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ERROR ESTIMATES OF THE FINITE ELEMENT METHOD WITH WEIGHTED BASIS FUNCTIONS FOR A SINGULARLY PERTURBED CONVECTION-DIFFUSION EQUATION*

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Abstract

In this paper, we establish a convergence theory for a finite element method with weighted basis functions for solving singularly perturbed convection-diffusion equations. The stability of this finite element method is proved and an upper bound $\mathcal{O}(h|\ln\varepsilon|^{3/2})$ for errors in the approximate solutions in the energy norm is obtained on the triangular Bakhvalov-type mesh. Numerical results are presented to verify the stability and the convergent rate of this finite element method.

Mathematics subject classification: 65N30, 35J20.
Key words: Convergence, Singular perturbation, Convection-diffusion equation, Finite element method.

1. Introduction

It is known that singularly perturbed convection-diffusion problems contain sharp boundary layers so that the application of a standard finite element or finite difference method to such a problem often results in spurious oscillation. To avoid non-physical numerical solutions, many special finite element techniques have been developed, including upwind finite element [1, 4], Petrov-Galerkin finite element [7], streamline diffusion finite element methods [2, 8, 9], and exponentially fitted finite elements [18, 21–23]. However, these methods do not always give accurate results, especially when a diffusion coefficient has the same magnitude as that of mesh size. In [12], Li et al presented a weighted basis finite element method. Since the basis functions with weighted factors are consistent with the direction of flow and have the nature of exponential fitting near the boundary layers, numerical solutions obtained by applying this finite element method is non-oscillatory. Although the method proposed in [12] is promising from its numerical performance, except for a simple error bound of order $\mathcal{O}(h^{1/2}|\ln\varepsilon|)$ in [14] the mathematical understanding of the method is very limited. Regarding about the convergent results on layer-adapted meshes, streamline diffusion finite element or standard finite element methods can give uniformly optimal convergent rate, the reader is referred to [2,3,11,16,24–28]. Moreover, spectral methods have been proposed to resolve the bounding layers, which are shown very effective, see, e.g., [29, 30].

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In this work, a combination of the standard linear finite element method and the weighted basis finite element method is investigated for solving two-dimensional convection-dominated problems. This combination is used in conjunction with an anisotropic mesh refinement technique, i.e., a convection-diffusion equation is discretized by the weighted finite element method in a region containing the boundary layers and by the standard finite element method on a regular triangulation of the subregion away from the layers. As shown in [12], the standard basis function can be regarded as a special case of the weighted basis function. Therefore, this combination of two finite element methods is framed as a weighted basis finite element method which reduces to either the standard finite element or the weighted basis finite element method by a judicious choice of weights. Because the weighted basis functions are continuous across the interface between the two subregions, the resulting finite element space is conforming. This conformity allows us to analyze the method using conventional finite element analysis techniques. This is in contrast to a nonconforming method with which a sophisticated technique needs to be used to deal. In this paper, we will prove the stability of this finite element method and establish an upper error bound for the approximate solutions by the method on the triangular Bakhvalov-type mesh. We will also show that the error bound is almost independent of ε. We comment that, although the problem considered in this work is two-dimensional and linear, the idea can be extended to higher dimensional and/or nonlinear problems [5, 13].

Throughout this paper, we use C as a generic positive constant which is independent of the small parameter ε and the mesh size. The rest of our paper is organized as follows. Section 2 describes the continuous problems and some preliminaries. The finite element formulation with weighted basis functions is presented in Section 3. In Section 4, the stability of this finite element method is shown and the error estimate in an energy norm is established. The numerical examples will be given in Section 5 to demonstrate the convergent rate and the stability of this finite element method.

2. Weighted Basis Functions on the Triangular Mesh

Consider the following singularly perturbed problem with a small positive parameter ε in two-dimensional space,

$$\nabla\cdot(-\varepsilon\nabla v+\mathbf{b}(X)v)+\mu(X)v=f(X),\qquad X\in\Omega\subset\mathbb{R}^2,\tag{2.1}$$

$$v\big|_{\partial\Omega}=0,\tag{2.2}$$

where $X=(x,y)^T$, $\Omega=(0,1)\times(0,1)$ and $\partial\Omega$ denotes the boundary of Ω.

In what follows, we will use conventional notation for function sets and spaces. More specifically, we use $L^2(\Omega)$ to denote the space of all square-integrable functions on Ω with the inner product $(\cdot,\cdot)$ and $C^k(\Omega)$ (or $C^k(\bar{\Omega})$) to denote the set of functions which, along with its up to kth derivatives are continuous on Ω (or $\bar{\Omega}$. The usual kth order Sobolev space is denoted by $H^k(\Omega)$ and we put $H_0^1(\Omega)=\{v\in H^1(\Omega): v(X)=0 \text{ on } \partial\Omega\}$.

For the coefficient functions, we assume that $\mathbf{b}(X)\in(C^1(\bar{\Omega}))^2$, $\mu(X)\in C(\bar{\Omega})\cap H^1(\Omega)$ and $f(X)\in L^\infty(\Omega)$. We also assume that $\mathbf{b}(X)$ satisfies

$$\frac{1}{2}\nabla\cdot\mathbf{b}+\mu(X)\geq\alpha>0,\qquad X\in\Omega,\tag{2.3}$$

where α is a positive constant. This condition (2.3) has been used in many existing works on uniform convergence analysis such as [18–20, 22, 25]. In fact, when ε is sufficiently small, the condition (2.3) can be easily satisfied by a simple transformation $v(x,y) = \exp(Kx+Ky)u(x,y)$ for a suitable positive constant K, see [19, 25]. Although the existence and uniqueness of the solutions to both of the continuous problem and the finite element problem do not need this condition, it will be used in the definition of the energy norm and the proof of the error estimates. For simplicity, we also assume that two components of $\mathbf{b}$ are bounded below by two positive constants $\underline{b}_1, \underline{b}_2$ such that

$$b_1(X) \geq \underline{b}_1 > 0, \quad b_2(X) \geq \underline{b}_2 > 0, \qquad \text{in } \Omega. \tag{2.4}$$

In this case, the solution to (2.1) and (2.2) has two exponential boundary layers with width $\mathcal{O}(\varepsilon)$ at boundaries $x = 1$ and $y = 1$. However, to avoid the appearance of singularity at the corners of Ω [10], f is often assumed to satisfy the following compatibility conditions

$$\begin{aligned} &f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0, \\ &\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0,0) = 0, \qquad 1 \leq i+j \leq 3, \end{aligned}$$

see [15, Theorem 5.1] and [25, Lemma 2.1] for details.

The variational problem corresponding to (2.1) and (2.2) is illustrated below.

