4}\sum_{i=2}^{n}\triangle t Re_{red}^{-1}||\nabla(\frac{\mathbf{e}_{i+1}^h - \mathbf{e}_{i-1}^h}{\triangle t})||_0^2\\
&\quad + \frac{1}{2}\sum_{i=2}^{n}\triangle t||(I-\pi_h)(\frac{\eta_{i+1}^h - \eta_{i-1}^h}{\triangle t})||_0^2\\
\leq& C\{Re_{red}(h^2 + \triangle t^4) + \nu_T H^2 + \nu\triangle t^4
\end{aligned}$$

$$+\sum_{i=1}^{n}\triangle t(Re_{red}+Re_{red}\frac{h^2+\triangle t^2+\nu_T H}{h\triangle t})||\frac{\mathbf{e}_i^h-\mathbf{e}_{i-1}^h}{\triangle t}||_0^2\}. \tag{4.74}$$

In the following, we estimate $||\frac{\mathbf{e}_2^h-\mathbf{e}_1^h}{\triangle t}||_0$. The bound of $||\frac{\mathbf{e}_2^h-\mathbf{e}_1^h}{\triangle t}||_0$ depend upon the way the first time step is taken.

$$\begin{aligned}
&||\frac{\mathbf{e}_2^h-\mathbf{e}_1^h}{\triangle t}||_0^2-||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2+\frac{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})\triangle t}{2}||\nabla(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})||_0^2+\frac{\triangle t}{2}||(I-\pi_h)(\frac{\eta_2^h-\eta_0^h}{\triangle t})||_0^2\\
=&-(\frac{\xi_2-2\xi_1+\xi_0}{\triangle t},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})-S(\frac{\xi_2-\xi_0}{2},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})-b_s(A[\mathbf{u}_1^h,\mathbf{u}_0^h],\zeta_{1+\frac{1}{2}},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})\\
&+b_s(\frac{\mathbf{u}_1^h+\mathbf{u}_0^h}{2},\zeta_{\frac{1}{2}},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})-b_s(A[\zeta_1,\zeta_0],\frac{\mathbf{u}(t_2)+\mathbf{u}(t_1)}{2},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})\\
&+b_s(\zeta_{\frac{1}{2}},\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})+T_2(\mathbf{u},p,\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t},\frac{\eta_2^h-\eta_0^h}{\triangle t})-T_1(\mathbf{u},p,\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t},\frac{\eta_2^h-\eta_0^h}{\triangle t})\\
&-G(\frac{p(t_2)+p(t_1)}{2},\frac{\eta_2^h-\eta_0^h}{\triangle t})+G(\frac{p(t_1)+p(t_0)}{2},\frac{\eta_2^h-\eta_0^h}{\triangle t}).
\end{aligned}\tag{4.75}$$

The bilinear terms are treated in the same way except for the nonlinear terms. Therefore, we go directly to treatment the nonlinear terms,

$$\begin{aligned}
&|-b_s(A[\mathbf{u}_1^h,\mathbf{u}_0^h],\zeta_{1+\frac{1}{2}},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})+b_s(\frac{\mathbf{u}_1^h+\mathbf{u}_0^h}{2},\zeta_{\frac{1}{2}},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})|\\
\leq&|b_s(\frac{3}{2}\mathbf{u}_1^h-\frac{1}{2}\mathbf{u}_0^h,\frac{\zeta_2-\zeta_0}{2},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})|+|b_s(\mathbf{u}_1^h-\mathbf{u}_0^h,\zeta_{\frac{1}{2}},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})|.
\end{aligned}\tag{4.76}$$

We estimate the first nonlinear term of (4.75). The term is treated as (4.69).

