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CONVERGENCE AND SUPERCONVERGENCE OF A NONCONFORMING FINITE ELEMENT ON ANISOTROPIC MESHES

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Abstract. The main aim of this paper is to study the error estimates of a nonconforming finite element for general second order problems, in particular, the superconvergence properties under anisotropic meshes. Some extrapolation results on rectangular meshes are also discussed. Finally, numerical results are presented, which coincides with our theoretical analysis perfectly.

Key Words. nonconforming finite element, anisotropic meshes, superconvergence, extrapolation.

1. Introduction

It is well-known that regular assumption or quasi-uniform assumption [10, 13] of finite element meshes is a basic condition in the analysis of finite element approximation both for conventional conforming and nonconforming elements. However, with the advances of the finite element methods and its applications to other fields and more complex problems, the above regular or quasi-uniform assumption becomes quite a restriction in practice for some problems in the finite element methods. For example, the solution may have anisotropic behavior in part of the domain, that is to say, the solution varies significantly only in certain directions. Such problems are frequently encountered in perturbed convection-diffusion-reaction equations where boundary or interior layers appear. In such cases, it is more effective to use anisotropic meshes with a small mesh size in the direction of the rapid variation of the solution and a larger mesh size in the perpendicular direction. Consider a bounded convex domain $\Omega \subset R^2$. Let $\mathcal{J}_h$ be a family of meshes of Ω. Denote the diameter of an element K and the diameter of the inscribed circle of K by h_K and ρ_K, respectively. $h = \max\limits_{K\in\mathcal{J}_h} h_K$. It is assumed in the classical finite element theory that $\frac{h_K}{\rho_K} \le C$, where C be a positive constant independent of K and the function considered. Such assumption is no longer valid in the case of anisotropic meshes. Conversely, anisotropic elements are characterized by $\frac{h_K}{\rho_K} \to \infty$ as $h \to 0$. Some early papers have been written to prove error estimates under more general conditions (refer to [7, 25]). Recently, much attention is paid to FEMs with anisotropic meshes. In particular, for anisotropic rectangular meshes we refer to Acosta [1, 2], Apel [3, 4, 5, 6], Chen [16, 17, 31, 38], Duran [22, 23], Shenk [37] and references therein, and for narrow quadrilateral meshes to Zenisek [45]. But to our best knowledge, there are few papers focused on the nonconforming elements under anisotropic meshes.

On the other hand, researchers have observed that for certain classes of problems the rate of convergence of the finite element solution and/or its derivatives at some

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special points exceeds the best global rate. This phenomenon has been termed " superconvergence" and has been analyzed mathematically because of its practical importance in engineering computations. Also some postprocessing methods have been developed to improve the accuracy of finite element solution. Many superconvergence results about conforming FEMs have been obtained, see e.g., [14, 26, 29, 43]. Do the superconvergence results of conforming elements still hold for those nonconforming ones? The answer is affirmative. In [15, 39], the superconvergence of Wilson element is studied and the superconvergence estimate of the gradient error on the centers of elements is obtained. Recently, some superconvergentce results of rotated $\mathcal{Q}_1$ type elements are derived for quasi-uniform meshes in [30, 32]. On the other hand, Wang [44] proposed a least-square surface fitting method to obtain the superconvergence under quasi-uniform mesh assumption. The main feature of their method is to apply an L^2 projection on a coarser mesh with size $\tau = \mathrm{O}(h^\alpha)$ ($\alpha \in (0,1)$). At the same time, extrapolation is widely used in the finite element method. Interested readers are referred to [8, 9, 29, 35, 36] for extrapolation results of the conforming linear and bilinear element.

In this work, we first study the anisotropic interpolation error on anisotropic affine quadrilateral meshes of a five-node nonconforming finite element proposed by [24, 30], and the optimal consistency error is derived by a detailed analysis for anisotropic affine quadrilaterals, which extends the results of [38] for the rectangular meshes. We comment that since the interpolation of the original rotated $\mathcal{Q}_1$ element does not satisfy the anisotropic interpolation properties, reference [4] deals with the modified anisotropic rectangular rotated $\mathcal{Q}_1$ element with the shape space $span\{1, \xi, \eta, \xi^2\}$. There has been other works for the modified rotated $\mathcal{Q}_1$ element (cf. [5, 19, 28]).

In section §3, following the technique developed in [30, 38], we obtain a higher order $O(h^2)$ of consistency error under anisotropic rectangular meshes. Based on this fact and some other higher order error estimates proved in this section, a superconvergent approximation between the interpolation of the exact solution and the finite element solution is derived. Then a superconvergent estimate on the centers of elements is obtained, and the global superconvergence $O(h^2)$ for the gradient of the solution is also derived with the aid of a suitable postprocessing method.

In section §4, for regular rectangular meshes, we study some error expansions for the five-node nonconforming element. Based on these expansions, we obtain a sharp error estimates $O(h^3)$ by extrapolations. In the last section, some numerical examples are presented to validate our theoretical analysis.

Finally, we recall some notations and terminology (or refer to [10, 13]). Let $(\cdot,\cdot)$ denote the usual L^2 -inner product and $\|u\|_{r,p,\Omega}$ (resp. $|u|_{r,p,\Omega}$) be the usual norm (resp. semi-norm) for the Sobolev space $W^{r,p}(\Omega)$. When $p = 2$, denote $W^{2,r}(\Omega)$ by $H^r(\Omega)$. Throughout this paper, C will be used as a generic positive constant, which is independent of h_K, and may be independent of the aspect ratio $\frac{h_K}{\rho_K}$ in §2 and §3.

2. Error estimates for general second order problems under anisotropic affine quadrilaterals

Let $\widehat{K} = [-1,1] \times [-1,1]$ be the reference element. Its four vertices are: $\widehat{a}_1 = (-1,-1), \widehat{a}_2 = (1,-1), \widehat{a}_3 = (1,1), \widehat{a}_4 = (-1,1)$, and its four sides are $\widehat{l}_1 = \widehat{a}_1\widehat{a}_2, \widehat{l}_2 = \widehat{a}_2\widehat{a}_3, \widehat{l}_3 = \widehat{a}_3\widehat{a}_4, \widehat{l}_4 = \widehat{a}_4\widehat{a}_1$.

The nonconforming five-node element[24,30] $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ on $\widehat{K}$ is defined as follows:

$$\widehat{\Sigma} = \{\widehat{v}_1, \widehat{v}_2, \widehat{v}_3, \widehat{v}_4, \widehat{v}_5\}, \qquad \widehat{P} = span\{1, \xi, \eta, \varphi(\xi), \varphi(\eta)\}, \tag{2.1}$$

where $\widehat{v}_i = \frac{1}{|\widehat{l}_i|}\int_{\widehat{l}_i}\widehat{v}d\widehat{s}$, $i = 1,2,3,4$. $\widehat{v}_5 = \frac{1}{|\widehat{K}|}\int_{\widehat{K}}\widehat{v}d\xi d\eta, \varphi(t) = \frac{1}{2}(3t^2-1)$.

It can be easily checked that the interpolation defined above is properly posed, the interpolation function is as follows:

$$\widehat{\Pi}\widehat{v} = \widehat{v}_5+\frac{1}{2}(\widehat{v}_2-\widehat{v}_4)\xi+\frac{1}{2}(\widehat{v}_3-\widehat{v}_1)\eta+\frac{1}{2}(\widehat{v}_2+\widehat{v}_4-2\widehat{v}_5)\varphi(\xi)+\frac{1}{2}(\widehat{v}_3+\widehat{v}_1-2\widehat{v}_5)\varphi(\eta). \tag{2.2}$$

The following lemma shows that the interpolation $\widehat{\Pi}$ defined as (2.2) possesses the anisotropic properties[6,16].

Lemma 2.1. *The interpolation operator $\widehat{\Pi}$ has the anisotropic interpolation properties, i.e., for $|\alpha| = 1$,*

$$\|\widehat{D}^\alpha(\widehat{v}-\widehat{\Pi}\widehat{v})\|_{0,\widehat{K}} \le C|\widehat{D}^\alpha\widehat{v}|_{1,\widehat{K}}. \tag{2.3}$$

Proof. When $\alpha = (1,0)$,

$$\widehat{D}^\alpha\widehat{\Pi}\widehat{v} = \frac{\partial\widehat{\Pi}\widehat{v}}{\partial\xi} = \frac{1}{2}(\widehat{v}_2-\widehat{v}_4)+\frac{1}{2}(\widehat{v}_2+\widehat{v}_4-2\widehat{v}_5)\varphi^{'}(\xi) \tag{2.4}$$

Noticed that $r = \dim\widehat{D}^\alpha\widehat{P} = 2$. Obviously, $\{1,\varphi^{'}(\xi)\}$ is a basis of $\widehat{D}^\alpha\widehat{P}$. Let

$$\widehat{D}^\alpha\widehat{\Pi}\widehat{v} = \beta_1+\beta_2\varphi^{'}(\xi),$$

where

$$\begin{aligned}
\beta_1 &= \frac{1}{2}(\widehat{v}_2-\widehat{v}_4) = \frac{1}{4}(\int_{\widehat{l}_2}\widehat{v}(1,\eta)d\eta-\int_{\widehat{l}_4}\widehat{v}(-1,\eta)d\eta) = \frac{1}{|\widehat{K}|}\int_{\widehat{K}}\frac{\partial\widehat{v}}{\partial\xi}d\xi d\eta,\\
\beta_2 &= \frac{1}{2}(\widehat{v}_2+\widehat{v}_4-2\widehat{v}_5) = \frac{1}{4}(\int_{\widehat{l}_2}\widehat{v}(1,\eta)d\eta+\int_{\widehat{l}_4}\widehat{v}(-1,\eta)d\eta-\int_{\widehat{K}}\widehat{v}(\xi,\eta)d\xi d\eta)\\
&= \frac{1}{|\widehat{K}|}\int_{\widehat{K}}\xi\frac{\partial\widehat{v}}{\partial\xi}d\xi d\eta.
\end{aligned}$$

For any $\widehat{w}\in H^1(\widehat{K})$, define

$$F_1(\widehat{w}) = \frac{1}{|\widehat{K}|}\int_{\widehat{K}}\widehat{w}d\xi d\eta, \quad F_2(\widehat{w}) = \frac{1}{|\widehat{K}|}\int_{\widehat{K}}\xi\widehat{w}d\xi d\eta. \tag{2.5}$$

Apparently, $F_j\in(H^1(\widehat{K}))^{'}, j = 1,2$. Employing the basic anisotropic interpolation theorem[16] yields

$$\|\widehat{D}^\alpha(\widehat{v}-\widehat{\Pi}\widehat{v})\|_{0,\widehat{K}} \le C|\widehat{D}^\alpha\widehat{v}|_{1,\widehat{K}}.$$

Similarly, we can prove that (2.3) is valid for $\alpha = (0,1)$. This completes the proof.

