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A PRIORI ERROR ESTIMATES OF A COMBINED MIXED FINITE ELEMENT AND DISCONTINUOUS GALERKIN METHOD FOR COMPRESSIBLE MISCIBLE DISPLACEMENT WITH MOLECULAR DIFFUSION AND DISPERSION*

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Abstract

A combined approximation for a kind of compressible miscible displacement problems including molecular diffusion and dispersion in porous media is studied. Mixed finite element method is applied to the flow equation, and the transport one is solved by the symmetric interior penalty discontinuous Galerkin method (SIPG). To avoid the inconvenience of the cut-off operator in [3, 21], some induction hypotheses different from the ones in [6] are used. Based on interpolation projection properties, a priori hp error estimates are obtained. Comparing with the existing error analysis that only deals with the diffusion case, the current work is more complicated and more significant.

Mathematics subject classification: 65M12, 65M60.
Key words: A priori error, Mixed finite element, Discontinuous Galerkin, Compressible miscible displacement.

1. Introduction

We consider the following single-phase, miscible displacement problem of one compressible fluid by another in porous media:

$$d(c)\frac{\partial p}{\partial t} + \nabla\cdot\boldsymbol{u} = d(c)\frac{\partial p}{\partial t} - \nabla\cdot(a(c)\nabla p) = q, \quad (x,t)\in\Omega\times J, \tag{1.1}$$

$$\phi\frac{\partial c}{\partial t} + b(c)\frac{\partial p}{\partial t} + \boldsymbol{u}\cdot\nabla c - \nabla\cdot(\boldsymbol{D}(\boldsymbol{u})\nabla c) = (\widehat{c}-c)q, \quad (x,t)\in\Omega\times J, \tag{1.2}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad (x,t)\in\partial\Omega\times J, \tag{1.3}$$

$$\boldsymbol{D}(\boldsymbol{u})\nabla c\cdot\boldsymbol{n} = 0, \quad (x,t)\in\partial\Omega\times J, \tag{1.4}$$

$$p(x,0) = p_0(x), \quad x\in\Omega, \tag{1.5}$$

$$c(x,0) = c_0(x), \quad x\in\Omega, \tag{1.6}$$

where Ω is a polygonal and bounded domain in $\mathbb{R}^n$ ($n = 1, 2$ *or* 3) with boundary $\partial\Omega$, $J = (0,T]$, $\boldsymbol{n}$ denotes the unit outward normal vector to $\partial\Omega$; $\boldsymbol{u}(x,t)$ represents the Darcy velocity of the mixture and $p(x,t)$ is the fluid pressure in the fluid mixture; $c(x,t)$ is the solvent concentration of interested species measured in amount of species per unit volume of the fluid mixture, $\phi(x)$

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is the effective porosity of the medium and is bounded above and below by positive constants, $\boldsymbol{D}(\boldsymbol{u})$ denotes a diffusion or dispersion tensor which has contributions from molecular diffusion and mechanical dispersion. Moreover,

$$\boldsymbol{D}(\boldsymbol{u}) = d_m\boldsymbol{I} + |\boldsymbol{u}|\Big(\alpha_l\boldsymbol{E}(\boldsymbol{u}) + \alpha_t\big(\boldsymbol{I} - \boldsymbol{E}(\boldsymbol{u})\big)\Big),$$

where $\boldsymbol{E}(\boldsymbol{u})$ is the tensor that projects onto the $\boldsymbol{u}$ direction, whose (i,j) component is

$$(\boldsymbol{E}(\boldsymbol{u}))_{i,j} = \frac{u_iu_j}{|\boldsymbol{u}|^2};$$

d_m is the molecular diffusivity and is assumed to be strictly positive; α_l and α_t are the longitudinal and transverse dispersion respectively, and are assumed to be nonnegative. The imposed external total flow rate q is a sum of sources and sinks. That is to say, $q = q^+ + q^-$, where $q^+ = \max(q,0)$ and $q^- = \min(q,0)$. q and $\frac{\partial q}{\partial t}$ are assumed to be bounded. The notation $\widehat{c}$ is the specified injected concentration c_w at sources if $q > 0$ and is the resident concentration c at sinks if $q < 0$. We assume that p, ∇p, c and ∇c are essentially bounded.

The coefficients $a(c)$, $b(c)$ and $d(c)$ are defined as:

$$a(c) = \frac{k(x)}{\mu(c)}, \quad b(c) = \phi(x)c_1(z_1 - z_1c_1 - z_2c_2), \quad d(c) = \phi(x)(z_1c_1 + z_2c_2),$$

where $c = c_1 = 1 - c_2$, $\mu(c)$ represents the viscosity, z_j denotes the constant compressibility factor for the jth component $(j = 1,2)$, $k(x)$ is the permeability of the medium. $a(c)$ and $d(c)$ have positive lower and upper bounds,

$$0 < a_* < a(c) < a^* \text{ and } 0 < d_* < d(c) < d^*,$$

$b(c)$ is bounded. In addition, $\frac{\partial a(c)}{\partial c}$ is uniformly bounded and Lipschitz continuous with respect to c.

It is well known that the mixed finite element (MFE) method can obtain the same optimal order of convergence for both the pressure and the Darcy velocity and has been widely used in the numerical simulation for porous media problems [8–10].

Recently, M. F. Wheeler, B. Rivière and S. Sun have devoted to using discontinuous Galerkin (DG) solver for problems in porous media [16,20]. V. Dolejsi and M. Feistauer, have investigated DG approximation for convection-diffusion problems (see [7, 12, 13]). DG methods belong to a class of non-conforming methods (see [3, 5, 15, 18, 23–25]) and they solve the differential equations by piecewise polynomial functions over a finite element space without any requirement on inter-element continuity – however, continuity on inter-element boundaries together with boundary conditions is weakly enforced through the bilinear form. DG is very attractive for practical numerical simulations because of its physical and numerical properties. Firstly, it is flexible which allows for general non-conforming meshes with variable degrees of approximation. Secondly, it is locally mass conservative and the average of the trace of the fluxes along an element edge is continuous. Thirdly, it has less numerical diffusion and can deal with rough coefficient problems. Finally, it is easier for the *hp*-adaptivity because the information over cell boundaries is almost decoupled.

To approximate to the exact solution of (1.1)–(1.6), we shall make use of a combined mixed finite element and DG method.

Many scholars have contributed to numerical approximations to miscible displacement problems [4, 14]. Unfortunately, there are very few literature dealing with DG methods. In [21] a continuous in time scheme consisting of the mixed finite element and nonsymmetric interior penalty Galerkin method for the incompressible miscible displacement problem in porous media was given, and in [22] continuous in time schemes of primal discontinuous Galerkin methods with interior penalty for the incompressible miscible displacement problem were proposed. Compressible case was discussed in [3, 6], but only the dispersion-free case ($\boldsymbol{D}(\boldsymbol{u}) = d_m\boldsymbol{I}$) is considered. The authors have derived a priori error of a discontinuous Galerkin approximation for a kind of compressible miscible displacement problems in [26]. In the current paper, a priori error estimates of a combined mixed finite element and discontinuous Galerkin method are given for the completely compressible case with molecular diffusion and dispersion. During the error analysis, the induction hypotheses are used as tools, instead of the cut-off operator employed in [3, 21] where it is necessary to choose properly the positive constant appearing in the operator. Moreover, the induction hypotheses here are different from the ones in [6].

The paper is organized as follows. In Section 2, we introduce the combined mixed finite element and discontinuous Galerkin method. In Section 3, error estimates are given. Proofs of the induction hypotheses are presented in Section 4.

2. A Combined MFE/DG Method

2.1. Notations

Let $\mathcal{T}_h$ be a family of quasi-uniform (which means that the element is convex and that there exists $\lambda > 0$ such that if h_E is the diameter of $E \in \mathcal{T}_h$, then each of the sub-triangles (for $n = 2$) or sub-tetrahedra (for $n = 3$) of element E contains a ball of radius λh_E in its interior), and possibly non-conforming finite element partitions of Ω composed of triangles or quadrilaterals if $n = 2$, or tetrahedra, prisms or hexahedra if $n = 3$.

Let Γ_h be the set of all interior edges (for 2 dimensional domain) or faces (for 3 dimensional domain) for $\mathcal{T}_h$. Let $\boldsymbol{n}$ be the outward unit normal vector on each edge or face $\gamma \in \Gamma_h \cup \partial\Omega$. We assume that $h = \max_{E\in\mathcal{T}_h} h_E$ the maximal element diameter over all elements with the common edge or face $\gamma \in \Gamma_h \cup \partial\Omega$.

The usual Sobolev inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|_{m,\Omega}$ on Ω are used. Similar notations are suitable for the element E and face or edge γ. Specially, $\|\cdot\|$ stands for $\|\cdot\|_{0,\Omega}$. For the sake of convenience, the notations dx and dt in $\int \cdot dx$ and $\int \cdot dt$ are omitted. We use $\int_g \cdot$ $(g = E, \gamma, \Omega)$ and $\int_0^t \cdot$ to represent the integrals in space $\int \cdot dx$ and the time integrals $\int \cdot dt$, respectively.

For $s \geq 0$, we define the following broken Sobolev space

$$H^s(\mathcal{T}_h) = \Big\{ v \in L^2(\Omega) : v|_E \in H^s(E), E \in \mathcal{T}_h \Big\}. \tag{2.1}$$

Let $E_i \in \mathcal{T}_h, E_j \in \mathcal{T}_h$ and $\gamma = \partial E_i \cap \partial E_j \in \Gamma_h$ with $\boldsymbol{n}$ exterior to E_i. For $v \in H^s(\mathcal{T}_h), s > 1/2$, the average $\{v\}$ of v on γ and the jump $[v]$ of v across γ are defined as follows:

$$\{v\} = \frac{(v|_{E_i})|_\gamma + (v|_{E_j})|_\gamma}{2}, \quad [v] = (v|_{E_i})|_\gamma - (v|_{E_j})|_\gamma.$$

We set the discontinuous finite element space:

$$D_r(\mathcal{T}_h) = \Big\{ v \in L^2(\Omega) : v|_E \in \mathrm{P}_r(E), E \in \mathcal{T}_h \Big\}, \tag{2.2}$$

where $\mathrm{P}_r(E)$ denotes the space of polynomials of total degree less than or equal to r on E.

Next, define the spaces

$$V = H(\text{div};\Omega) = \Big\{ \boldsymbol{u} \in (L^2(\Omega))^n, \text{div}\boldsymbol{u} \in L^2(\Omega) \Big\}, \tag{2.3}$$

$$V^0 = \Big\{ \boldsymbol{u} \in H(\text{div};\Omega), \boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega} = 0 \Big\}, \tag{2.4}$$

$$W = L^2(\Omega). \tag{2.5}$$

Let the approximation subspace $V_k(\mathcal{T}_h) \times W_k(\mathcal{T}_h)$ of $V \times W$ be the kth ($k \geq 0$) order Raviart-Thomas space (RT_k) of the partition $\mathcal{T}_h$. We define $V_k^0(\mathcal{T}_h) = V_k(\mathcal{T}_h) \cap V^0$.

