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A P-VERSION TWO LEVEL SPLINE METHOD FOR SEMI-LINEAR ELLIPTIC EQUATIONS*

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Abstract

A novel two level spline method is proposed for semi-linear elliptic equations, where the two level iteration is implemented between a pair of hierarchical spline spaces with different orders. The new two level method is implementation in a manner of p-adaptivity. A coarse solution is obtained from solving the model problem in the low order spline space, and the solution with higher accuracy are generated subsequently, via one step Newton or monidifed Newton iteration in the high order spline space. We also derive the optimal error estimations for the proposed two level schemes. At last, the illustrated numerical results confirm our error estimations and further research topics are commented.

Mathematics subject classification: 65N30, 65M55.
Key words: P-version, Two level method, Spline methods, Semi-linear, Error estimation.

1. Introduction

The finite element methods are widely used for its convenience and efficiency in the construction of the finite element spaces. Practically, any spline space can be viewed and used as the finite element space [7, 13]. Several applications appeared in numerical solution for Partial Differential Equations (PDEs) in recent years. Lai and Wenston [6] suggest a spline method for steady state Navier-Stokes equation in spline spaces, where the Newton iteration is employed to resolve the nonlinearity. Speleers and Dierckx [12], Li and Wang [8] prefer constructing such spline space in explicit manners. At the same time, Awanou and Lai [1, 2] generalize their method to three dimensional cases, which is also referred as spline element method. Their research illustrates that the general definition for the spline space is so flexible that it is convenient in many complex cases, as well as fulfilling the high order and/or high smoothness requirements in finite element applications.

It is also well accepted that one can obtain high resolution when the mesh size is small or the degree of the spline space is large enough, however, it could be rather time and storage consuming. Two level methods can effectively reduce the costs arising from the refinement of the finite element spaces. In many applications, the two level methods are implemented via mesh refinement, hence it is also named as two-grid methods. It has been proved by Xu et. al [9,14,15] that the two-grid methods have optimal convergence rates for finite element solution of the semi-linear elliptic equations. Recently, we developed a two-grid method for the spline methods [11], where the two grid acceralation is proved to be effective in the circumstance of spline methods.

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On the other hand, it is true that the spline method with local p-refinement, which means raising the degree of spline space locally, can reduce the costs in the linear cases [5], however, the local p-adaptivity is sensitive and some parameters have to be adjusted when applying to different problems. In the current research, we are interested in the spline methods with global p-refinement, and the two level method is considered to reduce the costs arising from global p-refinement. It is potentially more competitive than mesh refinement strategy and we refer it as a p-version two level spline method.

The rest of this paper is organized as following. In the subsequent section, we explain the details of the new two level spline methods for the semi-linear elliptic equation, and the corresponding optimal error estimations are derived in Section 3. Numerical examples are illustrated in Section 4, where the results verify the accuracy of the proposed schemes. Some conclusions and remarks are figured out in the last section. For the ease of reference, let us introduce some basic notations first. Denote $W_p^m(\Omega)$ as the standard Sobolev space equipped with p-norm $\|\cdot\|_{m,p}$ and $|\cdot|_{m,p}$ is the corresponding semi-norm. As the traditional settings, $W_2^m(\Omega) := H^m(\Omega)$ and $H_0^1(\Omega)$ is referred as its restriction with zero boundary. We also assume $\|\cdot\| = \|\cdot\|_{0,2}$ and $\|\cdot\|_m = \|\cdot\|_{m,2}$ as usual.

2. The P-version two Level Spline Methods

2.1. The spline spaces

Let $\triangle$ be a regular triangulation. Define a bivariate spline space $S_d^r(\triangle)$ on $\triangle$ as

$$S_d^r(\triangle) := \left\{ s \in C^r(\Omega), s|_t \in \mathbf{P}^d, t \in \triangle \right\},$$

where $\mathbf{P}^d$ is the space of bivariate polynomials with degree d and smoothness $r \geq 0$. Such spline space $S_d^r(\triangle)$ with smoothness r exists provided that the degree $d \geq 3r+2$. For the proof of the existence, we refer to [7].