Given a general affine quadrilateral K with four vertices a_i and sides $l_i = \overline{a_ia_{i+1}}, i = 1,2,3,4, a_5 = a_1$, let h_{K1} denote the length of the longest edge of K and $h_{K2} = |K|/h_{K1}$ is the corresponding height. Then the affine mapping $F_K : \widehat{K} \longrightarrow K$ can be written as

$$\begin{cases} x = b_1+b_{1,1}\xi+b_{1,2}\eta,\\ y = b_2+b_{2,1}\xi+b_{2,2}\eta\end{cases} \tag{2.6}$$

For simplicity, we assume $\Omega\subset R^2$ to be a convex polygon composed by a family of affine quadrilaterals meshes $\mathcal{J}_h$ which needs not satisfy the regular and inverse assumption conditions, but satisfies the maximal angle condition[6]. Furthermore, the axes can be rotated such that the elements satisfy the coordinate system condition[6]. The definition of the maximal angle condition and coordinate system condition are listed as follows.

Maximal angle condition: There is a constant $\sigma_* < \pi$ (independent of h and K)

such that the maximal interior angle σ of any element K is bounded by σ_*, i.e., $\sigma < \sigma_*$.

Coordinate system condition: The angle ϑ between the longer sides and the x-axis is bounded by $|\sin\vartheta| \le \frac{h_{K2}}{h_{K1}}$.

Based on reference [6], we have the following results.

Lemma 2.2[6]**.** *Assume that an affine quadrilateral element K satisfies the maximal angle condition and the coordinate system condition. Then the entries of the matrix $B = (b_{i,j})_{i,j=1}^2 \in R^{2\times 2}$ and of its inverse $B^{-1} = (b_{i,j}^{-1})_{i,j=1}^2 \in R^{2\times 2}$ satisfy the following conditions:*

$$|b_{i,j}| \le \min\{h_{Ki}, h_{Kj}\}, \quad i,j = 1,2, \tag{2.7}$$

$$|b_{i,j}^{-1}| \le \min\{h_{Ki}^{-1}, h_{Kj}^{-1}\}, \quad i,j = 1,2. \tag{2.8}$$

Lemma 2.3[6]**.** *Let α be a multiple-index, then*

$$\sum_{|\alpha|=m} |\widehat{D}^\alpha \widehat{v}| \le C \sum_{|s|=m} h_K^s |D^s v|, \quad |\widehat{D}^\alpha \widehat{v}| \le C h_K^\alpha \sum_{|s|=|\alpha|} |D^s v|, \tag{2.9}$$

$$\sum_{|\alpha|=m} |D^\alpha v| \le C \sum_{|s|=m} h_K^{-s} |\widehat{D}^s \widehat{v}|, \tag{2.10}$$

where $h_K^s = h_{K1}^{s_1} h_{K2}^{s_2}, s = (s_1, s_2)$, and h_K is used as diameter of element K in §1.

Define the finite element space as

$$V_h = \{v_h | \widehat{v_h} = v_h|_K \circ F_K \in \widehat{P},\ \int_F [v_h] ds = 0, F \subset \partial K, \forall K \in \mathcal{J}_h,\}$$

where we denote sides of elements by F and by $[v]$ the jump of the function v on the sides F. For boundary sides we identify $[v]$ with v.

Let the general element K be a quadrilateral in $x - y$ plane, the interpolate operator is defined as

$$\Pi_K : H^2(K) \to \widehat{P} \circ F_K^{-1}, \Pi_K v = (\widehat{\Pi}\widehat{v}) \circ F_K^{-1}, \quad \Pi_h : H^2(\Omega) \to V_h, \Pi_h|_K = \Pi_K.$$

We consider the following general second-order elliptic boundary value problem

$$\begin{cases} \mathcal{L}u = -\sum_{i,j=1}^2 \partial_j(\alpha_{ij}\partial_i u) + \sum_{i=1}^2 \alpha_i \partial_i u + \gamma u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

where $\partial_i u = \frac{\partial u}{\partial x_i}, (x_1, x_2) = (x, y)$, the coefficients $\alpha_{ij}, \alpha_i \in W^{1,\infty}(\Omega), 1 \le i,j \le 2$ and $\gamma \ge 0$, the right hand term $f \in L^2(\Omega)$.

We assume that the differential operator $\mathcal{L}$ is uniformly elliptic, i.e., there exists a positive constant C such that

$$C^{-1}(\xi_1^2 + \xi_2^2) \le \sum_{i,j=1}^2 \alpha_{ij}\xi_i\xi_j \le C(\xi_1^2 + \xi_2^2)$$

for all points $(x, y) \in \overline{\Omega}$ and real vectors $(\xi_1, \xi_2) \in R^2$.

Let $V = H_0^1(\Omega)$, then the weak form of (2.11) is:

$$\begin{cases} \text{Find } u \in V, \text{ such that} \\ a(u, v) = f(v), \ \forall\ v \in V \end{cases} \tag{2.12}$$

where

$$\begin{cases} a(u,v) = \int_\Omega \left(\sum_{i,j=1}^2 \alpha_{ij}\partial_i u \partial_j v + \sum_{i=1}^2 \alpha_i \partial_i u v + \gamma uv \right) dxdy, \\ f(v) = \int_\Omega fv dxdy. \end{cases}$$

The approximation of (2.12) reads as follows:

$$\begin{cases} \text{Find } u_h \in V_h, \text{ such that} \\ a_h(u_h, v_h) = f(v_h), \ \forall \ v_h \in V_h \end{cases} \tag{2.13}$$

with

$$a_h(u_h, v_h) = \sum_{K\in\mathcal{J}_h} \int_K \left(\sum_{i,j=1}^2 \alpha_{ij}\partial_i u_h \partial_j v_h + \sum_{i=1}^2 \alpha_i \partial_i u_h v_h + \gamma u_h v_h \right) dxdy.$$

Set

$$\|\cdot\|_h = \left(\sum_{K\in\mathcal{J}_h} |\cdot|_{1,K}^2 \right)^{\frac{1}{2}}. \tag{2.14}$$

Then it is easy to see that $\|\cdot\|_h$ is the norm over V_h .

The following theorem is the main results of this section.

Theorem 2.1. *Let $\mathcal{J}_h$ be a partition of $\overline{\Omega}$ by affine quadrilaterals which satisfies the maximal angle condition, but may not satisfy the regular conditions. Let u and u_h be the solution of problem (2.11) and (2.13) respectively. Then there hold*

$$\|u - u_h\|_h \le Ch(|u|_{2,\Omega} + |u|_{1,\Omega}) \tag{2.15}$$

and

$$\|u - u_h\|_{0,\Omega} \le Ch^2(|u|_{2,\Omega} + |u|_{1,\Omega}). \tag{2.16}$$

Proof. The second Strang's lemma[10,13] reads as

$$\|u - u_h\|_h \le C\left(\inf_{v_h\in V_h} \|u - v_h\|_h + \sup_{v_h\in V_h} \frac{|a_h(u, v_h) - (f, v_h)|}{\|v_h\|_h} \right). \tag{2.17}$$

Now we consider the first term on the right hand of (2.17), i.e., the interpolation error.

By using of (2.3), (2.9) and (2.10), the approximation error an be estimated as

$$
\begin{aligned}
\inf_{v_h\in V_h}\|u-v_h\|_h &\le \|u-\Pi_h u\|_h \\
&= \left(\sum_{K\in\mathcal{J}_h}|u-\Pi_K u|_{1,K}^2\right)^{\frac12} \\
&= \left(\sum_{K\in\mathcal{J}_h}\sum_{|\alpha|=1}\|D^\alpha(u-\Pi_K u)\|_{0,K}^2\right)^{\frac12} \\
&= \left(\sum_{K\in\mathcal{J}_h}\sum_{|\alpha|=1}h_K^{-2\alpha}(h_{K1}h_{K2})\|\widehat{D}^\alpha(\widehat{u}-\widehat{\Pi}\widehat{u})\|_{0,\widehat{K}}^2\right)^{\frac12} \\
&\le C\left(\sum_{K\in\mathcal{J}_h}\sum_{|\alpha|=1}h_K^{-2\alpha}(h_{K1}h_{K2})|\widehat{D}^\alpha\widehat{u}|_{1,\widehat{K}}^2\right)^{\frac12} \qquad (2.18)\\
&= C\left(\sum_{K\in\mathcal{J}_h}\sum_{|\alpha|=1}h_K^{-2\alpha}(h_{K1}h_{K2})\sum_{|\beta|=1}\|\widehat{D}^{\alpha+\beta}\widehat{u}\|_{0,\widehat{K}}^2\right)^{\frac12} \\
&\le C\left(\sum_{K\in\mathcal{J}_h}\sum_{\substack{|\alpha|=1\\|\beta|=1}}h_K^{2\beta}\|D^{\alpha+\beta}u\|_{0,K}^2\right)^{\frac12} \\
&\le C\left(\sum_{K\in\mathcal{J}_h}\sum_{|\beta|=1}h_K^{2\beta}|D^\beta u|_{1,K}^2\right)^{\frac12} \le Ch|u|_{2,\Omega}.
\end{aligned}
$$

Then we turn to the second term on the right hand of (2.17), i.e., the consistency error. By Green's formula, it follows from the conventional techniques of consistency error estimates[27,38] that,

(2.19)

$$
\begin{aligned}
E_h(u,v_h) = a_h(u,v_h)-f(v_h) &= \sum_{K\in\mathcal{J}_h}\int_{\partial K}\sum_{i,j=1}^{2}\alpha_{ij}\frac{\partial u}{\partial x_i}n_j v_h ds \\
&= \sum_{K\in\mathcal{J}_h}\sum_{m=1}^{4}\int_{l_m}\sum_{i,j=1}^{2}(\alpha_{ij}\frac{\partial u}{\partial x_i}-P_{0m}(\alpha_{ij}\frac{\partial u}{\partial x_i}))(v_h-P_{0m}v_h)n_{mj}ds \\
&= \sum_{K\in\mathcal{J}_h}\sum_{m=1}^{4}\sum_{i,j=1}^{2}A_{ij}^m.
\end{aligned}
$$

where $P_{0m}v=\frac{1}{|l_m|}\int_{l_m}vds, \quad m=1,2,3,4.$ $n_i=(n_{i1},n_{i2})$ denotes the usual unit normal of side l_i.

