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A NOTE ON THE NONCONFORMING FINITE ELEMENTS FOR ELLIPTIC PROBLEMS*

Boran Gao
School of Mathematical Sciences, Peking University, Beijing 100871, China
Email: ggggbr@163.com
Shuo Zhang
ICMSEC, AMSS, Chinese Academy of Science, Beijing 100190, PR China
Email: szhang@lsec.cc.ac.cn
Ming Wang
LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China
Email: mwang@math.pku.edu.cn

Abstract

In this paper, a class of rectangular finite elements for $2m$-th-oder elliptic boundary value problems in n-dimension ($m, n \geq 1$) is proposed in a canonical fashion, which includes the $(2m-1)$-th Hermite interpolation element ($n = 1$), the n-linear finite element ($m = 1$) and the Adini element ($m = 2$). A nonconforming triangular finite element for the plate bending problem, with convergent order $\mathcal{O}(h^2)$, is also proposed.

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1. Introduction

When the conforming finite element is used for numerically discretizing the elliptic problem, the convergence of the numerical solution to the exact solution depends on the approximation of the finite element space only. But the strong continuity requirement makes it difficult to construct such a conforming finite element. The idea of nonconforming finite element lies in that such difficulty can be overcome by loosing the request on the continuity. However, the loss of continuity will bring in the so-called consistent error, and some fundamental continuity of the finite element space is still necessary for well-posedness and convergence. This is the reason that most of the finite elements, conforming or nonconforming, were constructed case by case, depending on the order of the problem and sometimes the dimensions (cf. [1–3, 5, 7, 8, 12, 14, 15, 17]). A unified approach of constructing finite elements for general problems is still of theoretical and practical interest. Recently, a class of finite elements was discussed in a canonical fashion in [16], for all n-dimensional $2m$-th-order elliptic problem with $n \geq m \geq 1$. The well-known nonconforming linear element for the second-order problem and the Morley element for fourth-order problem are examples of this class. The class of finite elements is established on simplices, and makes use of the piecewise polynomials of the lowest degree. The nodal parameters are the natural ones to guarantee the fundamental continuity, and the consistency error can be controlled simultaneously.

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In this paper, we will discuss the choice of nodal parameters that can be used to construct nonconforming finite elements, with admissible consistency error. We will first propose a class of rectangular finite elements for n-dimensional $2m$-th-oder problems ($m, n \geq 1$) in a canonical fashion. The degrees of freedom are the values of function and all derivatives up to $(m-1)$-th-oder at all vertices of n-rectangle. The basic fundamental continuity is guaranteed and an $\mathcal{O}(h)$ convergence rate is shown. The $(2m-1)$-th Hermite interpolation element ($n = 1$), the n-linear finite element ($m = 1$) and the Adini element ($m = 2$) all belong to this class.

As almost all of the nonconforming finite elements are convergent in energy norm with order $\mathcal{O}(h)$, and the consistency error is the main limit, we will discuss the possibility of improving the convergence rate by strengthening the continuity of the finite element space. We choose the plate bending problem as an example. There have been successful attempts via other approaches, like conforming finite element, quasi-conforming finite elements (cf. [4,6,11]) and the double set parameter element (cf. [9]). But most nonconforming element for the plate bending problem, such as the Morley element [8], two Veubake elements [12], the NZT element [14], the rectangle Morley element (cf. [15]) and the Adini element (cf. [1]), are convergent with order $\mathcal{O}(h)$. In this work, a new nonconforming plate element will be given, with a convergence rate of $\mathcal{O}(h^2)$ in energy norm.

Finally, based on the new plate element, a new Zienkiewicz-type element will be deduced and reported for comparison. The new Zienkiewicz-type element is convergent for the plate bending problem with order $\mathcal{O}(h)$. Its consistent error is of order $\mathcal{O}(h^2)$ which is better than the two dimensional Zienkiewicz-type element proposed in [14]. In fact, the phenomenon that the consistency error can perform better than the approximation error has seldom been reported in literatures.

The paper is organized as follows. The rest of this section gives some basic notations. Section 2 gives the description of the class of rectangular finite elements. Section 3 gives the description of the new plate elements. Section 4 shows their convergence. Section 5 gives some numerical results for the new plate element.

Let n be a positive integer. Given a nonnegative integer k and a bounded domain $G \subset R^n$ with boundary ∂G, let $H^k(G)$, $H_0^k(G)$, $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$ denote the usual Sobolev spaces, norm and semi-norm respectively. Let $(\cdot,\cdot)$ denote the inner product of $L^2(\Omega)$.

