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ERROR REDUCTION, CONVERGENCE AND OPTIMALITY FOR ADAPTIVE MIXED FINITE ELEMENT METHODS FOR DIFFUSION EQUATIONS*

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Abstract

Error reduction, convergence and optimality are analyzed for adaptive mixed finite element methods (AMFEM) for diffusion equations without marking the oscillation of data. Firstly, the quasi-error, i.e. the sum of the stress variable error and the scaled error estimator, is shown to reduce with a fixed factor between two successive adaptive loops, up to an oscillation. Secondly, the convergence of AMFEM is obtained with respect to the quasi-error plus the divergence of the flux error. Finally, the quasi-optimal convergence rate is established for the total error, i.e. the stress variable error plus the data oscillation.

Mathematics subject classification: 65N30, 65N15, 65N12, 65N50.
Key words: Adaptive mixed finite element method, Error reduction, Convergence, Quasi-optimal convergence rate.

1. Introduction and Main Results

Let Ω be a bounded polygonal in $\mathbb{R}^2$. We consider the following diffusion problem with homogeneous Dirichlet boundary value:

$$\begin{cases} -\mathrm{div}(A\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the diffusion tensor $A \in L^\infty(\Omega;\mathbb{R}^{2\times 2})$ is a symmetric and uniformly positive definite matrix, and $f \in L^2(\Omega)$. The choice of boundary conditions is made for ease of presentation, since similar results are valid for other boundary conditions.

Adaptive methods for the numerical solution of PDEs are now standard tools in science and engineering to achieve better accuracy with minimum degrees of freedom. The adaptive procedure of (1.1) consists of loops of the form

$$SOLVE \rightarrow ESTIMATE \rightarrow MARK \rightarrow REFINE. \tag{1.2}$$

A posteriori error estimation (ESTIMATE) is an essential ingredient of adaptivity. We refer to [1, 2, 7, 17, 30] for related works on this topic. The analysis of convergence and optimality of the whole algorithm (1.2) is still in its infancy.

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The convergence analysis of standard adaptive finite element method (AFEM) started with Döfler [16]. Döfler introduced a crucial marking, and proved the strict energy error reduction of the standard AFEM for the Laplacian under the condition that the initial mesh $\mathcal{T}_0$ satisfies a fineness assumption. Morin et al. [24, 25] showed that such strict energy error reduction can not be expected in general. Introducing the concept of data oscillation and the interior node property, they proved convergence of the standard AFEM without fineness restriction on $\mathcal{T}_0$ which is valid only for A in (1.1) being piecewise constant on $\mathcal{T}_0$. Inspired by the work by Chen and Feng [11], Mekchay and Nochetto [22] extended this result to general second elliptic operators and proved that the standard AFEM is a contraction for the total error, namely the sum of the energy error and oscillation. Recently, Cascon, et al. [10] presented a new error notion, the so-called *quasi-error*, namely the sum of the energy error and the scaled estimator, and showed without the interior node property for the self-adjoint second elliptic problem that the quasi-error is strictly reduced by the standard AFEM even though each term may not be. Very recently, in [20, 21] Hu et al. first proved the convergence of adaptive conforming and nonconforming finite element methods without marking the oscillation of data.

Besides convergence, optimality is another important issue in AFEM which was first addressed by Binev et al. [4] and further studied by Stevenson [28, 29], who showed optimality without additional coarsening required in [4, 5]. These papers [4, 5, 28, 29] are restricted to Laplace operator and rely on suitable marking by data oscillation and the interior node property. Cascon et al. [10] succeeded in establishing quasi-optimality of the AFEM without both the assumption of the interior node property and marking by data oscillation for the self-adjoint second elliptic operator. Very recently, in [20,21] Hu et al. first analyzed the optimality of adaptive conforming or nonconforming finite element methods without using the algorithm that separates the error and the reduction of data oscillation.

However, for the convergence and optimality of AMFEM, the present works are carried out only for Poisson equations: In [8], Carstensen and Hoppe proved the error reduction and convergence for only the lowest-order Raviart-Thomas element. Chen et al. [12] showed the convergence of the quasi-error and the optimality of the flux error while marking the data oscillation. In [3, 9, 18], the convergence and optimality were analyzed for only the lowest-order Raviart-Thomas element where the local refinement was performed by using simply either the estimators or the data oscillation term.

Since the approximation of the mixed finite element methods is a saddle point of the corresponding energy, there is no orthogonality available, as is one of main difficulties for the convergence and optimality of AMFEM. Since the stress variable is of interest in many applications, we especially concern the stress variable error. In this paper, our main contribution is that we develop a novel technique and show, for more general elliptic problems and more general mixed elements, the reduction property of the quasi-error (i.e., the saturation property), the convergence of the quasi-error plus the divergence of the flux error, and the quasi-optimal convergence rate of the total error with only the Dörfler Marking and without marking the oscillation.

To summarize our main results, let $\{\mathcal{T}_k, (M_k, L_k), p_k, \eta_k\}_{k\geq 0}$ be the sequence of the meshes, a pair of finite element spaces with $\mathrm{div} M_k = L_k$, the approximation solutions, the estimators produced by AMFEM in the k-th step. We prove in Section 5 that the quasi-error uniformly reduces with a fixed rate between two successive meshes, up to an oscillation of data f, namely

$$\mathcal{E}_{k+1}^2 + \gamma\eta_{k+1}^2 \leq \alpha^2(\mathcal{E}_k^2 + \gamma\eta_k^2) + C\mathrm{osc}^2(f, \mathcal{T}_k),$$

where $\alpha \in (0,1)$, $\gamma > 0$, $\mathcal{E}_k^2 := ||A^{-1/2}(p-p_k)||^2_{L^2(\Omega)}$, and $\text{osc}(f,\mathcal{T}_k)$ is the oscillation of f over $\mathcal{T}_k$ (see Section 2.3). We point out here that in some cases, even though the stress variable error is monotone, strict error reduction for $\mathcal{E}_k$ may fail. On the other hand, the residual estimator $\eta_k := \eta_k(p_k,\mathcal{T}_k)$ exhibits strict reduction even when $p_k = p_{k+1}$ but no monotone behavior in general. The orthogonality for the divergence of the flux leads to the convergence result:

$$\mathcal{E}_{k+1}^2 + \gamma_1 ||h_{k+1}\text{div}(p-p_{k+1})||^2_{L^2(\Omega)} + \gamma_2 \eta_{k+1}^2 \le C_{\text{in}}\beta^{2(k+1)},$$

where constants $\gamma_1, \gamma_2 > 0$, $0 < \beta < 1$, and C_{in} denotes the error on the initial mesh, and h_{k+1} is the mesh-size function with respect to $\mathcal{T}_{k+1}$.

Since all decisions of AMFEM in MARK are based on the estimator η_k, a decay rate for the true error is closely related to the quality of the estimator, which is described by the global lower bound $\eta_k^2 \lesssim \mathcal{E}_k^2 + \text{osc}_k^2$. Hereafter, following the idea in [10,22], we refer to $(\mathcal{E}_k^2 + \text{osc}_k^2)^{1/2}$ as the *total error*. The lower bound demonstrates that the estimator is controlled by the error except up to an oscillation term and one can observe the difference between $\mathcal{E}_k$ and η_k only when oscillation is large. Furthermore, from the upper bound $\mathcal{E}_k^2 \lesssim \eta_k^2$ and $\text{osc}_k^2 \le \eta_k^2$ it follows $\mathcal{E}_k^2 + \text{osc}_k^2 \lesssim \eta_k^2$. This implies that the total error, which is the quantity reduced by AMFEM, is controlled by the estimator. Since the estimator itself is an upper bound for the quasi-error, in view of the global lower bound it holds $\mathcal{E}_k^2 + \text{osc}_k^2 \approx \eta_k^2 \approx \mathcal{E}_k^2 + \gamma\eta_k^2$.