We concentrate on a general element K, and for all i, j, m ,

(2.20)

$$\begin{aligned} A_{ij}^m &= \int_{l_m}\left(\alpha_{ij}\frac{\partial u}{\partial x_i} - P_{0m}(\alpha_{ij}\frac{\partial u}{\partial x_i})\right)(v_h - P_{0m}v_h)\,n_{mj}ds \\ &= |l_m|n_{mj}\int_{\widehat{l}_m}\left(\widehat{\alpha_{ij}\frac{\partial u}{\partial x_i}} - \widehat{P_{0m}}(\widehat{\alpha_{ij}\frac{\partial u}{\partial x_i}})\right)\left(\widehat{v}_h - \widehat{P_{0m}}\widehat{v}_h\right)d\widehat{s} \\ &\le C|l_m|n_{mj}|\widehat{\alpha_{ij}\frac{\partial u}{\partial x_i}}|_{1,\widehat{K}}|\widehat{v}_h|_{1,\widehat{K}} \\ &\le C|l_m|n_{mj}\left(\sum_{|\beta|=1}\|\widehat{D}^\beta(\widehat{\alpha_{ij}\frac{\partial u}{\partial x_i}})\|_{0,\widehat{K}}^2\right)^{\frac12}\left(\sum_{|\beta|=1}\|\widehat{D}^\beta\widehat{v}_h\|_{0,\widehat{K}}^2\right)^{\frac12} \\ &\le C\frac{|l_m|n_{mj}}{h_{K1}h_{K2}}\left(\sum_{|\beta|=1}h_K^{2\beta}\|D^\beta(\alpha_{ij}\frac{\partial u}{\partial x_i})\|_{0,K}^2\right)^{\frac12}\left(\sum_{|\beta|=1}h_K^{2\beta}\|D^\beta v_h\|_{0,K}^2\right)^{\frac12} \\ &\le C\frac{|l_m|n_{mj}}{h_{K1}h_{K2}}\left(\sum_{|\beta|=1}h_K^{2\beta}(|\frac{\partial u}{\partial x_i}|_{1,K}^2 + \|\frac{\partial u}{\partial x_i}\|_{0,K}^2)\right)^{\frac12}\left(\sum_{|\beta|=1}h_K^{2\beta}\|D^\beta v_h\|_{0,K}^2\right)^{\frac12}. \end{aligned}$$

If $|l_m|n_{mj} \le Ch_{K2}$, we can obtain

$$A_{ij}^m \le C\left(\sum_{|\beta|=1}h_K^{2\beta}(|\frac{\partial u}{\partial x_i}|_{1,K}^2 + \|\frac{\partial u}{\partial x_i}\|_{0,K}^2)\right)^{\frac12}|v_h|_{1,K}, \tag{2.21}$$

Unfortunately, when $|l_m|n_{mj} \le Ch_{K1}$ and $\frac{h_{K1}}{h_{K2}} \longrightarrow \infty$ (which is the case in anisotropic meshes), we will get the following pessimistic estimate,

$$A_{ij}^m \le C\frac{h_{K1}}{h_{K2}}\left(\sum_{|\beta|=1}h_K^{2\beta}(|\frac{\partial u}{\partial x_i}|_{1,K}^2 + \|\frac{\partial u}{\partial x_i}\|_{0,K}^2)\right)^{\frac12}|v_h|_{1,K}. \tag{2.22}$$

Therefore, we have to develop a different trick to derive the optimal consistency error estimates for anisotropic meshes. $\forall i, j = 1, 2$, let us consider A_{ij}^1 and A_{ij}^3 together, we expect that there holds some cancellation between the two opposite sides $l_1, l_3 \subset \partial K$.

Since $|l_1| = |l_3|, n_{1j} = -n_{3j}$, we have

$$\begin{aligned} &A_{ij}^1 + A_{ij}^3 \\ &= \int_{l_1}\left(\alpha_{ij}\frac{\partial u}{\partial x_i} - P_{01}(\alpha_{ij}\frac{\partial u}{\partial x_i})\right)(v_h - P_{01}v_h)\,n_{1j}ds \\ &+ \int_{l_3}\left(\alpha_{ij}\frac{\partial u}{\partial x_i} - P_{03}(\alpha_{ij}\frac{\partial u}{\partial x_i})\right)(v_h - P_{03}v_h)\,n_{3j}ds \\ &= |l_1|n_{1j}[\int_{-1}^1\left(\widehat{\alpha_{ij}\frac{\partial u}{\partial x_i}}(\xi,-1) - \widehat{P_{01}}(\widehat{\alpha_{ij}\frac{\partial u}{\partial x_i}})\right)\left(\widehat{v}_h(\xi,-1) - \widehat{P_{01}}\widehat{v}_h\right)d\xi \\ &- \int_{-1}^1\left(\widehat{\alpha_{ij}\frac{\partial u}{\partial x_i}}(\xi,1) - \widehat{P_{03}}(\widehat{\alpha_{ij}\frac{\partial u}{\partial x_i}})\right)\left(\widehat{v}_h(\xi,1) - \widehat{P_{03}}\widehat{v}_h\right)d\xi]. \end{aligned} \tag{2.23}$$

Moreover, noticed $\frac{\partial \widehat{v}_h}{\partial \xi}$ is a constant on $\widehat{K}$, then

$$\begin{aligned}\widehat{v}_h(\xi,-1)-\widehat{P_{01}\widehat{v}_h} &= \frac{1}{2}\int_{-1}^{1}\widehat{v}_h(\xi,-1)dt-\frac{1}{2}\int_{-1}^{1}\widehat{v}_h(t,-1)dt\\ &= \frac{1}{2}\int_{-1}^{1}\int_{t}^{\xi}\frac{\partial \widehat{v}_h}{\partial z}(z,-1)dzdt=\frac{1}{2}\int_{-1}^{1}\int_{t}^{\xi}\frac{\partial \widehat{v}_h}{\partial z}(z,1)dzdt\\ &= \widehat{v}_h(\xi,1)-\widehat{P_{03}\widehat{v}_h},\end{aligned} \tag{2.24}$$

Set $\widehat{V}=\widehat{\alpha_{ij}\frac{\partial u}{\partial x_i}}$, then

$$\begin{aligned}&A^1_{ij}+A^3_{ij}\\ &=|l_1|n_{1j}\int_{-1}^{1}\left(\widehat{V}(\xi,-1)-\widehat{V}(\xi,1)-\widehat{P_{01}\widehat{V}}+\widehat{P_{03}\widehat{V}}\right)d\xi\int_{-1}^{1}\int_{t}^{\xi}\frac{\partial \widehat{v}_h}{\partial z}dzdt\\ &=|l_1|n_{1j}\int_{-1}^{1}\left(-\int_{-1}^{1}\frac{\partial \widehat{V}(\xi,\zeta)}{\partial \zeta}d\zeta+\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial \widehat{V}(z,\zeta)}{\partial \zeta}dzd\zeta\right)d\xi\int_{-1}^{1}\int_{t}^{\xi}\frac{\partial \widehat{v}_h}{\partial z}dzdt\\ &\le Ch_{K1}\Big\|\frac{\partial \widehat{V}}{\partial \zeta}\Big\|_{0,\widehat{K}}\Big\|\frac{\partial \widehat{v}_h}{\partial z}\Big\|_{0,\widehat{K}}\\ &\le Ch_{K1}\Big\|\frac{\partial(\widehat{\alpha_{ij}\frac{\partial u}{\partial x_i}})}{\partial \zeta}\Big\|_{0,\widehat{K}}\Big\|\frac{\partial \widehat{v}_h}{\partial z}\Big\|_{0,\widehat{K}}\\ &\le C\frac{1}{h_{K1}h_{K2}}h_{K1}h_K^{(0,1)}\Big|\alpha_{ij}\frac{\partial u}{\partial x_i}\Big|_{1,K}h_{K1}\Big|v_h\Big|_{1,K}\\ &\le Ch_{K1}\left(|\frac{\partial u}{\partial x_i}|_{1,K}+\|\frac{\partial u}{\partial x_i}\|_{0,K}\right)\Big|v_h\Big|_{1,K}\end{aligned} \tag{2.25}$$

A combination of (2.19) and (2.25) yields

$$E_h(u,v_h)\le Ch(|u|_{2,\Omega}+|u|_{1,\Omega})\|v_h\|_h. \tag{2.26}$$

Then (2.15) follows from (2.17), (2.18) and (2.26), and an application of A-N technique[10,13] completes the proof.

3. Anisotropic superconvergence analysis

In this section, we focus on the superconvergence behavior of the element. For the sake of simplicity, we consider problem (2.11) with the assumption that $\alpha_{ij}=0, i\neq j$. Let $\Omega\subset R^2$ be a convex polygon composed by a family of rectangular meshes $\mathcal{J}_h$ which needs not satisfy the regular conditions. For any $K\in\mathcal{J}_h$, denote its barycenter by (x_K,y_K), the length of the edges parallel to x-axis and y-axis by $2h_{K1}, 2h_{K2}$ respectively. Then the mapping $F_K:\widehat{K}\longrightarrow K$ is defined as

$$\begin{cases}x=x_K+h_{K1}\xi,\\ y=y_K+h_{K2}\eta.\end{cases} \tag{3.1}$$

Now, we prepare to derive a superclose result.