We will use α, β, γ to denote n dimensional multi-indexes. Define

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

A finite element can be represented by a triple (T, P_T, D_T) with T the geometric shape, P_T the shape function space and D_T the vector of degrees of freedom, provided that D_T is P_T-unisolvent (see [5]).

Let Ω be a bounded polyhedron domain of R^n. For mesh size h with $h \to 0$, let $\mathcal{T}_h$ be a partition of Ω corresponding to a finite element (T, P_T, D_T), and let V_h, V_{h0} be the finite element spaces corresponding to the element and $\mathcal{T}_h$. Throughout this paper, we assume that $\{\mathcal{T}_h\}$ is shape regular.

For a subset $B \subset R^n$ and a nonnegative integer r, let $P_r(B)$ be the space of all polynomials defined on B with degree not greater than r, and $Q_r(B)$ the space of all polynomials with degree in each variable not greater than r.

2. A Class of Rectangular Finite Elements

Let m be a positive integer. This section is devoted to the rectangular finite element for the $2m$-th-oder elliptic boundary value problem in n-dimension.

Let T be an n-rectangle with each edge parallel to some coordinate axis respectively. Then there exist n positive numbers $h_1, h_2, \cdots, h_n$, such that,

$$T = \left\{ x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \;\middle|\; x_i = x_i^0 + \xi_i h_i, -1 \le \xi_i \le 1,\ 1 \le i \le n \right\}, \tag{2.1}$$

where x^0 is the center point of T. Define

$$\xi_i = \frac{1}{h_i}(x_i - x_i^0), \qquad 1 \le i \le n, \tag{2.2}$$

and set $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^{\mathrm{T}}$. Denote 2^n vertices of T by $a_j,\ 1 \le j \le 2^n$, and $(\xi_1, \xi_2, \cdots, \xi_n)^{\mathrm{T}}$ corresponding to a_j by $\Xi_j = (\xi_{1j}, \xi_{2j}, \cdots, \xi_{nj})^{\mathrm{T}}$.

For $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^{\mathrm{T}}$ and multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, define

$$\xi^\alpha = \prod_{i=1}^n \xi_i^{\alpha_i}, \qquad \alpha! = \prod_{i=1}^n \alpha_i!.$$

Set

$$\varphi_j = \frac{1}{2^n}\prod_{i=1}^n (1 + \xi_{ij}\xi_i), \qquad 1 \le j \le 2^n, \tag{2.3}$$

$$P_{T,m} = \ \mathrm{span}\left\{ \varphi_j \xi^{2\alpha} \;\middle|\; 1 \le j \le 2^n,\ |\alpha| < m \right\}. \tag{2.4}$$

Then $\varphi_j,\ 1 \le j \le 2^n$, form a basis of $Q_1(T)$, and

$$P_{T,m} = \ \mathrm{span}\left\{ p\,\xi^{2\alpha} \;\middle|\; p \in Q_1(T),\ |\alpha| < m \right\}. \tag{2.5}$$

The rectangular finite element of order m is defined by the triple (T, P_T, D_T) as follows,

1. T is the n-rectangle described by (2.1);
2. $P_T = P_{T,m}$;
3. the components of $D_T(v)$ for any $v \in C^{m-1}(T)$ are

$$\partial^\alpha v(a_j), \qquad |\alpha| < m,\ 1 \le j \le 2^n.$$

Lemma 2.1. *For the rectangular finite element of order m, D_T is P_T-unisolvent and $P_{2m-1}(T) \subset P_T$.*

Proof. It is obvious that the dimensions of D_T and P_T are all $2^n C_{n+m-1}^{m-1}$. For $1 \le j \le 2^n$, set

$$\varphi_{j,\alpha} = \frac{\Xi_j^\alpha}{\alpha! 2^{|\alpha|}}(\xi_1^2 - 1)^{\alpha_1}(\xi_2^2 - 1)^{\alpha_2} \cdots (\xi_n^2 - 1)^{\alpha_n}\varphi_j, \qquad |\alpha| < m. \tag{2.6}$$

Write the partial derivative with respect to ξ as

$$\partial_\xi^\alpha = \frac{\partial^{|\alpha|}}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n}}.$$

It can be verified that

$$\partial_\xi^\beta \varphi_{j,\alpha}(a_k) = \begin{cases} 1, & \beta = \alpha \quad \text{and} \quad j = k, \\ 0, & \text{otherwise}, \end{cases} \qquad 1 \le j, \quad k \le 2^n, \quad |\beta| \le |\alpha| < m. \tag{2.7}$$