In short, the behavior of AMFEM is intrinsically bonded to the total error, which measures the approximability of both the flux $p = A\nabla u$ and data encoded in the oscillation term. Note that when $A^{-1}p_h$ is a piecewise polynomial vector, oscillation will reduce to approximation of the right-hand side term f of (1.1) (see Section 2.3). In general case, approximation of data A in osc_k^2 couples in nonlinear fashion with the discrete solutions p_k.

In Section 6, we introduce an approximation class $\mathbb{A}_s$ (see [14, 15]) based on the total error. Using a quasi-monotonicity property of oscillation and a localized discrete upper bound, we prove the following quasi-optimal convergence rate for the AMFEM in terms of DOFs by assuming the marking parameter $\theta \in (0, \theta_*)$ with $0 < \theta_* < 1$ (see Theorem 6.2):

$$(\mathcal{E}_k^2 + \gamma_2 \text{osc}^2_{\mathcal{T}_k}(p_k,\mathcal{T}_k))^{1/2} \le \Theta(s,\theta,\mathbb{A}_s)(\#\mathcal{T}_k - \#\mathcal{T}_0)^{-s}.$$

The rest of this paper is organized as follows. Section 2 gives some preliminaries and details on notations. Section 3 shows an efficiency result of the a posteriori error estimator. Section 4 provides some auxiliary results for convergence and optimality. We derive in section 5 the error reduction and convergence of AMFEM, and prove in Section 6 the quasi-optimal convergence rate of AMFEM.

2. Preliminaries and Notations

2.1. Weak formulation

By splitting (1.1) into two equations, the mixed formulation is given as

$$\begin{cases} -\text{div}\, p = f \quad \text{and} \quad p = A\nabla u & \text{in} \quad \Omega \\ \qquad u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

Since the diffusion tensor A is symmetric and uniformly positive definite, by the Lax-Milgram theorem, there exists a unique solution $u \in H_0^1(\Omega)$ to the problem (2.1). Moreover, the weak formulation of (2.1) reads as: Find $(p,u) \in H(\mathrm{div},\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} &(A^{-1}p, q)_{0,\Omega} + (\mathrm{div}\ q, u)_{0,\Omega} = 0 \qquad \text{for all} \quad q \in H(\mathrm{div},\Omega),\\ &(\mathrm{div}\ p, v)_{0,\Omega} = -(f, v)_{0,\Omega} \qquad\qquad \text{for all} \quad v \in L^2(\Omega), \end{aligned} \tag{2.2}$$

where $H(\mathrm{div},\Omega) := \{q \in L^2(\Omega)^2 : \mathrm{div}\ q \in L^2(\Omega)\}$, and $(\cdot,\cdot)_{0,\Omega}$ denotes L^2 inner product on Ω.

Let $\mathcal{T}_h$ be a shape regular triangulation of the domain Ω in the sense of [13], and let M_h and L_h denote finite dimensional subspaces of $H(\mathrm{div},\Omega)$ and $L^2(\Omega)$, respectively, such as well-known Raviart-Thomas (RT), Brezzi-Douglas-Marini (BDM), and Brezzi-Douglas-Fortin-Marini (BDFM) mixed finite element spaces (see [6]). Note that M_h and L_h are indeed the spaces of the piecewise polynomials of total degree at most $l+1$ and l with $l \geq 0$, respectively. In the step $SOLVE$ a mixed finite element method reads as: Find $(p_h, u_h) \in M_h \times L_h$ such that

$$\begin{aligned} &(A^{-1}p_h, q_h)_{0,\Omega} + (\mathrm{div}\ q_h, u_h)_{0,\Omega} = 0 \qquad \text{for all} \quad q_h \in M_h,\\ &(\mathrm{div}\ p_h, v_h)_{0,\Omega} = -(f_h, v_h)_{0,\Omega} \qquad\qquad \text{for all} \quad v_h \in L_h, \end{aligned} \tag{2.3}$$

where f_h is the L^2- projection of f over L_h.

It is well-known that the existence and uniqueness of the solution of (2.2) hold true, and that the discrete problem (2.3) has a unique solution since a discrete inf-sup-condition is satisfied by these discrete spaces M_h and L_h (cf. [6]). Suppose that the module $SOLVE$ outputs a pair of discrete solutions over $\mathcal{T}_h$, namely, $(p_h, u_h) = SOLVE(\mathcal{T}_h)$.

In what follows, for each $T \in \mathcal{T}_h$ we denote the mesh size $h_T := |T|^{1/2}$ with $|T|$ the area of T. Let ε_h be the set of element edges in $\mathcal{T}_h$, $J(v)|_E := (v|_{T_+})|_E - (v|_{T_-})|_E$ denote the jump of $v \in H^1(\bigcup \mathcal{T}_h)$ over an interior edge $E := T_+ \cap T_-$ of length $h_E := \mathrm{diam}(E)$, shared by the two neighboring (closed) triangles $T_\pm \in \mathcal{T}_h$. Especially, $J(v)|_E := (v|_T)|_E$ if $E = \overline{T} \cap \partial\Omega$. Furthermore, for $T \in \mathcal{T}_h$, we denote by ω_T the union of all elements in $\mathcal{T}_h$ sharing one point with T, and define the patch of $E \in \varepsilon_h$ by

$$\omega_E := \bigcup \{T \in \mathcal{T}_h : E \subset \overline{T}\}.$$

Denote $\Gamma_h := \bigcup \varepsilon_h$, and note that $J : H^1(\bigcup \mathcal{T}_h) \to L^2(\Gamma_h)$ be an operator with $H^1(\bigcup \mathcal{T}_h) := \{v \in L^2(\Omega) : \forall T \in \mathcal{T}_h, v|_T \in H^1(T)\}$.

Throughout the paper, the local versions of the differential operators $\mathrm{div}, \nabla, \mathrm{curl}$ are understood in the distribution sense, i.e., in $D'(\Omega)$, namely, $\mathrm{div}_h, \mathrm{curl}_h : H^1(\bigcup \mathcal{T}_h)^2 \to L^2(\Omega)$ and $\nabla_h : H^1(\bigcup \mathcal{T}_h) \to L^2(\Omega)^2$ are defined such that, e.g., $\mathrm{div}_h v|_T := \mathrm{div}(v|_T)$ in $D'(T)$, for all $T \in \mathcal{T}_h$.

2.2. A posteriori error estimators

For all $E \in \varepsilon_h$, let τ be the unit tangential vector along E, and $p_h \in M_h$ be the approximation solution to the flux of (2.3) with respect to the triangulation $\mathcal{T}_h$. For convenience we define the stress variable error by

$$\mathcal{E}_h^2 := ||A^{-1/2}(p - p_h)||^2_{L^2(\Omega)},$$

and define the local and global error estimators (see [6]) respectively as

$$\eta^2_{\mathcal{T}_h}(p_h,T) := h_T^2||f+\text{div }p_h||^2_{L^2(T)} + h_T^2||\text{curl}(A^{-1}p_h)||^2_{L^2(T)} + h_T||J(A^{-1}p_h\cdot\tau)||^2_{L^2(\partial T)},$$

$$\eta^2_{\mathcal{T}_h}(p_h,\mathcal{T}_h) := \sum_{T\in\mathcal{T}_h}\eta^2_{\mathcal{T}_h}(p_h,T).$$

Here $\text{curl}\psi := \frac{\partial\psi_2}{\partial x_1} - \frac{\partial\psi_1}{\partial x_2}$ for $\psi = (\psi_1,\psi_2)^T$. Note that in this paper, the Curls of a scalar function ϕ are involved as

$$\text{Curl}\phi := (-\frac{\partial\phi}{\partial x_2},\frac{\partial\phi}{\partial x_1})^T.$$

We assume that, for a given triangulation $\mathcal{T}_h$ and a corresponding discrete solution $p_h\in M_h$, the module $ESTIMATE$ for the stress variable outputs the indicators

$$\{\eta^2_{\mathcal{T}_h}(p_h,T)\}_{T\in\mathcal{T}_h} = ESTIMATE(p_h,\mathcal{T}_h).$$

From [6], for the stress variable error the estimate

$$\mathcal{E}_h^2 \le C_1\eta^2_{\mathcal{T}_h}(p_h,\mathcal{T}_h) \tag{2.4}$$

holds with C_1 a constant independent of the mesh size.