Theorem 3.1. *Let $\mathcal{J}_h$ be a family of anisotropic rectangular meshes, and u, u_h, I_hu are the same as in Theorem 2.1,* $u\in H^3(\Omega)\cap H^1_0(\Omega), \alpha_{ii}\in W^{2,\infty}(\Omega), \alpha_i\in W^{1,\infty}(\Omega), i=1,2, \gamma\in L^\infty(\Omega)$, then there *holds the following superclose property*

$$\|\Pi_hu-u_h\|_h\le Ch^2(|u|_{3,\Omega}+|u|_{2,\Omega}+|u|_{1,\Omega}). \tag{3.2}$$

Proof. It can be proved easily that

$$\begin{aligned}C\|\Pi_hu-u_h\|_h^2 &\le a_h(\Pi_hu-u_h,\Pi_hu-u_h)\\ &= a_h(\Pi_hu-u,\Pi_hu-u_h)+a_h(u-u_h,\Pi_hu-u_h).\end{aligned} \tag{3.3}$$

Set $v_h = \Pi_h u - u_h$,$w = \Pi_h u - u$. Let us consider $a_h(w, v_h)$ first,

$$
\begin{aligned}
a_h(w, v_h) &= \sum_{K\in\mathcal{J}_h}\int_K\left(\sum_{i=1}^{2}\alpha_{ii}\frac{\partial w}{\partial x_i}\frac{\partial v_h}{\partial x_i}+\sum_{i=1}^{2}\alpha_i\frac{\partial w}{\partial x_i}v_h+\gamma w v_h\right)dxdy\\
&= \sum_{K\in\mathcal{J}_h}\int_K\left(\sum_{i=1}^{2}P_0\alpha_{ii}\frac{\partial w}{\partial x_i}\frac{\partial v_h}{\partial x_i}+\sum_{i=1}^{2}P_0\alpha_i\frac{\partial w}{\partial x_i}v_h+\gamma w v_h\right)dxdy\\
&+ \sum_{K\in\mathcal{J}_h}\int_K\left(\sum_{i=1}^{2}(\alpha_{ii}-P_0\alpha_{ii})\frac{\partial w}{\partial x_i}\frac{\partial v_h}{\partial x_i}+\sum_{i=1}^{2}(\alpha_i-P_0\alpha_i)\frac{\partial w}{\partial x_i}v_h\right)dxdy\\
&= I_1 + I_2,
\end{aligned} \tag{3.4}
$$

where $P_0 v = \frac{1}{|K|}\int_K v dxdy$.

$$
\begin{aligned}
I_1 &= \sum_{K\in\mathcal{J}_h}\int_K\left(\sum_{i=1}^{2}P_0\alpha_{ii}\frac{\partial w}{\partial x_i}\frac{\partial v_h}{\partial x_i}+\sum_{i=1}^{2}P_0\alpha_i\frac{\partial w}{\partial x_i}v_h+\gamma w v_h\right)dxdy\\
&= I_{11} + I_{12} + I_{13}.
\end{aligned} \tag{3.5}
$$

For any rectangular element K, when $i = 1$ (the case $i = 2$ can be treated similarly), noticing that $\frac{\partial^2 v_h}{\partial x_1^2} = const$, $\frac{\partial v_h}{\partial x_1}|_{l_j} = const, j = 2, 4$, then by Green's formula and the definition of the interpolant Π_K,

$$
\int_K\frac{\partial w}{\partial x_1}\frac{\partial v_h}{\partial x_1}dxdy = -\int_K\frac{\partial^2 v_h}{\partial x_1^2}wdxdy + \sum_{j=1}^{4}\int_{l_j}\frac{\partial v_h}{\partial x_1}wdx_2 = 0. \tag{3.6}
$$

Therefore,

$$
I_{11} = 0. \tag{3.7}
$$

I_{12} can be decomposed as

$$
\begin{aligned}
I_{12} &= \sum_{K\in\mathcal{J}_h}\int_K\sum_{i=1}^{2}P_0\alpha_i\frac{\partial w}{\partial x_i}v_h dxdy\\
&= \sum_{K\in\mathcal{J}_h}\int_K\sum_{i=1}^{2}P_0\alpha_i\frac{\partial w}{\partial x_i}(v_h - P_0 v_h)dxdy - \sum_{K\in\mathcal{J}_h}\int_K\sum_{i=1}^{2}P_0\alpha_i\frac{\partial w}{\partial x_i}P_0 v_h dxdy\\
&= I_{12}^1 + I_{12}^2.
\end{aligned} \tag{3.8}
$$

By (2.18), we have

$$
I_{12}^1 \le Ch\|w\|_h\|v_h\|_h \le Ch^2|u|_{2,\Omega}\|v_h\|_h \tag{3.9}
$$

By Green's formula and the definition of Π_h

$$
I_{12}^2 = \sum_{K\in\mathcal{J}_h}\sum_{i=1}^{2}P_0\alpha_i P_0 v_h\sum_{j=1}^{4}\int_{l_j}wds = 0. \tag{3.10}
$$

Proceeding along the same line of I_{12}, one can obtain

$$
I_{13} \le Ch^2|u|_{2,\Omega}\|v_h\|_h. \tag{3.11}
$$

Therefore we have bounded I_1 as

$$
I_1 \le Ch^2(|u|_{2,\Omega} + |u|_{3,\Omega})\|v_h\|_h. \tag{3.12}
$$

As to I_2, it is easy to show that

$$
I_2 \le Ch^2|u|_{2,\Omega}\|v_h\|_h. \tag{3.13}
$$

Consequently,

$$a_h(w, v_h) \le Ch^2(|u|_{2,\Omega} + |u|_{3,\Omega})\|v_h\|_h. \tag{3.14}$$

Now we bound $a_h(u-u_h, v_h) = E_h(u, v_h)$. Let us study further on the consistency error for anisotropic rectangular meshes. In fact,

$$\begin{aligned} a_h(u-u_h, v_h) &= \sum_{K\in\mathcal{J}_h} \int_{\partial K} \sum_{i=1}^{2} \alpha_{ii} \frac{\partial u}{\partial x_i} n_i v_h ds \\ &= \sum_{K\in\mathcal{J}_h} \sum_{m=1}^{4} \int_{l_m} \sum_{i=1}^{2} \alpha_{ii} \frac{\partial u}{\partial x_i} (v_h - P_{0m} v_h) n_{mi} ds \\ &= \sum_{K\in\mathcal{J}_h} \sum_{m=1}^{4} \int_{l_m} \sum_{i=1}^{2} B_{ii}^m. \end{aligned} \tag{3.15}$$

Consider any rectangular element K with center (x_K, y_K) and length h_{K1}, h_{K2} in x and y direction respectively. Due to the similarity, we only study the case $i = 1$. In this case, $B_{11}^1 = B_{11}^3 = 0$, and

$$B_{11}^2 + B_{11}^4 = \int_{l_2} \alpha_{11} \frac{\partial u}{\partial x} (v_h - P_{02} v_h)\, dy - \int_{l_4} \alpha_{11} \frac{\partial u}{\partial x} (v_h - P_{04} v_h)\, dy. \tag{3.16}$$

Due to the shape space of the element, there hold

$$(v_h - P_{02} v_h)\Big|_{l_2} = (v_h - P_{04} v_h)\Big|_{l_4} \tag{3.17}$$

and

$$\int_K \Big(v_h(x_K + h_{K1}, y) - P_{02} v_h(x_K + h_{K1}, y)\Big) dxdy = 0. \tag{3.18}$$

Set $U = \alpha_{11} \frac{\partial u}{\partial x}$. Then,

$$\begin{aligned} B_{11}^2 + B_{11}^4 &= \int_K \frac{\partial U}{\partial x} \Big(v_h(x_K + h_{K1}, y) - P_{02} v_h(x_K + h_{K1}, y)\Big) dxdy \\ &= \int_K \left(\frac{\partial U}{\partial x} - P_0 \frac{\partial U}{\partial x}\right) \Big(v_h(x_K + h_{K1}, y) - P_{02} v_h(x_K + h_{K1}, y)\Big) dxdy \\ &\le Ch_K^2 \left(|u|_{3,K} + |u|_{2,K} + |u|_{1,K}\right) |v_h|_{1,K}. \end{aligned} \tag{3.19}$$

So, we can obtain

$$a_h(u - u_h, v_h) \le Ch^2(|u|_{3,\Omega} + |u|_{2,\Omega} + |u|_{1,\Omega})\|v_h\|_h \tag{3.20}$$

Then the proof follows from (3.3), (3.14) and (3.20).

Remark 3.1. As noted in [30], the original five-node element proposed in [24] with $\varphi(t) = \frac{1}{2}(5t^4 - 3t^2)$ does not satisfy the superclose result. However, We will point out that this does not influence the superconvergent properties of the original five-node element, which will be addressed elsewhere.

Remark 3.2. Based on this theorem, by the interpolation theory and the inverse inequality $\|v_h\|_{0,\infty,\Omega} \le C|logh|^{\frac{1}{2}}\|v_h\|_h$, the maximum norm error estimate

$$\begin{aligned} \|u - u_h\|_{0,\infty,\Omega} &\le \|u - \Pi_h u\|_{0,\infty,\Omega} + \|\Pi_h u - u_h\|_{0,\infty,\Omega} \\ &\le Ch^2|u|_{2,\infty,\Omega} + C|logh|^{\frac{1}{2}}\|\Pi_h u - u_h\|_h \\ &\le Ch^2|logh|^{\frac{1}{2}}(|u|_{2,\infty,\Omega} + |u|_{1,\Omega} + |u|_{2,\Omega} + |u|_{3,\Omega}) \end{aligned}$$

follows. Compared with the general results presented on [33], the above maximum norm error estimate is a sharp estimate.

The following theorem is a pointwise superconvergence result.

Theorem 3.2. *Under the same assumptions as in Theorem 3.1, then the gradient ∇u_h has superconvergence estimate on the central point O_K of element K, i.e.,*

$$\left(\sum_{K\in\mathcal{J}_h}|(\nabla u-\nabla u_h)(O_K)|^2h_{K1}h_{K2}\right)^{\frac{1}{2}}\leq Ch^2(|u|_{3,\Omega}+|u|_{2,\Omega}+|u|_{1,\Omega}). \tag{3.21}$$

Proof. We only need to prove

$$\left(\sum_{K\in\mathcal{J}_h}|(D^\alpha u-D^\alpha\Pi_h u)(O_K)|^2h_{K1}h_{K2}\right)^{\frac{1}{2}}\leq Ch^2|u|_{3,\Omega},\quad |\alpha|=1 \tag{3.22}$$

and

$$\left(\sum_{K\in\mathcal{J}_h}|(\bigtriangledown\Pi_h u-\bigtriangledown u_h)(O_K)|^2h_{K1}h_{K2}\right)^{\frac{1}{2}}\leq Ch^2(|u|_{3,\Omega}+|u|_{2,\Omega}+|u|_{1,\Omega}). \tag{3.23}$$

For (3.22), we only discuss the case $\alpha=(1,0)$. From §2 we know that

$$\widehat{D}^\alpha\widehat{\Pi}\widehat{u}=F(\widehat{D}^\alpha\widehat{u}), \tag{3.24}$$

where $F=F_1+F_2$, F_1,F_2 are the functionals defined as in (2.5).