There are different representation forms of a polynomial with degree d. In the implementation of the multivariate spline spaces, the B-form is preferred to use. Let a triangle $t \in \triangle$ with vertices (v_1, v_2, v_3) and $(\lambda_1, \lambda_2, \lambda_3)$ be the barycentric coordinates of any point (x, y) with respect to triangle t, then any polynomial p with degree d can be represented by

$$p := \sum_{i+j+k=d} c_{ijk} B_{ijk}(\lambda_1, \lambda_2, \lambda_3), \tag{2.1}$$

where $\{c_{i,j,k}\}_{i+j+k=d}$ are called B-coefficients and $B_{ijk}(\lambda_1, \lambda_2, \lambda_3) = \dfrac{d!}{i!j!k!}\lambda_1^i \lambda_2^j \lambda_3^k$ are the bivariate Bernstein Polynomials for all $i+j+k=d$.

Now we consider the global C^r smoothness conditions for a spline on Δ. As a matter of fact, the C^1 case is enough here, and one can refer to [7] for more general C^r cases. It is sufficient to consider the case crossing the common edge between the adjacent patches. Let the triangle t with vertices v_1, v_2, v_3 and the triangle t' with vertices v_4, v_3, v_2 in $\triangle$ sharing one common edge e. Assume that $\lambda := (\lambda_1, \lambda_2, \lambda_3)$ is the barycentric coordinates of v_4 with respect to t, $\{c_{i,j,k}\}_{i+j+k=d}$ and $\{c'_{i,j,k}\}_{i+j+k=d}$ are the B-coefficients for any spline s on t and t' respectively, then $s \in S_d^1(\triangle)$ if and only if the condition

$$c'_{1,j,k} = \lambda_1 c_{1,j,k} + \lambda_2 c_{0,j+1,k} + \lambda_3 c_{0,j,k+1}, \qquad \forall j+k=d$$

hold, which is also called C^1 smoothness condition for s. When the common edge e runs over all the inner edge of $\triangle$, we get a series of linear equations as the constrains

$$H\mathbf{c} = \mathbf{0}, \tag{2.2}$$

where $\mathbf{c}$ is the vector of B-coefficients of the spline on all triangles. For the ease of representation, we refer (2.2) as the smoothness matrix in the rest of this paper. Then for any spline $s \in S_d^r(\triangle)$, we can represent it with its B-coefficient vector $\mathbf{c}$ satisfying both (2.1) and (2.2). We can also find the an error estimation for quasi-interpolation in $S_d^r(\triangle)$, which is useful in evaluating the approximation errors for the spline solutions later.

Lemma 2.1. *Let two constants d and $r \geq 0$ satisfying $d \geq 3r+2$, and suppose $\triangle$ is a regular triangulation of a domain Ω. Then for any $f \in W_p^{d+1}(\Omega)$, there exists a spline $Qf \in S_d^r(\triangle)$, such that*

$$||D_x^\alpha D_y^\beta (f - Qf)||_{0,p} \leq Ch^{d+1-\alpha-\beta}|f|_{d+1,p} \tag{2.3}$$

where $0 \leq \alpha + \beta \leq d, 1 \leq p \leq +\infty$ and $h = |\triangle|$ is the mesh size.

2.2. The spline approximation for the semi-linear problem

We are interested in the semi-linear elliptic equation with $u = 0$ on the boundary. It is remarkable that other types of boundary conditions can also be considered in a similar way.

$$-\Delta u + f(x,u) = 0, \quad x \in \Omega, \tag{2.4}$$

where $\Omega \subset R^2$ is a convex polygonal domain, $f(x,u)$ is nonlinear with respect to u and continuous with respect to both x and u. It is pointed out that the above equation has at least one solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ when the linearized operator $L_u \equiv -\Delta + f_u(x,u)$ is non-singular [14]. Then the weak solution for the above model is to find $u \in H_0^1(\Omega) \cap H^2(\Omega)$, satisfying

$$a(u,v) + F(u,v) = 0, \quad \forall v \in H_0^1(\Omega), \tag{2.5}$$

where $a(u,v) = (\nabla u, \nabla v)$ and $F(u,v) = (f(x,u),v)$.