Throughout the paper, we denote by $K, K_i (i \in N)$ generic positive constants that are independent of h, r and k, but might depend on the solution of PDEs. They may take different values at different occurrences. And ε will denote a fixed positive constant that can be chosen arbitrary small.

2.2. The continuous in time scheme

For $\forall \psi \in D_r(\mathcal{T}_h)$, we define the bilinear form $B(\boldsymbol{u}; c, \psi)$ and the linear functional $L(c,\psi)$:

$$\begin{aligned} B(\boldsymbol{u}; c, \psi) = & \sum_{E\in\mathcal{T}_h} \int_E \boldsymbol{D}(\boldsymbol{u})\nabla c\cdot\nabla\psi - \sum_{\gamma\in\Gamma_h}\int_\gamma \{\boldsymbol{D}(\boldsymbol{u})\nabla c\cdot\boldsymbol{n}\}[\psi] \\ & - \sum_{\gamma\in\Gamma_h}\int_\gamma \{\boldsymbol{D}(\boldsymbol{u})\nabla \psi\cdot\boldsymbol{n}\}[c] + \sum_{E\in\mathcal{T}_h}\int_E (\boldsymbol{u}\cdot\nabla c)\psi + J_0^\sigma(c,\psi), \\ L(c,\psi) = & \int_\Omega (\widehat{c} - c) q\psi. \end{aligned} \tag{2.6}$$

where h_γ denotes the size of γ,

$$J_0^\sigma(c,\psi) = \sum_{\gamma\in\Gamma_h} \frac{r^2\sigma_\gamma}{h_\gamma}\int_\gamma [c][\psi]$$

is the interior penalty term. σ is a discrete positive function that takes constant value σ_γ on the edge or face γ, and is bounded below by $\sigma_* > 0$ and above by σ^*.

The continuous in time numerical scheme–MFE/DG approximation to the solution of the equations (1.1)-(1.6) which solves the flow equation by the mixed finite element method and the concentration equation by SIPG, a primal discontinuous Galerkin method, is written: Finding $\boldsymbol{U} \in L^\infty(J; V_k^0(\mathcal{T}_h))$, $P \in L^\infty(J; W_k(\mathcal{T}_h))$ and $C \in L^\infty(J; D_r(\mathcal{T}_h))$ s.t.

$$\Big(d(C)\frac{\partial P}{\partial t}, w\Big) + (\nabla\cdot\boldsymbol{U}, w) = (q, w), \qquad \forall w \in W_k(\mathcal{T}_h), \tag{2.7}$$

$$(\alpha(C)\boldsymbol{U}, \boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, P) = 0, \qquad \forall \boldsymbol{v} \in V_k^0(\mathcal{T}_h), \tag{2.8}$$

$$\Big(\phi\frac{\partial C}{\partial t}, \psi\Big) + \Big(b(C)\frac{\partial P}{\partial t}, \psi\Big) + B(\boldsymbol{U}; C, \psi) = L(C,\psi), \quad \forall \psi \in D_r(\mathcal{T}_h), \tag{2.9}$$

with initial values $C(x,0) = \widetilde{c}_0$ and $\boldsymbol{U}(x,0) = \boldsymbol{U}_0$ which satisfies

$$(\alpha(c)\boldsymbol{U}_0, \boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, \widetilde{p}_0) = 0, \quad \forall \boldsymbol{v} \in V_k^0(\mathcal{T}_h),$$

where $\alpha(c) = 1/a(c)$, $\widetilde{c}_0$ and $\widetilde{p}_0$ are the interpolant of c_0 and p_0, respectively.

3. Error Estimates for the Combined MFE/DG Approximation

3.1. Interpolation projections and induction hypotheses

We define the interpolants $\widetilde{\boldsymbol{u}}$ and $\widetilde{p}$ of functions $\boldsymbol{u}$ and p as follows:

$$\Big(d(c)\frac{\partial p}{\partial t}, w\Big) + (\nabla\cdot\widetilde{\boldsymbol{u}}, w) = (q, w), \quad \forall w \in W_k(\mathcal{T}_h), \tag{3.1}$$

$$(\alpha(c)\widetilde{\boldsymbol{u}}, \boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, \widetilde{p}) = 0, \quad \forall \boldsymbol{v} \in V_k^0(\mathcal{T}_h), \tag{3.2}$$

$$(\widetilde{p}, 1) = (p, 1). \tag{3.3}$$

Let $\boldsymbol{\rho} = \boldsymbol{u} - \widetilde{\boldsymbol{u}}$, $\boldsymbol{\sigma} = \widetilde{\boldsymbol{u}} - \boldsymbol{U}$, $\eta = p - \widetilde{p}$, $\pi = \widetilde{p} - P$ and assume that $\boldsymbol{\sigma}(0) = 0$, $\pi(0) = 0$. Following the method in [10], we easily find that the following projection error equations

$$(\nabla\cdot\boldsymbol{\rho}, w) = 0, \quad \forall w \in W_k(\mathcal{T}_h),$$
$$(\alpha(c)\boldsymbol{\rho}, \boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, \eta) = 0, \quad \forall \boldsymbol{v} \in V_k^0(\mathcal{T}_h),$$

and

$$\Big(\nabla\cdot\frac{\partial \boldsymbol{\rho}}{\partial t}, w\Big) = 0, \quad \forall w \in W_k(\mathcal{T}_h),$$
$$\Big(\alpha(c)\frac{\partial \boldsymbol{\rho}}{\partial t}, \boldsymbol{v}\Big) - (\nabla\cdot\boldsymbol{v}, \frac{\partial \eta}{\partial t}) = -\Big(\frac{\partial \alpha(c)}{\partial c}\frac{\partial c}{\partial t}\boldsymbol{\rho}, v\Big), \quad \forall \boldsymbol{v} \in V_k^0(\mathcal{T}_h)$$

are satisfied. For $\boldsymbol{\rho}$ and η, we have

$$\|\boldsymbol{\rho}\| + \|\eta\| \le K \sum_{E\in\mathcal{T}_h} \frac{h_E^{\min(k+1,\omega_E-1)}}{k^{\omega_E-1/2}} \|p\|_{\omega_E,E}, \tag{3.4}$$

$$\|\frac{\partial \boldsymbol{\rho}}{\partial t}\| + \|\frac{\partial \eta}{\partial t}\| \le K \sum_{E\in\mathcal{T}_h} \frac{h_E^{\min(k+1,\omega_E-1)}}{k^{\omega_E-1/2}} \Big(\|p\|_{\omega_E,E} + \|\frac{\partial p}{\partial t}\|_{\omega_E,E}\Big), \tag{3.5}$$

where k is the order of the RT_k spaces. Because $\|\boldsymbol{\rho}\|_{H(\mathrm{div};\Omega)}$ does not appear in the following estimates and the L^2 norm estimates instead of the $H(\mathrm{div};\Omega)$ norm ones are needed, we lower the regularity of p (from $k+3$ to $k+2$, i.e. ω_E can be $k+2$).

In order to estimate the error of the transport equation, the following known hp approximation results [1, 2, 6] are used. For $E \in \mathcal{T}_h$, $w \in H^{\lambda}(\mathcal{T}_h)$, there exists a sequence $z_r^h \in \mathrm{P}_r(E)$, $\quad r = 1, 2, \cdots$ (where $\mathrm{P}_r(E)$ denotes the polynomial of degree less than or equal to r on E), and there exists a constant K depending on λ but independent of w, r, h_E, such that for $0 \le q \le \lambda$ and $\mu = \min(r+1, \lambda)$,

$$||w - z_r^h||_{q,E} \le K\frac{h_E^{\mu-q}}{r^{s-q}}||w||_{\lambda,E}, \quad \lambda \ge 0, \tag{3.6}$$

$$||w - z_r^h||_{\delta,\partial E} \le K\frac{h_E^{\mu-\delta-1/2}}{r^{\lambda-\delta-1/2}}||w||_{\lambda,E}, \quad \lambda > \frac{1}{2} + \delta, \quad \delta = 0, 1. \tag{3.7}$$

In our work, we will use the above estimates for $q = 1$ and $q = 2$. At the same time, the hp approximation results for the function w being the function which is the derivative with respect to the time variable t.

Let $\widetilde{c}$ be the interpolant of c, satisfying the above optimal hp approximation properties. Let $\zeta = c - \widetilde{c}$ and $\xi = \widetilde{c} - C$ and we assume that $\xi(0) = 0$.

During the analysis, the induction hypotheses are needed, which can be proved in section 4.

$$h^{-n/2}\|\boldsymbol{\sigma}\| \to 0, \qquad \text{if} \quad h \to 0, \tag{3.8}$$

$$h^{-n/2}\|\frac{\partial \boldsymbol{\sigma}}{\partial t}\|_{L^2(L^2(\Omega))} \to 0, \quad \text{if} \quad h \to 0. \tag{3.9}$$

Also, we shall make use of the following trace inequalities and inverse inequalities [18, 19].

Lemma 3.1. *For $\forall v \in H^1(E)$, we have*

$$\|v\|_{0,\partial E}^2 \leq K\Big(h_E^{-1}\|v\|_{0,E}^2 + h_E|v|_{1,E}^2\Big),$$

$$\|\nabla v \cdot \boldsymbol{n}\|_{0,\partial E}^2 \leq K\Big(h_E^{-1}\|\nabla v\|_{0,E}^2 + \|\nabla v\|_{0,E}\|\nabla^2 v\|_{0,E}\Big),$$

where $\boldsymbol{n}$ is the unit normal vector on an edge or face ∂E of E.

Lemma 3.2. *Let χ be a polynomial of degree r on the element E. Then,*

$$\|\chi\|_{0,\partial E} \leq Krh_E^{-1/2}\|\chi\|_{0,E},$$

$$\|\nabla\chi \cdot \boldsymbol{n}\|_{0,\partial E} \leq Krh_E^{-1/2}\|\nabla\chi\|_{0,E},$$

where $\boldsymbol{n}$ is the unit normal vector on an edge or face ∂E of E.