Define

$$\psi_{j,\alpha} = \varphi_{j,\alpha} \tag{2.8}$$

when $|\alpha| = m - 1$, and

$$\psi_{j,\alpha} = \varphi_{j,\alpha} - \sum_{k=1}^{2^n} \sum_{|\beta|=|\alpha|+1}^{m-1} \partial_\xi^\beta \varphi_{j,\alpha}(a_k) \psi_{k,\beta} \tag{2.9}$$

when $|\alpha| < m - 1$. Then

$$\partial_\xi^\beta \psi_{j,\alpha}(a_k) = \begin{cases} 1, & \beta = \alpha \text{ and } j = k, \\ 0, & \text{otherwise}, \end{cases} \qquad 1 \le j, k \le 2^n, \; |\alpha|, |\beta| < m. \tag{2.10}$$

Therefore, $h_1^{\alpha_1} \cdots h_n^{\alpha_n} \psi_{j,\alpha}$ $(1 \le j \le 2^n, \; |\alpha| < m)$ are the basis functions corresponding to the degrees of freedom since $\partial_\xi^\alpha = h_1^{\alpha_1} \cdots h_n^{\alpha_n} \partial^\alpha$. Thus, we obtain that D_T is P_T-unisolvent.

Now we show that $P_{2m-1}(T) \subset P_{T,m}$. Let $p \in P_{2m-1}(T)$, then p can be written as

$$p = \sum_{|\alpha| \le 2m-1} C_\alpha x^\alpha,$$

with C_α constants. For term $C_\alpha x^\alpha$ with $|\alpha| \le 2m - 1$, define β and γ by

$$\beta_i = \begin{cases} \dfrac{\alpha_i}{2}, & \alpha_i \text{ is even}, \\ \dfrac{\alpha_i - 1}{2}, & \alpha_i \text{ is odd}, \end{cases} \qquad \gamma_i = \begin{cases} 0, & \alpha_i \text{ is even}, \\ 1, & \alpha_i \text{ is odd}, \end{cases} \qquad 1 \le i \le n.$$

Then $\alpha = 2\beta + \gamma$, so that $C_\alpha x^\alpha = C_\alpha x^\gamma x^{2\beta} \in P_{T,m}$ by (2.5) and the fact that $x^\gamma \in Q_1(T)$ and $|\beta| < m$. Consequently, $p \in P_{T,m}$. □

For the rectangular finite element of order m, the corresponding finite element spaces V_h and V_{h0} are defined as follows. $V_h = \{v \in L^2(\Omega) \mid v|_T \in P_{T,m}, \forall T \in T_h, \; \partial^\alpha v, \; |\alpha| < m$, are continuous at all vertices of elements in $\mathcal{T}_h\}$. $V_{h0} = \{v \in V_h \mid \partial^\alpha v, \; |\alpha| < m$, vanish at all vertices of elements in $\mathcal{T}_h$ which are belonging to $\partial\Omega$ $\}$.

Remark 2.1. The rectangular finite element of order m is just the $(2m-1)$-th-oder Hermite interpolation element when $n = 1$, the n-linear finite element when $m = 1$ and the Adini element when $m = 2$. For the $2m$-th-oder problems, the rectangular finite element of order m is conforming when $m = 1$ or $n = 1$, otherwise it is nonconforming. The rectangular finite element of order m can be viewed as the natural and reasonable generalizations of the one dimensional $(2m-1)$-th-oder Hermite interpolation element to higher dimensions or the n-linear finite element to higher order problems. This generalization shows that the conforming elements and the nonconforming elements are in same category.

Now let $\Pi_{T,m}$ be the corresponding interpolation operator to the rectangular finite element of order m.

Lemma 2.2. *For the rectangular finite element of order m,*

$$\int_T \frac{\partial}{\partial x_i}\Big(\partial^\beta p - \Pi_{T,1}\partial^\beta p\Big)\,\mathrm{d}x = 0, \qquad 1 \le i \le n, \quad |\beta| < m, \quad \forall p \in P_{T,m}. \tag{2.11}$$

Proof. Let $p \in P_{T,m}$, $1 \le i \le n$ and $|\beta| < m$. We know by (2.4) that $\partial^\beta p$ is a linear combination of the following functions,

$$F_{j,\alpha} = \partial^\beta\big((\xi_1^2-1)^{\alpha_1}(\xi_2^2-1)^{\alpha_2}\cdots(\xi_n^2-1)^{\alpha_n}\varphi_j\big), \qquad 1 \le j \le 2^n, \quad |\alpha| < m.$$