Problem 2.1. *Find $v \in H_0^1(\Omega)$ such that for all $w \in H_0^1(\Omega)$,*

$$A(v,w) = (f,w), \tag{2.5}$$

where $A(\cdot,\cdot)$ is a bilinear form on $(H_0^1(\Omega))^2$ defined by

$$A(v,w) = (\varepsilon \nabla v - \mathbf{b}v, \nabla w) + (\mu(X)v, w). \tag{2.6}$$

Let $||\cdot||_\varepsilon$ be the norm defined on $H_0^1(\Omega)$ by

$$||v||_\varepsilon = \left(\varepsilon(\nabla v, \nabla v) + \left(\left(\frac{1}{2} \nabla \cdot \mathbf{b} + \mu(X) \right) v, v \right) \right)^{\frac{1}{2}}.$$

It is easy to see that the norm $||\cdot||_\varepsilon$ is true on $H_0^1(\Omega)$ due to the condition (2.3). For any $u \in H_0^1(\Omega)$, we have

$$A(u,u) = \varepsilon(\nabla u, \nabla u) + \left(\left(\frac{1}{2} \nabla \cdot \mathbf{b} + \mu(X) \right) u, u \right) = ||u||_\varepsilon^2.$$

Then the condition (2.3) guarantees that the bilinear function $A(\cdot,\cdot)$ is coercive and therefore the variational problem 2.1 has a unique solution in $H_0^1(\Omega)$.

For a triangle T with vertices X_i, X_j, X_k in the anti-clockwise direction, the standard linear basis functions satisfy

$$\varphi_l(X_m) = \delta_{lm},$$

where δ_{lm} is the Kronecker delta function. By virtue of the Bernoulli function

$$B(s) = \begin{cases} \dfrac{s}{e^s - 1}, & \text{if } s \neq 0, \\ 1, & \text{if } s = 0, \end{cases}$$

for a given function $\widetilde{\mathbf{b}}(X)$ we can define the weighted factor $m_l(X)$ corresponding to $\varphi_l(X)$ $(l = i, j, k)$ as

$$m_l(X) = B(-\widetilde{\mathbf{b}}^t (X - X_l)/\varepsilon). \tag{2.7}$$

Using these weighted factors, one obtains weighted basis functions on T

$$\widetilde{\varphi}_l(X) = \frac{m_l(X)\varphi_l(X)}{m_i(X)\varphi_i(X) + m_j(X)\varphi_j(X) + m_k(X)\varphi_k(X)}, \qquad l = i, j, k, \tag{2.8}$$

where $\widetilde{\varphi}_l$ has the same support as φ_l.

By the definition (2.8), $\widetilde{\varphi}_l$ $(l = i, j, k)$ have the following properties (see [12]):

$$\widetilde{\varphi}_i(X_l) = \delta_{il}, \qquad 0 \le \widetilde{\varphi}_i \le 1,$$

and on $\overline{T}$

$$\widetilde{\varphi}_i + \widetilde{\varphi}_j + \widetilde{\varphi}_k = 1. \tag{2.9}$$

For a smooth function u, we define a flux $\widetilde{\mathbf{g}}(u)$ corresponding to the function $\widetilde{\mathbf{b}}(X)$ as

$$\widetilde{\mathbf{g}}(u) = -\varepsilon \nabla u + \widetilde{\mathbf{b}} u. \tag{2.10}$$

As shown in [12] , we can give the approximations $\widetilde{\mathbf{g}}_l$ to $\widetilde{\mathbf{g}}(\widetilde{\varphi}_l)$ $(l = i, j, k)$:

$$\begin{aligned}
\widetilde{\mathbf{g}}_i =& -\varepsilon \begin{pmatrix} x - x_j & y - y_j \\ x - x_k & y - y_k \end{pmatrix}^{-1} \begin{pmatrix} B(\widetilde{\mathbf{b}}^t (X - X_j)/\varepsilon) \\ B(\widetilde{\mathbf{b}}^t (X - X_k)/\varepsilon) \end{pmatrix} \tilde{\varphi}_i(X) && \text{(2.11a)} \\
=& -(\varepsilon/2S_T) \begin{pmatrix} y - y_k & -(y - y_j) \\ -(x - x_k) & x - x_j \end{pmatrix} \begin{pmatrix} B(\widetilde{\mathbf{b}}^t (X - X_j)/\varepsilon) \\ B(\widetilde{\mathbf{b}}^t (X - X_k)/\varepsilon) \end{pmatrix} \big(\tilde{\varphi}_i(X)/\varphi_i(X)\big), \\
\widetilde{\mathbf{g}}_j =& -(\varepsilon/2S_T) \begin{pmatrix} y - y_i & -(y - y_k) \\ -(x - x_i) & x - x_k \end{pmatrix} \begin{pmatrix} B(\widetilde{\mathbf{b}}^t (X - X_k)/\varepsilon) \\ B(\widetilde{\mathbf{b}}^t (X - X_i)/\varepsilon) \end{pmatrix} \big(\tilde{\varphi}_j(X)/\varphi_j(X)\big), && \text{(2.11b)} \\
\widetilde{\mathbf{g}}_k =& -(\varepsilon/2S_T) \begin{pmatrix} y - y_j & -(y - y_i) \\ -(x - x_j) & x - x_i \end{pmatrix} \begin{pmatrix} B(\widetilde{\mathbf{b}}^t (X - X_i)/\varepsilon) \\ B(\widetilde{\mathbf{b}}^t (X - X_j)/\varepsilon) \end{pmatrix} \big(\tilde{\varphi}_k(X)/\varphi_k(X)\big). && \text{(2.11c)}
\end{aligned}$$

In the above definitions, S_T is the measure of the element T. It can be shown that fluxes and their approximations satisfy the following [12]:

$$\widetilde{\mathbf{g}}(\tilde{\varphi}_i) + \widetilde{\mathbf{g}}(\tilde{\varphi}_j) + \widetilde{\mathbf{g}}(\tilde{\varphi}_k) = \widetilde{\mathbf{b}}, \tag{2.12a}$$

$$\widetilde{\mathbf{g}}_i + \widetilde{\mathbf{g}}_j + \widetilde{\mathbf{g}}_k = \widetilde{\mathbf{b}}. \tag{2.12b}$$

3. The Galerkin Finite Element Formulation

In this section, we consider the weighted basis finite element method on a triangular mesh with the refinement in boundary layers. Let

$$\delta_1 = \frac{\beta}{\underline{b}_1} \varepsilon |\ln \varepsilon|, \qquad \delta_2 = \frac{\beta}{\underline{b}_2} \varepsilon |\ln \varepsilon|, \tag{3.1}$$

where $\beta \ge 2$ is a constant. As shown in Fig. 3.1, we divide the region Ω into four subregions $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ given respectively as

$$\begin{aligned}
\Omega_1 =& (0, 1 - \delta_1) \times (0, 1 - \delta_2), \quad \Omega_2 = (0, 1 - \delta_1) \times (1 - \delta_2, 1), \\
\Omega_3 =& (1 - \delta_1, 1) \times (1 - \delta_2, 1), \quad \Omega_4 = (1 - \delta_1, 1) \times (0, 1 - \delta_2).
\end{aligned}$$