2.3. Oscillation of data

For an integer $n\ge l+1$, we denote by Π_n^2 the L^2-best approximation operator onto the set of piecewise polynomials of degree $\le n$ over $T\in\mathcal{T}_h$ or $E\in\varepsilon_h$, denote by id the identity operator, and set $P_n^2 := \text{id}-\Pi_n^2$. We define the oscillation of data as

$$\text{osc}^2_{\mathcal{T}_h}(p_h,T) := h_T^2||P_n^2\text{curl}(A^{-1}p_h)||^2_{L^2(T)} + h_T||P^2_{n+1}J(A^{-1}p_h\cdot\tau)||^2_{L^2(\partial T)} + h_T^2||f-f_h||^2_{L^2(T)} \quad \text{for all } T\in\mathcal{T}_h.$$

Especially, for any subset $\mathcal{T}_h'\subset\mathcal{T}_h$, we set

$$\text{osc}^2_{\mathcal{T}_h}(p_h,\mathcal{T}_h') := \sum_{T\in\mathcal{T}_h'}\text{osc}^2_{\mathcal{T}_h}(p_h,T) \quad\text{and}\quad \text{osc}_h^2 := \text{osc}^2_{\mathcal{T}_h}(p_h,\mathcal{T}_h).$$

We also define the oscillation of f as

$$\text{osc}^2(f,\mathcal{T}_h) := ||h(f-f_h)||^2_{L^2(\Omega)}.$$

Remark 2.1. Let $\mathcal{T}_h$ be a triangulation, $q_h\in M_h$ be given. By substituting p_h with q_h in the definitions of $\eta_{\mathcal{T}_h}(p_h,T)$ and $\text{osc}_{\mathcal{T}_h}(p_h,T)$, we can see that the indicator $\eta_{\mathcal{T}_h}(q_h,T)$ controls oscillation $\text{osc}_{\mathcal{T}_h}(q_h,T)$, i.e., $\text{osc}_{\mathcal{T}_h}(q_h,T)\le\eta_{\mathcal{T}_h}(q_h,T)$ for all $T\in\mathcal{T}_h$. In addition, for the stress variables, the definitions of the error indicator and oscillation are fully localized to T, which means $\eta_{\mathcal{T}_H}(q_H,T)=\eta_{\mathcal{T}_h}(q_H,T)$ and $\text{osc}_{\mathcal{T}_H}(q_H,T)=\text{osc}_{\mathcal{T}_h}(q_H,T)$ for any refinement $\mathcal{T}_h$ of $\mathcal{T}_H$ with $T\in\mathcal{T}_h\cap\mathcal{T}_H$ and $q_H\in M_H$. Moreover, a combination of the monotonicity of local mesh sizes and properties of the local L^2-projection yields

$$\eta_{\mathcal{T}_h}(q_H,\mathcal{T}_h)\le\eta_{\mathcal{T}_H}(q_H,\mathcal{T}_H) \quad\text{and}\quad \text{osc}_{\mathcal{T}_h}(q_H,\mathcal{T}_h)\le\text{osc}_{\mathcal{T}_H}(q_H,\mathcal{T}_H) \quad \forall q_H\in M_H.$$

We note that in this paper, the triangulation $\mathcal{T}_h$ means a refinement of $\mathcal{T}_H$, all notations with respect to the mesh $\mathcal{T}_H$ are defined similarly. Throughout the rest of the paper we use the notation $A\lesssim B$ to represent $A\le CB$ with a mesh-size independent, generic constant $C>0$. Moreover, $A\approx B$ abbreviates $A\lesssim B\lesssim A$.

2.4. Adaptive algorithm

In the $MARK$ step of (1.2), by relying on Dörfler marking, we select the elements to mark according to the indicators for the stress variables, namely, given a grid $\mathcal{T}_H$ with the set of indicators $\{\eta_{\mathcal{T}_H}(p_H, T)\}_{T\in\mathcal{T}_H}$ and marking parameter $\theta \in (0, 1]$, the module $MARK$ outputs a subset of making elements $\mathcal{M}_H \subset \mathcal{T}_H$, i.e.,

$$\mathcal{M}_H = MARK(\{\eta_{\mathcal{T}_H}(p_H, T)\}_{T\in\mathcal{T}_H}, \mathcal{T}_H, \theta),$$

such that $\mathcal{M}_H$ satisfies Dörfler property

$$\eta_{\mathcal{T}_H}(p_H, \mathcal{M}_H) \geq \theta\eta_{\mathcal{T}_H}(p_H, \mathcal{T}_H). \tag{2.5}$$

In the $REFINE$ step of (1.2), we suppose that the refinement rule, such as the newest vertex bisection [23], is guaranteed to produce conforming and shape regular mesh. Given two fixed integers $b_0 > b \geq 1$, a mesh $\mathcal{T}_H$, and a subset $\mathcal{M}_H \subset \mathcal{T}_H$ of marked elements, a conforming triangulation $\mathcal{T}_h$ is output by

$$\mathcal{T}_h = REFINE(\mathcal{T}_H, \mathcal{M}_H),$$

where all elements of $\mathcal{M}_H$ are at least bisected b times and at most b_0 times. Note that not only marked elements get refined but also additional elements are refined to recovery the conformity of triangulations. Let $\mathcal{R} := \mathcal{R}_{\mathcal{T}_H\to\mathcal{T}_h} := \mathcal{T}_H/(\mathcal{T}_H \cap \mathcal{T}_h)$ denote the set of refined elements, which means $\mathcal{M}_H \subset \mathcal{R}_{\mathcal{T}_H\to\mathcal{T}_h}$. Here, we quote a result about complexity of refinement, its proof can be found in [29].

Lemma 2.1. (Complexity of refinement) *Assume that $\mathcal{T}_0$ satisfies condition (b) of section* 4 *in* [29]. *Let $\{\mathcal{T}_k\}_{k\geq 0}$ with cardinality $\#\mathcal{T}_k$ be any conforming triangulation sequence refined from a shape regular triangulation $\mathcal{T}_0$, $\mathcal{T}_{k+1}$ be generated from $\mathcal{T}_k$ by $\mathcal{T}_{k+1} = REFINE(\mathcal{T}_k, \mathcal{M}_k)$ with a subset $\mathcal{M}_k \subset \mathcal{T}_k$. Then it holds*

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \sum_{j=0}^{k-1} \#\mathcal{M}_j \quad \text{for all } k \geq 1.$$

We now describe the algorithm of the AMFEM of the stress variables. In doing this, we replace the subscript H (or h) by an iteration counter called $k \geq 0$. Let $\mathcal{T}_0$ be a shape regular triangulation, $\eta_0 := \eta_{\mathcal{T}_0}(p_{0,\mathcal{T}_0})$ denote the error indicator onto the initial mesh $\mathcal{T}_0$, with a marking parameter $\theta \in (0, 1]$. The basic loop of AMFEM is then given by the following iterations:

AMFEM

Set $k = 0, \eta_k = \eta_0$ and iterate

(1) $(p_k, u_k) = SOLVE(\mathcal{T}_k)$;

(2) $\{\eta_k(p_k, T)\}_{T\in\mathcal{T}_k} = ESTIMATE(p_k, \mathcal{T}_k)$;

(3) $\mathcal{M}_k = MARK(\{\eta_k(p_k, T)\}_{T\in\mathcal{T}_k}, \mathcal{T}_k, \theta)$;

(4) $\mathcal{T}_{k+1} = REFINE(\mathcal{T}_k, \mathcal{M}_k), k = k + 1$.