For $1 \le j \le 2^n$ and $|\alpha| < m$, set

$$f_i = \frac{\mathrm{d}^{\beta_i}}{\mathrm{d}\xi_i^{\beta_i}}(\xi_i^2-1)^{\alpha_i}, \qquad g_i = \frac{\mathrm{d}^{\beta_i}}{\mathrm{d}\xi_i^{\beta_i}}\Big(\xi_i(\xi_i^2-1)^{\alpha_i}\Big).$$

Then $F_{j,\alpha}$ can be written as the sum of such terms that each term has two factors, one is f_i or g_i, and another is independent of component ξ_i. Define

$$G_i(T) = \Big\{v \in C^\infty(T) \;\Big|\; \int_T \frac{\partial v}{\partial \xi_i}\,\mathrm{d}x = 0,\ v(a_j) = 0,\ 1 \le j \le 2^n\Big\}.$$

a) $\beta_i < \alpha_i$. In this case, $\dfrac{\mathrm{d}f_i}{\mathrm{d}\xi_i}$ is just the Legendre polynomial of ξ_i or its integral. Hence

$$\int_{-1}^{1}\frac{\mathrm{d}f_i}{\mathrm{d}\xi_i}\,\mathrm{d}\xi_i = 0.$$

On the other hand, f_i vanishes when $\xi_i = \pm 1$. Then $f_i \in G_i(T)$.

b) $\beta_i \ge \alpha_i$ and $\alpha_i \le 1$. In this case, $f_i \in Q_1(T)$.

c) $\beta_i \ge \alpha_i \ge 2$. In this case, we have

$$\begin{aligned} f_i =& \frac{\mathrm{d}^{\beta_i-1}}{\mathrm{d}\xi_i^{\beta_i-1}}\Big(2\alpha_i\xi_i(\xi_i^2-1)^{\alpha_i-1}\Big)\\ =& \frac{\mathrm{d}^{\beta_i-2}}{\mathrm{d}\xi_i^{\beta_i-2}}\Big(2\alpha_i(\xi_i^2-1)^{\alpha_i-1} + 4\alpha_i(\alpha_i-1)\xi_i^2(\xi_i^2-1)^{\alpha_i-2}\Big)\\ =& C_1\frac{\mathrm{d}^{\beta_i-2}}{\mathrm{d}\xi_i^{\beta_i-2}}(\xi_i^2-1)^{\alpha_i-1} + C_2\frac{\mathrm{d}^{\beta_i-2}}{\mathrm{d}\xi_i^{\beta_i-2}}(\xi_i^2-1)^{\alpha_i-2}, \end{aligned}$$

where C_1 and C_2 are constants. Repeating the same argument, we can read f_i as the linear combination of terms satisfying case a) or case b). Then $f_i \in G_i(T) + Q_1(T)$.

d) For g_i, we have

$$g_i = \frac{1}{2(\alpha_i+1)}\frac{\mathrm{d}^{\beta_i+1}}{\mathrm{d}\xi_i^{\beta_i+1}}(\xi_i^2-1)^{\alpha_i+1}.$$

Then $g_i \in G_i(T) + Q_1(T)$ by the discussion from case a) to case c).

Finally, we conclude that $\partial^\beta p \in G_i(T) + Q_1(T)$. Therefore, $\partial^\beta p - \Pi_{T,1}\partial^\beta p \in G_i(T) + Q_1(T)$. Since $\partial^\beta p - \Pi_{T,1}\partial^\beta p$ vanishes at the vertices of T, we have that $\partial^\beta p - \Pi_{T,1}\partial^\beta p \in G_i(T)$, and (2.11) is proved. □

3. C^0 Nonconforming Plate Elements

In this section, we will focus on the plate bending problem, and consider the nonconforming finite element. Let $n = 2$.

Given a triangle T, its vertices are denoted by a_i, $1 \le i \le 3$. The side of T opposite to a_i is denoted by F_i, its unit outer normal by ν_{F_i} and its measure by $|F_i|$, $1 \le i \le 3$. Let $\lambda_1, \lambda_2, \lambda_3$ be the barycentric coordinates of T. Denote

$$\begin{cases} \tilde{q}_1 = 2\Big(5(\lambda_1 - \lambda_1^2 - 2\lambda_2\lambda_3) - 1\Big)\lambda_1\lambda_2\lambda_3, \\ \tilde{q}_2 = 2\Big(5(\lambda_2 - \lambda_2^2 - 2\lambda_1\lambda_3) - 1\Big)\lambda_1\lambda_2\lambda_3, \\ \tilde{q}_3 = 2\Big(5(\lambda_3 - \lambda_3^2 - 2\lambda_1\lambda_2) - 1\Big)\lambda_1\lambda_2\lambda_3. \end{cases} \tag{3.1}$$

Set

$$P_3^+(T) = P_3(T) + \operatorname{span}\{\tilde{q}_1, \tilde{q}_2, \tilde{q}_3\}. \tag{3.2}$$

It is obvious that

$$\tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3 = -6\lambda_1\lambda_2\lambda_3,$$

so the dimension of $P_3^+(T)$ is at most twelve.