3.2. A priori error estimate for the flow equation

Subtracting (2.7)-(2.8) from (3.1)-(3.2) respectively, we get

$$\big(d(C)\frac{\partial \pi}{\partial t}, w\big) + (\nabla\cdot\boldsymbol{\sigma}, w) = \big(\big(d(C)-d(c)\big)\frac{\partial \widetilde{p}}{\partial t}, w\big) - \big(d(c)\frac{\partial \eta}{\partial t}, w\big), \ \ \forall w \in W_k(\mathcal{T}_h), \tag{3.10}$$

$$\big(\alpha(C)\boldsymbol{\sigma}, \boldsymbol{v}\big) - (\nabla\cdot\boldsymbol{v}, \pi) = \big(\big(\alpha(C)-\alpha(c)\big)\widetilde{\boldsymbol{u}}, \boldsymbol{v}\big), \qquad \forall\, \boldsymbol{v} \in V_k^0(\mathcal{T}_h). \tag{3.11}$$

Taking $w = \frac{\partial \pi}{\partial t}$ in (3.10) leads to

$$\Big(d(C)\frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t}\Big) + (\nabla\cdot\boldsymbol{\sigma}, \frac{\partial \pi}{\partial t}) = \Big(\big(d(C)-d(c)\big)\frac{\partial \widetilde{p}}{\partial t}, \frac{\partial \pi}{\partial t}\Big) - \Big(d(c)\frac{\partial \eta}{\partial t}, \frac{\partial \pi}{\partial t}\Big). \tag{3.12}$$

By differentiating (3.11) with respect to the time variable t and choosing $\boldsymbol{v} = \boldsymbol{\sigma}$, we have

$$(\frac{\partial(\alpha(C)\boldsymbol{\sigma})}{\partial t}, \boldsymbol{\sigma}\big) - (\nabla\cdot\boldsymbol{\sigma}, \frac{\partial \pi}{\partial t}) = \Big(\frac{\partial}{\partial t}\Big(\big(\alpha(C)-\alpha(c)\big)\widetilde{\boldsymbol{u}}\Big), \boldsymbol{\sigma}\Big). \tag{3.13}$$

Recall that

$$\Big(\frac{\partial}{\partial t}\big(\alpha(C)\boldsymbol{\sigma}\big), \boldsymbol{\sigma}\Big) = \frac{1}{2}\frac{d}{dt}\big(\alpha(C)\boldsymbol{\sigma}, \boldsymbol{\sigma}\big) + \frac{1}{2}\Big(\frac{\partial \alpha(C)}{\partial C}\frac{\partial C}{\partial t}\boldsymbol{\sigma}, \boldsymbol{\sigma}\Big).$$

Substitute the above equality into (3.13) to obtain

$$\frac{1}{2}\frac{d}{dt}\big(\alpha(C)\boldsymbol{\sigma}, \boldsymbol{\sigma}\big) - (\nabla\cdot\boldsymbol{\sigma}, \frac{\partial \pi}{\partial t}) = \Big(\frac{\partial}{\partial t}\Big(\big(\alpha(C)-\alpha(c)\big)\widetilde{\boldsymbol{u}}\Big), \boldsymbol{\sigma}\Big) - \frac{1}{2}\Big(\frac{\partial \alpha(C)}{\partial C}\frac{\partial C}{\partial t}\boldsymbol{\sigma}, \boldsymbol{\sigma}\Big). \tag{3.14}$$

Adding (3.12) and (3.14), we see that

$$\begin{aligned}&\Big(d(C)\frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t}\Big) + \frac{1}{2}\frac{d}{dt}\big(\alpha(C)\boldsymbol{\sigma}, \boldsymbol{\sigma}\big)\\ =&\Big(\big(d(C)-d(c)\big)\frac{\partial \widetilde{p}}{\partial t}, \frac{\partial \pi}{\partial t}\Big) - \big(d(c)\frac{\partial \eta}{\partial t}, \frac{\partial \pi}{\partial t}\big) + \Big(\frac{\partial}{\partial t}\Big(\big(\alpha(C)-\alpha(c)\big)\widetilde{\boldsymbol{u}}\Big), \boldsymbol{\sigma}\Big) - \frac{1}{2}\Big(\frac{\partial \alpha(C)}{\partial C}\frac{\partial C}{\partial t}\boldsymbol{\sigma}, \boldsymbol{\sigma}\Big).\end{aligned}$$

Next, we shall bound the right hand side of the above equation by virtue of Cauchy-Schwartz inequality, Cauchy's inequality with ε, the boundedness of p, the assumption of $a(c)$, and the induction hypothesis (3.8).

$$\Big|\Big((d(C)-d(c))\frac{\partial\widetilde{p}}{\partial t},\frac{\partial\pi}{\partial t}\Big)\Big|\le\varepsilon\Big\|\frac{\partial\pi}{\partial t}\Big\|^2+K\Big(||\zeta||^2+||\xi||^2\Big),$$
$$\Big|\Big(d(c)\frac{\partial\eta}{\partial t},\frac{\partial\pi}{\partial t}\Big)\Big|\le\varepsilon\Big\|\frac{\partial\pi}{\partial t}\Big\|^2+K\Big\|\frac{\partial\eta}{\partial t}\Big\|^2.$$

Note that $\|\frac{\partial\widetilde{\boldsymbol{u}}}{\partial t}\|_\infty$ is bounded [10]. So,

$$\begin{aligned}\Big|\Big(\frac{\partial}{\partial t}\Big((\alpha(C)-\alpha(c))\widetilde{\boldsymbol{u}}\Big),\boldsymbol{\sigma}\Big)\Big|\le&\Big|\Big(\widetilde{\boldsymbol{u}}\big(\frac{\partial\alpha(C)}{\partial C}\frac{\partial C}{\partial t}-\frac{\partial\alpha(c)}{\partial c}\frac{\partial c}{\partial t}\big)+\big(\alpha(C)-\alpha(c)\big)\frac{\partial\widetilde{\boldsymbol{u}}}{\partial t},\boldsymbol{\sigma}\Big)\Big|\\ \le&\frac{\varepsilon}{2}\Big\|\frac{\partial\xi}{\partial t}\Big\|^2+K\Big(\Big\|\frac{\partial\zeta}{\partial t}\Big\|^2+||\zeta||^2+||\xi||^2+\|\boldsymbol{\sigma}\|^2\Big).\end{aligned}$$

If h is sufficiently small, by using the induction hypothesis (3.8), we have

$$\begin{aligned}\Big|\Big(\frac{\partial\alpha(C)}{\partial C}\frac{\partial C}{\partial t}\boldsymbol{\sigma},\boldsymbol{\sigma}\Big)\Big|=&\Big|\Big(\frac{\partial\alpha(C)}{\partial C}\frac{\partial\widetilde{c}}{\partial t}\boldsymbol{\sigma},\boldsymbol{\sigma}\Big)-\Big(\frac{\partial\alpha(C)}{\partial C}\frac{\partial\xi}{\partial t}\boldsymbol{\sigma},\boldsymbol{\sigma}\Big)\Big|\\ \le&K\|\boldsymbol{\sigma}\|^2+Kh^{-n/2}\Big\|\frac{\partial\xi}{\partial t}\Big\|\cdot\|\boldsymbol{\sigma}\|^2\le\frac{\varepsilon}{2}\Big\|\frac{\partial\xi}{\partial t}\Big\|^2+K\|\boldsymbol{\sigma}\|^2.\end{aligned}$$

Note that $\sigma(\cdot,0)=0$. Collecting the above bounds and integrating with respect to t yields

$$\int_0^t\Big\|\frac{\partial\pi}{\partial t}\Big\|^2+\|\boldsymbol{\sigma}\|^2(t)\le\quad\varepsilon\int_0^t\Big\|\frac{\partial\xi}{\partial t}\Big\|^2+K\int_0^t\Big(\Big\|\frac{\partial\zeta}{\partial t}\Big\|^2+\Big\|\frac{\partial\eta}{\partial t}\Big\|^2+||\zeta||^2+||\xi||^2+\|\boldsymbol{\sigma}\|^2\Big).\quad(3.15)$$

3.3. A priori error estimate for the transport equation

Let $(p,\boldsymbol{u},c)$ be the solution of (1.1)-(1.6). They satisfy the following weak formulation in the discontinuous finite element space $D_r(\mathcal{T}_h)$:

$$\Big(\phi\frac{\partial c}{\partial t},\psi\Big)+\Big(b(c)\frac{\partial p}{\partial t},\psi\Big)+B(\boldsymbol{u};c,\psi)=L(c,\psi),\qquad\forall\psi\in D_r(\mathcal{T}_h),\ \ t\in J.\qquad(3.16)$$

Note that [10]

$$(\widehat{c}-c)-(\widehat{C}-C)=\begin{cases}-(\zeta+\xi), & \text{if } q>0,\\ 0, & \text{if } q<0.\end{cases}$$

Subtract (2.9) from (3.16) to get

$$\begin{aligned}&\Big(\phi\frac{\partial(\zeta+\xi)}{\partial t},\psi\Big)+\Big(\big(b(c)-b(C)\big)\frac{\partial p}{\partial t},\psi\Big)+\Big(b(C)\frac{\partial(\eta+\pi)}{\partial t},\psi\Big)\\ &+\sum_{E\in\mathcal{T}_h}\int_E\big(\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{U})\big)\nabla c\cdot\nabla\psi-\sum_{\gamma\in\Gamma_h}\int_\gamma\{\big(\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{U})\big)\nabla c\cdot\boldsymbol{n}\}[\psi]\\ &-\sum_{\gamma\in\Gamma_h}\int_\gamma\{\big(\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{U})\big)\nabla\psi\cdot\boldsymbol{n}\}[c]+\sum_{E\in\mathcal{T}_h}\int_E(\boldsymbol{u}\cdot\nabla c-\boldsymbol{U}\cdot\nabla C)\psi\\ &+\sum_{\gamma\in\Gamma_h}\frac{r^2\sigma_\gamma}{h_\gamma}\int_\gamma[c-C][c-C][\psi]+\sum_{E\in\mathcal{T}_h}\int_E\boldsymbol{D}(\boldsymbol{U})\nabla(\zeta+\xi)\cdot\nabla\psi\\ &-\sum_{\gamma\in\Gamma_h}\int_\gamma\{\boldsymbol{D}(\boldsymbol{U})\nabla(\zeta+\xi)\cdot\boldsymbol{n}\}[\psi]-\sum_{\gamma\in\Gamma_h}\int_\gamma\{\boldsymbol{D}(\boldsymbol{U})\nabla\psi\cdot\boldsymbol{n}\}[\zeta+\xi]\end{aligned}$$

$$= -\int_{\Omega}(\zeta+\xi)q^{+}\psi, \qquad \forall\psi\in D_r(\mathcal{T}_h), t\in J.$$