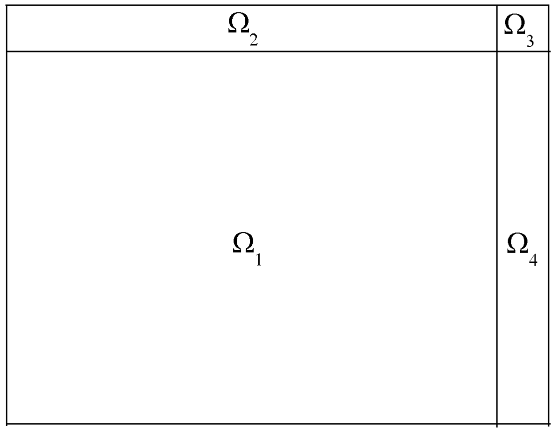

Fig. 3.1. Ω and its subregions.

The region Ω is triangulated as in Fig. 3.1. The subregions $\overline{\Omega}_1$ and $\overline{\Omega}_2 \cup \overline{\Omega}_3 \cup \overline{\Omega}_4$ are triangulated separately. We assume that the triangulation of $\overline{\Omega}_1$with the mesh parameter h is regular. Let (N_1+1) be the number of nodes on $\overline{\Omega}_1 \cap \overline{\Omega}_2$. Then $x-$direction subinterval $[1-\delta_1, 1]$ of $\overline{\Omega}_4$ is partitioned into N_1 mesh intervals by inverting the function $\exp(-\underline{b}_1(1-x)/(\beta\varepsilon))$. We specify the x_i in $\overline{\Omega}_4$, for $i = N_1, \cdots, 2N_1$, so that $\{\exp(-\underline{b}_1(1-x)/(\beta\varepsilon))\}_i$ is a linear function in i, i.e., we set

$$\exp\big(-\underline{b}_1(1-x_i)/(\beta\varepsilon)\big) = Ai + D$$

and choosing the unknowns A and D so that $x_{N_1} = 1-\delta_1$ and $x_{2N_1} = 1$. This gives

$$x_i = 1 + \frac{\beta}{\underline{b}_1}\varepsilon \ln\left(\frac{1-e^{-1}}{N} i + 2e^{-1} - 1\right), \qquad i = N_1+1, \cdots, 2N_1.$$

An analogous formula can be given for the mesh points y_j in subinterval $[1-\delta_2, 1]$ of $\overline{\Omega}_2$. To triangulate the L-shaped subregions $\overline{\Omega}_2 \cup \overline{\Omega}_3 \cup \overline{\Omega}_4$, we first divide it into rectangles using lines $x = x_i$ or $y = y_j$ parallel or perpendicular to one of the axes. Note that, in this partition, the $y-$coordinates of the latitude lines in $\overline{\Omega}_2$ and the $x-$coordinates of the longitude lines in $\overline{\Omega}_4$ are determined by the mesh nodes of the triangulation for $\overline{\Omega}_1$ on the boundary segments $\overline{\Omega}_1 \cap \overline{\Omega}_2$ and $\overline{\Omega}_1 \cap \overline{\Omega}_4$. As shown in Fig. 3.2, each of the rectangles is then divided into two triangles by one of its diagonals. The triangulations for $\overline{\Omega}_1$ and $\overline{\Omega}_2 \cup \overline{\Omega}_3 \cup \overline{\Omega}_4$ form the mesh T_h on Ω. This global triangulation satisfies that it is regular on $\overline{\Omega}_1$ and $\overline{\Omega}_3$ and it contains long, thin triangles on $\overline{\Omega}_2$ and $\overline{\Omega}_4$. A typical case is displayed in Fig. 3.2. Moreover, The triangular refinement in boundary layers $\overline{\Omega}_2 \cup \overline{\Omega}_3 \cup \overline{\Omega}_4$ must be of Bakhvalov-type such that the projection of the diameter of any triangle in $\overline{\Omega}_2 \cup \overline{\Omega}_3$ onto the $y-$direction is $\mathcal{O}(\varepsilon h|\ln\varepsilon|)$, and the projection of the diameter of any triangle in $\overline{\Omega}_3 \cup \overline{\Omega}_4$ onto the $x-$direction is $\mathcal{O}(\varepsilon h|\ln\varepsilon|)$.

As the width δ_2 of $\Omega_2 \cup \Omega_3$ is defined in (3.1), the projection of the diameter of any triangle in $\Omega_2 \cup \Omega_3$ onto the $y-$direction is smaller than δ_2. Similarly, the projections of the diameters of triangles in $\Omega_4 \cup \Omega_3$ onto the $x-$direction are smaller than δ_1.

Although the weighted basis finite element method adopted in [12] can deal with boundary layers well, it costs more CPU time than the standard finite element method in smooth solution

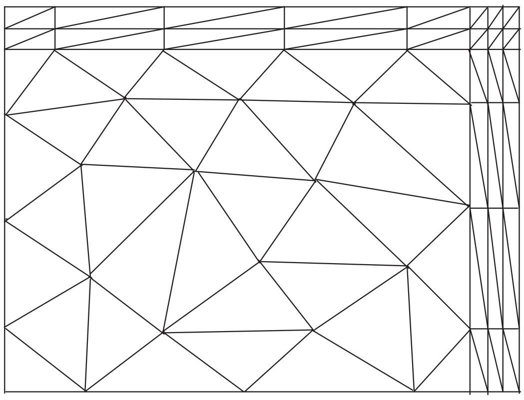

Fig. 3.2. A sample mesh with boundary layer refinement.

subregions. An efficient method is the combination of the two methods. This strategy is carried out by giving different weights $m_i(X)$ in (2.7) and choosing different $\widetilde{\mathbf{b}}(X)$ in the four subregions.

$$\widetilde{\mathbf{b}}(X) = 0, \qquad \text{if } X \in \overline{\Omega}_1, \tag{3.2a}$$
$$\widetilde{\mathbf{b}}(X) = (0, b_2(X))^t, \qquad \text{if } X \in \overline{\Omega}_2, \tag{3.2b}$$
$$\widetilde{\mathbf{b}}(X) = \mathbf{b}(X), \qquad \text{if } X \in \overline{\Omega}_3, \tag{3.2c}$$
$$\widetilde{\mathbf{b}}(X) = (b_1(X), 0)^t, \qquad \text{if } X \in \overline{\Omega}_4. \tag{3.2d}$$