The new plate element is defined by (T, P_T, D_T) with

1. T is a triangle;
2. $P_T = P_3^+(T)$;
3. the components of $D_T(v)$ for any $C^1(T)$ are:

$$\begin{cases} v(a_j), \dfrac{1}{|F_j|}\displaystyle\int_{F_j} \frac{\partial v}{\partial \nu_{F_j}} \mathrm{d}s, & 1 \le j \le 3, \\ (a_j - a_i)^{\mathrm{T}} \nabla v(a_i), & 1 \le i \ne j \le 3, \end{cases} \tag{3.3}$$

   where ∇ is the gradient operator.

Define, for $1 \le i \ne j \ne k \le 3$,

$$\begin{cases} q_i = \dfrac{\tilde{q}_i}{\|\nabla\lambda_i\|}, \\ p_i = 3\lambda_i^2 - 2\lambda_i^3 + \displaystyle\sum_{\substack{1 \le l \le 3 \\ l \ne i}} \frac{\nabla\lambda_i^{\mathrm{T}}\nabla\lambda_l}{\|\nabla\lambda_l\|} q_l, \\ p_{ij} = \lambda_i^2\lambda_j + 10\lambda_i(\lambda_j - \lambda_k)\lambda_1\lambda_2\lambda_3. \end{cases} \tag{3.4}$$

Let δ_{ij} be the Kronecker delta. It can be verified that q_i, p_i, $p_{ij} \in P_3^+(T)$, and

$$\begin{cases} q_i(a_k) = 0, & (a_l - a_k)^{\mathrm{T}}\nabla q_i(a_k) = 0, & \dfrac{1}{|F_k|}\displaystyle\int_{F_k} \frac{\partial q_i}{\partial \nu_{F_k}} \mathrm{d}s = \delta_{ik}, \\ p_i(a_k) = \delta_{ik}, & (a_l - a_k)^{\mathrm{T}}\nabla p_i(a_k) = 0, & \dfrac{1}{|F_k|}\displaystyle\int_{F_k} \frac{\partial p_i}{\partial \nu_{F_k}} \mathrm{d}s = 0, \\ p_{ij}(a_k) = 0, & (a_l - a_k)^{\mathrm{T}}\nabla p_{ij}(a_k) = \delta_{ik}\delta_{jl}, & \dfrac{1}{|F_k|}\displaystyle\int_{F_k} \frac{\partial p_{ij}}{\partial \nu_{F_k}} \mathrm{d}s = 0, \end{cases} \tag{3.5}$$

when $1 \le i \ne j \le 3$ and $1 \le k \ne l \le 3$. Hence q_i, p_i and p_{ij} are the nodal basis functions with respect to the degrees of freedom. Therefore, the dimension of $P_3^+(T)$ is 12 and D_T is P_T-unisolvent.

One can verify that

$$\int_{F_k} \lambda_l \frac{\partial q_i}{\partial \nu_{F_k}} \,\mathrm{d}s = \frac{|F_k|}{2}\delta_{ik}, \qquad \int_{F_k} \lambda_l \frac{\partial p_i}{\partial \nu_{F_k}} \,\mathrm{d}s = 0. \tag{3.6}$$

when $1 \le i \le 3$ and $1 \le k \ne l \le 3$, and that

$$\nabla p_{ij}|_{F_i} \equiv 0, \qquad \int_{F_k} \lambda_i \frac{\partial p_{ij}}{\partial \nu_{F_k}} \,\mathrm{d}s = -\int_{F_k} \lambda_l \frac{\partial p_{ij}}{\partial \nu_{F_k}} \,\mathrm{d}s = \frac{|F_k|}{12}\nu_{F_k}^{\mathrm{T}} \nabla \lambda_j, \tag{3.7}$$

when $1 \le i \ne j \le 3$, $1 \le k \ne i \le 3$ and $1 \le l \ne k, i \le 3$.