The spline method is to find an approximation $u_h \in S_d^r(\triangle) \cap H^1(\Omega)$, which satisfy the zero boundary condition and

$$a(u_h, v_h) + F(u_h, v_h) = 0, \quad \forall v_h \in S_d^r(\triangle) \cap H_0^1(\Omega). \tag{2.6}$$

It is possible to find a spline approximation for specified partial differential equation as well as the standard finite element method.

Let us consider the numerical method for solving equation (2.6) when only one single spline space is considered. Due to the nonlinearity of operator $F(\cdot,\cdot)$, the traditional Newton iteration can be applied. For the ease of representation, we define an operator

$$A(w;u,v) = a(u,v) + N(w;u,v), \tag{2.7}$$

where $N(u;w,v) = (f_u(x,u)w,v)$. Then for any given tolerance $\epsilon > 0$ and an initial guess $u_h^0 \in S_d^r(\triangle)$, each step of the Newton iteration is required to solve the following linear equation

$$A(u_h^m; u_h^{m+1}, v_h) = N(u_h^m; u_h^m, v_h) - F(u_h^m, v_h), \quad \forall v_h \in S_d^r(\triangle) \cap H_0^1(\Omega) \tag{2.8}$$

until $\|u_h^{m+1} - u_h^m\| \leq \epsilon$, where the superscription m means the value for m's step of the Newton iteration. The above Newton iteration has second order convergence rate in common, however, it is still rather time consuming for certain large value of d. People are willing to reduce the costs of the iteration considerably by two level method. Before that, we want to prove that the following error estimation holds for the spline solutions of semi-linear elliptic equation:

Theorem 2.1. *Assuming that* $u \in W_p^{d+1}(\Omega), 2 \leq p < \infty$ *is the solution of* (2.4) *and* $u_h \in S_d^r(\triangle)$ *is a spline solution for* (2.6), *then*

$$\|u - u_h\|_{0,p} + h\|u - u_h\|_{1,p} \leq Ch^{d+1}, \tag{2.9}$$

where C *is a constant independent on* h.

Proof. It has been proved in [15] that the weak form (2.5) is coesive. According to Lemma 2.1, if u_h is a solution of (2.6) in the spline space $S_d^r(\triangle) \cap H_0^2(\Omega)$, then we have

$$\|u - u_h\|_{1,p} \leq C \inf_{\chi \in S_d^r(\triangle)} \|u - \chi\|_{1,p} \leq Ch^d,$$

where the mesh size h is sufficient small and C is a const independent on h for any $2 \leq p \leq +\infty$. It is notable that C should be related with $\|u\|_{1,p}$, however, the regularity of u make it possible to absorb this into a const, so all the constant C have the same meaning in the rest of this paper. Then following the analysis in [15], it is true that

$$\|u - u_h\|_{0,p} \leq C\Big(h\|u - u_h\|_{1,p} + \|u - u_h\|_{1,2s}^2\Big),$$

where $2 \leq p < +\infty$ and $s > 2p/(p+2)$. If $p = 2$, for arbitrary $\epsilon > 0$,

$$\|u - u_h\|_{0,2} \leq C\Big(h\|u - u_h\|_{1,2} + \|u - u_h\|_{1,2+\epsilon}^2\Big) \leq Ch^{d+1}.$$

For $2 < p < +\infty$, since $4p/(p+2) < p$, we obtain

$$\|u - u_h\|_{0,p} \leq C\Big(h\|u - u_h\|_{1,p} + \|u - u_h\|_{1,p}^2\Big).$$

Then (2.9) holds due to above two estimations.

2.3. The new two level spline methods

Two level methods require two hierarchical finite element spaces, namely, the coarse space and the fine space. The fundamental idea is to solve a "hard" problem(for example, nonlinear and not symmetric) in the coarse space and subsequently a "easy" problem(linear, symmetric positive problem) in the fine space. Since the size of the coarse space is much smaller than that of the fine space in two dimensional cases, the cost for solving a nonlinear problem in the coarse space is relatively cheap. So that considerable quantity of computational costs can be saved without any losing in accuracy. Practically, the most common method for constructing hierarchical finite element spaces is mesh refinement, while in the literature of spline methods, the fine spaces can also be got via degree elevation of the spline spaces.