Lemma 3.1. *The basis functions* (2.8) *with weights defined by* $\widetilde{\mathbf{b}}(X)$ *in* (3.2) *are continuous.*

Proof. If the node X_i is in $\Omega_1, \Omega_2, \Omega_3$ or Ω_4, it is easy to see that the basis function $\widetilde{\varphi}_i(X)$ defined by (2.8) with the weights (3.2) is continuous. Therefore, only the case for node X_i on the interface between different subregions is shown. Without loss of generality, we consider that X_i is on the intersection between $\overline{\Omega}_2$and $\overline{\Omega}_3$. Let triangle T_1 with vertices $X_i, X_j, X_{k_1}$and T_2 with vertices X_i, X_{k_2}, X_j belong to $\overline{\Omega}_2$ and $\overline{\Omega}_3$, respectively. The two triangles have a common edge $\overline{X_iX_j}$. From (2.8) and (3.2) we have

$$\widetilde{\varphi}_i(X)\big|_{T_1} = \frac{B\big(b_2(y-y_i)/\varepsilon\big)\varphi_i}{B\big(b_2(y-y_i)/\varepsilon\big)\varphi_i + B\big(b_2(y-y_j)/\varepsilon\big)\varphi_j + B\big(b_2(y-y_{k_1})/\varepsilon\big)\varphi_{k_1}},$$
$$\widetilde{\varphi}_i(X)\big|_{T_2} = \frac{B\big(\mathbf{b}\cdot(X-X_i)/\varepsilon\big)\varphi_i}{B\big(\mathbf{b}\cdot(X-X_i)/\varepsilon\big)\varphi_i + B\big(\mathbf{b}\cdot(X-X_j)/\varepsilon\big)\varphi_j + B\big(\mathbf{b}\cdot(X-X_{k_2})/\varepsilon\big)\varphi_{k_2}}.$$

If $X \in \overline{X_iX_j}$, then

$$\varphi_{k_1}(X) = \varphi_{k_2}(X) = 0,$$
$$\mathbf{b}\cdot(X - X_l) = b_2(y - y_l), \qquad l = i, j.$$

We then have

$$\widetilde{\varphi}(X)\big|_{T_1} = \widetilde{\varphi}(X)\big|_{T_2}, \qquad \text{if } X \in \overline{X_iX_j}.$$

If $X \notin \overline{X_iX_j}$, then $\widetilde{\varphi}(X)$ is continuous at X due to the continuity of $\widetilde{\varphi}(X)|_{T_1}$and $\widetilde{\varphi}(X)|_{T_2}$. □

Remark 3.1. Although different weights are adopted in different subregions, all weighted basis functions are continuous. Therefore, our finite element method is still conforming.

Replacing $\widetilde{\mathbf{b}}$ by $\mathbf{b}$ in (2.10), the flux corresponding to u and $\mathbf{b}$ is denoted by $\mathbf{g}$, i.e.,

$$\mathbf{g}(u) = -\varepsilon\nabla u + \mathbf{b}u. \tag{3.3}$$

Let

$$\overline{\mathbf{g}}_l = \widetilde{\mathbf{g}}_l + (\mathbf{b}-\widetilde{\mathbf{b}})\widetilde{\varphi}_l, \qquad l = i, j, k. \tag{3.4}$$

Then $\overline{\mathbf{g}}_l$ is also regarded as an approximation to $\mathbf{g}(\widetilde{\varphi}_l)$. By $\widetilde{\mathbf{g}}_l$ defined by (2.11a)-(2.11c) we have

Lemma 3.2. *The flux* $\mathbf{g}(\widetilde{\varphi}_l)$ *and its approximation* $\overline{\mathbf{g}}_l$ *satisfy*

$$|\mathbf{g}(\widetilde{\varphi}_l)-\overline{\mathbf{g}}_l| \le Ch_T, \qquad l = i, j, k, \tag{3.5}$$

where h_T *the shortest edge of the element* T.

Proof. If $X_l \in \overline{\Omega}_1$ and $X \in \Omega_1$, then $\widetilde{\mathbf{b}} = 0$ and $\widetilde{\varphi}_l$ is reduced to φ_l. By computation we get $\widetilde{\mathbf{g}}_l = -\varepsilon\nabla\varphi_l = -\varepsilon\nabla\widetilde{\varphi}_l$. So $\mathbf{g}(\widetilde{\varphi}_l) = \overline{\mathbf{g}}_l$, i.e. the inequality (3.5) holds.

If $X_l \in \overline{\Omega}_3$ and $X \in \Omega_3$, then $\widetilde{\mathbf{b}} = \mathbf{b}$ and $\overline{\mathbf{g}}_l = \widetilde{\mathbf{g}}_l$. Following Theorem 4.3 of Li et al. [12], we have $|\mathbf{g}(\widetilde{\varphi}_l)-\overline{\mathbf{g}}_l| \le Ch_T$.

Furthermore, for the case in Ω_2, we have $\widetilde{\mathbf{b}} = (0,b_2)^t$. By the definition of flux $\mathbf{g}(\widetilde{\varphi}_l)$ and $\overline{\mathbf{g}}_l$ in (3.4), we get

$$\begin{aligned}\big|\mathbf{g}(\widetilde{\varphi}_l)-\overline{\mathbf{g}}_l\big| &= \big|(-\varepsilon\nabla\widetilde{\varphi}_l + \mathbf{b}\widetilde{\varphi}_l) - (\widetilde{\mathbf{g}}_l + (\mathbf{b}-\widetilde{\mathbf{b}})\widetilde{\varphi}_l)\big| \\ &= \big|(-\varepsilon\nabla\widetilde{\varphi}_l + \widetilde{\mathbf{b}}\widetilde{\varphi}_l) - \widetilde{\mathbf{g}}_l\big| \\ &= \big|\widetilde{\mathbf{g}}(\widetilde{\varphi}_l)-\widetilde{\mathbf{g}}_l\big|.\end{aligned}$$

It follows from Theorem 4.3 in [12], that

$$\big|\mathbf{g}(\widetilde{\varphi}_l)-\overline{\mathbf{g}}_l\big| \le \big|\widetilde{\mathbf{g}}(\widetilde{\varphi}_l)-\widetilde{\mathbf{g}}_l\big| \le Ch_T.$$

For the case in Ω_4, its proof is similar. This completes the proof of Lemma 3.2. □

Moreover, by (2.8) and (2.12) we can get

$$\mathbf{g}(\widetilde{\varphi}_i) + \mathbf{g}(\widetilde{\varphi}_j) + \mathbf{g}(\widetilde{\varphi}_k) = \mathbf{b}, \tag{3.6a}$$

$$\overline{\mathbf{g}}_i + \overline{\mathbf{g}}_j + \overline{\mathbf{g}}_k = \mathbf{b}. \tag{3.6b}$$

Let T_h denote a triangular mesh on Ω. The set of vertices of T_h not on $\partial\Omega$ is denoted by $\{X_i\}_1^N$. Corresponding to the partition T_h, the finite element space is $V_h = span\{\widetilde{\varphi}_1, \widetilde{\varphi}_2, \cdots, \widetilde{\varphi}_N\} \subset H_0^1(\Omega)$. The finite element method corresponding to (2.1)-(2.2) is define as follows.