Taking $\psi=\frac{\partial\xi}{\partial t}$ in the above equation, we have

$$\begin{aligned}
&\Big(\phi\frac{\partial\xi}{\partial t},\frac{\partial\xi}{\partial t}\Big)+\sum_{E\in\mathcal{T}_h}\int_E \boldsymbol{D}(\boldsymbol{U})\nabla\xi\cdot\nabla\frac{\partial\xi}{\partial t}+\frac{1}{2}\frac{d}{dt}\Big(\int_{\Omega}q^{+}\xi^2+J_0^{\sigma}(\xi,\xi)\Big)\\
=&-\Big(\phi\frac{\partial\zeta}{\partial t},\frac{\partial\xi}{\partial t}\Big)+\Big((b(C)-b(c))\frac{\partial p}{\partial t},\frac{\partial\xi}{\partial t}\Big)-\Big(b(C)\frac{\partial\eta}{\partial t},\frac{\partial\xi}{\partial t}\Big)-\Big(b(C)\frac{\partial\pi}{\partial t},\frac{\partial\xi}{\partial t}\Big)\\
&+\sum_{\gamma\in\Gamma_h}\int_{\gamma}\{(\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{U}))\nabla c\cdot\boldsymbol{n}\}\,[\frac{\partial\xi}{\partial t}]-\sum_{E\in\mathcal{T}_h}\int_E(\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{U}))\nabla c\cdot\nabla\frac{\partial\xi}{\partial t}\\
&+\sum_{\gamma\in\Gamma_h}\int_{\gamma}\Big\{(\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{U}))\nabla\frac{\partial\xi}{\partial t}\cdot\boldsymbol{n}\Big\}[c]-\sum_{E\in\mathcal{T}_h}\int_E(\boldsymbol{\rho}+\boldsymbol{\sigma})\cdot\nabla c\frac{\partial\xi}{\partial t}\\
&-\sum_{E\in\mathcal{T}_h}\int_E\boldsymbol{U}\cdot\nabla(\zeta+\xi)\frac{\partial\xi}{\partial t}-\sum_{E\in\mathcal{T}_h}\int_E\boldsymbol{D}(\boldsymbol{U})\nabla\zeta\cdot\nabla\frac{\partial\xi}{\partial t}+\sum_{\gamma\in\Gamma_h}\int_{\gamma}\{\boldsymbol{D}(\boldsymbol{U})\nabla(\zeta+\xi)\cdot\boldsymbol{n}\}\Big[\frac{\partial\xi}{\partial t}\Big]\\
&-J_0^{\sigma}\Big(\zeta,\frac{\partial\xi}{\partial t}\Big)+\sum_{\gamma\in\Gamma_h}\int_{\gamma}\Big\{\boldsymbol{D}(\boldsymbol{U})\nabla\frac{\partial\xi}{\partial t}\cdot\boldsymbol{n}\Big\}[\zeta+\xi]-\int_{\Omega}\zeta q^{+}\frac{\partial\xi}{\partial t}+\frac{1}{2}\int_{\Omega}\frac{\partial q^{+}}{\partial t}\xi^2\\
\equiv&\sum_{i=1}^{15}T_i, \qquad \forall\psi\in D_r(\mathcal{T}_h), t\in J,
\end{aligned}\tag{3.17}$$

where we have used the following equalities

$$\begin{aligned}
&\int_{\Omega}q^{+}\xi\frac{\partial\xi}{\partial t}=\frac{1}{2}\frac{d}{dt}\int_{\Omega}q^{+}\xi^2-\frac{1}{2}\int_{\Omega}\frac{\partial q^{+}}{\partial t}\xi^2,\\
&\sum_{\gamma\in\Gamma_h}\int_{\gamma}\frac{r_{\gamma}^2\sigma_{\gamma}}{h_{\gamma}}[\xi]\Big[\frac{\partial\xi}{\partial t}\Big]=\frac{1}{2}\frac{d}{dt}\Big(\sum_{\gamma\in\Gamma_h}\int_{\gamma}\frac{r_{\gamma}^2\sigma_{\gamma}}{h_{\gamma}}[\xi][\xi]\Big)=\frac{1}{2}\frac{d}{dt}J_0^{\sigma}(\xi,\xi),\\
&\boldsymbol{u}\cdot\nabla c-\boldsymbol{U}\cdot\nabla C=(\boldsymbol{u}-\boldsymbol{U})\cdot\nabla c+\boldsymbol{U}\cdot\nabla(c-C)=(\boldsymbol{\rho}+\boldsymbol{\sigma})\cdot\nabla c+\boldsymbol{U}\cdot\nabla(\zeta+\xi).
\end{aligned}$$

Note that

$$\|\boldsymbol{U}\|_{\infty}\le\|\widetilde{\boldsymbol{u}}\|_{\infty}+\|\boldsymbol{\sigma}\|_{\infty}\le\|\widetilde{\boldsymbol{u}}\|_{\infty}+Kh^{-n/2}\|\boldsymbol{\sigma}\|,\tag{3.18}$$

$$\|\frac{\partial\boldsymbol{U}}{\partial t}\|_{L^2(L^{\infty}(\Omega))}\le\Big\|\frac{\partial\widetilde{\boldsymbol{u}}}{\partial t}\Big\|_{L^2(L^{\infty}(\Omega))}+Kh^{-n/2}\Big\|\frac{\partial\boldsymbol{\sigma}}{\partial t}\Big\|_{L^2(L^2(\Omega))},\tag{3.19}$$

where the inverse inequality $\|\boldsymbol{v}\|_{\infty}\le Kh^{-n/2}\|\boldsymbol{v}\|, \forall\boldsymbol{v}\in V_k$ is used. Due to the induction hypotheses (3.8) and (3.9), when h is chosen to be sufficiently small, both the second item in (3.18) and the second item in (3.19) tend to zero. Furthermore, according to [10], $\|\widetilde{\boldsymbol{u}}\|_{\infty}$ and $\|\frac{\partial\widetilde{\boldsymbol{u}}}{\partial t}\|_{L^2(L^{\infty}(\Omega))}$ are bounded. Therefore,

$$\|\boldsymbol{U}\|_{\infty}\le M, \quad \Big\|\frac{\partial\boldsymbol{U}}{\partial t}\Big\|_{L^2(L^{\infty}(\Omega))}\le M,$$

where M is some positive constant. Note that $\boldsymbol{u}-\boldsymbol{U}=\boldsymbol{\rho}+\boldsymbol{\sigma}$. It is obvious that

$$\|\boldsymbol{u}-\boldsymbol{U}\|\le\|\boldsymbol{\rho}\|+\|\boldsymbol{\sigma}\|,\tag{3.20}$$

$$\Big\|\frac{\partial(\boldsymbol{u}-\boldsymbol{U})}{\partial t}\Big\|\le\Big\|\frac{\partial\boldsymbol{\rho}}{\partial t}\Big\|+\Big\|\frac{\partial\boldsymbol{\sigma}}{\partial t}\Big\|.\tag{3.21}$$

Next, let us bound the terms T_i, $i=1,\cdots,15$. By virtue of Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
|T_1| &\le \varepsilon\Big|\Big|\frac{\partial\xi}{\partial t}\Big|\Big|^2 + K\Big|\Big|\frac{\partial\zeta}{\partial t}\Big|\Big|^2,\\
|T_2| &\le \varepsilon\Big|\Big|\frac{\partial\xi}{\partial t}\Big|\Big|^2 + K\Big(||\zeta||^2+||\xi||^2\Big),\\
|T_3| &\le \varepsilon\Big|\Big|\frac{\partial\xi}{\partial t}\Big|\Big|^2 + K\Big|\Big|\frac{\partial\eta}{\partial t}\Big|\Big|^2,\\
|T_4| &\le \varepsilon\Big|\Big|\frac{\partial\xi}{\partial t}\Big|\Big|^2 + K_2\Big|\Big|\frac{\partial\pi}{\partial t}\Big|\Big|^2,\\
|T_8| &\le \varepsilon\Big|\Big|\frac{\partial\xi}{\partial t}\Big|\Big|^2 + K\Big(\|\boldsymbol{\rho}\|^2+\|\boldsymbol{\sigma}\|^2\Big),\\
|T_9| &\le \varepsilon\Big|\Big|\frac{\partial\xi}{\partial t}\Big|\Big|^2 + K\Big(||\nabla\zeta||^2+||\nabla\xi||^2\Big),\\
|T_{14}| &\le \varepsilon\Big|\Big|\frac{\partial\xi}{\partial t}\Big|\Big|^2 + K||\zeta||^2, \quad |T_{15}| \le K||\xi||^2.
\end{aligned}
$$

Recall that

$$|\boldsymbol{D}(\boldsymbol{U})| \le d_m + \max(\alpha_l,\alpha_t)|\boldsymbol{U}| \le M$$

according to the definition of $\boldsymbol{D}(\boldsymbol{u})$. By reason of the setting $\xi(x,0)=0$, integrate T_{10} with respect to t and apply integration by parts to get

$$
\begin{aligned}
&\Big|\int_0^t T_{10}\Big| = \Big|\int_0^t \sum_{E\in\mathcal{T}_h}\int_E \boldsymbol{D}(\boldsymbol{U})\nabla\zeta\cdot\nabla\frac{\partial\xi}{\partial t}\Big|\\
\le&\Big|\sum_{E\in\mathcal{T}_h}\int_E \boldsymbol{D}(\boldsymbol{U})\nabla\zeta\cdot\nabla\xi(t)\Big| + \Big|\int_0^t\sum_{E\in\mathcal{T}_h}\int_E \boldsymbol{D}(\boldsymbol{U})\nabla\xi\cdot\nabla\frac{\partial\zeta}{\partial t}\Big| + \Big|\int_0^t\sum_{E\in\mathcal{T}_h}\int_E \frac{\partial\boldsymbol{D}(\boldsymbol{U})}{\partial t}\nabla\xi\cdot\nabla\zeta\Big|\\
\le&\varepsilon\sum_{E\in\mathcal{T}_h}\int_E|\nabla\xi|^2(t) + K\sum_{E\in\mathcal{T}_h}\int_E|\nabla\zeta|^2(t) + K\int_0^t\sum_{E\in\mathcal{T}_h}\int_E|\nabla\xi|^2\\
&\quad + K\int_0^t\sum_{E\in\mathcal{T}_h}\int_E\big|\nabla\frac{\partial\zeta}{\partial t}\big|^2 + K\int_0^t\|\boldsymbol{U}\|_\infty\|\frac{\partial\boldsymbol{U}}{\partial t}\|_\infty\|\nabla\xi\|\cdot\|\nabla\zeta\|\\
\le&K\int_0^t\Big(||\nabla\xi||^2+||\nabla\zeta||^2\Big).
\end{aligned}
$$

For T_{11}–T_{13}, by virtue of integration by parts, Cauchy-Schwartz inequality, boundedness of σ_γ^{-1} and the inverse inequalities in Lemma 3.2, we obtain