Set $\widehat{w}=\widehat{D}^\alpha\widehat{u}$ and $\widehat{l}(\widehat{w})=(\widehat{w}-\widehat{F}(\widehat{w}))(\widehat{O})$, where $\widehat{O}$ is the central point of $\widehat{K}$. Then it is not difficult to verify that

$$\widehat{l}(\widehat{w})=0,\quad \forall\widehat{w}\in P_1(\widehat{K}).$$

Noticed that $H^2(\widehat{K})\hookrightarrow C^0(\widehat{K})$, then

$$|\widehat{l}(\widehat{w})|\leq C\|\widehat{w}\|_{2,\widehat{K}},\qquad \forall\ \widehat{w}\in H^2(\widehat{K}).$$

An application of the usual Bramble-Hilbert Lemma yields

$$|\widehat{l}(\widehat{w})|\leq C|\widehat{w}|_{2,\widehat{K}},$$

which, by virtue of the scaling argument, we have

$$|(D^\alpha u-D^\alpha\Pi_h u)(Z)|^2h_{K1}h_{K2}\leq Ch_K^4|D^\alpha u|^2_{2,K}. \tag{3.25}$$

This implies (3.22).

Noticing the results (3.24), (3.2), together with the scaling argument and the equivalence of norms over the reference element $\widehat{K}$, we can prove (3.23), which completes the proof of Theorem 3.2.

Now, we will use a proper postprocessing interpolation operator to get anisotropic global superconvergence. For this purpose, we furthermore assume that $\mathcal{J}_h$ is obtained from $\mathcal{J}_{2h}$ (where $\mathcal{J}_{2h}$ is an anisotropic rectangular partition of Ω) by dividing each element M of $\mathcal{J}_{2h}$ into four congruent rectangles K_1,K_2,K_3,K_4, refer to Figure 3.1.

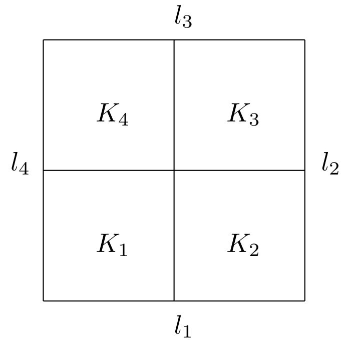

Figure 3.1: $M = \cup_{i=1}^{4} K_i$

Define an operator I_{2h} on the partition $\mathcal{J}_{2h}$, $I_{2h}|_M = I_M$, $I_M = \widehat{I} \circ F_M^{-1}$ and I_M is defined on M as

$$\left\{ \begin{array}{l} I_M u|_M \in P_2(M), \\ \int_{l_i} (I_M u - u) = 0, i = 1,2,3,4, \\ \int_{K_1 \cup K_3} (I_M u - u) = 0, \int_{K_2 \cup K_4} (I_M u - u) = 0. \end{array} \right. \tag{3.26}$$

The interpolation operator on the reference element $\widehat{I}$ is expressed as

$$\begin{aligned} \widehat{I}\widehat{v} = & \frac{\widehat{v}_5 + \widehat{v}_6}{2} + \frac{\widehat{v}_2 - \widehat{v}_4}{2}\xi + \frac{\widehat{v}_3 - \widehat{v}_1}{2}\eta + 2(\widehat{v}_5 - \widehat{v}_6)\xi\eta \\ & + \frac{\widehat{v}_2 + \widehat{v}_4 - \widehat{v}_5 - \widehat{v}_6}{2}\varphi(\xi) + \frac{\widehat{v}_1 + \widehat{v}_3 - \widehat{v}_5 - \widehat{v}_6}{2}\varphi(\eta), \end{aligned} \tag{3.27}$$

where $\widehat{v}_i = \frac{1}{|\widehat{l}_i|} \int_{\widehat{l}_i} \widehat{v} d\widehat{s}, \quad i = 1,2,3,4, \quad \widehat{v}_5 = \frac{1}{2}(\int_{\widehat{K_1}} + \int_{\widehat{K_3}}) \widehat{v} d\xi d\eta, \widehat{v}_6 = \frac{1}{2}(\int_{\widehat{K_2}} + \int_{\widehat{K_4}}) \widehat{v} d\xi d\eta$ and $\widehat{K_i} = K_i \circ F_M, i = 1,2,3,4, F_M : M \longrightarrow \widehat{K}$.

The following lemma shows the postprocessing operator $\widehat{I}$ satisfies the anisotropic interpolation properties.

Lemma 3.1. *For* $|\alpha| = 1$, *there holds*

$$\|\widehat{D}^\alpha(\widehat{v} - \widehat{I}\widehat{v})\|_{0,\widehat{K}} \leq C|\widehat{D}^\alpha \widehat{v}|_{2,\widehat{K}}, \quad \forall \widehat{v} \in H^3(\widehat{K}). \tag{3.28}$$

Proof. We only prove (3.28) for $\alpha = (1,0)$. A direct calculation gives

$$\begin{aligned} \widehat{D}^\alpha \widehat{I}\widehat{v} = & \frac{\widehat{v}_2 - \widehat{v}_4}{2} + \frac{3(\widehat{v}_2 + \widehat{v}_4 - \widehat{v}_5 - \widehat{v}_6)\xi}{2} + 2(\widehat{v}_5 - \widehat{v}_6)\eta \\ = & \gamma_1 + \gamma_2 \xi + \gamma_3 \eta, \end{aligned} \tag{3.29}$$

where

$$\gamma_1 = \frac{\widehat{v}_2 - \widehat{v}_4}{2} = \frac{1}{|\widehat{K}|} \int_{\widehat{K}} \widehat{D}^\alpha \widehat{v} d\xi d\eta = F_3(\widehat{D}^\alpha \widehat{v}),$$

$$\gamma_2 = \frac{3(\widehat{v}_2+\widehat{v}_4-\widehat{v}_5-\widehat{v}_6)}{2} = \frac{3}{2}[\frac{1}{2}\int_{-1}^{1}\widehat{v}(1,\eta)d\eta + \frac{1}{2}\int_{-1}^{1}\widehat{v}(-1,\eta)d\eta$$
$$-\frac{1}{2}(\int_{-1}^{0}\int_{-1}^{0}\widehat{v}(\xi,\eta)d\xi d\eta + \int_{0}^{1}\int_{0}^{1}\widehat{v}(\xi,\eta)d\xi d\eta)$$
$$-\frac{1}{2}(\int_{-1}^{0}\int_{0}^{1}\widehat{v}(\xi,\eta)d\xi d\eta + \int_{0}^{1}\int_{-1}^{0}\widehat{v}(\xi,\eta)d\xi d\eta)]$$
$$= \frac{3}{2}[(\frac{1}{2}\int_{-1}^{0}\widehat{v}(1,\eta)d\eta - \frac{1}{2}\int_{-1}^{0}\int_{-1}^{0}\widehat{v}(\xi,\eta)d\xi d\eta)$$
$$+(\frac{1}{2}\int_{-1}^{0}\widehat{v}(-1,\eta)d\eta - \frac{1}{2}\int_{0}^{1}\int_{-1}^{0}\widehat{v}(\xi,\eta)d\xi d\eta)$$
$$+(\frac{1}{2}\int_{0}^{1}\widehat{v}(1,\eta)d\eta - \frac{1}{2}\int_{-1}^{0}\int_{0}^{1}\widehat{v}(\xi,\eta)d\xi d\eta)$$
$$+(\frac{1}{2}\int_{0}^{1}\widehat{v}(-1,\eta)d\eta - \frac{1}{2}\int_{0}^{1}\int_{0}^{1}\widehat{v}(\xi,\eta)d\xi d\eta)]$$
$$= \frac{3}{4}[\int_{\xi}^{1}\int_{-1}^{0}\int_{-1}^{0}\widehat{D}^{\alpha}\widehat{v}d\xi d\eta d\zeta - \int_{-1}^{\xi}\int_{0}^{1}\int_{-1}^{0}\widehat{D}^{\alpha}\widehat{v}d\xi d\eta d\zeta$$
$$+\int_{\xi}^{1}\int_{-1}^{0}\int_{0}^{1}\widehat{D}^{\alpha}\widehat{v}d\xi d\eta d\zeta - \int_{-1}^{\xi}\int_{0}^{1}\int_{0}^{1}\widehat{D}^{\alpha}\widehat{v}d\xi d\eta d\zeta]$$
$$= F_4(\widehat{D}^{\alpha}\widehat{v}),$$

$$\gamma_3 = 2(\widehat{v}_5-\widehat{v}_6) = \int_{-1}^{0}\int_{-1}^{0}\widehat{v}(\xi,\eta)d\xi d\eta + \int_{0}^{1}\int_{0}^{1}\widehat{v}(\xi,\eta)d\xi d\eta$$
$$-\int_{-1}^{0}\int_{0}^{1}\widehat{v}(\xi,\eta)d\xi d\eta - \int_{0}^{1}\int_{-1}^{0}\widehat{v}(\xi,\eta)d\xi d\eta)$$
$$= \int_{-1}^{0}\int_{-1}^{0}\widehat{v}(\xi,\eta)d\xi d\eta - \int_{-1}^{0}\widehat{v}(0,\eta)d\xi d\eta + \int_{-1}^{0}\widehat{v}(0,\eta)d\xi d\eta - \int_{0}^{1}\int_{-1}^{0}\widehat{v}(\xi,\eta)d\xi d\eta$$
$$+\int_{0}^{1}\int_{0}^{1}\widehat{v}(\xi,\eta)d\xi d\eta - \int_{0}^{1}\widehat{v}(0,\eta)d\xi d\eta + \int_{0}^{1}\widehat{v}(0,\eta)d\xi d\eta - \int_{-1}^{0}\int_{0}^{1}\widehat{v}(\xi,\eta)d\xi d\eta$$
$$= \int_{0}^{\xi}\int_{-1}^{0}\int_{-1}^{0}\widehat{D}^{\alpha}\widehat{v}d\xi d\eta d\zeta - \int_{0}^{\xi}\int_{0}^{1}\int_{-1}^{0}\widehat{D}^{\alpha}\widehat{v}d\xi d\eta d\zeta$$
$$+\int_{0}^{\xi}\int_{0}^{1}\int_{0}^{1}\widehat{D}^{\alpha}\widehat{v}d\xi d\eta d\zeta - \int_{0}^{\xi}\int_{-1}^{0}\int_{0}^{1}\widehat{D}^{\alpha}\widehat{v}d\xi d\eta d\zeta$$
$$= F_5(\widehat{D}^{\alpha}\widehat{v}).$$

Where F_3, F_4, F_5 are functionals defined over $H^2(\widehat{K})$.