Problem 3.1. *Find a* $v_h \in V_h$ *such that for any* $w_h \in V_h$,

$$A(v_h, w_h) = (\mathbf{g}(v_h), \nabla w_h) + (\mu(X)v_h, w_h) = (f, w_h), \tag{3.7}$$

where $A(\cdot,\cdot)$ *is the same bilinear form defined in* (2.6).

The following assumption is needed for error estimates.

Assumption 3.1. *The solution v to problem* (2.1)-(2.2) *can be decomposed into four parts v_l ($l = 1, 2, 3, 4$), i.e.,*

$$v = v_1 + v_2 + v_3 + v_4, \tag{3.8}$$

where v_1 satisfies

$$||v_1||_{k,\infty,\Omega} \le C, \qquad \text{for } k = 0, 1, 2, \tag{3.9}$$

and v_l ($l = 2, 3, 4$) satisfy

$$\left|\frac{\partial^{i+j} v_2}{\partial x^i \partial y^j}\right| \le C\varepsilon^{-j} \exp\left(-\frac{\underline{b}_2(1-y)}{\varepsilon}\right), \tag{3.10}$$

$$\left|\frac{\partial^{i+j} v_3}{\partial x^i \partial y^j}\right| \le C\varepsilon^{-i-j} \exp\left(-\frac{\underline{b}_1(1-x)}{\varepsilon}\right) \exp\left(-\frac{\underline{b}_2(1-y)}{\varepsilon}\right), \tag{3.11}$$

$$\left|\frac{\partial^{i+j} v_4}{\partial x^i \partial y^j}\right| \le C\varepsilon^{-i} \exp\left(-\frac{\underline{b}_1(1-x)}{\varepsilon}\right), \tag{3.12}$$

for $0 \le i + j \le 2$.

In the above assumption, v_1 is globally smooth and uniformly bounded in Ω, while v_2, v_3 and v_4 contain boundary layers in Ω_2, Ω_3 and Ω_4, respectively. Sufficient conditions for the existence of this decomposition have been discussed in many literatures [6, 11, 15, 17, 22, 25]. The following lemma shows that v and all its first and second partial derivatives are uniformly bounded in Ω_1.

Lemma 3.3. *if $\beta \ge 2$, then the solution v to* (2.1)-(2.2) *satisfies*

$$||v_l||_{k,\infty,\Omega_1} \le C, \qquad l = 1, 2, 3, 4, \quad k = 0, 1, 2; \tag{3.13}$$

$$||v||_{k,\Omega_1} \le C, \qquad k = 0, 1, 2. \tag{3.14}$$

Proof. The proof of this lemma follows directly from (3.9)-(3.12) in Assumption 3.1. □

4. Error Estimates

Let $v_l = v(X_l)$ and v^I be the V_h-interpolation of the exact solution v to the problem (2.1)-(2.2), i.e.,

$$v^I(X) = \sum_{l=1}^{N} v_l \widetilde{\varphi}_l(X).$$

Then we have

$$A(v - v_h, v - v_h) = A(v - v_h, v - v^I) + A(v - v_h, v^I - v_h). \tag{4.1}$$

Furthermore, because v and v_h satisfy the variational problems (2.5) and (3.7) respectively, one can get the following statement

$$A(v - v_h, v^I - v_h) = 0$$

by noting the fact $(v^I - v_h) \in V_h$. So the term $A(v-v_h, v-v^I)$ in (4.1) is needed to estimate. For the first term on the right-hand side (RHS) of (4.1), we have

$$\begin{aligned} A(v-v_h, v-v^I) =&(\varepsilon\nabla(v-v_h), \nabla(v-v^I)) - (\mathbf{b}(v-v_h),\nabla(v-v^I)) \\ &+ (\mu(X)(v-v_h), (v-v^I)) \\ =&(\varepsilon\nabla(v-v_h), \nabla(v-v^I)) + (\mathbf{b}(v-v^I), \\ &\nabla(v-v_h)) + ((\mu(X)+\nabla\cdot\mathbf{b})(v-v_h), (v-v^I)). \end{aligned} \tag{4.2}$$

To obtain upper error bounds on the RHS of (4.2), we first analyze the error between v and its interpolation v^I which is given in the following lemma.

Lemma 4.1. *Let v be the solution to* (2.1) *and v^I be the interpolation of v in V_h. Then*

$$||v(X) - v^I(X)||_{L^\infty, \overline{\Omega}_1^c} \le Ch|\ln\varepsilon|, \tag{4.3}$$

where $\overline{\Omega}_1^c = \overline{\Omega}_2 \cup \overline{\Omega}_3 \cup \overline{\Omega}_4$.

Proof. Without loss of generality, we assume that X belongs to a triangle T in Ω_2. The proofs for other cases are similar. Let X_i, X_j and X_k denote the three vertices of the triangle T, as depicted in Fig. 4.1, we have

$$\begin{aligned} |v(X) - v^I(X)| \le&|v(X) - v_i| + |v_i - v^I(X)| \\ =&|v(X) - v_i| + |(v_j - v_i)\widetilde{\varphi}_j| + |(v_k - v_i)\widetilde{\varphi}_k|. \end{aligned}$$

By the properties of weighted basis functions, we know that $0 \le \widetilde{\varphi}_l \le 1 (l = i, j, k)$. Therefore, we have from the above inequality

$$|v(X) - v^I(X)| \le |v(X) - v_i| + |(v_j - v_i)| + |(v_k - v_i)|. \tag{4.4}$$

As shown in Fig. 4.1, we assume, without loss of generality, that X_iX_j is the horizontal edge in T let X' denote the intersection of X_iX_j and its perpendicular passing through X. By Assumption 3.1, we have

$$\begin{aligned} |v(X) - v_i| \le&|v(X) - v(X')| + |v(X') - v_i| \\ =&\left|\int_{X'X} \frac{\partial v}{\partial y} dy\right| + \left|\int_{X_iX'} \frac{\partial v}{\partial x} dx\right| \\ \le&C\Big(|X'X|/\varepsilon + |X_iX'|\Big) \le Ch|\ln\varepsilon|. \end{aligned} \tag{4.5}$$