$$
\begin{aligned}
&\Big|\int_0^t(T_{11}+T_{13})\Big|\\
=&\Big|\int_0^t\sum_{\gamma\in\Gamma_h}\int_\gamma\{\boldsymbol{D}(\boldsymbol{U})\nabla(\zeta+\xi)\cdot\boldsymbol{n}\}\Big[\frac{\partial\xi}{\partial t}\Big] + \int_0^t\sum_{\gamma\in\Gamma_h}\int_\gamma\Big\{\boldsymbol{D}(\boldsymbol{U})\nabla\frac{\partial\xi}{\partial t}\cdot\boldsymbol{n}\Big\}[\zeta+\xi]\Big|\\
\le&\Big|\sum_{\gamma\in\Gamma_h}\int_\gamma\{\boldsymbol{D}(\boldsymbol{U})\nabla\zeta\cdot\boldsymbol{n}\}[\xi](t)\Big| + \Big|\int_0^t\sum_{\gamma\in\Gamma_h}\int_\gamma\Big\{\boldsymbol{D}(\boldsymbol{U})\nabla\frac{\partial\zeta}{\partial t}\cdot\boldsymbol{n}\Big\}[\xi]\Big|\\
&+\Big|\int_0^t\sum_{\gamma\in\Gamma_h}\int_\gamma\Big\{\frac{\partial\boldsymbol{D}(\boldsymbol{U})}{\partial t}\nabla\zeta\cdot\boldsymbol{n}\Big\}[\xi]\Big| + \Big|\sum_{\gamma\in\Gamma_h}\int_\gamma\{\boldsymbol{D}(\boldsymbol{U})\nabla\xi\cdot\boldsymbol{n}\}[\zeta](t)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\int_0^t \sum_{\gamma\in\Gamma_h}\int_\gamma \{\boldsymbol{D}(\boldsymbol{U})\nabla\xi\cdot\boldsymbol{n}\}\left[\frac{\partial\zeta}{\partial t}\right]\right| + \left|\int_0^t \sum_{\gamma\in\Gamma_h}\int_\gamma \left\{\frac{\partial \boldsymbol{D}(\boldsymbol{U})}{\partial t}\nabla\xi\cdot\boldsymbol{n}\right\}[\zeta]\right| \\
&+\left|\sum_{\gamma\in\Gamma_h}\int_\gamma \{\boldsymbol{D}(\boldsymbol{U})\nabla\xi\cdot\boldsymbol{n}\}[\xi](t)\right| + \left|\int_0^t \sum_{\gamma\in\Gamma_h}\int_\gamma \left\{\frac{\partial \boldsymbol{D}(\boldsymbol{U})}{\partial t}\nabla\xi\cdot\boldsymbol{n}\right\}[\xi]\right| \\
\equiv &\sum_{i=1}^{8} F_{1i}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
F_{11} \le & K\sum_{\gamma\in\Gamma_h}\int_\gamma \left(\frac{r^2\sigma_\gamma}{h_\gamma}\right)^{-1}\{\nabla\zeta\cdot\boldsymbol{n}\}^2(t) + \varepsilon\sum_{\gamma\in\Gamma_h}\int_\gamma \frac{r^2\sigma_\gamma}{h_\gamma}[\xi]^2(t) \\
\le & K\sum_{\gamma\in\Gamma_h} r^{-2}h_\gamma \big\|\nabla\zeta\cdot\boldsymbol{n}\big\|_{0,\gamma}^2(t) + \varepsilon J_0^\sigma(\xi,\xi)(t), \\
F_{12} \le & K\int_0^t \sum_{\gamma\in\Gamma_h}\int_\gamma \left(\frac{r^2\sigma_\gamma}{h_\gamma}\right)^{-1}\left\{\nabla\frac{\partial\zeta}{\partial t}\cdot\boldsymbol{n}\right\}^2 + K\int_0^t \sum_{\gamma\in\Gamma_h}\int_\gamma \frac{r^2\sigma_\gamma}{h_\gamma}[\xi]^2 \\
\le & K\int_0^t \sum_{\gamma\in\Gamma_h} r^{-2}h_\gamma \left\|\nabla\frac{\partial\zeta}{\partial t}\cdot\boldsymbol{n}\right\|_{0,\gamma}^2 + K\int_0^t J_0^\sigma(\xi,\xi), \\
F_{13} \le & K\int_0^t \|\boldsymbol{U}\|_\infty \left\|\frac{\partial\boldsymbol{U}}{\partial t}\right\|_\infty \sum_{\gamma\in\Gamma_h}\int_\gamma \big|\{\nabla\zeta\cdot\boldsymbol{n}\}[\xi]\big| \\
\le & K\int_0^t \sum_{\gamma\in\Gamma_h}\int_\gamma \left(\frac{r^2\sigma_\gamma}{h_\gamma}\right)^{-1}\{\nabla\zeta\cdot\boldsymbol{n}\}^2 + K\int_0^t \sum_{\gamma\in\Gamma_h}\int_\gamma \frac{r^2\sigma_\gamma}{h_\gamma}[\xi]^2 \\
\le & K\int_0^t \sum_{\gamma\in\Gamma_h} r^{-2}h_\gamma \big\|\nabla\zeta\cdot\boldsymbol{n}\big\|_{0,\gamma}^2 + K\int_0^t J_0^\sigma(\xi,\xi), \\
F_{14} \le & K\sum_{\gamma\in\Gamma_h}\int_\gamma \left(\frac{r^2\sigma_\gamma}{h_\gamma}\right)^{-1}\{\nabla\xi\cdot\boldsymbol{n}\}^2(t) + K\sum_{\gamma\in\Gamma_h}\int_\gamma \frac{r^2\sigma_\gamma}{h_\gamma}[\zeta]^2(t) \\
\le & K_1(\min_{\gamma\in\Gamma_h}\sigma_\gamma)^{-1}\sum_{E\in\mathcal{T}_h}\int_E |\nabla\xi|^2(t) + K\sum_{\gamma\in\Gamma_h}\frac{r^2}{h_\gamma}\big\|[\zeta]\big\|_{0,\gamma}^2(t), \\
F_{15} \le & K\int_0^t \sum_{\gamma\in\Gamma_h}\int_\gamma \left(\frac{r^2\sigma_\gamma}{h_\gamma}\right)^{-1}\{\nabla\xi\cdot\boldsymbol{n}\}^2 + K\int_0^t \sum_{\gamma\in\Gamma_h}\int_\gamma \frac{r^2\sigma_\gamma}{h_\gamma}\left[\frac{\partial\zeta}{\partial t}\right]^2 \\
\le & K\int_0^t \sum_{E\in\mathcal{T}_h}\int_E |\nabla\xi|^2 + K\int_0^t \sum_{\gamma\in\Gamma_h}\frac{r^2}{h_\gamma}\left\|\left[\frac{\partial\zeta}{\partial t}\right]\right\|_{0,\gamma}^2, \\
F_{16} \le & K\int_0^t \|\boldsymbol{U}\|_\infty \left\|\frac{\partial\boldsymbol{U}}{\partial t}\right\|_\infty \sum_{\gamma\in\Gamma_h}\int_\gamma \{\nabla\xi\cdot\boldsymbol{n}\}[\zeta]\big| \\
\le & K\int_0^t \sum_{\gamma\in\Gamma_h}\int_\gamma \left(\frac{r^2\sigma_\gamma}{h_\gamma}\right)^{-1}\{\nabla\xi\cdot\boldsymbol{n}\}^2 + K\int_0^t \sum_{\gamma\in\Gamma_h}\int_\gamma \frac{r^2\sigma_\gamma}{h_\gamma}[\zeta]^2 \\
\le & K(\min_{\gamma\in\Gamma_h}\sigma_\gamma)^{-1}\int_0^t \sum_{E\in\mathcal{T}_h}\int_E |\nabla\xi|^2 + K\int_0^t \sum_{\gamma\in\Gamma_h}\frac{r^2}{h_\gamma}\big\|[\zeta]\big\|_{0,\gamma}^2,
\end{aligned}
$$

$$
\begin{aligned}
F_{17} \le & K \sum_{\gamma\in\Gamma_h} \int_\gamma \left(\frac{r^2\sigma_\gamma}{h_\gamma}\right)^{-1} \{\nabla\xi\cdot\boldsymbol{n}\}^2(t) + \varepsilon \sum_{\gamma\in\Gamma_h} \int_\gamma \frac{r^2\sigma_\gamma}{h_\gamma}[\xi]^2(t) \\
\le & K_1(\min_{\gamma\in\Gamma_h}\sigma_\gamma)^{-1} \sum_{E\in\mathcal{T}_h} \int_E |\nabla\xi|^2(t) + \varepsilon J_0^\sigma(\xi,\xi)(t), \\
F_{18} \le & K \int_0^t \|\boldsymbol{U}\|_\infty \left\|\frac{\partial \boldsymbol{U}}{\partial t}\right\|_\infty \sum_{\gamma\in\Gamma_h} \int_\gamma \{\nabla\xi\cdot\boldsymbol{n}\}\,[\xi]\Big| \\
\le & K(\min_{\gamma\in\Gamma_h}\sigma_\gamma)^{-1} \int_0^t \sum_{E\in\mathcal{T}_h} \int_E |\nabla\xi|^2 + K\int_0^t J_0^\sigma(\xi,\xi).
\end{aligned}
$$

Integrating T_{12} by parts with respect to the time variable t, we see that

$$
\begin{aligned}
\Big|\int_0^t T_{12}\Big| = & \Big| \sum_{\gamma\in\Gamma_h} \frac{r^2\sigma_\gamma}{h_\gamma} \int_\gamma [\zeta][\xi](t) - \int_0^t \sum_{\gamma\in\Gamma_h} \frac{r^2\sigma_\gamma}{h_\gamma} \int_\gamma [\frac{\partial\zeta}{\partial t}][\xi]\Big| \\
\le & \varepsilon J_0^\sigma(\xi,\xi)(t) + K \sum_{\gamma\in\Gamma_h} \frac{r^2}{h_\gamma} \|[\zeta]\|_{0,\gamma}^2(t) \\
& + K\int_0^t J_0^\sigma(\xi,\xi) + K\int_0^t \sum_{\gamma\in\Gamma_h} \frac{r^2}{h_\gamma}\left(\big\|[\zeta]\big\|_{0,\gamma}^2 + \left\|\left[\frac{\partial\zeta}{\partial t}\right]\right\|_{0,\gamma}^2\right).
\end{aligned}
$$

Noting that $[c] = 0$, we find $T_7 = 0$. It remains to estimate the terms T_5–T_6. Apply the integration by parts to get

$$
\begin{aligned}
T_5 + T_6 = & \frac{d}{dt} \sum_{\gamma\in\Gamma_h} \int_\gamma \{\big(\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{U})\big)\nabla c\cdot\boldsymbol{n}\}[\xi] - \sum_{\gamma\in\Gamma_h} \int_\gamma \{\big(\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{U})\big)\nabla\frac{\partial c}{\partial t}\cdot\boldsymbol{n}\}[\xi] \\
& - \sum_{\gamma\in\Gamma_h} \int_\gamma \{\frac{\partial}{\partial t}\big(\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{U})\big)\nabla c\cdot\boldsymbol{n}\}[\xi] + \Big(\frac{\partial}{\partial t}\big(\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{U})\big)\nabla c, \nabla\xi\Big) \\
& - \frac{d}{dt}\Big(\big(\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{U})\big)\nabla c, \nabla\xi\Big) + \Big(\big(\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{U})\big)\nabla\frac{\partial c}{\partial t}, \nabla\xi\Big) \\
\equiv & \sum_{i=1}^6 I_i,
\end{aligned}
$$

According to the definition of $\boldsymbol{D}(\boldsymbol{u})$, Cauchy-Schwartz inequality with ε, Lemma 3.2 and integrating I_1, I_2, I_5, I_6 with respect to t yields

$$
\begin{aligned}
\int_0^t |I_1| \le & K \sum_{\gamma\in\Gamma_h} \int_\gamma \Big((\frac{r^2\sigma_\gamma}{h_\gamma})^{-1} \big\|\boldsymbol{u} - \boldsymbol{U}\big\|(t) + \frac{r^2\sigma_\gamma}{h_\gamma}[\xi]^2(t)\Big) \\
\le & K_1\sigma_\gamma^{-1}(\|\boldsymbol{\rho}\|^2(t) + \|\boldsymbol{\sigma}\|^2(t)) + \varepsilon J_0^\sigma(\xi,\xi)(t), \\
\int_0^t |I_2| \le & K\int_0^t \|\nabla\frac{\partial c}{\partial t}\|_\infty \sum_{\gamma\in\Gamma_h} \big\|\boldsymbol{u} - \boldsymbol{U}\big\| \cdot \|[\xi]\| \\
\le & K\sigma_\gamma^{-1}\int_0^t (\|\boldsymbol{\rho}\|^2 + \|\boldsymbol{\sigma}\|^2) + K\int_0^t J_0^\sigma(\xi,\xi), \\
\int_0^t |I_5| \le & K\big\|\boldsymbol{u} - \boldsymbol{U}\big\| \cdot \|\nabla c\|_\infty \cdot \|\nabla\xi\|(t) \\
\le & K_2(\|\boldsymbol{\rho}\|^2(t) + \|\boldsymbol{\sigma}\|^2(t)) + \varepsilon\big\|\nabla\xi\big\|^2(t),
\end{aligned}
$$

$$\int_0^t |I_6| \le K\int_0^t \big\|\boldsymbol{u}-\boldsymbol{U}\big\|\cdot\|\nabla\frac{\partial c}{\partial t}\|_\infty\cdot\|\nabla\xi\| \\ \le K\int_0^t (\|\boldsymbol{\rho}\|^2+\|\boldsymbol{\sigma}\|^2)+K\int_0^t \big|\big|\nabla\xi\big|\big|^2.$$