By Cauchy-Schwarz inequality and the trace theorem, we can show that

$$|F_i(\widehat{v})| \le C\|\widehat{v}\|_{1,\widehat{K}} \le C\|\widehat{v}\|_{2,\widehat{K}}, \quad i = 3,4,5,$$

i.e., $F_i, i = 3,4,5$ are bounded linear functionals on $H^2(\widehat{K})$. Then an application of the basic anisotropic interpolation theorem[16] yields the desired result.

Lemma 3.2. *The interpolation operator have the following properties:*

$$I_{2h}\Pi_h u = I_{2h}u, \tag{3.30}$$

$$\|I_{2h}u - u\|_h \le Ch^2|u|_{3,\Omega}, \tag{3.30}$$

$$\|I_{2h}v_h\|_h \le C\|v_h\|_h, \forall v_h \in V_h. \tag{3.31}$$

Proof. (3.30) is obvious and (3.31) can be obtained proceeding along with the same lines of Lemma 2.1. So we only need to prove (3.32).

Thanks to the equivalence of norms over the finite dimensional space, we have

$$|\gamma_i| \leq C\|\hat{D}^\alpha \widehat{v}_h\|_{2,\widehat{K}} \leq C\|\widehat{D}^\alpha \widehat{v}_h\|_{0,\widehat{K}}, i = 1, 2, 3, \quad \forall v_h \in V_h.$$

Then

$$\begin{aligned}\|D^\alpha I_{2h} v_h\|_{0,K} &= h_K^{-\alpha}(h_{K1}h_{K2})^{\frac{1}{2}}\|\widehat{D}^\alpha \widehat{I}\widehat{v}_h\|_{0,\widehat{K}} \\ &\leq Ch_K^{-\alpha}(h_{K1}h_{K2})^{\frac{1}{2}}\sum_{i=1}^{3}|\gamma_i| \leq C\|D^\alpha v_h\|_{0,K}, \forall v_h \in V_h.\end{aligned}$$

Hence

$$\|I_{2h}v_h\|_h = \left(\sum_K \sum_{|\alpha|=1} \|D^\alpha I_{2h}v_h\|_{0,K}^2\right)^{\frac{1}{2}} \leq C\|v_h\|_h, \forall v_h \in V_h,$$

where the desired result is obtained.

Then we can get the following superconvergence theorem easily.

Theorem 3.3. *Under the above hypothesis, we have*

$$\|u - I_{2h}u_h\|_h \leq Ch^2(|u|_{3,\Omega} + |u|_{2,\Omega} + |u|_{1,\Omega}). \tag{3.32}$$

Proof. Noticing that $I_{2h}\Pi_h u = I_{2h}u$, then

$$\begin{aligned}\|u - I_{2h}u_h\|_h \leq& \|u - I_{2h}\Pi_h u\|_h + \|I_{2h}(\Pi_h u - u_h)\|_h \\ \overset{(3.32)}{\leq}& \|u - I_{2h}u\|_h + C\|\Pi_h u - u_h\|_h \\ \overset{(3.31)(3.2)}{\leq}& Ch^2(|u|_{3,\Omega} + |u|_{2,\Omega} + |u|_{1,\Omega}).\end{aligned} \tag{3.33}$$

Remark 3.3. We comment that the conventional superconvergence analysis is based on the quasi-uniform assumption on the meshes. However, here our analysis has avoided the regular assumption and inverse assumption on the meshes, i.e., the constant C appeared in our estimate is independent of h_K/ρ_K and h/h_K.

Remark 3.4. In fact, the meshes $\mathcal{J}_h$ is not necessarily as in Figure 3.1 if the quasi-uniform assumption is satisfied. That is to say, for the case $\mathcal{J}_h$ is obtained from $\mathcal{J}_{2h}$ by dividing each element M of $\mathcal{J}_{2h}$ into four different rectangles, we can still obtain (3.33) with the constant C dependent on h_K/ρ_K and h/h_K as in conventional analysis.

4. Extrapolation results

In this section, we assume that $\alpha_{ij} = C_i\delta_i^j$, where δ_i^j is the Kronecker index , $C_i = const, i = 1, 2,\ \alpha_i = 0, \gamma \in W^{1,\infty}(\Omega)$. The meshes consider in this section is regular rectangular meshes.

Lemma 4.1. *For any $v_h \in V_h$, there holds*

$$\begin{aligned}a_h(\Pi_h u - u_h, v_h) =& \int_\Omega \left(\frac{\alpha_{11}h_{K2}^2}{3} + \frac{\alpha_{22}h_{K1}^2}{3}\right)\frac{\partial^4 u}{\partial x^2 \partial y^2} v_h dxdy \\ &+ O(h^3)\|u\|_{4,\Omega}\|v_h\|_h.\end{aligned} \tag{4.1}$$

Proof. Let us consider $a_h(\Pi_h u - u, v_h)$ first,
(4.2)
$$\begin{aligned} a_h(\Pi_h u - u, v_h) &= \sum_{K\in\mathcal{J}_h}\int_K \left(\sum_{i=1}^{2}\alpha_{ii}\frac{\partial(\Pi_h u - u)}{\partial x_i}\frac{\partial v_h}{\partial x_i} + \gamma(\Pi_h u - u)v_h\right)dxdy \\ &= J_1 + J_2. \end{aligned}$$

It can be checked easily that

$$J_1 = 0. \tag{4.3}$$

J_2 can be decomposed as

$$\begin{aligned} J_2 = &\sum_{K\in\mathcal{J}_h}\int_K [P_0\gamma(\Pi_h u - u)(v_h - P_0 v_h) + P_0\gamma P_0 v_h(\Pi_h u - u) \\ &+ (\gamma - P_0\gamma)(\Pi_h u - u)(v_h - P_0 v_h) + (\gamma - P_0\gamma)(\Pi_h u - u)P_0 v_h]dxdy \\ = &J_{21} + J_{22} + J_{23} + J_{24}. \end{aligned} \tag{4.4}$$

Then we have

$$J_{21} \le Ch^3|u|_{2,\Omega}\|v_h\|_h, \quad J_{22} = 0, \quad J_{23} \le Ch^4|u|_{2,\Omega}\|v_h\|_h, \tag{4.5}$$

and by the discrete Poincaré inequality (refer to [11, 20, 39, 40]),

$$J_{24} \le Ch^3|u|_{2,\Omega}\|v_h\|_{0,\Omega} \le Ch^3|u|_{2,\Omega}\|v_h\|_h. \tag{4.6}$$

So,

$$a_h(\Pi_h u - u, v_h) \le O(h^3)|u|_{2,\Omega}\|v_h\|_h. \tag{4.7}$$

Now, let us consider $a_h(u - u_h, v_h)$ again, i.e., the consistency error. We only need to prove

$$\begin{aligned} a_h(u - u_h, v_h) = &\int_\Omega \left(\frac{\alpha_{11}h_{K2}^2}{3} + \frac{\alpha_{22}h_{K1}^2}{3}\right)\frac{\partial^4 u}{\partial x^2\partial y^2}v_h dxdy \\ &+ O(h^3)\|u\|_{4,\Omega}\|v_h\|_h. \end{aligned} \tag{4.8}$$

For this purpose, we turn back to (3.18) in §3. Set $V = \frac{\partial U}{\partial x}$, then

$$\begin{aligned} B_{11}^2 + B_{11}^4 &= \int_K V\left(v_h(x_K + h_{K1}, y) - P_{02}v_h(x_K + h_{K1}, y)\right)dxdy \\ &= h_{K1}h_{K2}\int_{\widehat{K}} \widehat{V}\left(\widehat{v}_h(1,\eta) - \widehat{P}_{02}\widehat{v}_h(1,\eta)\right)d\xi d\eta. \end{aligned}$$

For any fixed $\widehat{v}_h$, we define the functional

$$T(\widehat{V}) = \int_{\widehat{K}} \widehat{V}\left(\widehat{v}_h(1,\eta) - \widehat{P}_{02}\widehat{v}_h(1,\eta)\right)d\xi d\eta - \frac{1}{3}\int_{\widehat{K}}\frac{\partial\widehat{V}}{\partial\eta}\frac{\partial\widehat{v}_h}{\partial\eta}d\xi d\eta.$$

Obviously,

$$|T(\widehat{V})| \le C\|\frac{\partial\widehat{v}_h}{\partial\eta}\|_{0,\widehat{K}}\|\widehat{V}\|_{2,\widehat{K}}.$$

Hence $T \in H^2(\widehat{K})'$ and $\|T\| \le C\|\frac{\partial\widehat{v}_h}{\partial\eta}\|_{0,\widehat{K}}$. A detailed calculation shows that

$$T(\widehat{V}) = 0, \quad \forall\widehat{V} \in P_1(\widehat{K}). \tag{4.9}$$

Then an application of Bramble-Hilbert lemma yields

$$T(\widehat{V}) \le C\|\frac{\partial\widehat{v}_h}{\partial\eta}\|_{0,\widehat{K}}|\widehat{V}|_{2,\widehat{K}}. \tag{4.10}$$

So by the scaling argument and Green's formula,

$$
\begin{aligned}
B_{11}^2+B_{11}^4 &\le Ch_K^3\|u\|_{4,K}|v_h|_{1,K}+\frac{h_{K2}^2}{3}\int_K\frac{\partial V}{\partial y}\frac{\partial v_h}{\partial y}dxdy\\
&=O(h_K^3)\|u\|_{4,K}|v_h|_{1,K}+\frac{h_{K2}^2}{3}\int_K\frac{\partial^2 V}{\partial y^2}v_hdxdy\\
&\quad+\frac{h_{K2}^2}{3}(\int_{l_3}-\int_{l_1})\frac{\partial V}{\partial y}v_hdx.
\end{aligned}
\tag{4.11}
$$