Applying the above result to the two special cases when $X = X_j$ and $X = X_k$, we obtain

$$|v_j - v_i| \le Ch, \tag{4.6}$$

$$|v_k - v_i| \le Ch|\ln\varepsilon|. \tag{4.7}$$

Substituting (4.5)-(4.7) into (4.4), we get the inequality (4.3). This competes the proof of the lemma. □

Lemma 4.2. *If $\beta \ge 2$, then v and v^I satisfy the following*

$$||v(X) - v^I(X)||_{0,\Omega_1} \le Ch^2, \tag{4.8}$$

$$||v(X) - v^I(X)||_{0,\overline{\Omega}_1^c} \le Ch^2\varepsilon|\ln\varepsilon|^3. \tag{4.9}$$

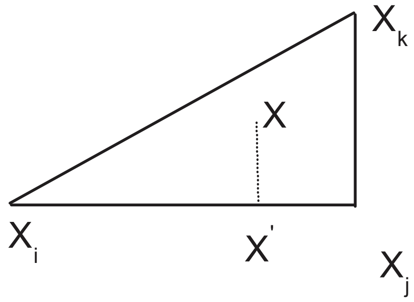

Fig. 4.1. The projection of X.

Proof. By Bramble-Hilbert lemma and Lemma 3.3, one gets (4.8). To show (4.9), we consider the element T in Ω_3. Let X_i, X_j and X_k denote the vertices of the triangle T. For X in T, we have

$$||v(X) - v^I(X)||_{0,T} \leq ||I_1||_{0,T} + ||I_2||_{0,T}, \tag{4.10}$$

where I_1 and I_2 are defined as

$$\begin{aligned} I_1 =& v(X) - (v_i\varphi_i + v_j\varphi_j + v_k\varphi_k), \\ I_2 =& [v_i(\varphi_i - \widetilde{\varphi}_i) + v_j(\varphi_j - \widetilde{\varphi}_j) + v_k(\varphi_k - \widetilde{\varphi}_k)]. \end{aligned}$$

For I_1, we have

$$\begin{aligned} I_1 =& v(X) - (v_i\varphi_i + v_j\varphi_j + v_k\varphi_k) \\ =& \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\bigg|_{X=\xi}(x-x_i)^2 + \frac{\partial^2 v}{\partial x \partial y}\bigg|_{X=\xi}(x-x_i)(y-y_i) + \frac{1}{2}\frac{\partial^2 v}{\partial y^2}\bigg|_{X=\xi}(y-y_i)^2, \end{aligned}$$

where $\xi \in T$. Using Assumption 3.1 and the fact $|X - X_i| \leq \varepsilon h |\ln \varepsilon|$, one gets

$$||I_1||_{0,T} \leq Ch^2 \ln^2 \varepsilon \sqrt{S_T}, \tag{4.11}$$

where S_T is the measure of T and $S_T \leq Ch^2\varepsilon^2 \ln^2 \varepsilon$ if $T \in \Omega_3$.

For I_2, using the definition of the weighted basis functions in (2.8) one can obtain

$$I_2 = \frac{(m_i - m_j)\varphi_i\varphi_j(v_i - v_j) + (m_j - m_k)\varphi_j\varphi_k(v_j - v_k) + (m_k - m_i)\varphi_k\varphi_i(v_k - v_i)}{m_i\varphi_i + m_j\varphi_j + m_k\varphi_k}.$$

Noting that $|m_p - m_q| \leq Ch|\ln \varepsilon|$, $|v_p - v_q| \leq Ch|\ln \varepsilon|$ $(p, q = i, j, k)$, we have

$$|I_2| \leq Ch^2 \ln^2 \varepsilon.$$

Furthermore,

$$||I_2||_{0,T} \leq Ch^2 \ln^2 \varepsilon \sqrt{S_T}, \tag{4.12}$$

Substituting (4.11)-(4.12) into (4.10) , one obtains

$$||v - v^I||_{0,T} \leq Ch^2 \ln^2 \varepsilon \sqrt{S_T}.$$

Then we have

$$||v - v^I||^2_{0,\Omega_3} = \sum_{T\in\Omega_3} ||v - v^I||^2_{0,T} \leq Ch^4 \ln^4 \varepsilon \sum_{T\in\Omega_3} S_T \leq Ch^4\varepsilon^2 |\ln \varepsilon|^6,$$

which implies

$$||v-v^I||_{0,\Omega_3} \le Ch^2\varepsilon|\ln\varepsilon|^3.$$

It is an analogue to show that

$$||v-v^I||_{0,\Omega_l} \le Ch^2\varepsilon|\ln\varepsilon|^3, \quad l=2,4.$$

Following the above two inequalities, we get the inequality (4.9). □

If $\varepsilon \ll 1$, then $\varepsilon|\ln\varepsilon|^3 < C$. In this case, so (4.9) becomes

$$||v-v^I||_{0,\overline{\Omega}_1^c} \le Ch^2. \tag{4.13}$$

By Lemmas 4.1 and 4.2, the interpolation error estimate of $v-v^I$ in the energy norm is obtained as follows.

Theorem 4.1. *The interpolation error $v-v^I$ satisfies*

$$||v-v^I||_\varepsilon^2 \le Ch\Big(h+\varepsilon|\ln\varepsilon|\Big). \tag{4.14}$$

Proof. Considering $(v-v^I)\in H_0^1$, we have

$$\begin{aligned}||v-v^I||_\varepsilon^2 =&\Big(\Big(\nabla\big(-\varepsilon\nabla(v-v^I)+\mathbf{b}(v-v^I)\big)+\mu(X)(v-v^I)\Big),\ (v-v^I)\Big)\\ =&\big(f,(v-v^I)\big)-\big(\nabla\cdot\mathbf{g}(v^I),(v-v^I)\big)-\big(\mu(X)v^I,\ (v-v^I)\big),\end{aligned} \tag{4.15}$$

where the flux $\mathbf{g}(v^I)$ is defined by (3.3).