In the spline methods, the polynomials' order is always large, which result in considerable costs not only in computation time but also in storage requirement. The two level methods can usually reduce the cost effectively, which is essential for the nonlinear iterations. The first task is to constructing two spline spaces $S_d^r(\triangle) \subset S_D^r(\triangle)$ with degree d and D satisfying $3r+2 \le d < D$, which we still refer them as the coarse space and the fine space. In the two level methods, Newton iteration is often used to solve the nonlinear equation (2.6) in the fine space, while the initial guess u_h of the iteration can be provided by the solution of (2.6) in the coarse space. As a matter of fact, only one single linearized problem need to be solved in the fine spline space $S_D^r(\triangle)$, as well as find a solution $u_N \in S_D^r(\triangle)$ satisfying

$$A(u_h; u_N, v_h) = N(u_h; u_h, v_h) - F(u_h, v_h), \quad \forall v_h \in S_D^r(\triangle) \cap H_0^1(\Omega). \tag{2.10}$$

It is obvious that the two level method is much cheaper than executing the nonlinear iterations in $S_D^r(\triangle)$ directly. Furthermore, the spline solution u_N obtained from above procedure has second order accuracy, which has the same convergence rate as those from the standard Newton iterations. So that we refer (2.10) as two level Newton Method. It is further proved in the next section that the following correction step yields a third order two level method

$$A(u_h; u_M, v_h) = N(u_h; u_N, v_h) - F(u_N, v_h), \quad \forall v_h \in S_D^r(\triangle) \cap H_0^1(\Omega). \tag{2.11}$$

For the easy of reference, let us refer (2.11) as the two level modified Newton method, as well as in the nonlinear iteration theory.

In this case, the basis of the spline spaces should be hierarchy with respect to their order, which plays an important role in transferring the numerical solutions from the low order spline space to the high order one. The B-form representation for bivariate polynomials has the so-called degree elevating algorithm(c.f. [7]), which can fulfill the requirements.

Algorithm 2.1. *Let p be a polynomial of degree d defined on a triangle t in the form of* (2.1)*, then p can be evaluated to a polynomial of degree $d+1$ with the form $p = \sum_{i+j+k=d+1} c'_{ijk} B_{ijk}(x,y)$, where*

$$c'_{ijk} = \left(\frac{i}{d+1} c_{i-1,j,k} + \frac{j}{d+1} c_{i,j-1,k} + \frac{k}{d+1} c_{i,j,k-1}\right), \tag{2.12}$$

for $i+j+k = d+1$, and the coefficients with negative subscripts are assumed to be zero.

It is mentionable that the above degree elevating algorithm is free from error for arbitrary choice of d and D. One can elevate the degree of a polynomial from d to D by repeatedly executing the algorithm. For the ease of reference, we would like to conclude the p-version two level spline methods into the following algorithm.

Algorithm 2.2. *Two level spline methods based on the Newton iterations*
Input:$\triangle$, d, D, r
Output:u_h, u_N, u_M

1. *Build two hierarchical spline spaces $S_d^r(\triangle)$ and $S_D^r(\triangle)$;*
2. *Obtain a low order solution $u_h \in S_d^r(\triangle)$ with any nonlinear iterations;*
3. *Elevate the degree of u_h from d to D via* (2.12) *in Algorithm* 2.1*;*
4. *Calculate $u_N \in S_D^r(\triangle)$ by one step Newton iteration* (2.10)*;*
5. *Obtain a higher accurate solution $u_M \in S_D^r(\triangle)$ by correction* (2.11).

It is remarkable that almost no extra costs arise in step 5 of Algorithm 2.2, since the coefficient matrix in step 5 is exactly the same with that in step 4. Actually, one can drop step 5 if the difference between d and D is not too big. More accurate error estimations are derived in the next section.