Similarly,

$$
\begin{aligned}
B_{22}^1+B_{22}^3 &= O(h_K^3)\|u\|_{4,K}|v_h|_{1,K}+\frac{h_{K1}^2}{3}\int_K\frac{\partial^2 V}{\partial x^2}v_hdxdy\\
&\quad+\frac{h_{K1}^2}{3}(\int_{l_2}-\int_{l_4})\frac{\partial V}{\partial x}v_hdy.
\end{aligned}
\tag{4.12}
$$

Hence, the summation of $K\in\mathcal{J}_h$ gives

$$
\begin{aligned}
a_h(u-u_h,v_h) &= \int_\Omega\left(\frac{\alpha_{11}h_{K2}^2}{3}+\frac{\alpha_{22}h_{K1}^2}{3}\right)\frac{\partial^4 u}{\partial x^2\partial y^2}v_hdxdy+O(h^3)\|u\|_{4,\Omega}\|v_h\|_h\\
&\quad+\sum_{K\in\mathcal{J}_h}\left(\frac{h_{K1}^2}{3}(\int_{l_2}-\int_{l_4})\frac{\partial V}{\partial x}v_hdy+\frac{h_{K2}^2}{3}(\int_{l_3}-\int_{l_1})\frac{\partial V}{\partial y}v_hdx\right).
\end{aligned}
\tag{4.13}
$$

Then a combination of the obvious result

$$
\sum_{K\in\mathcal{J}_h}\left(\frac{h_{K1}^2}{3}(\int_{l_2}-\int_{l_4})\frac{\partial V}{\partial x}v_hdy+\frac{h_{K2}^2}{3}(\int_{l_3}-\int_{l_1})\frac{\partial V}{\partial y}v_hdx\right)=O(h^3)\|u\|_{4,\Omega}\|v_h\|_h
$$

implies the desired result, which completes the proof. □

Now, we will prove the following error expansions.

Lemma 4.2. *There exists a function $\phi\in H^2(\Omega)$, such that*

$$
\|u_h-I_hu-h^2\phi_h\|_h\le Ch^3\|u\|_{4,\Omega},
\tag{4.14}
$$

where $\phi_h\in V_h$ is a nonconforming finite element projection of ϕ.

Proof. We define the linear functional

$$
F(v)=\int_\Omega\left(\frac{\alpha_{11}h_{K2}^2}{3h^2}+\frac{\alpha_{22}h_{K1}^2}{3h^2}\right)\frac{\partial^4 u}{\partial x^2\partial y^2}v_hdxdy
$$

and consider the following auxiliary problem

$$
a(\phi,v)=F(v),\quad \forall v\in H_0^1(\Omega).
\tag{4.15}
$$

By Lax-Milgram theorem, problem (4.15) exists a solution $\phi\in H^2(\Omega)$, and due to the regularity of elliptic equation

$$
\|\phi\|_{2,\Omega}\le C|u|_{4,\Omega}.
\tag{4.16}
$$

Let ϕ_h be a nonconforming finite element projection of ϕ, i.e.,

$$
\begin{cases}
\quad Find\ \phi_h\in V_h,\ such\ \ that\\
a_h(\phi_h,v_h)=F(v_h),\ \forall\ v_h\in V_h.
\end{cases}
\tag{4.17}
$$

Then by (4.1), we have

$$
a_h(u_h-I_hu-h^2\phi_h,v_h)=O(h^3)\|u\|_{4,\Omega}\|v_h\|_h.
\tag{4.18}
$$

Taking $v_h=u_h-I_hu-h^2\phi_h$, then

$$
C\|u_h-I_hu-h^2\phi_h\|_h^2\le a_h(u_h-I_hu-h^2\phi_h,u_h-I_hu-h^2\phi_h),
\tag{4.19}
$$

which implies (4.14).

Now, we define another postprocessing operator T_{3h}^3 as in §3. Assume the macroelement $M \in \mathcal{J}_{3h}$ consist of 9 subrectangles $K_i \in \mathcal{J}_h, i = 1, 2, ..., 9$ (refer to Figure 4.1). Then we choose the interpolant T_{3h}^3 as follows:

$$\begin{cases} T_{3h}^3 u|_M \in P_3(M), \\ \int_{l_i} (T_{3h}^3 u - u) = 0, i = 1, 2, 3, 4, \\ \int_{K_i} (T_{3h}^3 u - u) = 0, i = 1, 2, 3, 4, 5, 7. \end{cases} \tag{4.20}$$

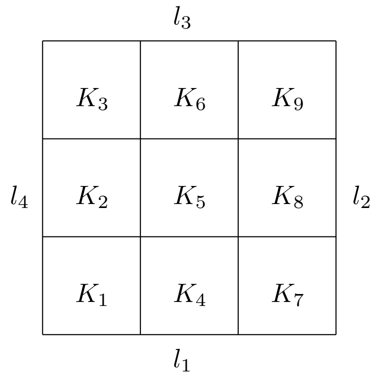

Figure 4.1: $M = \cup_{i=1}^9 K_i$

It can be checked that the interpolation defined as (4.20) is well-posed. Furthermore, it has the following properties:

$$\begin{cases} T_{3h}^3 \Pi_{\frac{3h}{2}} u = T_{3h}^3 u, \\ \|u - T_{3h}^3 u\|_h \le Ch^3 |u|_{4,\Omega}, \\ \|T_{3h}^3 v_h\|_h \le C\|v_h\|_h, \quad \forall v_h \in V_h. \end{cases} \tag{4.21}$$

Theorem 4.1. *Suppose $u_{\frac{h}{2}}, u_h$ to be the nonconforming finite element solution of the meshes $\mathcal{J}_{\frac{h}{2}}$ and $\mathcal{J}_h$, respectively. We can compute the extrapolant solution as*

$$\widetilde{u}_h = \frac{4}{3} T_{\frac{3h}{2}}^3 u_{\frac{h}{2}} - \frac{1}{3} T_{3h}^3 u_h, \tag{4.22}$$

then we have the following sharp error estimate:

$$\|u - \widetilde{u}_h\|_h \le Ch^3 \|u\|_{4,\Omega}. \tag{4.23}$$

Proof. By (4.21) and (4.14), we have

(4.24)

$$\begin{aligned}
\|u-\widetilde{u}_h\|_h &= \|\frac{4}{3}(u-T^3_{\frac{3h}{2}}u_{\frac{h}{2}})+\frac{1}{3}(T^3_{3h}u_h-u)\|_h \\
&= \|\frac{4}{3}(u-T^3_{\frac{3h}{2}}\Pi_{\frac{h}{2}}u)+\frac{4}{3}(T^3_{\frac{3h}{2}}\Pi_{\frac{h}{2}}u-T^3_{\frac{3h}{2}}u_{\frac{h}{2}}) \\
&+\frac{1}{3}(T^3_{3h}u_h-T^3_{3h}\Pi_h u)+\frac{1}{3}(T^3_{3h}\Pi_h u-u)\|_h \\
&= \|\frac{4}{3}(u-T^3_{\frac{3h}{2}}\Pi_{\frac{h}{2}}u)-\frac{4}{3}T^3_{\frac{3h}{2}}(u_{\frac{h}{2}}-\Pi_{\frac{h}{2}}u-(\frac{h}{2})^2\phi_{\frac{h}{2}}) \\
&+\frac{1}{3}T^3_{3h}(u_h-\Pi_h u-h^2\phi_h)+\frac{1}{3}(T^3_{3h}\Pi_h u-u)+\frac{h^2}{3}(T^3_{\frac{3h}{2}}\phi_{\frac{h}{2}}-T^3_{3h}\phi_h)\|_h \\
&\le C[\|u_{\frac{h}{2}}-\Pi_{\frac{h}{2}}u-(\frac{h}{2})^2\phi_{\frac{h}{2}}\|_h+\|u-T^3_{\frac{3h}{2}}u\|_h+\|u_h-\Pi_h u-h^2\phi_h\|_h \\
&+\|T^3_{3h}u-u\|_h+h^2\|T^3_{\frac{3h}{2}}(\phi_{\frac{h}{2}}-\phi)\|_h+h^2\|T^3_{\frac{3h}{2}}\phi-\phi\|_h \\
&+h^2\|\phi-T^3_{3h}\phi\|_h+h^2\|T^3_{3h}(\phi-\phi_h)\|_h] \\
&\le C(h^3\|u\|_{4,\Omega}+h^3\|\phi\|_{2,\Omega}) \\
&\le Ch^3\|u\|_{4,\Omega}
\end{aligned}$$

The proof is completed.

Remark 4.1. After we have submitted this paper, we have learned that the superconvergence of this element has been studied in [30] by Lin and his collaborators. However, the results of this paper are obtained for more general meshes and equations, which will be useful in the numerical analysis of perturbed convection-diffusion-reaction equations where anisotropic meshes are preferred.

5. Numerical experiments

In order to investigate the numerical behavior of the five-node nonconforming element, we consider the following Dirichlet elliptic boundary problem :

$$\begin{cases} -\triangle u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

with $\Omega = [0,1]\times[0,1]$, and the right hand side $f(x,y)$ is taken such that $u(x,y) = (1-e^{(-x(1-x)/\epsilon)})(1-e^{(-y(1-y)/\epsilon)})$ (refer to Figure 1 for the case $\epsilon = 0.05$) is the exact solution, which varies significantly near the boundary of Ω for small ε.

The unit square $\Omega = [0,1]\times[0,1]$ is subdivided in the following two fashions:

mesh 1: *Subdividing the boundary of Ω into n equal intervals along the $x-axis$ and $y-axis$, respectively. The mesh obtained in this way for $n = 8$ is illustrated in Figure 2;*

mesh 2: *Each edge of Ω is divided into n segments with $n+1$ points $(1-cos(\frac{i\pi}{n}))/2, i = 0,1,...,n$. The mesh obtained in this way for $n = 16$ is illustrated in Figure 3;*

The numerical results are listed in Table 5.1∼5.2. Herein, α denotes the convergence order, $SE_h = \left(\sum_{K\in\mathcal{J}_h}|(\nabla u-\nabla u_h)(O_K)|^2 h_{K1}h_{K2}\right)^{\frac{1}{2}}$.