Considering that f and $\mu(X)v^I$ are continuous and uniformly bounded, by Lemma 4.2 and the inequality (4.13) one gets

$$|(f,(v-v^I))_{\Omega_i}| \le Ch^2, \qquad (i=1,2,3,4), \tag{4.16}$$

$$|(\mu(X)v^I,(v-v^I))_{\Omega_i}| \le Ch^2, \quad (i=1,2,3,4),$$

$$|(\nabla\cdot\mathbf{g}(v^I),(v-v^I))_{\Omega_1}| \le Ch^2. \tag{4.17}$$

Furthermore, the flux $\mathbf{g}(v^I)$ in Ω_3 can be decomposed into two parts $\overline{\mathbf{g}}(v^I)$ and $\mathbf{R}(v^I)$. Due to equalities (3.6a)-(3.6b) and the fact that $v^I|_T = v_i\widetilde{\varphi}_i+v_j\widetilde{\varphi}_j+v_k\widetilde{\varphi}_k$, $\overline{\mathbf{g}}(v^I)$ in an element T with vertices X_i, X_j and X_k can be written as

$$\begin{aligned}\overline{\mathbf{g}}(v^I)|_T =&v_i\overline{\mathbf{g}}_i+v_j\overline{\mathbf{g}}_j+v_k\overline{\mathbf{g}}_k\\ =&v_i\mathbf{b}+(v_j-v_i)\overline{\mathbf{g}}_j+(v_k-v_i)\overline{\mathbf{g}}_k.\end{aligned} \tag{4.18}$$

Let

$$\mathbf{R}(v^I)=\mathbf{g}(v^I)-\overline{\mathbf{g}}(v^I).$$

Then, by Lemma 3.2 it satisfies

$$\mathbf{R}(v^I)(X_l)=0, \qquad l=i,j,k, \tag{4.19}$$

$$||\mathbf{R}(v^I)||_{0,\Omega_3} \le Ch\varepsilon^2|\ln\varepsilon|^2.$$

Combining (4.18) and (4.19), we have

$$\begin{aligned}
&|(\nabla\cdot\mathbf{g}(v^I),(v-v^I))_{\Omega_3}|\\
=&|(\nabla\cdot\overline{\mathbf{g}}(v^I),(v-v^I))_{\Omega_3}+(\nabla\cdot\mathbf{R}(v^I),(v-v^I))_{\Omega_3}|\\
\le&|(\nabla\cdot\overline{\mathbf{g}}(v^I),(v-v^I))_{\Omega_3}+(\mathbf{R}(v^I),\nabla(v-v^I))_{\Omega_3}|+Ch\varepsilon|\ln\varepsilon|\\
\le&\sum_{T\in\Omega_3}\Big(||\nabla\cdot(v_i\mathbf{b})||_{0,T}+||(v_j-v_i)\nabla\cdot(\overline{\mathbf{g}}_j)||_{0,T}+||(v_k-v_i)\nabla\cdot(\overline{\mathbf{g}}_k)||_{0,T}\Big)\\
&\cdot||v-v^I||_{0,\Omega_3}+||\mathbf{R}(v^I)||_{0,\Omega_3}||\nabla(v-v^I)||_{0,\Omega_3}+Ch\varepsilon|\ln\varepsilon|\\
\le&\sum_{T\in\Omega_3}\Big(||\nabla\cdot(v_i\mathbf{b})||_{0,T}+||(v_j-v_i)\nabla\cdot(\overline{\mathbf{g}}_j)||_{0,T}+||(v_k-v_i)\nabla\cdot(\overline{\mathbf{g}}_k)||_{0,T}\Big)\\
&\cdot||v-v^I||_{0,\Omega_3}+Ch\varepsilon|\ln\varepsilon|.
\end{aligned}$$

In the above deduction, we have used the fact that $||\nabla(v-v^I)||_{0,\Omega_3}$ is bounded, see [15, Theorem 3.2]. By Lemma 4.1 and direct computation, one can verify that $\nabla\cdot(v_i\mathbf{b})$,$(v_j-v_i)\nabla\cdot(\overline{\mathbf{g}}_j)$ and $(v_k-v_i)\nabla\cdot(\overline{\mathbf{g}}_k)$ are bounded. Therefore, following Lemma 4.2 we have

$$|(\nabla\cdot\mathbf{g}(v^I),(v-v^I))_{\Omega_3}|\le Ch\Big(h+\varepsilon|\ln\varepsilon|\Big). \tag{4.20}$$

Similarly, we can show that

$$\big|(\nabla\cdot\mathbf{g}(v^I),(v-v^I))_{\Omega_l}\big|\le Ch\Big(h+\varepsilon|\ln\varepsilon|\Big),\qquad l=2,4. \tag{4.21}$$

Combining (4.17), (4.20) and (4.21), we have

$$\big|(\nabla\cdot\mathbf{g}(v^I),(v-v^I))\big|_{\Omega}\le Ch\Big(h+\varepsilon|\ln\varepsilon|\Big). \tag{4.22}$$

Substituting (4.16) and (4.22) into (4.15), one obtains the estimate (4.14). □

By Theorem 4.1, the first term in (4.2) can be rewritten as

$$\big|\varepsilon(\nabla(v-v_h),\nabla(v-v^I))\big|\le Ch(h+\varepsilon|\ln\varepsilon|)+(\varepsilon/4)||v-v_h||_{1,\Omega}^2. \tag{4.23}$$

Then we continue the error analysis in (4.2) and turn to the convection term,

$$\begin{aligned}
&\big|(\mathbf{b}\cdot\nabla(v-v_h),(v-v^I))\big|\\
=&\big|(\mathbf{b}\cdot\nabla(v-v_h),(v-v^I))_{\Omega_1}\big|+\big|(\mathbf{b}\cdot\nabla(v-v_h),(v-v^I))_{\overline{\Omega}_1^c}\big|.
\end{aligned}\tag{4.24}$$

Considering that $\|\nabla(v-v_h)\|_{\Omega_1}$ is bounded in the smooth solution region Ω_1, one gets

$$|(\mathbf{b}\cdot\nabla(v-v_h),(v-v^I))_{\Omega_1}|\le C\,\|\nabla(v-v_h)\|_{\Omega_1}\left\|v-v^I\right\|_{\Omega_1}\le Ch^2. \tag{4.25}$$

Furthermore, we have

$$\begin{aligned}
&\big|(\mathbf{b}\cdot\nabla(v-v_h),(v-v^I))_{\overline{\Omega}_1^c}\big|\\
\le&C||v-v^I||_{L^\infty,\overline{\Omega}_1^c}||\nabla(v-v_h)||_{L^1,\overline{\Omega}_1^c}\\
\le&C||v-v^I||_{L^\infty,\overline{\Omega}_1^c}(\varepsilon|\ln\varepsilon|)^{1/2}||\nabla(v-v_h)||_{L^2,\overline{\Omega}_1^c}\\
\le&C|\ln\varepsilon|||v-v^I||^2_{L^\infty,\overline{\Omega}_1^c}+(\varepsilon/4)||\nabla(v-v_h)||^2_{L^2,\overline{\Omega}_1^c}\\
\le&C|\ln\varepsilon|||v-v^I||^2_{L^\infty,\overline{\Omega}_1^c}+(\varepsilon/4)||(v-v_h)||^2_{1,\overline{\Omega}}.
\end{aligned}\tag{4.26}$$