$$\begin{aligned}
&|b_s(\frac{3}{2}\mathbf{u}_1^h-\frac{1}{2}\mathbf{u}_0^h,\frac{\zeta_2-\zeta_0}{2},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})|\\
\leq&\frac{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})}{24}\triangle t||\nabla(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})||_0+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})}\triangle t||\nabla(\frac{\xi_2-\xi_0}{\triangle t})||_0^2\\
&+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})}\triangle t||\nabla(\mathbf{e}_1^h)||_0^2||\nabla(\frac{\xi_2-\xi_0}{\triangle t})||_0^2\\
&+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})}\triangle t(||\nabla(\xi_1)||_0^2+||\nabla(\xi_0)||_0^2)||\nabla(\frac{\xi_2-\xi_0}{\triangle t})||_0^2.
\end{aligned}\tag{4.77}$$

For the second nonlinear terms of (4.76), using lemma 4.2 and Young's inequality, we have

$$\begin{aligned}
&|b_s(\mathbf{u}_1^h-\mathbf{u}_0^h,\zeta_{\frac{1}{2}},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})|\\
\leq&|b_s(\mathbf{u}(t_1)-\mathbf{u}(t_0),\zeta_{\frac{1}{2}},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})|+|b_s(\mathbf{e}_1^h-\mathbf{e}_0^h,\zeta_{\frac{1}{2}},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})|+|b_s(\xi_1-\xi_0,\zeta_{\frac{1}{2}},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})|\\
\leq&\frac{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})}{24}\triangle t||\nabla(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})||_0+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})}\triangle t||\nabla(\frac{\xi_1-\xi_0}{\triangle t})||_0^2||\nabla(\zeta_{\frac{1}{2}})||_0^2\\
&+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})}\triangle t||\nabla(\zeta_{\frac{1}{2}})||_0^2+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})}\triangle th^{-1}||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2||\nabla(\zeta_{\frac{1}{2}})||_0^2.
\end{aligned}\tag{4.78}$$

For the third and forth nonlinear terms of (4.75), we assume that $||\nabla(\mathbf{u}_t(t))||_0,\ ||\nabla(\mathbf{u}(t))||_0$ are bounded for any $t\in[0,T]$.

$$\begin{aligned}&|-b_s(A[\zeta_1,\zeta_0],\frac{\mathbf{u}(t_2)+\mathbf{u}(t_1)}{2},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})+b_s(\frac{\zeta_1+\zeta_0}{2},\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\frac{e_2^h-e_0^h}{\triangle t})|\\ \le&|b_s(\zeta_1-\zeta_0,\frac{\mathbf{u}(t_2)+\mathbf{u}(t_1)}{2},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})|+|b_s(\frac{\zeta_1+\zeta_0}{2},\frac{\mathbf{u}(t_1)-\mathbf{u}(t_0)}{2},\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})|\\ \le&\frac{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})}{24}\triangle t||\nabla(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})||_0+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})}\triangle t||\nabla(\frac{\xi_1-\xi_0}{\triangle t})||_0^2\\ &+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})}\triangle t(||\nabla(\zeta_1)||_0^2+||\nabla(\zeta_0)||_0^2)+\frac{C}{\nu+\nu_{add}(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})}\triangle t||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2.\end{aligned}\tag{4.79}$$

We have

$$\begin{aligned}&||\frac{\mathbf{e}_2^h-\mathbf{e}_1^h}{\triangle t}||_0^2-||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2+\frac{1}{4}Re_{red}^{-1}\triangle t||\nabla(\frac{\mathbf{e}_2^h-\mathbf{e}_0^h}{\triangle t})||_0^2+\frac{1}{2}\triangle t||(I-\pi_h)(\frac{\eta_2^h-\eta_0^h}{\triangle t})||_0^2\\ \le&C\{Re_{red}(h^2+\triangle t^4)+\nu_T H^2+\nu\triangle t^4+\triangle t Re_{red}(1+\frac{h^2+\triangle t^2+\nu_T H}{h\triangle t})||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2\}.\end{aligned}\tag{4.80}$$