3. The Error Estimates for Two Level Spline Methods

In order to derivative the error estimations for the spline solutions obtained by Algorithm 2.2, we need an important lemma in the case of semi-linear elliptic problems.

Lemma 3.1. *Let u be a weak solution of the semi-linear elliptic problem in* (2.5)*. Then there exists a constant $\delta > 0$, such that for any $v \in S_d^r(\triangle) \cap H_0^1(\Omega)$ with $\|u - v\|_{0,\infty} < \delta$, there exists a constant $C(\delta)$ satisfying*

$$\sup_{\chi \in S_d^r(\triangle)} \frac{A(v; w, \chi)}{\|\chi\|_1} \geq C\|w\|_1, \quad \forall w \in S_d^r(\triangle), \tag{3.1}$$

where $A(v; w, \chi) = a(w, \chi) + N(v; w, \chi)$.

This result is a direct generation of the results in [14]. Since $S_d^r(\triangle)$ is a finite dimensional function space with full approximation power due to Lemma 2.1, the above result is soon followed from the results of [14]. Then we can get the error estimations both for the spline solutions u_N and u_M in a traditional way.

Theorem 3.1. *Assume that two spline spaces $S_d^r(\triangle)$ and $S_D^r(\triangle)$ are well defined with $3r+2 \leq d < D$. If u_N is the spline approximation obtained from Algorithm* 2.2 *and $u \in H^{2d+3}(\Omega)$ is the solution of the original model* (2.5)*, then*

$$\|u_N - u\|_1 \leq Ch^{\min\{D, 2d+2\}},$$

where h is the mesh size and C is a constant independent on h.

Proof. Let u_H be the solution obtained with Newton iteration (2.8) in one single spline space $S_D^r(\triangle)$, that is, u_H satisfies

$$a(u_H, v_h) = -F(u_H, v_h), \quad \forall v_h \in S_D^r(\triangle) \cap H_0^1(\Omega).$$

Hence $\|u_H - u\|_1 = Ch^D$ directly follows from Theorem 2.1. For convenience, let us assume that all $v_h \in S_D^r(\triangle) \cap H_0^1(\Omega)$ in this theorem. Because of the definition of functional $F(\cdot, \cdot)$ and $N(\cdot; \cdot, \cdot)$, we have

$$a(u_H, v_h) + N(u_h; u_H, v_h) = -(f(x, u_H) - f_u(x, u_h)u_H, v_h).$$

Substracting the above equation from (2.10), then we get

$$\begin{aligned} A(u_h; u_N - u_H, v_h) &:= a(u_N - u_H, v_h) + N(u_h; u_N - u_H, v_h) \\ &= (f(x, u_H) - f(x, u_h) - f_u(x, u_h)(u_H - u_h), v_h) \\ &= (\mu(u_H - u_h)^2, v_h), \end{aligned} \tag{3.2}$$

where $\mu = \int_0^1 (1-t) f_{uu}(x, u_h + t(u_H + u_h))dt$ is uniformly (with respect to h) bounded on $\bar{\Omega}$ when f is sufficient smooth. Following the Hölder inequality and the Sobolev inequality, it yields

$$(\mu(u_H - u_h)^2, v_h) \leq C\|(u_H - u_h)^2\|_{0,p/2}\|v_h\|_{0,\frac{p}{p-2}} \leq C\|u_H - u_h\|_{0,p}^2\|v_h\|_1 \tag{3.3}$$

for any $p \geq 2$. According to (3.1), if $\|u_h - u\|_{0,\infty} \leq \delta$, then the combination of (3.2) and (3.3) yields

$$\|u_N - u_H\|_1 \leq C\|u_H - u_h\|^2 \leq Ch^{2d+2}. \tag{3.4}$$

Since u_H is a direct approximation in single spline space $S^r_{2d+2}(\triangle)$, then (2.9) holds for u_H. When the triangle inequality is applied,

$$\begin{aligned} \|u_N - u\|_1 &\leq \|u_N - u_H\|_1 + \|u_H - u\|_1 \\ &\leq C_1 h^{2d+2} + C_2 h^D \leq Ch^{\min\{D,2d+2\}}, \end{aligned}$$

where C_1, C_2 and C are all constants independent of mesh size h. □

Theorem 3.2. *Assume $u \in H^{3d+4}(\Omega)$ and two spline spaces $S^r_d(\triangle)$ and $S^r_D(\triangle)$ are well defined with $3r + 2 \leq d < D$. If u_M is the spline approximation obtained from Algorithm* 2.2, *then*

$$\|u_M - u\|_1 \leq Ch^{\min\{D,3d+3\}},$$

where h is the mesh size and C is a constant independent on h.