Given $p \in P_3^+(T)$, it can be written as

$$p = \sum_{1 \le i \le 3} p(a_i) p_i + \sum_{1 \le i \le 3} \frac{1}{|F_i|} \int_{F_i} \frac{\partial p}{\partial \nu_{F_i}} \,\mathrm{d}F_i \, q_i + \sum_{1 \le i \ne j \le 3} (a_j - a_i)^{\mathrm{T}} \nabla p(a_i) p_{ij}.$$

Then for $1 \le i \ne j \ne k \le 3$, it can be computed by (3.6), (3.7) and the above equality that

$$\frac{1}{|F_i|} \int_{F_i} \lambda_j \frac{\partial p}{\partial \nu_{F_i}} \,\mathrm{d}s = \frac{1}{12}\Big(\frac{\partial p}{\partial \nu_{F_i}}(a_j) - \frac{\partial p}{\partial \nu_{F_i}}(a_k)\Big) + \frac{1}{2|F_i|} \int_{F_i} \frac{\partial p}{\partial \nu_{F_i}} \,\mathrm{d}s. \tag{3.8}$$

Given any edge F of $T \in \mathcal{T}_h$, denote its unit outer normal by ν_F. For any $v \in L^2(\Omega)$ with $v|_{\tilde{T}} \in H^1(\tilde{T})$, $\forall \tilde{T} \in \mathcal{T}_h$, we define the jump of v across F as follows:

$$[v]_F = v|_T - v|_{T'},$$

if $F = T \cap T'$ for some other $T' \in \mathcal{T}_h$, and

$$[v]_F = v|_T,$$

if $F = T \cap \partial\Omega$.

For the new element, define the corresponding finite element spaces V_h and V_{h0} as follows. $V_h = \{v \in L^2(\Omega) \mid v|_T \in P_3^+(T), \forall T \in T_h$, v and ∇v are continuous at all vertices of elements in $\mathcal{T}_h$, and for any edge F of T with $F \not\subset \partial\Omega$ the integral average of $\nu_F^{\mathrm{T}}[\nabla v]_F$ over F is zero$\}$; and $V_{h0} = \{v \in V_h \mid v$ and ∇v vanish at all vertices belonging to $\partial\Omega$, and for any edge F of T with $F \subset \partial\Omega$ the integral average of $\frac{\partial}{\partial \nu_F} v$ over F is zero$\}$.

We claim that $V_h \subset C^0(\bar{\Omega})$ and $V_{h0} \subset C_0^0(\Omega)$. Let $v_h \in V_h$, F be a common edge of $T, T' \in \mathcal{T}_h$. By the definition, $[v_h]_F$ is in $P_3(F)$, and it and its directive derivative along F are zero at two endpoints of F. Hence $[v_h]_F \equiv 0$, that is, $v_h \in C^0(\bar{\Omega})$. Similarly, we can show that $v_h \in C_0^0(\Omega)$ when $v_h \in V_{h0}$.

By (3.8), the definitions of V_h and V_{h0} and the fact that $V_h \subset C^0(\bar{\Omega})$ and $V_{h0} \subset C_0^0(\Omega)$, we obtain the following lemma.

Lemma 3.1. *If F is a common edge of distinct $T, T' \in \mathcal{T}_h$, then*

$$\int_F p\, [\nabla v_h]_F \,\mathrm{d}s = 0, \qquad \forall p \in P_1(F), \quad \forall v_h \in V_h. \tag{3.9}$$

If an edge F of $T \in \mathcal{T}_h$ is on $\partial\Omega$ then

$$\int_F p\,[\nabla v_h]_F \,\mathrm{d}s = 0, \qquad \forall p \in P_1(F), \quad \forall v_h \in V_{h0}. \tag{3.10}$$

From the new plate element given above, we can deduce a new Zienkiewicz-type element. For $1 \le i \le 3$, define

$$\phi_i(v) = \frac{1}{|F_i|}\int_{F_i} \frac{\partial v}{\partial \nu_{F_i}}\,\mathrm{d}s - \frac{1}{2}\sum_{1\le j\le 3, j\ne i} \frac{\partial v}{\partial \nu_{F_i}}(a_j), \qquad \forall v \in C^1(T). \tag{3.11}$$

Set

$$P_T^z = \Big\{\, p \in P_3^+(T) \;\Big|\; \phi_1(p) = \phi_2(p) = \phi_3(p) = 0 \Big\}. \tag{3.12}$$

Observing the fact that $P_2(T) \subset P_3^+(T)$ and $\phi_1(p) = \phi_2(p) = \phi_3(p) = 0, \forall p \in P_2(T)$, we have $P_2(T) \subset P_T^z$.