Assume $\frac{h^2+\triangle t^2+\nu_T H}{h\triangle t}\le C$. Then it follows from the discrete Gronwall lemma

$$\begin{aligned}&||\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_n^h}{\triangle t}||_0^2-||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2+\frac{1}{4}Re_{red}^{-1}\sum_{i=1}^n\triangle t||\nabla(\frac{\mathbf{e}_{i+1}^h-\mathbf{e}_{i-1}^h}{\triangle t})||_0^2\\ &+\frac{1}{2}\sum_{i=1}^n\triangle t||(I-\pi_h)(\frac{\eta_{i+1}^h-\eta_{i-1}^h}{\triangle t})||_0^2\\ \le&C\{Re_{red}(h^2+\triangle t^4)+\nu_T H^2+\nu\triangle t^4\}.\end{aligned}\tag{4.81}$$

Finally, we estimate the bound of $||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2$,

$$\begin{aligned}&(\frac{\zeta_1-\zeta_0}{\triangle t},\mathbf{v}^h)+B^*((\zeta_{\frac{1}{2}},\frac{p(t_1)+p(t_0)}{2}-\frac{p_1^h+p_0^h}{2});(\mathbf{v}^h,q^h))+S(\zeta_{\frac{1}{2}},\mathbf{v}^h)\\ =&-b_s(\frac{\mathbf{u}_1^h+\mathbf{u}_0^h}{2},\zeta_{\frac{1}{2}},\mathbf{v}^h)-b_s(\frac{\zeta_1+\zeta_0}{2},\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\mathbf{v}^h)+T_0(\mathbf{u},p;\mathbf{v}^h,q^h),\end{aligned}\tag{4.82}$$

where

$$\begin{aligned}&T_0(\mathbf{u},p;\mathbf{v}^h,q^h)\\ =&(\frac{\mathbf{u}(t_1)-\mathbf{u}(t_0)}{\triangle t}-\mathbf{u}_t(t_{\frac{1}{2}}),\mathbf{v}^h)+b_s(\mathbf{u}(t_{\frac{1}{2}}),\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2}-\mathbf{u}(t_{\frac{1}{2}}),\mathbf{v}^h)\\ &-b_s(\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2}-\mathbf{u}(t_{\frac{1}{2}}),\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\mathbf{v}^h)+S(\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\mathbf{v}^h)\\ &+a(\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2}-\mathbf{u}(t_{\frac{1}{2}}),\mathbf{v}^h)-b(\frac{p(t_1)+p(t_0)}{2}-p(t_{\frac{1}{2}}),\mathbf{v}^h)\\ &+G(\frac{p(t_1)+p(t_0)}{2},q^h).\end{aligned}\tag{4.83}$$

Set $\mathbf{v}^h=\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}\in\mathbf{V}_h$, $q^h=\frac{\eta_1^h-\eta_0^h}{\triangle t}\in Q_h$ in (4.82) to obtain

$$||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2+\beta(\sqrt{\nu}||\nabla(\frac{\mathbf{e}_1^h+\mathbf{e}_0^h}{2})||_0+\frac{1}{\sqrt{\nu}}||\frac{\phi_1^h+\phi_0^h}{2}||_0)(\sqrt{\nu}||\nabla(\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})||_0$$

$$+\frac{1}{\sqrt{\nu}}||\frac{\phi_1^h-\phi_0^h}{\triangle t}||_0)+\frac{\nu_T}{2\triangle t}||\nabla(\mathbf{e}_1^h)||_0^2$$

$$\leq -(\frac{\xi_1-\xi_0}{\triangle t},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})-S(\frac{\xi_1+\xi_0}{2},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})-b_s(\frac{\mathbf{u}_1^h+\mathbf{u}_0^h}{2},\zeta_{\frac{1}{2}},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})$$
$$-b_s(\frac{\zeta_1+\zeta_0}{2},\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})+T_0(\mathbf{u},p,\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t},\frac{\eta_1^h-\eta_0^h}{\triangle t})$$
$$-G(\frac{p(t_1)+p(t_0)}{2},\frac{\eta_1^h-\eta_0^h}{\triangle t}). \tag{4.84}$$