Proof. Let u_H being the spline solution of (2.6) obtained by Newton iteration in a single spline space $S^r_D(\triangle)$, which satisfies

$$a(u_H, v_h) = -F(u_H, v_h), \quad \forall v_h \in S^r_D(\triangle) \cap H^1_0(\Omega).$$

Hence $\|u_H - u\|_1 = Ch^D$ directly follows from Theorem 2.1. According to the definition of $F(\cdot,\cdot)$ and $N(\cdot;\cdot,\cdot)$, the Taylor's formulae of $f(x, u_H)$ at u_h leads to

$$\begin{aligned} &A(u_h; u_H, v_h) \\ :=\ & a(u_H, v_h) + N(u_h; u_H, v_h) \\ =\ & -\big(f(x, u_H), v_h\big) + \big(f_u(x, u_h)u_H, v_h\big) \\ =\ & -\big(f(x, u_h) - f_u(x, u_h)u_h + \frac{1}{2} f_{uu}(x, u_h)(u_H - u_h)^2, v_h\big) + \big(O(u_H - u_h)^3, v_h\big). \end{aligned}$$

Here we always assume $v_h \in S^r_D(\triangle) \cap H^1_0(\Omega)$. Similarly, the following arguments hold according to (2.11)

$$\begin{aligned} &A(u_h; u_M, v_h) \\ :=\ & a(u_M, v_h) + N(u_h; u_M, v_h) \\ =\ & \big(-f(x, u_N) + f_u(x, u_h)u_N, v_h\big) \\ =\ & -\big(f(x, u_h) - f_u(x, u_h)u_h + \frac{1}{2} f_{uu}(x, u_h)(u_N - u_h)^2, v_h\big) + \big(O(u_N - u_h)^3, v_h\big). \end{aligned}$$

The substraction between above two equations gives

$$\begin{aligned} &A(u_h; u_M - u_H, v_h) \\ :=& a(u_M - u_H, v_h) + N(u_h; u_M - u_H, v_h) \\ \leq& \frac{1}{2} f_{uu}(x, u_h)\big((u_H - u_h)^2 - (u_N - u_h)^2, v_h\big) + \big(O(u_H - u_h)^3, v_h\big) + \big(O(u_N - u_h)^3, v_h\big). \end{aligned} \tag{3.5}$$

Using the Hölder inequality and the Sobolev inequality, we have

$$\begin{aligned}&\big((u_H-u_h)^2-(u_N-u_h)^2,v_h\big)\\ =&\big((u_N-u_H)(u_N+u_H-2u_h),v_h\big)\\ \le& C\|(u_N-u_H)\|_{0,4}\|(u_N+u_H-2u_h)\|_{0,2}\|v_h\|_{0,4}\\ \le& C\|u_N-u_H\|_1\Big(\|u_N-u_H\|+2\|u_H-u_h\|\Big)\|v_h\|_1,\end{aligned}$$

and according to the estimation (3.4),

$$\begin{aligned}&\big((u_H-u_h)^2-(u_N-u_h)^2,v_h\big)\\ \le& Ch^{2d+2}(C_1h^{2d+2}+C_2h^{d+1})\|v_h\|_1\le Ch^{3d+3}\|v_h\|_1.\end{aligned}$$