We note that the AMFEM for the stress variables is a standard algorithm in which it employs only the error estimator $\{\eta_{\mathcal{T}_k}(p_k, T)\}_{T\in\mathcal{T}_k}$, does not mark the oscillation, and does not need the interior node property.

3. Efficiency of Estimator

In [6] the efficiency of the proposed estimator was derived under the constraint that A^{-1} is a piecewise polynomial matrix. Here, by introducing the oscillation of data, we shall show the efficiency without the constraint on A^{-1}.

Using the property of L^2-best approximation and standard arguments we easily have the following two lemmas.

Lemma 3.1. *Let $p_h \in M_h$ be an approximation solution to the stress variable of* (2.3)*, P_n^2 denote the operator defined in Section* 2.3*, and $\epsilon := p - p_h$ be the error of the flux. Then, for all $T \in \mathcal{T}_h$, it holds*

$$h_T||\mathrm{curl}(A^{-1}p_h)||_{L^2(T)} \lesssim ||A^{-1/2}\epsilon||_{L^2(T)} + h_T||P_n^2\mathrm{curl}(A^{-1}p_h)||_{L^2(T)}. \tag{3.1}$$

Lemma 3.2. *Let $p_h \in M_h$ be the approximation solution to the flux of* (2.3)*, P_{n+1}^2 denote the operator defined in Section* 2.3*, and ϵ be the same as in Lemma* 3.1*. Then, for all $E \in \varepsilon_h$, it holds*

$$\begin{aligned} &h_E^{1/2}||J(A^{-1}p_h\cdot\tau)||_{L^2(E)} \\ \lesssim &h_E^{1/2}||P_{n+1}^2 J(A^{-1}p_h\cdot\tau)||_{L^2(E)} + h_E||P_n^2\mathrm{curl}_h(A^{-1}p_h)||_{L^2(\omega_E)} + ||A^{-1/2}\epsilon||_{L^2(\omega_E)}. \end{aligned} \tag{3.2}$$

We now prove the efficiency of the estimator $\eta_{\mathcal{T}_h}(p_h, \mathcal{T}_h)$ by using the above two lemmas.

Theorem 3.1. *Let $p \in H(\mathrm{div}, \Omega)$ and $p_h \in M_h$ be the solutions of* (2.2) *and* (2.3)*, respectively, and let $\eta_{\mathcal{T}_h}(p_h, \mathcal{T}_h)$, $\mathcal{E}_h$, and osc_h be defined as in Sections* 2.2 *and* 2.3*. Then, for the estimator of the stress variables for the RT, BDM, and BDFM elements, there exists a constant C_2 independent of mesh-size such that*

$$C_2\eta_{\mathcal{T}_h}(p_h, \mathcal{T}_h)^2 \le \mathcal{E}_h^2 + \mathrm{osc}_h^2. \tag{3.3}$$

Proof. Since $f + \mathrm{div}\ p_h = f - f_h$, combining Lemmas 3.1-3.2, and summing over all $T \in \mathcal{T}_h$ and $E \in \varepsilon_h$, we obtain the desired result (3.3). $\square$

4. Auxiliary Results for Convergence and Optimality

4.1. Quasi-orthogonality

Following the same line as in the proof of (Lemma 4.2, [12]), we have the following result:

Lemma 4.1. *Let $\mathcal{T}_h$ and $\mathcal{T}_H$ be two nested triangulations, Π_{L_H} be the $L^2(\Omega)$ $-$projection operator onto L_H, and $(p_h, u_h) \in M_h \times L_h$ be the solutions of* (2.3)*. Then for any $T \in \mathcal{T}_H$, there exists a positive constant C_0 depending only on the shape regularity of $\mathcal{T}_H$ such that*

$$||u_h - \Pi_{L_H}u_h||_{L^2(T)} \le \sqrt{C_0}H_T||A^{-1/2}p_h||_{L^2(T)}. \tag{4.1}$$

In order to prove the quasi-orthogonality, we need to introduce a pair of auxiliary solutions. We denote by f_H the L^2-projection of f over L_H, and consider the following problem: Find $(\widetilde{p}_h, \widetilde{u}_h) \in M_h \times L_h$ such that

$$\begin{aligned} &(A^{-1}\widetilde{p}_h, q_h)_{0,\Omega} + (\mathrm{div}\ q_h, \widetilde{u}_h)_{0,\Omega} = 0 && \text{for all} \quad q_h \in M_h, \\ &(\mathrm{div}\ \widetilde{p}_h, v_h)_{0,\Omega} = -(f_H, v_h)_{0,\Omega} && \text{for all} \quad v_h \in L_h. \end{aligned} \tag{4.2}$$

In fact, the solution $(\widetilde{p}_h, \widetilde{u}_h)$ of this auxiliary problem may be regarded as another approximation to the flux and displacement (p, u).

Lemma 4.2. *Let $\mathcal{T}_h$ and $\mathcal{T}_H$ be two nested triangulations,* $\mathrm{osc}(f_h, \mathcal{T}_H)$ *denote the oscillation of f_h over $\mathcal{T}_H$, p_h and $\widetilde{p}_h$ be the solution of* (2.3) *and* (4.2)*, respectively. Then there exists a constant C_0 depending only on the shape regularity of $\mathcal{T}_H$ such that*

$$||A^{-1/2}(p_h - \widetilde{p}_h)||_{L^2(\Omega)} \le \sqrt{C_0}\mathrm{osc}(f_h, \mathcal{T}_H.) \tag{4.3}$$

Proof. The conclusion follows from Lemma 4.1 and the same line as in the proof of (Theorem 4.3, [12]). □

We state the property of quasi-orthogonality as follows.

Theorem 4.1. (Quasi-orthogonality) *Given $f \in L^2(\Omega)$ and two nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, let p_h and p_H be the solutions of* (2.3) *with respect to $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively. Then it holds*

$$(A^{-1}(p - p_h), p_h - p_H)_{0,\Omega} \le C_0^{1/2}||A^{-1/2}(p - p_h)||_{L^2(\Omega)}\mathrm{osc}(f_h, \mathcal{T}_H). \tag{4.4}$$

Furthermore, for any $\delta_1 > 0$, it holds

$$\begin{aligned}&(1 - \delta_1)||A^{-1/2}(p - p_h)||^2_{L^2(\Omega)}\\ \le&||A^{-1/2}(p - p_H)||^2_{L^2(\Omega)} - ||A^{-1/2}(p_h - p_H)||^2_{L^2(\Omega)} + C_0\delta_1^{-1}\mathrm{osc}^2(f_h, \mathcal{T}_H).\end{aligned} \tag{4.5}$$

Proof. Let $(\widetilde{p}_h, \widetilde{u}_h)$ solve the problem (4.2). Then we have

$$\begin{aligned}&(A^{-1}(p - p_h), \widetilde{p}_h - p_H)_{0,\Omega}\\ =& - (\mathrm{div}(\widetilde{p}_h - p_H), u - \widetilde{u}_h)_{0,\Omega} = (f_H - f_H, u - \widetilde{u}_h)_{0,\Omega} = 0.\end{aligned} \tag{4.6}$$

From the above identity (4.6) and Lemma 4.2, we obtain

$$\begin{aligned}(A^{-1}(p - p_h), p_h - p_H)_{0,\Omega} &= (A^{-1}(p - p_h), p_h - \widetilde{p}_h)_{0,\Omega}\\ &\le ||A^{-1/2}(p - p_h)||_{L^2(\Omega)}||A^{-1/2}(p_h - \widetilde{p}_h)||_{L^2(\Omega)}\\ &\le C_0^{1/2}||A^{-1/2}(p - p_h)||_{L^2(\Omega)}\mathrm{osc}(f_h, \mathcal{T}_H),\end{aligned} \tag{4.7}$$

which implies the first result (4.4). Furthermore, notice that

$$\begin{aligned}||A^{-1/2}(p - p_h)||^2_{L^2(\Omega)} = ||A^{-1/2}(p - p_H)||^2_{L^2(\Omega)} &- ||A^{-1/2}(p_h - p_H)||^2_{L^2(\Omega)}\\ &- 2(A^{-1}(p - p_h), p_h - p_H)_{0,\Omega}.\end{aligned} \tag{4.8}$$