For I_3 and I_4, we have

$$\int_0^t |I_3| \le K\sum_{\gamma\in\Gamma_h}\left(\left\|\frac{\partial}{\partial t}(\boldsymbol{u}-\boldsymbol{U})\right\|_{L^2(L^2(\gamma))}+\|\boldsymbol{u}-\boldsymbol{U}\|_{L^\infty(L^2(\gamma))}\|\boldsymbol{U}\|_{L^\infty(L^\infty(\gamma))}\left\|\frac{\partial \boldsymbol{U}}{\partial t}\right\|_{L^2(L^\infty(\gamma))}\right) \\ \cdot\|\nabla c\|_{L^\infty(L^\infty(\Omega))}\cdot\sum_{\gamma\in\Gamma_h}\|[\xi]\|_{L^2(L^2(\gamma))} \\ \le K\int_0^t J_0^\sigma(\xi,\xi)+K\sum_{E\in\mathcal{T}_h}\frac{h_E^{2\min(k+1,\omega_E-1)}}{k^{2\omega_E-1}}\left(\|p\|_{\omega_E,E}+\left\|\frac{\partial p}{\partial t}\right\|_{\omega_E,E}\right)^2,$$

$$\int_0^t |I_4| \le K\left(\left\|\frac{\partial}{\partial t}(\boldsymbol{u}-\boldsymbol{U})\right\|_{L^2(L^2(\Omega))}+\|\boldsymbol{u}-\boldsymbol{U}\|_{L^\infty(L^2(\Omega))}\|\boldsymbol{U}\|_{L^2(L^\infty(\Omega))}\left\|\frac{\partial \boldsymbol{U}}{\partial t}\right\|_{L^2(L^\infty(\Omega))}\right) \\ \cdot\|\nabla c\|_{L^\infty(L^\infty(\Omega))}\cdot\|\nabla\xi\|_{L^2(L^2(\Omega))} \\ \le K\int_0^t \big|\big|\nabla\xi\big|\big|^2+K\sum_{E\in\mathcal{T}_h}\frac{h_E^{2\min(k+1,\omega_E-1)}}{k^{2\omega_E-1}}\Big(\|p\|_{\omega_E,E}+\|\frac{\partial p}{\partial t}\|_{\omega_E,E}\Big)^2,$$

where Cauchy-Schwartz inequality, the estimates (3.4)-(3.5), (3.20)-(3.21), and the induction hypotheses (3.8)-(3.9) are used.

Recalling that $\xi(x,0)=0$, we have

$$\int_0^t\sum_{E\in\mathcal{T}_h}\int_E \boldsymbol{D}(\boldsymbol{U})\nabla\xi\cdot\nabla\frac{\partial\xi}{\partial t}=\frac{1}{2}\sum_{E\in\mathcal{T}_h}\int_E\boldsymbol{D}(\boldsymbol{U})\nabla\xi\cdot\nabla\xi(t)-\int_0^t\sum_{E\in\mathcal{T}_h}\int_E\frac{\partial\boldsymbol{D}(\boldsymbol{U})}{\partial t}\nabla\xi\cdot\nabla\xi.$$

Using the definition of $\boldsymbol{D}(\boldsymbol{u})$, we have

$$\sum_{E\in\mathcal{T}_h}\int_E\boldsymbol{D}(\boldsymbol{U})\nabla\xi\cdot\nabla\xi(t)\ge(d_m+\alpha_t|\boldsymbol{U}|)\sum_{E\in\mathcal{T}_h}\int_E|\nabla\xi|^2(t),$$

$$\Big|\int_0^t\sum_{E\in\mathcal{T}_h}\int_E\frac{\partial\boldsymbol{D}(\boldsymbol{U})}{\partial t}\nabla\xi\cdot\nabla\xi\Big|\le\int_0^t\|\boldsymbol{U}\|_\infty\|\frac{\partial\boldsymbol{U}}{\partial t}\|_\infty\sum_{E\in\mathcal{T}_h}\int_E\Big|\nabla\xi\cdot\nabla\xi\Big|\le K\int_0^t||\nabla\xi||^2.$$

Let σ_γ be large enough, so that $4K_1\le\min_{\gamma\in\Gamma_h}\sigma_\gamma$. Integrating with respect to t for (3.17) and combining all the above inequalities, we deduce the following estimation

$$\int_0^t\Big|\Big|\frac{\partial\xi}{\partial t}\Big|\Big|^2+||\nabla\xi||^2(t)+||\xi||^2(t)+J_0^\sigma(\xi,\xi)(t) \\ \le K_2\int_0^t\Big|\Big|\frac{\partial\pi}{\partial t}\Big|\Big|^2+K\int_0^t\left(\|\boldsymbol{\rho}\|^2+\|\boldsymbol{\sigma}\|^2+\big|\big|\zeta\big|\big|^2+\big|\big|\xi\big|\big|^2+\big|\big|\nabla\zeta\big|\big|^2+\big|\big|\nabla\xi\big|\big|^2+\left\|\frac{\partial\zeta}{\partial t}\right\|^2\right. \\ \left.+\left\|\frac{\partial\eta}{\partial t}\right\|^2+\left\|\nabla\frac{\partial\zeta}{\partial t}\right\|^2\right)+K\|\nabla\zeta\|^2(t)+\left(K_2+\frac{1}{4}\right)(\|\boldsymbol{\rho}\|^2(t)+\|\boldsymbol{\sigma}\|^2(t)) \\ +K\int_0^t J_0^\sigma(\xi,\xi)+K\int_0^t\sum_{\gamma\in\Gamma_h}\left(\frac{h_\gamma}{r^2}\right)\left(\left\|\nabla\zeta\cdot\boldsymbol{n}\right\|_{0,\gamma}^2+\left\|\nabla\frac{\partial\zeta}{\partial t}\cdot\boldsymbol{n}\right\|_{0,\gamma}^2\right)$$

$$+K\sum_{\gamma\in\Gamma_h}\left(\frac{h_\gamma}{r^2}\|\nabla\zeta\cdot\boldsymbol{n}\|_{0,\gamma}^2(t)+\frac{r^2}{h_\gamma}\|[\zeta]\|_{0,\gamma}^2(t)\right)+K\int_0^t\sum_{\gamma\in\Gamma_h}\frac{r^2}{h_\gamma}\left(\|[\zeta]\|_{0,\gamma}^2+\left\|\left[\frac{\partial\zeta}{\partial t}\right]\right\|_{0,\gamma}^2\right)$$
$$+K\sum_{E\in\mathcal{T}_h}\frac{h_E^{2\min(k+1,\omega_E-1)}}{k^{2\omega_E-1}}\left(\|p\|_{\omega_E,E}+\left\|\frac{\partial p}{\partial t}\right\|_{\omega_E,E}\right)^2. \tag{3.22}$$

3.4. A priori error estimate for the coupled system

We shall state and prove the final result, which estimates the coupled system of flow and transport.

Theorem 3.1. *Let the integers λ, μ and ω be the regularity orders of functions c, $\frac{\partial c}{\partial t}$ and p and they take values λ_E, μ_E and ω_E on element E, respectively. The integer r_E denotes the order of discontinuous finite element space for c on element E and k is the order of RT_k space. We assume that*

$$h_E^{\min(r_E,\lambda_E-1,\mu_E-1,\omega_E-1,k+1)}=o(h^{n/2}),\quad \forall E\in\mathcal{T}_h. \tag{3.23}$$

Let $(p,\boldsymbol{u},c)$ be the solution of (1.1)-(1.6), which satisfy the following regularity requirements:

$$p\in L^2(J;H^\omega(\mathcal{T}_h)),\quad \frac{\partial p}{\partial t}\in L^2(J;H^\omega(\mathcal{T}_h)),$$
$$c\in L^2(J;H^\lambda(\mathcal{T}_h)),\quad \frac{\partial c}{\partial t}\in L^2(J;H^\mu(\mathcal{T}_h)).$$

We also assume that p, ∇p, c and ∇c are essentially bounded. Then there exists a positive constant K independent of h_E, r_E and k, such that

$$\begin{aligned}&\Big(\int_0^t\Big\|\frac{\partial E_p}{\partial t}\Big\|^2\Big)^{1/2}+||E_p||(t)+\Big(\int_0^t\Big\|\frac{\partial E_c}{\partial t}\Big\|^2\Big)^{1/2}\\&\quad+||\nabla E_c||(t)+||E_c||(t)+\big(J_0^\sigma(E_c,E_c)(t)\big)^{1/2}+||\boldsymbol{E_u}||(t)\\&\le K\sum_{E\in\mathcal{T}_h}\frac{h_E^{\min(k+1,\omega_E-1)}}{k^{\omega_E-1/2}}\Big(\|p\|_{\omega_E,E}+\|\frac{\partial p}{\partial t}\|_{\omega_E,E}\Big)\\&\quad+K\sum_{E\in\mathcal{T}_h}\Big(\frac{h_E^{\min(r_E,\lambda_E-1)}}{r_E^{\lambda_E-3/2}}||c||_{\lambda_E,E}+\frac{h_E^{\min(r_E,\mu_E-1)}}{r_E^{\mu_E-3/2}}\Big\|\frac{\partial c}{\partial t}\Big\|_{\mu_E,E}\Big).\end{aligned}$$