From Tables 5.2, we can see that the optimal energy error in norm between u and u_h is obtained under large aspect ratio ($\frac{h_K}{\rho_K} = \frac{\sqrt{m^2+n^2}}{m}$). It shows that the optimal error estimates are independent of h_K, $\max_{K\in\mathcal{J}_h}\{h_K/\rho_K\}$ and $\max_{K\in\mathcal{J}_h}\{h/h_K\}$, which means that we can get the same order of error estimates whether the subdivision satisfies the regular assumption or not. Moreover, the numerical result of mesh 2 is better than that of mesh 1, which shows that the anisotropic meshes are more attractive than the regular meshes for some special cases.

On the other hand, from Table 5.1 $\sim$ 5.2, we can see that the superconvergence behaviors of the numerical solution are also coincide with our theoretical analysis. It can be seen that the postprocessing errors $\|u-I_{2h}u_h\|_h \ll \|u-u_h\|_h$, $\|u-\widetilde{u}_h\|_h \ll \|u-u_h\|_h$, taking Table 5.1 for 96×96 meshes as an example, $\|u-u_h\|_h$ is 17 times as $\|u-I_{2h}u_h\|_h$ and even 414 times as $\|u-\widetilde{u}_h\|_h$, How remarkable the numerical results are ! However, the additional computations are not hard and the cost is cheap.

Lastly, we also compute with the rotated $\mathcal{Q}_1$ element $(R\mathcal{Q}_1)$ and modified $\mathcal{Q}_1$ element $(MR\mathcal{Q}_1)$ by Apel [4, 5]. A comparison between the results for mesh 2 of these elements is made, please refer to Figure 3-5. We can see that the rotated Q_1 element $(R\mathcal{Q}_1)$ is also convergent on anisotropic meshes by the numerical results. Moreover, the superconvergence at the central points of elements is still valid for $R\mathcal{Q}_1$ and modified $\mathcal{Q}_1$ element$(MR\mathcal{Q}_1)$ by Apel [4,5]. We believe that these are only technical problems. However, the superclose result for $R\mathcal{Q}_1$ and $MR\mathcal{Q}_1$ does not hold, and the numerical results of the two elements are both worse than that of the five-node element (FN).

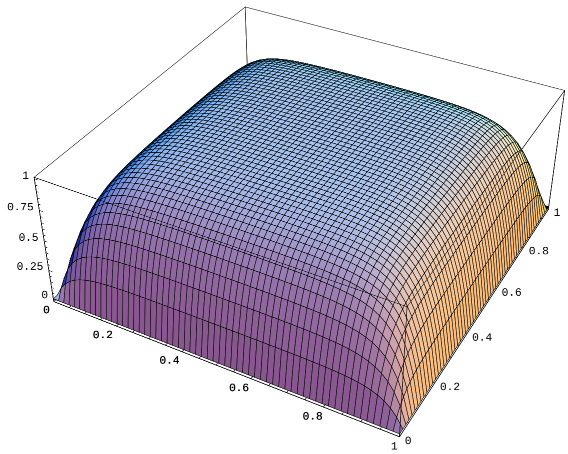

FIGURE 1. the exact solution u for case $\varepsilon = 0.05$

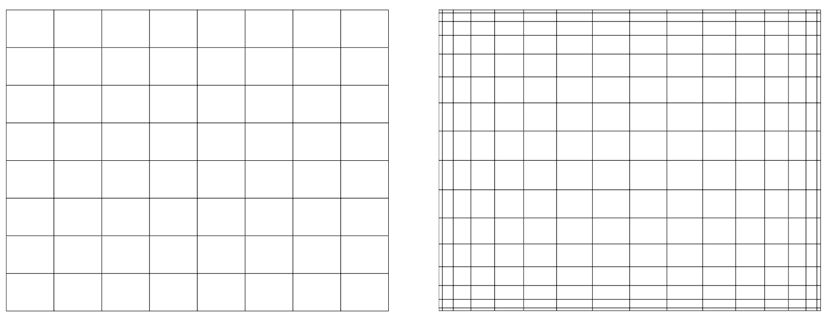

FIGURE 2. mesh 1 for case $n = 8$ (left) and mesh 2 for case $n = 16$ (right)

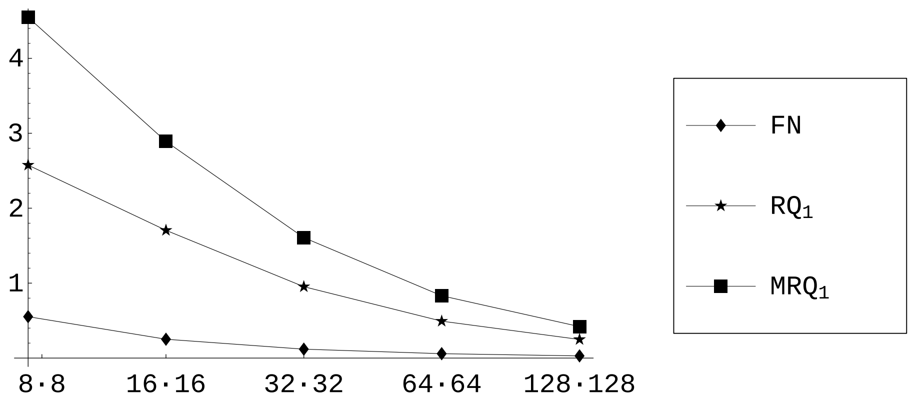

FIGURE 3. the numerical results $\|u - u_h\|_h$ for FN, RQ_1, MRQ_1 on mesh 2

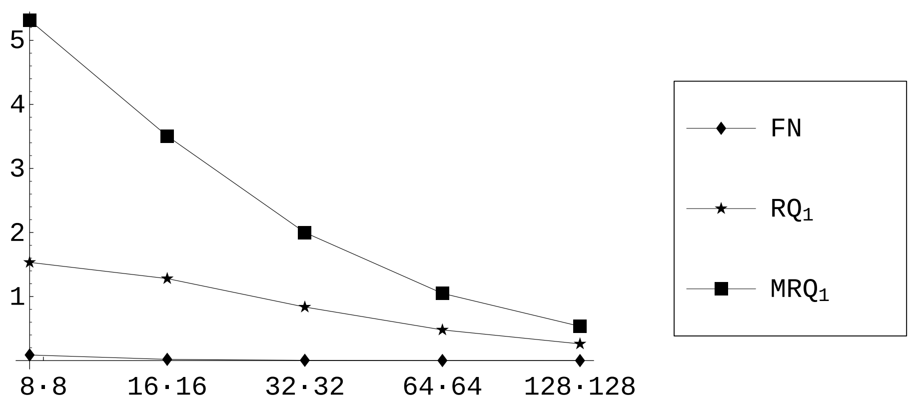

FIGURE 4. the numerical results $\|\Pi_h u - u_h\|_h$ for FN, RQ_1, MRQ_1 on mesh 2

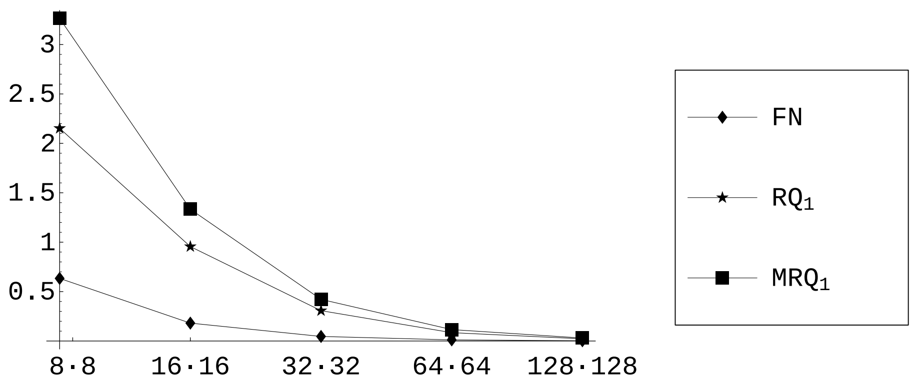

FIGURE 5. the numerical results SE_h for FN, RQ_1, MRQ_1 on mesh 2

Table 5.1: Five-node nonconforming element on mesh 1

$n \times n$	6×6	12×12	24×24	48×48	96×96
$\|u - u_h\|_h$	1.61764745	0.79301153	0.35239943	0.16547887	0.08111326
α	\	1.02848339	1.17013025	1.09056461	1.02863741
$\|\Pi_h u - u_h\|_h$	0.39708376	0.14182994	0.04015942	0.01027563	0.00251715
α	\	1.48528123	1.82035184	1.96651149	2.02936387
SE_h	2.47888224	0.97567347	0.29143627	0.07688414	0.01949847
α	\	1.34521949	1.74321795	1.92242253	1.97932518
$\|u - I_{2h} u_h\|_h$	0.48715268	0.21841877	0.06884013	0.01847473	0.00470690
α	\	1.15727711	1.66577518	1.89769661	1.97270417
$\|u - \widetilde{u}_h\|_h$	0.31281563	0.06987871	0.01120638	0.00151361	0.00019573
α	\	2.16238761	2.64053278	2.88825219	2.95102458

Table 5.2: Five-node nonconforming element on mesh 2

$n \times n$	8×8	16×16	32×32	64×64	128×128
$\|u - u_h\|_h$	0.55249775	0.25135539	0.11893743	0.05851057	0.02913152
α	\	1.13623953	1.07952571	1.02343357	1.00611627
$\|\Pi_h u - u_h\|_h$	0.08759189	0.01915686	0.00449336	0.00110538	0.00027523
α	\	2.19293618	2.09199452	2.02325225	2.02325225
SE_h	0.63364048	0.18108744	0.04627087	0.01163454	0.00291286
α	\	1.80697799	1.96851027	1.99169004	1.99790597
$\|u - I_{2h} u_h\|_h$	0.17265555	0.05556576	0.01486130	0.00383666	0.00096558
α	\	1.63562879	1.90263586	1.95363751	1.99038649
h	0.270598	0.137950	0.069309	0.034696	0.017353
$\max_{K \in \mathcal{J}_h} \{h_K/\rho_K\}$	7.109732	14.358751	28.786978	57.608674	115.234703
$\max_{K \in \mathcal{J}_h} \{h/h_K\}$	5.027339	10.53170	20.355408	40.735484	81.483240

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