Then for any $\delta_1 > 0$, from (4.7) and Young's inequality we have

$$\begin{aligned}||A^{-1/2}(p - p_h)||^2_{L^2(\Omega)} \le ||A^{-1/2}(p - p_H)||^2_{L^2(\Omega)} &- ||A^{-1/2}(p_h - p_H)||^2_{L^2(\Omega)}\\ &+ \delta_1||A^{-1/2}(p - p_h)||^2_{L^2(\Omega)} + \frac{C_0}{\delta_1}\mathrm{osc}^2(f_h, \mathcal{T}_H),\end{aligned}$$

which implies the estimate (4.5). □

Although the oscillation of f_h over the triangulation $\mathcal{T}_H$ appears in the estimate of quasi-orthogonality, it is dominated by $\mathrm{osc}(f, \mathcal{T}_H)$. We refer to [12] for the proof of the following observation.

Lemma 4.3. *Let f_h denote the L^2-projection of f over L_h. Then it holds*

$$\mathrm{osc}(f_h, \mathcal{T}_H) \le \mathrm{osc}(f, \mathcal{T}_H).$$

4.2. Reduction of estimators and oscillation

In this subsection, we aim at the reduction of the estimators and oscillation. To this end, we relate the error indicators and oscillation of two nested triangulations to each other. The link involves weighted maximum-norms of the inverse matrix, A^{-1}, and its oscillation.

For a nonnegative integer $m = n - l$, any given triangulation $\mathcal{T}_H$, and $v \in L^\infty(\Omega)$, we denote by $\Pi_m^\infty v$ the best $L^\infty(\Omega)-$approximation of v in the space of piecewise polynomials of degree $\leq m$, and denote by ω_T the patch of element T defined in Section 2.1. We further set

$$\begin{aligned}
&\Pi_{-1}^\infty v := 0, \quad P_m^\infty v := (id - \Pi_m^\infty)v,\\
&\eta^2_{\mathcal{T}_H}(A^{-1}, T) := H_T^2(||\text{Curl } A^{-1}||^2_{L^\infty(T)} + H_T^{-2}||A^{-1}||^2_{L^\infty(\omega_T)}) \ \text{ for all } \ T \in \mathcal{T}_H,\\
&\text{osc}^2_{\mathcal{T}_H}(A^{-1}, T) := H_T^2(||P_{m-1}^\infty \text{Curl } A^{-1}||^2_{L^\infty(T)} + H_T^{-2}||P_m^\infty A^{-1}||^2_{L^\infty(\omega_T)}).
\end{aligned}$$

Noticing that P_m^∞ is defined elementwise, for any subset $\mathcal{T}_H' \subset \mathcal{T}_H$ we set

$$\eta_{\mathcal{T}_H}(A^{-1}, \mathcal{T}_H') := \max_{T \in \mathcal{T}_H'} \eta_{\mathcal{T}_H}(A^{-1}, T), \ \text{osc}_{\mathcal{T}_H}(A^{-1}, \mathcal{T}_H') := \max_{T \in \mathcal{T}_H'} \text{osc}_{\mathcal{T}_H}(A^{-1}, T).$$

Remark 4.1. (Monotonicity) The use of best approximation in L^∞ in the definition of $\eta_{\mathcal{T}_H}$ $(A^{-1}, \mathcal{T}_H)$ and $\text{osc}_{\mathcal{T}_H}(A^{-1}, \mathcal{T}_H)$ implies the following monotonicity: for any refinement $\mathcal{T}_h$ of $\mathcal{T}_H$, it holds

$$\eta_{\mathcal{T}_h}(A^{-1}, \mathcal{T}_h) \leq \eta_{\mathcal{T}_H}(A^{-1}, \mathcal{T}_H) \ \text{ and } \ \text{osc}_{\mathcal{T}_h}(A^{-1}, \mathcal{T}_h) \leq \text{osc}_{\mathcal{T}_H}(A^{-1}, \mathcal{T}_H).$$

To avoid any smoothness assumptions on the diffusion coefficient matrix, we need to quote a result about implicit interpolation, whose proof can be found in [10]. Let m be a nonnegative integer, ι be a positive integer, and ω be a one- or two-dimensional simplex. We denote $\tilde{P}_m^j := id - \Pi_m^j$, where $\Pi_m^j : L^j(\omega, \mathbb{R}^\iota) \to P_m(\omega, \mathbb{R}^\iota)$ is the operator of best L^j-approximation in ω for $j = 2, \infty$.

Lemma 4.4. (Implicit interpolation) *Let m and n be two nonnegative integer. Then for all $v \in L^\infty(\omega, \mathbb{R}^\iota)$, $V \in P_n(\omega, \mathbb{R}^\iota)$ and $m \geq n$, it holds*

$$||\tilde{P}_m^2(vV)||_{L^2(\omega)} \leq ||\tilde{P}_{m-n}^\infty v||_{L^\infty(\omega)}||V||_{L^2(\omega)}. \tag{4.9}$$

Lemma 4.5. *Let $\mathcal{T}_H$ be a triangulation. For all $T \in \mathcal{T}_H$ and any pair of discrete functions $\sigma_H, \tau_H \in M_H$, there exists a constant $\bar{\Lambda}_1 > 0$ depending only on the shape regularity of $\mathcal{T}_0$, the polynomial degree $l+1$, and the eigenvalues of A^{-1}, such that*

$$\eta_{\mathcal{T}_H}(\sigma_H, T) \leq \eta_{\mathcal{T}_H}(\tau_H, T) + \bar{\Lambda}_1 \eta_{\mathcal{T}_H}(A^{-1}, T)||A^{-1/2}(\sigma_H - \tau_H)||_{L^2(\omega_T)}, \tag{4.10}$$

$$\text{osc}_{\mathcal{T}_H}(\sigma_H, T) \leq \text{osc}_{\mathcal{T}_H}(\tau_H, T) + \bar{\Lambda}_1 \text{osc}_{\mathcal{T}_H}(A^{-1}, T)||A^{-1/2}(\sigma_H - \tau_H)||_{L^2(\omega_T)}. \tag{4.11}$$

Proof. We only prove the second estimate (4.11), since the first one (4.10) is somewhat simpler and can be derived similarly. We denote by $L^2(\Gamma_H)$ the square integrable function spaces on $\Gamma_H := \bigcup \varepsilon_H$. The jump of the tangential component defines a linear mapping $J : M_H \to L^2_{\Gamma_H}$ by $J(q_H) = J(A^{-1}q_H \cdot \tau)$ for all $q_H \in M_H$ from M_H into $L^2_{\Gamma_H}$. Recalling $\tilde{P}_n^2 = id - \Pi_n^2$ with Π_n^2 being the L^2-projection, denoting $q_H := \sigma_H - \tau_H$ and using the triangle inequality, we have

$$\begin{aligned}
\text{osc}_{\mathcal{T}_H}(\sigma_H, T) \leq\ & \text{osc}_{\mathcal{T}_H}(\tau_H, T) + H_T||\tilde{P}_n^2 \text{curl}(A^{-1}q_H)||_{L^2(T)}\\
&+ H_T^{1/2}||\tilde{P}_{n+1}^2 J(A^{-1}q_H \cdot \tau)||_{L^2(\partial T)}.
\end{aligned} \tag{4.12}$$