Proof. Multiplying (3.15) by (K_2+1) and adding it to (3.22), we get

$$\begin{aligned}&\int_0^t\Big\|\frac{\partial\pi}{\partial t}\Big\|^2+\Big\|\boldsymbol{\sigma}\Big\|^2(t)+\int_0^t\Big\|\frac{\partial\xi}{\partial t}\Big\|^2+||\nabla\xi||^2(t)+||\xi||^2(t)+J_0^\sigma(\xi,\xi)(t)\\&\le K\int_0^t\Big(\Big\|\frac{\partial\zeta}{\partial t}\Big\|^2+\Big\|\frac{\partial\eta}{\partial t}\Big\|^2+||\zeta||^2+||\xi||^2+\|\boldsymbol{\rho}\|^2+\|\boldsymbol{\sigma}\|^2\Big)\\&\quad+K\int_0^t\Big(||\nabla\zeta||^2+||\nabla\xi||^2+\Big\|\frac{\partial\zeta}{\partial t}\Big\|^2+\Big\|\nabla\frac{\partial\zeta}{\partial t}\Big\|^2\Big)+K\int_0^t J_0^\sigma(\xi,\xi)\\&\quad+K\sum_{E\in\mathcal{T}_h}\int_E|\nabla\zeta|^2(t)+K\int_0^t\sum_{\gamma\in\Gamma_h}\Big(\frac{r^2}{h_\gamma}\Big)^{-1}\Big(\|\nabla\zeta\cdot\boldsymbol{n}\|_{0,\gamma}^2+\Big\|\nabla\frac{\partial\zeta}{\partial t}\cdot\boldsymbol{n}\Big\|_{0,\gamma}^2\Big)\\&\quad+K\int_0^t\sum_{\gamma\in\Gamma_h}\frac{r^2}{h_\gamma}\Big(\|[\zeta]\|_{0,\gamma}^2+\Big\|\Big[\frac{\partial\zeta}{\partial t}\Big]\Big\|_{0,\gamma}^2\Big)+K\sum_{\gamma\in\Gamma_h}\Big(h_\gamma r^{-2}\|\nabla\zeta\cdot\boldsymbol{n}\|_{0,\gamma}^2(t)+h_\gamma^{-1}r^2\|[\zeta]\|_{0,\gamma}^2(t)\Big)\\&\quad+K\sum_{E\in\mathcal{T}_h}\frac{h_E^{2\min(k+1,\omega_E-1)}}{k^{2\omega_E-1}}\Big(\|p\|_{\omega_E,E}+\Big\|\frac{\partial p}{\partial t}\Big\|_{\omega_E,E}\Big)^2.\end{aligned}$$

Note that the following trivial inequality

$$\|v(t)\|^2 = \int_0^t \frac{d}{dt}\|v(t)\|^2 = \int_0^t 2\|v(t)\| \cdot \Big\|\frac{\partial v(t)}{\partial t}\Big\| \le \varepsilon \int_0^t \Big\|\frac{\partial v}{\partial t}\Big\|^2 + K\int_0^t \|v\|^2$$

holds for any v which satisfies $v(0) = 0$ and recall the facts $\xi(\cdot, 0) = 0$ and $\pi(\cdot, 0) = 0$. Applying this inequality to ξ and π, we obtain

$$\begin{aligned}
&\int_0^t \Big\|\frac{\partial \pi}{\partial t}\Big\|^2 + \Big\|\boldsymbol{\sigma}\Big\|^2(t) + \int_0^t \Big\|\frac{\partial \xi}{\partial t}\Big\|^2 + ||\nabla\xi||^2(t) + ||\xi||^2(t) + \|\pi\|^2(t) + J_0^\sigma(\xi,\xi)(t) \\
\le& K\int_0^t \Big(\Big\|\frac{\partial \zeta}{\partial t}\Big\|^2 + \Big\|\frac{\partial \eta}{\partial t}\Big\|^2 + ||\zeta||^2 + ||\xi||^2 + ||\pi||^2 + \|\boldsymbol{\rho}\|^2 + \|\boldsymbol{\sigma}\|^2\Big) \\
&+ K\int_0^t \Big(||\nabla\zeta||^2 + ||\nabla\xi||^2 + \Big\|\frac{\partial \zeta}{\partial t}\Big\|^2 + \Big\|\nabla\frac{\partial \zeta}{\partial t}\Big\|^2\Big) \\
&+ K\int_0^t J_0^\sigma(\xi,\xi) + K\sum_{E\in\mathcal{T}_h}\int_E |\nabla\zeta|^2(t) + K\int_0^t \sum_{\gamma\in\Gamma_h}\Big(\frac{r^2}{h_\gamma}\Big)^{-1}\Big(\|\nabla\zeta\cdot\boldsymbol{n}\|_{0,\gamma}^2 + \Big\|\nabla\frac{\partial\zeta}{\partial t}\cdot\boldsymbol{n}\Big\|_{0,\gamma}^2\Big) \\
&+ K\int_0^t \sum_{\gamma\in\Gamma_h}\frac{r^2}{h_\gamma}\Big(\|[\zeta]\|_{0,\gamma}^2 + \Big\|\Big[\frac{\partial\zeta}{\partial t}\Big]\Big\|_{0,\gamma}^2\Big) + K\sum_{\gamma\in\Gamma_h}\Big(h_\gamma r^{-2}\|\nabla\zeta\cdot\boldsymbol{n}\|_{0,\gamma}^2(t) + h_\gamma^{-1}r^2\|[\zeta]\|_{0,\gamma}^2(t)\Big) \\
&+ K\sum_{E\in\mathcal{T}_h}\frac{h_E^{2\min(k+1,\omega_E-1)}}{k^{2\omega_E-1}}\Big(\|p\|_{\omega_E,E} + \Big\|\frac{\partial p}{\partial t}\Big\|_{\omega_E,E}\Big)^2.
\end{aligned}$$

Applying Lemma 3.1, Lemma 3.2, the projection error estimates (3.4)-(3.5), the approximation properties (3.6)-(3.7), and Gronwall's inequality to the above equation, we find

$$\begin{aligned}
&\int_0^t \Big\|\frac{\partial \pi}{\partial t}\Big\|^2 + \|\boldsymbol{\sigma}\|^2(t) + \int_0^t \Big\|\frac{\partial \xi}{\partial t}\Big\|^2 + ||\nabla\xi||^2(t) + ||\xi||^2(t) + \|\pi\|^2(t) + J_0^\sigma(\xi,\xi)(t) \\
\le& K\Big(\sum_{E\in\mathcal{T}_h}\frac{h_E^{\min(k+1,\omega_E-1)}}{k^{\omega_E-1/2}}\Big(\|p\|_{\omega_E,E} + \Big\|\frac{\partial p}{\partial t}\Big\|_{\omega_E,E}\Big)\Big)^2 \\
&+ K\Big(\sum_{E\in\mathcal{T}_h}\Big(\frac{h_E^{\min(r_E,\lambda_E-1)}}{r_E^{\lambda_E-3/2}}||c||_{\lambda_E,E} + \frac{h_E^{\min(r_E,\mu_E-1)}}{r_E^{\mu_E-3/2}}\Big\|\frac{\partial c}{\partial t}\Big\|_{\mu_E,E}\Big)\Big)^2. \qquad (3.24)
\end{aligned}$$

By means of the triangle inequality and the estimates for $\boldsymbol{\rho}$, η and ζ, we get the desired result, which completes the proof. □

Remark 3.1. Theorem 3.1 gives an error estimate in $L^2(H^1)$ and $L^\infty(L^2)$ norm for concentration, and also gives a $L^\infty(L^2)$ rate of convergence for velocity.

4. Proofs of the Induction Hypotheses

We need to verify the induction hypotheses (3.8)-(3.9). Using the assumption (3.23) and the inequality (3.24), we find that when h tends to zero,

$$\begin{aligned}
h^{-n/2}\|\boldsymbol{\sigma}\| \le& Kh^{-n/2}\Big(\sum_{E\in\mathcal{T}_h}\frac{h_E^{\min(k+1,\omega_E-1)}}{k^{\omega_E-1/2}}\Big(\|p\|_{\omega_E,E} + \Big\|\frac{\partial p}{\partial t}\Big\|_{\omega_E,E}\Big)\Big) \\
&+ Kh^{-n/2}\Big(\sum_{E\in\mathcal{T}_h}\Big(\frac{h_E^{\min(r_E,\lambda_E-1)}}{r_E^{\lambda_E-3/2}}||c||_{\lambda_E,E} + \frac{h_E^{\min(r_E,\mu_E-1)}}{r_E^{\mu_E-3/2}}\Big\|\frac{\partial c}{\partial t}\Big\|_{\mu_E,E}\Big)\Big) \to 0.
\end{aligned}$$

This shows the induction hypothesis (3.8). To prove the induction hypothesis (3.9), we shall use the estimate of $\frac{\partial \boldsymbol{\sigma}}{\partial t}$. Differentiating Eqs. (3.10) and (3.11) with respect to the time variable t and taking $w = \frac{\partial \pi}{\partial t}$ and $\boldsymbol{v} = \frac{\partial \boldsymbol{\sigma}}{\partial t}$ respectively, leads to

$$\frac{1}{2}\frac{d}{dt}\Big(d(C)\frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t}\Big) + \Big(\nabla \cdot \frac{\partial \boldsymbol{\sigma}}{\partial t}, \frac{\partial \pi}{\partial t}\Big)$$
$$= \Big(\frac{\partial\big((d(C)-d(c))\frac{\partial \widetilde{p}}{\partial t}\big)}{\partial t}, \frac{\partial \pi}{\partial t}\Big) - \Big(\frac{\partial d(c)}{\partial c}\frac{\partial c}{\partial t}\frac{\partial \eta}{\partial t} + d(c)\frac{\partial^2 \eta}{\partial t^2}, \frac{\partial \pi}{\partial t}\Big) - \frac{1}{2}\Big(\frac{\partial d(C)}{\partial C}\frac{\partial C}{\partial t}\frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t}\Big),$$
$$\Big(\alpha(C)\frac{\partial \boldsymbol{\sigma}}{\partial t}, \frac{\partial \boldsymbol{\sigma}}{\partial t}\Big) - \Big(\nabla \cdot \frac{\partial \boldsymbol{\sigma}}{\partial t}, \frac{\partial \pi}{\partial t}\Big) = \Big(\frac{\partial\big((\alpha(C)-\alpha(c))\widetilde{\boldsymbol{u}}\big)}{\partial t}, \frac{\partial \boldsymbol{\sigma}}{\partial t}\Big) - \Big(\frac{\partial \alpha(C)}{\partial C}\frac{\partial C}{\partial t}\boldsymbol{\sigma}, \frac{\partial \boldsymbol{\sigma}}{\partial t}\Big).$$

Consequently,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(d(C)\frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t}\Big) + \Big(\alpha(C)\frac{\partial \boldsymbol{\sigma}}{\partial t}, \frac{\partial \boldsymbol{\sigma}}{\partial t}\Big)\\
=&\Bigg(\frac{\partial\Big((d(C)-d(c))\frac{\partial \widetilde{p}}{\partial t}\Big)}{\partial t}, \frac{\partial \pi}{\partial t}\Bigg) - \Big(\frac{\partial d(c)}{\partial c}\frac{\partial c}{\partial t}\frac{\partial \eta}{\partial t} + d(c)\frac{\partial^2 \eta}{\partial t^2}, \frac{\partial \pi}{\partial t}\Big) + \Big(\frac{\partial\big((\alpha(C)-\alpha(c))\widetilde{\boldsymbol{u}}\big)}{\partial t}, \frac{\partial \boldsymbol{\sigma}}{\partial t}\Big)\\
&-\frac{1}{2}\Big(\frac{\partial d(C)}{\partial C}\frac{\partial C}{\partial t}\frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t}\Big) - \Big(\frac{\partial \alpha(C)}{\partial C}\frac{\partial C}{\partial t}\boldsymbol{\sigma}, \frac{\partial \boldsymbol{\sigma}}{\partial t}\Big)\\
\equiv&\sum_{i=1}^{5} Y_i.
\end{aligned} \tag{4.1}$$