Applying the Cauchy-Schwartz inequality and Young's inequality to the bilinear terms on the right hand side of (4.84), we have

$$|-(\frac{\xi_1-\xi_0}{\triangle t},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})|\leq\frac{1}{8}||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2+2||\frac{\xi_1-\xi_0}{\triangle t}||_0^2, \tag{4.85}$$

$$|-S(\frac{\xi_1+\xi_0}{2},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})|\leq\frac{1}{2\beta}\nu_{add}(\frac{\xi_1+\xi_0}{2})||\nabla(\frac{\xi_1+\xi_0}{2})||_0^2+\frac{\beta}{2}\nu_{add}(\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})||\nabla(\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})||_0^2. \tag{4.86}$$

We start with the first nonlinear term of (4.84). Adding and subtracting the quantity $b_s(\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\zeta_{\frac{1}{2}},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})$, and using Lemma 4.2, followed by Young's inequality, we have

$$|-b_s(\frac{\mathbf{u}_1^h+\mathbf{u}_0^h}{2},\zeta_{\frac{1}{2}},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})|$$
$$\leq|b_s(\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\frac{\xi_1+\xi_0}{2},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})|+|b_s(\frac{\xi_1+\xi_0}{2},\frac{\xi_1+\xi_0}{2},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})|$$
$$+|b_s(\frac{\mathbf{e}_1^h+\mathbf{e}_0^h}{2},\frac{\xi_1+\xi_0}{2},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})|$$
$$\leq\frac{1}{8}||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2+C(||\nabla(\xi_1)||^2+||\nabla(\xi_0)||_0^2)+C||\nabla(\mathbf{e}_1^h)||_0^2(||\nabla(\xi_1)||_0^2+||\nabla(\xi_0)||_0^2)$$
$$+C(||\nabla(\xi_1)||_0^4+||\nabla(\xi_0)||_0^4). \tag{4.87}$$

$$|-b_s(\frac{\zeta_1+\zeta_0}{2},\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})|\leq\frac{1}{8}||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2+C(||\nabla(\zeta_1)||_0^2+||\nabla(\zeta_0)||_0^2). \tag{4.88}$$

Using Taylor expansion, we have

$$T_0(\mathbf{u},p;\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t},\frac{\eta_1^h-\eta_0^h}{\triangle t})-G(\frac{p(t_1)+p(t_0)}{2},\frac{\eta_1^h-\eta_0^h}{\triangle t})$$
$$=C\triangle t^2(\mathbf{u}_{ttt}(t_\theta),\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})+C\triangle t^2b_s(\mathbf{u}(t_{\frac{1}{2}}),\mathbf{u}_{tt}(t_\theta),\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})$$
$$+C\triangle t^2b_s(\mathbf{u}_{tt}(t_\theta),\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})+S(\frac{\mathbf{u}(t_1)+\mathbf{u}(t_0)}{2},\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})$$
$$+C\triangle t^2a(\mathbf{u}_{tt}(t_\theta),\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})+C\triangle t^2b(p_{tt}(t_\theta),\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})$$
$$\leq\frac{1}{8}||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2+\frac{\nu+\nu_{add}(\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})}{4}\beta||\nabla(\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})||_0^h+\frac{\nu}{4}\beta||\nabla(\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})||_0^h$$
$$+\frac{\nu_{add}(\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})}{2}\beta||\nabla(\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})||_0^2+C(\triangle t^4+\nu_TH^2) \tag{4.89}$$

for some $\theta \in (0,1)$. Combining (4.84)-(4.89), we have

$$\frac{1}{2}||\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t}||_0^2+\min\{\beta,1\}\frac{\nu+\nu_{add}(\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})}{4}\triangle t||\nabla(\frac{\mathbf{e}_1^h-\mathbf{e}_0^h}{\triangle t})||_0^2\le C\{h^2+\triangle t^4+\nu_T H^2\}. \quad (4.90)$$