The new Zienkiewicz-type element is defined by (T, P_T, D_T) with

1. T is a triangle;
2. $P_T = P_T^z$;
3. the components of $D_T(v)$ for any $v \in C^1(T)$ are:

$$\begin{cases} v(a_j), & 1 \le j \le 3, \\ (a_j - a_i)^{\mathrm{T}}\nabla v(a_i), & 1 \le i \ne j \le 3. \end{cases} \tag{3.13}$$

It is easy to verify that D_T is P_T-unisolvent.

For the new Zienkiewicz-type element, the corresponding finite element spaces V_h^z and V_{h0}^z are defined as follows. $V_h^z = \{v \in L^2(\Omega) \mid v|_T \in P_T^z, \forall T \in T_h,\ v$ and ∇v are continuous at all vertices of elements in $\mathcal{T}_h\}$, $V_{h0}^z = \{v \in V_h^z \mid v$ and ∇v vanish at all vertices belonging to $\partial\Omega\}$.

The difference between the new Zienkiewicz-type element here and the two dimensional one proposed in [14] is their shape function spaces. The consistent term of the element here is of order $\mathcal{O}(h^2)$, while the consistent term of the element given in [14] is of order $\mathcal{O}(h)$.

4. Convergence Analysis

Let $f \in L^2(\Omega)$. We take the following boundary value problem as example to show the convergent result:

$$\begin{cases} (-1)^m \Delta^m u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \cdots = \dfrac{\partial^{m-1} u}{\partial \nu^{m-1}}\Big|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)^{\mathrm{T}}$ is the unit outer normal to $\partial\Omega$ and Δ is the standard Laplacian operator. Define

$$a(u, v) = \sum_{1\le j_1,\cdots,j_m\le n} \int_\Omega \frac{\partial^m u}{\partial x_{j_1}\cdots\partial x_{j_m}} \frac{\partial^m v}{\partial x_{j_1}\cdots\partial x_{j_m}}\,\mathrm{d}x. \tag{4.2}$$

Then the weak form of problem (4.1) is: find $u \in H_0^m(\Omega)$ such that

$$a(u,v) = (f,v), \qquad \forall v \in H_0^m(\Omega). \tag{4.3}$$

For nonnegative integer s and $\mathcal{T}_h$, define

$$H^s(\mathcal{T}_h) = \Big\{ v \in L^2(\Omega) \;\Big|\; v|_T \in H^s(T), \qquad \forall T \in \mathcal{T}_h \Big\}.$$

For $v, w \in H^m(\mathcal{T}_h)$, define

$$a_h(v,w) = \sum_{T\in\mathcal{T}_h} \sum_{1\le j_1,\cdots,j_m\le n} \int_T \frac{\partial^m v}{\partial x_{j_1}\cdots\partial x_{j_m}} \frac{\partial^m w}{\partial x_{j_1}\cdots\partial x_{j_m}}\,\mathrm{d}x. \tag{4.4}$$

The finite element method for problem (4.3) is: find $u_h \in V_{h0}$ such that

$$a_h(u_h, v_h) = (f, v_h), \qquad \forall v_h \in V_{h0}. \tag{4.5}$$

We introduce the following mesh dependent norm $\|\cdot\|_{s,h}$ and semi-norm $|\cdot|_{s,h}$:

$$\begin{cases} \|v\|_{s,h} = \Big(\sum\limits_{T\in\mathcal{T}_h} \|v\|_{s,T}^2\Big)^{1/2}, \\ |v|_{s,h} = \Big(\sum\limits_{T\in\mathcal{T}_h} |v|_{s,T}^2\Big)^{1/2}, \end{cases} \qquad \forall v \in H^s(\mathcal{T}_h).$$

For the nonconforming elements, the basic mathematical theory has been established (see [5, 7, 10, 13, 18]). We can use them to give the convergence analysis of our new elements.

For the finite elements given in previous two sections, one can verify the following statements by their constructions, Lemmas 2.1, 2.2 and 3.1:

- They all have the approximability.
- They all have the superapproximation.
- They all have the weak continuity.
- They all pass the patch test.
- They all pass the generalized patch test.

Then by the result in [10] or by the one in [13] we can obtain the following theorems.