We split the curl term as

$$\mathrm{curl}(A^{-1}q_H) = \mathrm{Curl}\ A^{-1}\cdot q_H + A^{-1} : \widetilde{\mathrm{curl}}\ q_H,$$

where Curl A^{-1} is a vector whose every component is the curl of the corresponding column vector of A^{-1}, and $\widetilde{\mathrm{curl}}\ q_H$ is a matrix whose column vector is the Curl of the corresponding component of q_H. Invoking Lemma 4.4 with $\omega = T$ and noticing that polynomial degree of q_H is $l+1$, we infer for the first term that

$$||\tilde{P}^2_n(Curl\ A^{-1}\cdot q_H)||_{L^2(T)} \lesssim ||\tilde{P}^\infty_{n-l-1}Curl\ A^{-1}||_{L^\infty(T)}||A^{-1/2}q_H||_{L^2(T)}. \tag{4.13}$$

Since $\widetilde{\mathrm{curl}}\ q_H$ is a polynomial of degree $\leq l$, applying (4.9) again in conjunction with an inverse inequality, we obtain for the second term that

$$\begin{aligned}
&||\tilde{P}^2_n(A^{-1} : \widetilde{\mathrm{curl}}\ q_H)||_{L^2(T)}\\
&\leq ||\tilde{P}^\infty_{n-l}A^{-1}||_{L^\infty(T)}||\widetilde{\mathrm{curl}}\ q_H||_{L^2(T)} \leq ||\tilde{P}^\infty_{n-l}A^{-1}||_{L^\infty(T)}|q_h|_{H^1(T)}\\
&\lesssim H_T^{-1}||\tilde{P}^\infty_{n-l}A^{-1}||_{L^\infty(T)}||A^{-1/2}q_H||_{L^2(T)}.
\end{aligned} \tag{4.14}$$

We now deal with the jump residual. Let $T' \in \mathcal{T}_H$ share an interior edge E with T. We write $J(A^{-1}q_H\cdot\tau) = ((A^{-1}q_H)|_T - (A^{-1}q_H)|_{T'})\cdot\tau$ and use the linearity of Π^2_{n+1}, Lemma 4.4 with $\omega = E$, and the inverse inequality $||q_H||_{L^2(E)} \lesssim H_T^{-1/2}||q_H||_{L^2(T)}$ to deduce

$$\begin{aligned}
&||\tilde{P}^2_{n+1}((A^{-1}q_H)|_T\cdot\tau)||_{L^2(E)}\\
&= ||(\tilde{P}^2_{n+1}(A^{-1}q_H|_T))\cdot\tau||_{L^2(E)} \leq ||\tilde{P}^2_{n+1}(A^{-1}q_H|_T)||_{L^2(E)}\\
&\leq ||\tilde{P}^\infty_{n-l}A^{-1}|_T||_{L^\infty(E)}||q_H||_{L^2(E)} \lesssim H_T^{-1/2}||\tilde{P}^\infty_{n-l}A^{-1}||_{L^\infty(T)}||q_H||_{L^2(T)}.
\end{aligned} \tag{4.15}$$

Since $\mathcal{T}_H$ is shape-regular, we can replace H_T' by H_T, a similar argument leads to

$$||\tilde{P}^2_{n+1}((A^{-1}q_H)|_{T'}\cdot\tau)||_{L^2(E)} \lesssim H_T^{-1/2}||\tilde{P}^\infty_{n-l}A^{-1}||_{L^\infty(T')}||q_H||_{L^2(T')}. \tag{4.16}$$

A combination of (4.15) and (4.16) yields

$$\begin{aligned}
&||\tilde{P}^2_{n+1}J(A^{-1}q_H\cdot\tau)||_{L^2(E)}\\
&= ||\tilde{P}^2_{n+1}(((A^{-1}q_H)|_T - (A^{-1}q_H)|_{T'})\cdot\tau)||_{L^2(E)}\\
&\leq ||\tilde{P}^2_{n+1}((A^{-1}q_H)|_T\cdot\tau)||_{L^2(E)} + ||\tilde{P}^2_{n+1}((A^{-1}q_H)|_{T'}\cdot\tau)||_{L^2(E)}\\
&\lesssim H_T^{-1/2}||\tilde{P}^\infty_{n-l}A^{-1}||_{L^\infty(\omega_E)}||A^{-1/2}q_H||_{L^2(\omega_E)}.
\end{aligned} \tag{4.17}$$

By summing over all edges of element T, from the above inequality (4.17), we get

$$||\tilde{P}^2_{n+1}J(A^{-1}q_H\cdot\tau)||_{L^2(\partial T)} \lesssim H_T^{-1/2}||\tilde{P}^\infty_{n-l}A^{-1}||_{L^\infty(\omega_T)}||A^{-1/2}q_H||_{L^2(\omega_T)}. \tag{4.18}$$

Finally, the desired result (4.11) follows from (4.12)-(4.14) and (4.18). $\square$

The following two corollaries are global forms of the above lemma.

Corollary 4.1. (Estimator reduction) *For a triangulation $\mathcal{T}_H$ with $\mathcal{M}_H \subset \mathcal{T}_H$, let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$ obtained by $\mathcal{T}_h := REFINE(\mathcal{T}_H, \mathcal{M}_H)$. Denote $\Lambda_1 := 3\bar{\Lambda}_1^2$ with $\bar{\Lambda}_1$ given in Lemma 4.5, and $\lambda := 1 - 2^{-b/2} > 0$ with b given in Section 2.4. Then it holds*

$$\begin{aligned}
\eta^2_{\mathcal{T}_h}(\sigma_h, \mathcal{T}_h) \leq &(1+\delta_2)\{\eta^2_{\mathcal{T}_H}(\sigma_H, \mathcal{T}_H) - \lambda\eta^2_{\mathcal{T}_H}(\sigma_H, \mathcal{M}_H)\}\\
&+ (1+\delta_2^{-1})\Lambda_1\eta^2_{\mathcal{T}_0}(A^{-1}, \mathcal{T}_0)||A^{-1/2}(\sigma_H - \sigma_h)||^2_{L^2(\Omega)}
\end{aligned} \tag{4.19}$$

for all $\sigma_H \in M_H, \sigma_h \in M_h$ and any $\delta_2 > 0$.

Proof. The desired result follows from Lemma 4.5, Remark 2.1, Remark 4.1, and the same line as in the proof of Corollary 3.4 in [10]. □

Corollary 4.2. (Perturbation of oscillation) *Let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$, and let Λ_1 be the same as in Corollary* 4.1. *Then for all $\sigma_H \in M_h,\ \sigma_h \in M_h$, it holds*

$$\begin{aligned}&\mathrm{osc}^2_{\mathcal{T}_H}(\sigma_H, \mathcal{T}_H \cap \mathcal{T}_h)\\ \le&2\mathrm{osc}^2_{\mathcal{T}_h}(\sigma_h, \mathcal{T}_H \cap \mathcal{T}_h) + 2\Lambda_1 \mathrm{osc}^2_{\mathcal{T}_0}(A^{-1}, \mathcal{T}_0)||A^{-1/2}(\sigma_h - \sigma_H)||^2_{L^2(\Omega)}.\end{aligned} \tag{4.20}$$

Proof. Remark 2.1 yields $\mathrm{osc}_{\mathcal{T}_H}(\sigma_H, T) = \mathrm{osc}_{\mathcal{T}_h}(\sigma_H, T)$ for all $T \in \mathcal{T}_H \cap \mathcal{T}_h$. Hence, by the estimate (4.11) and Young's inequality, we get

$$\mathrm{osc}^2_{\mathcal{T}_H}(\sigma_H, T) \le 2\mathrm{osc}^2_{\mathcal{T}_h}(\sigma_h, T) + 2\bar{\Lambda}_1^2 \mathrm{osc}^2_{\mathcal{T}_h}(A^{-1}, \mathcal{T})||A^{-1/2}(\sigma_h - \sigma_H)||^2_{L^2(\omega_T)}. \tag{4.21}$$

By summing (4.21) over $T \in \mathcal{T}_H \cap \mathcal{T}_h$ and using the monotonicity property $\mathrm{osc}_{\mathcal{T}_h}(A^{-1}, \mathcal{T}_h) \le \mathrm{osc}_{\mathcal{T}_0}(A^{-1}, \mathcal{T}_0)$ stated in Remark 4.1, the inequality (4.21) indicates the desired assertion. □

5. Convergence of AMFEM

In this section, we first prove that the quasi-error uniformly reduces with a fixed rate on two successive meshes, up to an oscillation term of f. This means AMFEM is a contraction with respect to the quasi-error. We then prove that AMFEM is convergent with respect to the quasi-error plus the divergence of the flux error. To this end, subsequently we replace the subscripts H, h respectively with iteration counters $k, k+1$, and denote by

$$\mathcal{E}_k := ||A^{-1/2}(p - p_k)||_{L^2(\Omega)} \quad \text{and} \quad \eta_k := \eta_{\mathcal{T}_k}(p_k, \mathcal{T}_k)$$

the stress variable error and the estimator over the whole mesh $\mathcal{T}_k$,respectively.