Next, we shall bound all the items on the right-hand side of (4.1). Using Cauchy-Schwartz inequality, Young's inequality with ε, we get

$$\begin{aligned}
|Y_1| =&\Big|\Big(\Big(\frac{\partial d(C)}{\partial C}\frac{\partial C}{\partial t} - \frac{\partial d(c)}{\partial c}\frac{\partial c}{\partial t}\Big)\frac{\partial \widetilde{p}}{\partial t} + (d(C)-d(c))\frac{\partial^2 \widetilde{p}}{\partial t^2}, \frac{\partial \pi}{\partial t}\Big)\Big|\\
\le&K\Big(\Big\|\frac{\partial \pi}{\partial t}\Big\|^2 + \Big\|\frac{\partial \xi}{\partial t}\Big\|^2 + \Big\|\frac{\partial \zeta}{\partial t}\Big\|^2 + ||\xi||^2 + ||\zeta||^2\Big),\\
|Y_2| \le&K\Big(\Big\|\frac{\partial \pi}{\partial t}\Big\|^2 + \Big\|\frac{\partial \eta}{\partial t}\Big\|^2\Big) + \varepsilon\Big\|\frac{\partial^2 \eta}{\partial t^2}\Big\|^2,\\
|Y_3| =&\Big|\Big(\widetilde{\boldsymbol{u}}\Big(\frac{\partial \alpha(C)}{\partial C}\frac{\partial C}{\partial t} - \frac{\partial \alpha(c)}{\partial c}\frac{\partial c}{\partial t}\Big) + (\alpha(C)-\alpha(c))\frac{\partial \widetilde{\boldsymbol{u}}}{\partial t}, \frac{\partial \boldsymbol{\sigma}}{\partial t}\Big)\Big|\\
\le&\varepsilon\|\frac{\partial \boldsymbol{\sigma}}{\partial t}\|^2 + K\Big(\Big\|\frac{\partial \xi}{\partial t}\Big\|^2 + \Big\|\frac{\partial \zeta}{\partial t}\Big\|^2 + ||\xi||^2 + ||\zeta||^2\Big).
\end{aligned}$$

By the assumption (3.23) and the inequality (3.24), we see that

$$h^{-n/2}\Big\|\frac{\partial \pi}{\partial t}\Big\|_{L^2(L^2(\Omega))} \to 0.$$

Thus, if h is chosen sufficiently small, then

$$\begin{aligned}
|Y_4| =&\Big|\frac{1}{2}\Big(\frac{\partial d(C)}{\partial C}\Big(\frac{\partial \widetilde{c}}{\partial t} - \frac{\partial \xi}{\partial t}\Big)\frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t}\Big)\Big|\\
\le&K\Big\|\frac{\partial \pi}{\partial t}\Big\|^2 + h^{-n/2}\Big\|\frac{\partial \xi}{\partial t}\Big\| \cdot \Big\|\frac{\partial \pi}{\partial t}\Big\|^2 \le K\Big(\Big\|\frac{\partial \pi}{\partial t}\Big\|^2 + \Big\|\frac{\partial \xi}{\partial t}\Big\|^2\Big).
\end{aligned}$$

Applying the induction hypothesis (3.8) to Y_5, we see that when h is chosen sufficiently small,

$$
\begin{aligned}
|Y_5| =&\left|\left(\frac{\partial\alpha(C)}{\partial C}\left(\frac{\partial\widetilde{c}}{\partial t}-\frac{\partial\xi}{\partial t}\right)\boldsymbol{\sigma},\frac{\partial\boldsymbol{\sigma}}{\partial t}\right)\right|\\
\le& K\left\|\frac{\partial\boldsymbol{\sigma}}{\partial t}\right\|\left(\|\boldsymbol{\sigma}\|+h^{-n/2}\|\boldsymbol{\sigma}\|\cdot\left\|\frac{\partial\xi}{\partial t}\right\|\right)\le\varepsilon\left\|\frac{\partial\boldsymbol{\sigma}}{\partial t}\right\|^2+K\|\boldsymbol{\sigma}\|^2+K\left\|\frac{\partial\xi}{\partial t}\right\|^2.
\end{aligned}
$$

Combining all the above inequalities yields

$$
\begin{aligned}
&\left\|\frac{\partial\boldsymbol{\sigma}}{\partial t}\right\|^2+\frac{d}{dt}\left(d(C)\frac{\partial\pi}{\partial t},\frac{\partial\pi}{\partial t}\right)\\
\le& K\left(\|\frac{\partial\pi}{\partial t}\|^2+\left\|\frac{\partial\xi}{\partial t}\right\|^2+\left\|\frac{\partial\zeta}{\partial t}\right\|^2+||\xi||^2+||\zeta||^2+\left\|\frac{\partial\eta}{\partial t}\right\|^2+\|\boldsymbol{\sigma}\|^2\right)+2\varepsilon\left\|\frac{\partial\boldsymbol{\sigma}}{\partial t}\right\|^2.
\end{aligned}
$$

Recall that $\boldsymbol{\sigma}(0)=0,\pi(0)=0$. Use Gronwall's lemma and the estimate (3.24) to get

$$
\begin{aligned}
\left\|\frac{\partial\boldsymbol{\sigma}}{\partial t}\right\|_{L^2(L^2(\Omega))}\le& K\left(\sum_{E\in\mathcal{T}_h}\frac{h_E^{\min(k+1,\omega_E-1)}}{k^{\omega_E-1/2}}\left(\|p\|_{\omega_E,E}+\left\|\frac{\partial p}{\partial t}\right\|_{\omega_E,E}\right)\right)\\
&+K\left(\sum_{E\in\mathcal{T}_h}\left(\frac{h_E^{\min(r_E,\lambda_E-1)}}{r_E^{\lambda_E-3/2}}\|c\|_{\lambda_E,E}+\frac{h_E^{\min(r_E,\mu_E-1)}}{r_E^{\mu_E-3/2}}\left\|\frac{\partial c}{\partial t}\right\|_{\mu_E,E}\right)\right).
\end{aligned}
$$

Thus, when h is chosen sufficiently small, the induction hypothesis (3.9) is satisfied by virtue of the assumption (3.23). □

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References

[1] I. Babuška and M. Suri, The optimal convergence rate of the p-version of the finite element method, *SIAM J. Numer. Anal.*, **24** (1987), 750-776.

[2] I. Babuška and M. Suri, The *h-p* version of the finite element method with quasiuniform meshes, *RAIRO Math. Model Numer. Anal.*, **21** (1987), 199-238.

[3] H. Chen and Y. Chen, A combined mixed finite element and discontinuous Galerkin method for compressible miscible displacement problem, *Natur. Sci. J. Xiangtan Univ.*, **26**:2 (2004), 119-126.

[4] A. Cheng, Optimal error estimate in $L^\infty(J;L^2(\Omega))$ norm of FEM for a model for compressible miscible displacement in porous media, *Numer. Math.*, **4**:2 (1995), 222-236.

[5] Y. Chen and J. Yang, A posteriori error estimation for a fully discrete discontinuous Galerkin approximation to a kind of singularly perturbed problems, *Finite Elem. Anal. Des.*, **43**:10 (2007), 757-770.

[6] M.R. Cui, A combined mixed and discontinuous Galerkin method for compressible miscible displacement problem in porous media, *J. Comput. Appl. Math.*, **198**:1 (2007), 19-34.

[7] V. Dolejsi, M. Feistauer, V. Sobotikova, Analysis of the discontinuous Galerkin method for nonlinear convection-diffusion problems, *Comput. Method. Appl. M.*, **194** (2005), 2709-2733.

[8] J. Douglas Jr., R.E. Ewing and M.F. Wheeler, The approximation of the pressure by a mixed method in the simulation of miscible displacement, *RAIRO Numer. Anal.*, **17** (1983), 17-33.

[9] J. Douglas Jr., R.E. Ewing and M.F. Wheeler, A time-discretization procedure for a mixed finite element approximation of miscible displacement in porous media, *RAIRO Numer. Anal.*, **17** (1983), 249-265.

[10] J. Douglas Jr. and J.E. Roberts, Numerical methods for a model for compressible miscible displacement in porous media, *Math. Comput.*, **41** (1983), 441-459.

[11] J. Douglas Jr. and J.E. Roberts, Global estimates for mixed finite element methods for second order elliptic equations, *Math. Comput.*, **44** (1985), 39-52.

[12] V. Dolejsi, M. Feistauer, Error estimates of the discontinuous Galerkin method for nonlinear nonstationary convection-diffusion problems, *Numer. Funct. Anal. Opt.*, **26**:3 (2005), 349-383.

[13] M. Feistauer, K. Svadlenka, Discontinuous Galerkin method of lines for solving nonstationary singularly perturbed linear problems, *J. Numer. Math.*, **12** (2004), 97-118.

[14] Q. Li, S. Chou, Mixed methods for compressible miscible displacement with the effect of molecular dispersion, *Acta Math. Sin.*, **11**:2 (1995), 123-138.

[15] J.T. Oden, I. Babuška, and C.E. Baumann, A discontinuous hp finite element method for diffusion problems, *J. Comput. Phys.*, **146** (1998), 491-516.

[16] B. Rivière, Discontinuous Galerkin Methods for Solving the Miscible Displacement Problem in Porous Media, *Ph.D. Thesis*, The University of Texas at Austin, Austin, 2000.

[17] B. Rivière, M.F. Wheeler, Discontinuous Galerkin methods for coupled flow and transport problems, *Commun. Numer. Methods En.*, **18** (2002), 63-68.

[18] B. Rivière, M.F. Wheeler, and V. Girault, A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems, *SIAM J. Numer. Anal.*, **39** (2001), 902-931.

[19] A. Romkes, S. Prudhomme, J.T. Oden, A priori error analysis of a stabilized discontinuous Galerkin method, *Comput. Math. Appl.*, **46** (2003), 1289-1311.

[20] S. Sun, Discontinuous Galerkin Methods for Reactive Transport in Porous Media, *Ph.D. Thesis*, The University of Texas at Austin, Austin, 2003.

[21] S. Sun, B. Rivière, M.F. Wheeler, A combined mixed finite element and discontinuous Galerkin method for miscible displacement problem in porous media, *Recent Progress in Computational and Applied PDEs* , Kluwer Academic Publishers, Plenum Press, Dordrecht, New York, (2002), 323-351.

[22] S. Sun, M.F. Wheeler, Discontinuous Galerkin methods for coupled flow and reactive transport problems, *Appl. Numer. Math.*, **52** (2005), 273-298.

[23] S. Sun and M.F. Wheeler, $L^2(H^1)$ norm a posteriori error estimation for discontinuous Galerkin approximations of reactive transport problems, *J. Sci. Comput.*, **22** (2005), 511-540.

[24] M.F. Wheeler, S. Sun, O. Eslinger, and B. Rivière, Discontinuous Galerkin method for modeling flow and reactive transport in porous media, in *Analysis and Simulation of Multified Problem*, Springer Verlag, August 2003, 37-58.

[25] J. Yang and Y. Chen, A unified a posteriori error analysis for discontinuous Galerkin approximations of reactive transport equations, *J. Comput. Math.* , **24**:3 (2006), 425-434.

[26] J. Yang and Y. Chen, A priori error analysis of a discontinuous Galerkin approximation for a kind of compressible miscible displacement problems, *Sci. China Math.*, DOI: 10.1007/s11425-010-3128-2.