Theorem 4.1. *Assume that $m, n \ge 1$. Let V_{h0} be the finite element space corresponding to the rectangular finite element of order m, and let u and u_h be the solutions of problems* (4.3) *and* (4.5) *respectively. Then*

$$\lim_{h\to 0} \|u - u_h\|_{m,h} = 0. \tag{4.6}$$

Theorem 4.2. *Assume that $m = n = 2$. Let V_{h0} be the finite element space corresponding to the new plate element or new Zienkiewicz-type element, and let u and u_h be the solutions of problems* (4.3) *and* (4.5) *respectively. Then*

$$\lim_{h\to 0} \|u - u_h\|_{2,h} = 0. \tag{4.7}$$

By the result in [13], we know that the error of the rectangular finite element of order m and the new Zienkiewicz-type are all order $\mathcal{O}(h)$. For the new plate element, we can obtain the following theorem by Lemma 3.1 and the usual technique dealing with the consistent term.

Theorem 4.3. *Assume that $m = n = 2$. Let V_{h0} be the finite element space corresponding to the new plate element, and let u and u_h be the solutions of problems* (4.3) *and* (4.5) *respectively. Then there exists a constant C independent of h such that*

$$\|u - u_h\|_{2,h} \leq Ch^2 |u|_{4,\Omega}, \tag{4.8}$$

when $u \in H^4(\Omega)$. In addition,

$$\|u - u_h\|_{1,\Omega} \leq Ch^3 |u|_{4,\Omega}, \tag{4.9}$$

when Ω is convex.

Remark 4.1. Let $k \geq 1$. The finite element space V_h corresponding to the rectangular element of order k is a subspace of $H^1(\Omega)$. Hence the element is convergent with order $\mathcal{O}(h^{2k-1})$ by Lemma 2.1 when it is applied to solving the second-oder problems. In general, the rectangular element of order k is a convergent nonconforming element for the $2m$-th-oder problem when $k \geq m$, which can be shown by Lemmas 2.1 and 2.2. In this situation, the finite element space V_{h0} should be defined accordingly.

5. Numerical Examples

In this section, we give some numerical results of the new plate element. Now let $m = n = 2$, $\Omega = (0, 1) \times (0, 1)$ and define

$$\begin{aligned} u_1(x) &= x_1^2 (x_1 - 1)^2 x_2^2 (x_2 - 1)^2, \\ u_2(x) &= \big(\sin(\pi x_1) \sin(\pi x_2)\big)^2, \\ u_3(x) &= e^{x_1 + x_2}. \end{aligned}$$

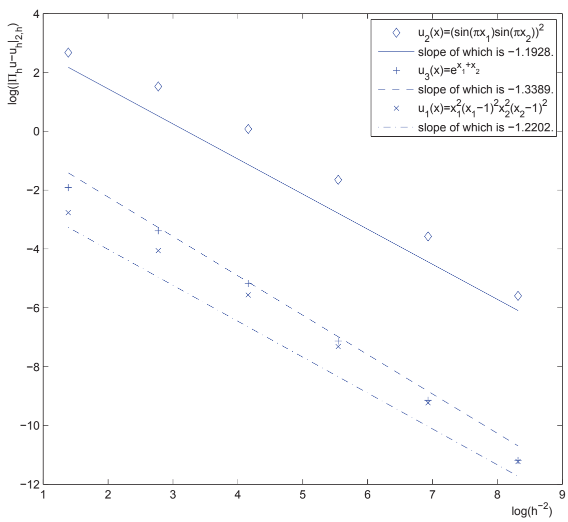

Fig. 5.1. The error: $|\Pi_h u - u_h|_{2,h}$

For $1 \le i \le 3$, set $f_i = \Delta^2 u_i$. Then u_i is the solution of problem:

$$\begin{cases} \Delta^2 u = f_i, & \text{in } \Omega, \\ u = u_i, \ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial u_i}{\partial \nu}, & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Problem (5.1) is a homogeneous Dirichlet boundary value problem when $i = 1, 2$, and a non-homogeneous one when $i = 3$.

For mesh size $h = 2^{-1}, 2^{-2}, \cdots$, Ω is divided into $h \times h$ squares, and each square is further divided into two triangles by the diagonal with a negative slash.

Let Π_h be the interpolation operator corresponding to new plate element and $\mathcal{T}_h$, and let u_h be the finite element solution corresponding to new plate element and triangulation $\mathcal{T}_h$. The numerical results of error term $|\Pi_h u - u_h|_{2,h}$ are shown in Fig. 5.1 with respect to mesh size h. It is seen that the error terms $|\Pi_h u - u_h|_{2,h}$ are of $\mathcal{O}(h^2)$ as h approaches 0. On the other hand, the interpolation error $|u - \Pi_h u|_{2,h}$ is of order $\mathcal{O}(h^2)$ at least. So that $|u - u_h|_{2,h}$ is at least two order of h as well.

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