Theorem 5.1. (Contraction property) *Given $\theta \in (0, 1]$, let $\{\mathcal{T}_k; (M_k, L_k); p_k\}_{k\ge0}$ be the sequence of meshes,a pair of finite element spaces, and the approximation solutions produced by AMFEM. Then there exits constants $\gamma > 0$, $0 < \alpha < 1$, and $C > 0$ depending solely on the shape-regularity of $\mathcal{T}_0$, b, $\eta_{\mathcal{T}_0(A^{-1},\mathcal{T}_0)}$, and the marking parameter θ, such that*

$$\mathcal{E}^2_{k+1} + \gamma\eta^2_{k+1} \le \alpha^2(\mathcal{E}^2_k + \gamma\eta^2_k) + C\mathrm{osc}^2(f, \mathcal{T}_k). \tag{5.1}$$

Proof. For convenience, we use the notations $E_k := p_k - p_{k+1}$, $\eta_k(\mathcal{M}_k) := \eta_{\mathcal{T}_k}(p_k, \mathcal{M}_k)$, $\eta_0(A^{-1}) := \eta_{\mathcal{T}_0}(A^{-1}, \mathcal{T}_0)$. Applying the estimator reduction (Corollary 4.1) to (4.5), we get for any $\bar{\gamma} \ge 0$,

$$\begin{aligned}(1-\delta_1)\mathcal{E}^2_{k+1} + \bar{\gamma}\eta^2_{k+1} \le \mathcal{E}^2_k - ||A^{-1/2}E_k||^2_{L^2(\Omega)} + \bar{\gamma}(1+\delta_2)\{\eta^2_k - \lambda\eta^2_k(\mathcal{M}_k)\}\\ + \bar{\gamma}(1+\delta_2^{-1})\Lambda_1\eta^2_0(A^{-1})||A^{-1/2}E_k||^2_{L^2(\Omega)} + \tfrac{C_0}{\delta_1}\mathrm{osc}^2(f_{k+1}, \mathcal{T}_k).\end{aligned}$$

In what follows we choose $\bar{\gamma} := 1/\left((1+\delta_2^{-1})\Lambda_1\eta^2_0(A^{-1})\right)$ so as to obtain

$$(1-\delta_1)\mathcal{E}^2_{k+1} + \bar{\gamma}\eta^2_{k+1} \le \mathcal{E}^2_k + \bar{\gamma}(1+\delta_2)\left(\eta^2_k - \lambda\eta^2_k(\mathcal{M}_k)\right) + \frac{C_0}{\delta_1}\mathrm{osc}^2(f_{k+1}, \mathcal{T}_k).$$

By using the reliable estimation (2.4) of the stress variable error, and invoking Dörfler marking property (2.5), the above inequality yields for any constant α,

$$\begin{aligned}
&(1-\delta_1)\mathcal{E}_{k+1}^2+\bar{\gamma}\eta_{k+1}^2\\
\le\,&\alpha^2(1-\delta_1)\mathcal{E}_k^2+(1-\alpha^2(1-\delta_1))\mathcal{E}_k^2+\bar{\gamma}(1+\delta_2)\{\eta_k^2-\lambda\eta_k^2(\mathcal{M}_k)\}+\frac{C_0}{\delta_1}\mathrm{osc}^2(f_{k+1},\mathcal{T}_k)\\
\le\,&\alpha^2(1-\delta_1)\mathcal{E}_k^2+(1-\alpha^2(1-\delta_1))C_1\eta_k^2+\bar{\gamma}(1+\delta_2)(1-\lambda\theta^2)\eta_k^2+\frac{C_0}{\delta_1}\mathrm{osc}^2(f_{k+1},\mathcal{T}_k)\\
\le\,&\alpha^2\{(1-\delta_1)\mathcal{E}_k^2+\frac{(1-\alpha^2(1-\delta_1))C_1+\bar{\gamma}(1+\delta_2)(1-\lambda\theta^2)}{\alpha^2}\eta_k^2\}+\frac{C_0}{\delta_1}\mathrm{osc}^2(f_{k+1},\mathcal{T}_k). \qquad (5.2)
\end{aligned}$$

We choose α such that $(1-\alpha^2(1-\delta_1))C_1+\bar{\gamma}(1+\delta_2)(1-\lambda\theta^2)=\alpha^2\bar{\gamma}$, which gives

$$\alpha^2=\frac{(1-\delta_1)C_1+\bar{\gamma}(\delta_1C_1/\bar{\gamma}+(1+\delta_2)(1-\lambda\theta^2))}{(1-\delta_1)C_1+\bar{\gamma}}.$$

We now choose δ_3 and δ_1 such that

$$\delta_2\le\lambda\theta^2/2(1-\lambda\theta^2)\quad\text{and}\quad\delta_1<\min\{1,\frac{(1-\lambda\theta^2)\delta_2\bar{\gamma}}{C_1}\}.$$

Then it follows

$$\begin{aligned}
&\delta_1C_1/\bar{\gamma}+(1+\delta_2)(1-\lambda\theta^2)\\
<&(1-\lambda\theta^2)\delta_2+(1+\delta_2)(1-\lambda\theta^2)\le(1-\lambda\theta^2)(1+\frac{\lambda\theta^2}{1-\lambda\theta^2})=1,
\end{aligned}$$

which leads to $\alpha^2<1$. Finally we set $\gamma=\bar{\gamma}/(1-\delta_1)$. Then the desired result (5.1) follows from (5.2) and lemma 4.3. □

We note that the oscillation, $\mathrm{osc}(f,\mathcal{T}_k)$, of the right-hand side term f measures intrinsic information missing in the average process associated with finite elements, but fails to detect fine structures of f. Since the oscillation occurs in the quasi-orthogonality for the stress variable, we shall use an equivalent term, $||h_k\mathrm{div}(p-p_k)||$, to offset it so as to derive the convergence of AMFEM without marking the oscillation.