Therefore

$$\begin{aligned}&||\frac{\mathbf{e}_{n+1}^h-\mathbf{e}_n^h}{\triangle t}||_0^2+\frac{1}{4}\sum_{i=1}^n\triangle t Re_{red}^{-1}||\nabla(\frac{\mathbf{e}_{i+1}^h-\mathbf{e}_{i-1}^h}{\triangle t})||_0^2+\frac{1}{2}\sum_{i=1}^n\triangle t||(I-\pi_h)(\frac{\eta_{i+1}^h-\eta_{i-1}^h}{\triangle t})||_0^2\\ \le& C(Re_{red},\Omega,\mathbf{u},p,T)\{h^2+\nu_T H^2+\triangle t^4\}. \end{aligned} \quad (4.91)$$

**Theorem 4.2.** *Let* $(\boldsymbol{u}^h,p^h)\in(\boldsymbol{V}_h,Q_h)$, $\boldsymbol{u}\in L^2(0,T;(H^2(\Omega))^2)$, $\boldsymbol{u}_t\in L^2(0,T;(H^2(\Omega))^2)$, $\boldsymbol{u}_{tt}\in L^2(0,T;(H^2(\Omega))^2)$, $p_{ttt}\in L^2(0,T;L^2(\Omega))$, $\boldsymbol{u}_{ttt}\in C^0(0,T;(L^2(\Omega))^2)$, $\boldsymbol{u}_{tttt}\in C^0(0,T;(L^2(\Omega))^2)$ *and* $h\sim\triangle t$. *Then there exists an* $C=C(Re_{red},\boldsymbol{u},p,T)<\infty$, *such that* $\forall n\in 0,1,\cdots,N-1$, *the error in Algorithm* 1 *satisfied*

$$\{\sum_{i=0}^n\triangle t||\frac{p(t_{i+1})+p(t_i)}{2}-\frac{p_i^h+p_{i+1}^h}{2}||_0^2\}^{\frac{1}{2}}\le C(Re_{red},\boldsymbol{u},p,T)(h+\nu_T^{\frac{1}{2}}H+\triangle t^2). \quad (4.92)$$

*Proof.* From (4.38), we can see that

$$\begin{aligned}&B*((\zeta_{n+1/2},\Phi_{n+1/2});(\mathbf{v}^h,q^h))\\ =&-(\frac{\zeta_{n+1}-\zeta_n}{\triangle t},\mathbf{v}^h)-S(\zeta_{n+1/2},\mathbf{v}^h)-b_s(A[\mathbf{u}_n^h,\mathbf{u}_{n-1}^h],\zeta_{n+1/2},\mathbf{v}^h)\\ &-b_s(A[\zeta_n,\zeta_{n-1}],\frac{\mathbf{u}(t_{n+1})+\mathbf{u}(t_n)}{2},v^h)+T_n(\mathbf{u},p;\mathbf{v}^h,q^h). \end{aligned} \quad (4.93)$$

Using Lemma 4.1, we get

$$\frac{1}{\sqrt{\nu}}||\Phi_{n+1/2}||_0\le\beta^{-1}\sup_{(\mathbf{v}^h,q^h)\in(\mathbf{V}_h,Q_h)}\frac{|B^*((\zeta_{n+1/2},\Phi_{n+1/2});(\mathbf{v}^h,q^h))|}{\sqrt{\nu}||\nabla\mathbf{v}^h||_0+\frac{1}{\sqrt{\nu}}||q^h||_0}. \quad (4.94)$$

From (4.94), using Lemma 4.4 and Theorem 4.1, we have

$$\{\sum_{i=0}^n\triangle t||\frac{p(t_{i+1})+p(t_i)}{2}-\frac{p_i^h+p_{i+1}^h}{2}||_0^2\}^{\frac{1}{2}}\le C(Re_{red},\mathbf{u},p,T)(h+\nu_T^{\frac{1}{2}}H+\triangle t^2).$$

This proves (4.92). □

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