By Lemma 4.1, (4.26) can be rewritten as

$$\big|(\mathbf{b}\cdot\nabla(v-v_h),(v-v^I))_{\overline{\Omega}_1^c}\big| \le Ch^2|\ln\varepsilon|^3+(\varepsilon/4)||(v-v_h)||^2_{1,\overline{\Omega}}. \tag{4.27}$$

Substituting (4.27) and (4.25) into (4.24), we get

$$\big|(\mathbf{b}\cdot\nabla(v-v_h),(v-v^I))_{\Omega}\big| \le Ch^2|\ln\varepsilon|^3+(\varepsilon/4)||(v-v_h)||^2_{1,\overline{\Omega}}. \tag{4.28}$$

Furthermore, combining (4.28), (4.23) with (4.2) and using the condition (2.3), we obtain

$$\begin{aligned}
&\big|A(v-v_h,v-v^I)\big|\\
\le& Ch(h+\varepsilon|\ln\varepsilon|+h|\ln\varepsilon|^3)+\frac{\varepsilon}{2}||(v-v_h)||^2_{1,\overline{\Omega}}+|(\mu+\nabla\cdot\mathbf{b})(v-v_h),(v-v^I)|\\
\le& Ch(h+\varepsilon|\ln\varepsilon|+h|\ln\varepsilon|^3)+\frac{\varepsilon}{2}||(v-v_h)||^2_{1,\overline{\Omega}}+\frac{\alpha}{2}||v-v_h||^2_{0,\overline{\Omega}}+C||v-v^I||^2_{0,\overline{\Omega}}\\
\le& Ch(h+\varepsilon|\ln\varepsilon|+h|\ln\varepsilon|^3)+\frac{1}{2}||v-v_h||^2_{\varepsilon}+Ch^2.
\end{aligned}$$

Finally, by the equality (4.1) and the definition of energy norm $||\cdot||_\varepsilon$, we obtain

$$||v-v_h||^2_\varepsilon \le Ch^2\Big(1+|\ln\varepsilon|^3+\varepsilon|\ln\varepsilon|/h\Big).$$

Summarizing the above analysis, we have the following theorem which contains the main result of the error analysis.

Theorem 4.2. *Let v and v_h be the solutions to Problems* 2.1 *and* 3.1, *respectively. If v satisfies Assumption* 3.1, *then v and v_h satisfy*

$$||v-v_h||_\varepsilon \le Ch\Big(1+|\ln\varepsilon|^3+\varepsilon|\ln\varepsilon|/h\Big)^{1/2}.$$

As can be seen, the upper error bound in Theorem 4.2 depends very weakly on ε. In terms of computation, $|\ln\varepsilon|$ can be approximately treated as a bounded quantity. For example, when $\varepsilon=10^{-15}$, $|\ln\varepsilon|^{3/2}<(34.6)^{3/2}$. Therefore, the theorem implies essentially that the error of $v-v_h$ is almost ε-uniformly bounded.

5. Numerical Results

To demonstrate the theoretical results, numerical experiments on two examples have been performed.

Example 5.1. Let us consider the two-dimensional convection-dominated problem defined by

$$\begin{aligned}
&\frac{\partial}{\partial x}\big(-\varepsilon v_x+(3-x)v\big)+\frac{\partial}{\partial y}\big(-\varepsilon v_y+(4-2y+y^2)v\big)+(4-2y)v=f(x,y),\\
&v\big|_{\partial\Omega}=0,
\end{aligned}$$

where $\Omega=(0,1)\times(0,1)$ and

$$f(x,y)=\frac{3}{2}\pi\cos\frac{\pi x}{2}+y^3\sin\frac{\pi x}{2}-\frac{\pi x}{2}\cos\frac{\pi x}{2}+12y^2-6y^3+3y^4.$$

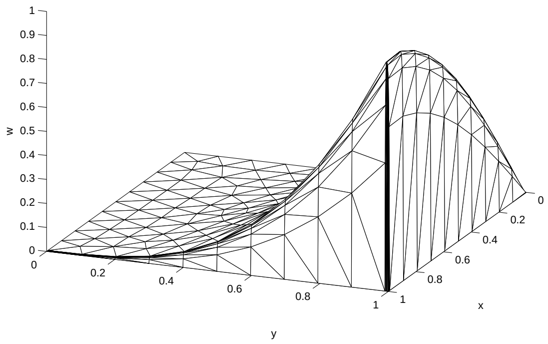

Fig. 5.1. The numerical solution of Example 5.1.

In our computation, we choose $\varepsilon = 10^{-3}$. The triangulation with refinement is similar to that in Fig. 3.2. In this mesh, there are 275 nodes and 488 elements, of which 135 nodes and 250 elements are used to refine the subregion Ω_1^c. We solve this example by the weighted basis finite element method and the numerical solution is depicted in Fig. 5.1.

Example 5.2. Consider the following advection-diffusion problem with the boundary condition $v|_{\partial\Omega} = 0$.

$$-\nabla \cdot (\varepsilon \nabla v - \mathbf{b} v) + 2v = f(x,y), \qquad \text{in } \Omega = (0,1) \times (0,1),$$

where $\mathbf{b} = (1,1)^t$, and

$$\begin{aligned} f(x,y) =& x\big(1 - e^{(x-1)/\varepsilon}\big)\Big(1 + e^{(y-1)/\varepsilon} + y\big(1 - e^{(y-1)/\varepsilon}\big)\Big) \\ &+ y\big(1 - e^{(y-1)/\varepsilon}\big)\Big(1 + e^{(x-1)/\varepsilon} + x\big(1 - e^{(x-1)/\varepsilon}\big)\Big). \end{aligned}$$

Its exact solution is

$$v(x,y) = xy\big(1 - e^{(x-1)/\varepsilon}\big)\big(1 - e^{(y-1)/\varepsilon}\big).$$

In this example, the triangulation of the computational region with the refinement is also similar to that in Fig. 3.2. The errors in energy norm for different values of ε are listed in Table 5.1 from which one can see the convergent rate of this finite element is about one.

Table 5.1: Computed errors for Example 5.2.

Error	$h = 1/5$	$h = 1/10$	$h = 1/20$	$h = 1/40$
$\varepsilon = 0.01$	0.1975	0.0719	0.0375	0.0189
$\varepsilon = 0.001$	0.2378	0.0881	0.0461	0.0247
$\varepsilon = 0.0001$	0.2847	0.1054	0.0550	0.0273

6. Conclusion

In this work, we presented an error analysis for a weighted basis finite element method on the triangular Bakhvalov-type mesh applied to a two-dimensional singularly perturbed problem.

This method is based on choosing different weights in the smooth solution domain and the boundary layers. The error bound of order $\mathcal{O}(h)$ is obtained which is almost independent of the diffusion coefficient ε. Numerical results were presented to demonstrate the accuracy and convergence of the method.

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