The estimation for the high order term is

$$\begin{aligned}&\big(O(u_N-u_h)^3,v\big)\\ \le& C\|(u_N-u_h)^3\|_{0,4/3}\|v_h\|_{0,4}\\ \le& C\|u_N-u_h\|_{0,4}^3\|v_h\|_1\le Ch^{3d+3}\|v_h\|_1,\end{aligned}$$

and $\big(O(u_H-u_h)^3,v_h\big)\le Ch^{3d+3}\|v_h\|_1$ is obtained in a similar way. According to (3.1), the equation (3.5) yields

$$\|u_M-u_H\|_1\le\sup_{v_h\in S_d^r(\triangle)}\frac{A(u_h;u_M-u_H,v_h)}{\|v_h\|_1}\le Ch^{3d+3},$$

where C_1, C_2 and C are all constants independent on h and $\|u_h-u\|_{0,\infty}\le\delta$. Finally,

$$\begin{aligned}\|u_M-u\|_1&\le\|u-u_H\|_1+\|u_M-u_H\|_1\\ &\le Ch^D+Ch^{3d+3}\le Ch^{\min\{D,3d+3\}}\end{aligned}$$

hold and the theorem follows. □

We remark that the error estimates for above two theorems are valid for all $d\le D$, and the highest possible accuracy is obtained by simply choosing d, which satisfy exactly $D=2d+2$ or $D=3d+3$ with respect to standard two level Newton method and modified Newton method. However, one is suggested to choose a proper D to balance the accuracy and the computational costs in practice. For example, when $d=3$, practical choices for the order of fine spline space should be 7 for the Newton method and 10 for the modified Newton method. This is also verified by numerical tests.

4. Numerical Examples

In this section, several numerical tests are given to demonstrate the efficiency of the proposed two level spline methods. All the numerical results are got in the environment of Matlab 7.1 and the computer is equipped with Intel Core i5 2.4Ghz CPU and 4GB memory.

Example 4.1. The first example is an elliptic equation with a reaction term u^3 and Dirichlet boundary condition [14], and the computational domain is defined as simple as the unit square square.

$$\begin{cases}-\Delta u+u^3=f, & x\in\Omega=(0,1)^2,\\ u=0, & x\in\partial\Omega.\end{cases}$$

For the accurate calculation of the errors, proper right hand function $f := f(x)$ is chosen so that $u = \sin(\pi x)\sin(\pi y)$ is promised to be the exact solution.

In all the computations, the uniform triangulation $\triangle_{1/8}$ with mesh size $h = \frac{1}{8}$ is used. In Table 4, we list the H^1 errors for the solutions obtained with one level Newton method, two level Newton method and the two level modified Newton method in three different columns respectively. Each computation stops when $\|\mathbf{c}_{d_1}^{m+1} - \mathbf{c}_{d_1}^{m}\| \leq \epsilon = 1e-12$. Various combinations of different orders low and high order spline spaces are considered for two level methods.

Table 4.1: H1 error for one level and two level algorithms in Example 4.1.

D	One Level $\|u_h - u\|_1$		Two Level Newton $\|u_N - u\|_1$			Two Level Modified Newton $\|u_M - u\|_1$	
	iter	$d = D$	$d = 2$	$d = 3$	$d = 4$	$d = 2$	$d = 3$
4	5	1.01e-04	1.01e-04	1.01e-04	—	1.01e-04	1.01e-04
5	5	3.52e-06	3.52e-06	3.52e-06	3.52e-06	3.52e-06	3.52e-06
6	5	1.01e-07	1.56e-07	1.08e-07	1.07e-07	1.07e-07	1.08e-07
7	5	2.79e-09	**1.13e-07**	2.80e-09	2.79e-09	2.79e-09	2.79e-09
8	6	6.49e-11	—	2.10e-10	6.49e-11	6.50e-11	6.49e-11
9	6	1.45e-12	—	**2.00e-10**	1.46e-12	3.79e-12	1.50e-12
10	8	5.58e-13	—	—	6.04e-13	**3.54e-12**	5.63e-13

It is illustrated that all three method have the same accuracy when certain relation between D and d are satisfied, and the best accuracy is archived when $D = 2d + 2$ for the two level Newton method and $D = 3d + 3$ for two level modified Newton method. These results verify the error estimations in Theorem (3.1) and Theorem (3.2). It is worth to figure out that a minor accuracy losing happens when $D = 2d + 2$ and $D = 3d + 3$ in this example, we believe it is caused by the different value of the constants C in our error estimations, which is only declared being optimal with respect to mesh size h. There is an important feature that the two level method only required one or two iterations on high order spline spaces, while the one level method cost 5-8 iterations in the same spline spaces to archive the same accuracy.

Example 4.2. In this example, we consider a practical problem which describes the electrostatic potential u in a charged body Ω,

$$\begin{cases} -\Delta u + e^u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

We regard the computational domain as $\Omega = \{(x, y)|x^2 + y^2 \leq 1\}$ and the analytic solution is chosen to be

$$u = 2\ln\left(\frac{B+1}{B\left(x^2+y^2\right)+1}\right),$$

where $B = -5+2\sqrt{6}$ is a properly choosed constant with physical meaning. The computational mesh $\triangle$ is chosen as plotted as in Fig. 4.1.

For the purpose of avoiding the pollution from the errors of the discrete boundaries, the mesh refine near the domain boundary is the most practical way. As an alternative choice, the using of curved boundary triangle element can also meet the requirements. We would like to leave this as another topic for practical using. Since the analytic form of u is known in the

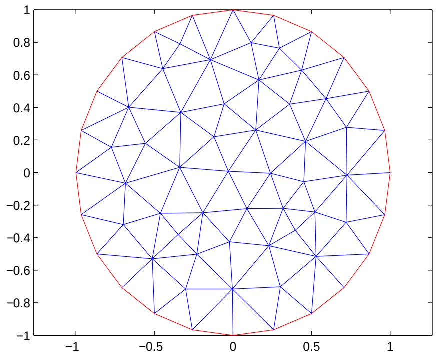

Fig. 4.1. Computational mesh $\triangle$.

Table 4.2: H1 and L2 errors for one level and two level algorithms in Example 4.2.

D	One Level $d=D$		Two Level Newton $\|u_N-u\|_1$			Two Level Modified Newton$\|u_M-u\|_1$	
	iter	$\|u_h-u\|_1$	$d=2$	$d=3$	$d=4$	$d=2$	$d=3$
2	4	1.36e-03	—	—	—	—	—
3	4	6.19e-05	6.19e-05	—	—	6.19e-05	6.19e-05
4	4	1.41e-06	1.41e-06	1.41e-06	—	1.41e-06	1.41e-06
5	4	5.27e-08	5.27e-08	5.27e-08	5.27e-08	5.27e-08	5.27e-08
6	4	1.51e-09	1.52e-09	1.51e-09	1.51e-09	1.51e-09	1.51e-09
7	4	5.34e-11	**1.57e-10**	5.34e-11	5.34e-11	5.34e-11	5.34e-11
8	4	1.71e-12	**1.48e-10**	1.72e-12	1.71e-12	1.71e-12	1.71e-12
9	5	1.43e-13	—	**3.21e-13**	1.60e-13	1.44e-13	1.43e-13

current example, we simply evaluate u with its analytic expression instead of imposing $u = 0$ everywhere on the boundary edges.

As same as in the previous example, the numerical results obtained in this case by both the one-level method and two-level methods are listed in Table 4.2, where the order of fine spline spaces are started from 2. We remark that larger value of D is prevent from the accuracy of the numerical quadrature formulae currently used in our implementation. Only a 73-points stable Gauss quadrature formulae is implemented according to [3], as for higher order one, it is an important research topic in the area of numerical analysis. However, we declare that our p-version two level methods are still efficient if one can find higher order numerical quadrature formulaes. Currently, we observe from Table 4.2 the same phenomenon with those in Example 4.1.

5. Conclusions

The proposed p-versions two level spline methods are proved to be effeicient and robust for solving the semi-linear elliptic equations in this paper. Due to the large scaling between the degrees of low order and high order spline spaces, the p-version two level methods are competitively the most effective choice for reducing the costs caused by using high order spline spaces. As for the further research, their applications in the stream-function form Navier-Stokes equations are undergoing investigations.

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