Theorem 5.2. (Convergence result) *Given $\theta\in(0,1]$, let $\{\mathcal{T}_k;(M_k,L_k);p_k)\}_{k\ge0}$ be the sequence of meshes, a pair of finite element spaces, and the approximation solutions produced by AMFEM, and denote C_{in} the error on the initial mesh by*

$$C_{\mathrm{in}}:=\mathcal{E}_0^2+\gamma_1||h_0\mathrm{div}(p-p_0)||_{L^2(\Omega)}^2+\gamma_2\eta_0^2.$$

Then there exits constants γ_1,γ_2,β with $\gamma_1,\gamma_2>0$ and $0<\beta<1$, depending solely on the shape-regularity of $\mathcal{T}_0$, b, $\eta_{\mathcal{T}_0(A^{-1},\mathcal{T}_0)}$, and the marking parameter θ, such that

$$\mathcal{E}_{k+1}^2+\gamma_1||h_{k+1}\mathrm{div}(p-p_{k+1})||_{L^2(\Omega)}^2+\gamma_2\eta_{k+1}^2\le C_{\mathrm{in}}\beta^{2(k+1)}. \qquad (5.3)$$

Proof. Notice that the second equation of (2.2) and (2.3) implies the following orthogonality for the divergence of the flux on each element

$$(\mathrm{div}(p-p_{k+1}),\mathrm{div}(p_{k+1}-p_k))_T=0\quad\text{for all}\quad T\in\mathcal{T}_{k+1}.$$

This orthogonality leads to

$$\begin{aligned}
&||\mathrm{div}(p-p_{k+1})||^2_{L^2(T)}\\
=&||\mathrm{div}(p-p_k)||^2_{L^2(T)} - ||\mathrm{div}(p_{k+1}-p_k)||^2_{L^2(T)} - 2(\mathrm{div}(p-p_{k+1}), \mathrm{div}(p_{k+1}-p_k))_T\\
=&||\mathrm{div}(p-p_k)||^2_{L^2(T)} - ||\mathrm{div}(p_{k+1}-p_k)||^2_{L^2(T)}.
\end{aligned} \tag{5.4}$$

Since the marked elements are at most bisected b_0 times, from the monotonicity of the mesh-size function, and summing (5.4) over all element $T \in \mathcal{T}_{k+1}$, we obtain

$$\begin{aligned}
&||h_{k+1}\mathrm{div}(p-p_{k+1})||^2_{L^2(\Omega)}\\
=&\sum_{T\in\mathcal{T}_{k+1}} h^2_{k+1}||\mathrm{div}(p-p_k)||^2_{L^2(T)} - \sum_{T\in\mathcal{T}_{k+1}} h^2_{k+1}||\mathrm{div}(p_{k+1}-p_k)||^2_{L^2(T)}\\
\le&\sum_{T\in\mathcal{T}_k}\{h^2_k||\mathrm{div}(p-p_k)||^2_{L^2(T)} - 2^{-(b_0+1)}h^2_k||\mathrm{div}(p_{k+1}-p_k)||^2_{L^2(T)}\}\\
=& ||h_k\mathrm{div}(p-p_k)||^2_{L^2(\Omega)} - 2^{-(b_0+1)}||h_k\mathrm{div}(p_{k+1}-p_k)||^2_{L^2(\Omega)}.
\end{aligned} \tag{5.5}$$

Let f_{k+1} and f_k denote the L^2-best approximation of f over L_{k+1} and L_k, respectively, and Π_{L_k} be the L^2-projection operator over L_k. Notice that div $p_{k+1} = -f_{k+1}$, div $p_k = -f_k$. From the property of the L^2-best approximation, we get

$$\begin{aligned}
\mathrm{osc}^2(f_{k+1},\mathcal{T}_k) =& \sum_{T\in\mathcal{T}_k} h^2_T||f_{k+1}-\Pi_{L_k}f_{k+1}||^2_{L^2(T)} \le \sum_{T\in\mathcal{T}_k} h^2_T||f_{k+1}-f_k||^2_{L^2(T)}\\
=&||h_k\mathrm{div}(p_{k+1}-p_k)||^2_{L^2(\Omega)}.
\end{aligned} \tag{5.6}$$

Applying (5.6) to (4.5), combining the estimator reduction (Corollary 4.1) and (5.5), and invoking Dörfler marking property (2.5), we get for any $\bar{\gamma}_1, \bar{\gamma}_2 > 0$,

$$\begin{aligned}
&(1-\delta_1)\mathcal{E}^2_{k+1} + \bar{\gamma}_1||h_{k+1}\mathrm{div}(p-p_{k+1})||^2_{L^2(\Omega)} + \bar{\gamma}_2\eta^2_{k+1}\\
\le& \mathcal{E}^2_k + \bar{\gamma}_1||h_k\mathrm{div}(p-p_k)||^2_{L^2(\Omega)} + \{\bar{\gamma}_2(1+\delta_3^{-1})\Lambda_1\eta^2_0(A^{-1}) - 1\}||A^{-1/2}E_k||^2_{L^2(\Omega)}\\
&+ (C_0\delta_1^{-1} - \bar{\gamma}_1 2^{-(b_0+1)})||h_k\mathrm{div}\ E_k||^2_{L^2(\Omega)} + \bar{\gamma}_2(1+\delta_3)(1-\lambda\theta^2)\eta^2_k,
\end{aligned}$$

where $\eta_0(A^{-1}) := \eta_{\mathcal{T}_0}(A^{-1}, \mathcal{T}_0)$ and $E_k := p_k - p_{k+1}$. We now choose $\bar{\gamma}_1$ and $\bar{\gamma}_2$ such that

$$C_0\delta_1^{-1} - \bar{\gamma}_1 2^{-(b_0+1)} = 0, \quad \bar{\gamma}_2(1+\delta_3^{-1})\Lambda_1\eta^2_0(A^{-1}) - 1 = 0.$$

This choice leads to

$$\begin{aligned}
&(1-\delta_1)\mathcal{E}^2_{k+1} + \bar{\gamma}_1||h_{k+1}\mathrm{div}(p-p_{k+1})||^2_{L^2(\Omega)} + \bar{\gamma}_2\eta^2_{k+1}\\
\le& \mathcal{E}^2_k + \bar{\gamma}_1||h_k\mathrm{div}(p-p_k)||^2_{L^2(\Omega)} + \bar{\gamma}_2(1+\delta_3)(1-\lambda\theta^2)\eta^2_k\\
\le& \beta_1^2\bigg(\mathcal{E}^2_k + \bar{\gamma}_1||h_k\mathrm{div}(p-p_k)||^2_{L^2(\Omega)} + \bar{\gamma}_2\eta^2_k\bigg) + (1-\beta_1^2)C_1\eta^2_k\\
&+ (1-\beta_1^2)\bar{\gamma}_1\eta^2_k + \bigg(\bar{\gamma}_2(1+\delta_3)(1-\lambda\theta^2) - \bar{\gamma}_2\beta_1^2\bigg)\eta^2_k.
\end{aligned}$$

We then choose δ_3 and β_1^2 such that

$$\delta_3 < \frac{\lambda\theta^2}{1-\lambda\theta^2}, \qquad \beta_1^2 := \frac{\bar{\gamma}_2(1+\delta_3)(1-\lambda\theta^2) + (\bar{\gamma}_1 + C_1)}{\bar{\gamma}_2 + (\bar{\gamma}_1 + C_1)} < 1,$$

so as to obtain

$$
\begin{aligned}
&(1-\delta_1)\mathcal{E}_{k+1}^2+\bar{\gamma}_1||h_{k+1}\mathrm{div}(p-p_{k+1})||^2_{L^2(\Omega)}+\bar{\gamma}_2\eta_{k+1}^2\\
\leq{}& \beta_1^2\bigg(\mathcal{E}_k^2+\bar{\gamma}_1||h_k\mathrm{div}(p-p_k)||^2_{L^2(\Omega)}+\bar{\gamma}_2\eta_k^2\bigg).
\end{aligned}
$$

Let $0<\beta_1^2<\beta^2<1$, and set $\gamma_3=1-\delta_1=\beta_1^2/\beta^2$. This means $\delta_1=1-\beta_1^2/\beta^2$. Whence, we have

$$
\begin{aligned}
&\gamma_3\mathcal{E}_{k+1}^2+\bar{\gamma}_1||h_{k+1}\mathrm{div}(p-p_{k+1})||^2_{L^2(\Omega)}+\bar{\gamma}_2\eta_{k+1}^2\\
\leq{}& \beta^2\bigg(\gamma_3\mathcal{E}_k^2+\gamma_3\bar{\gamma}_1||h_k\mathrm{div}(p-p_k)||^2_{L^2(\Omega)}+\gamma_3\bar{\gamma}_2\eta_k^2\bigg)\\
\leq{}& \beta^2\bigg(\gamma_3\mathcal{E}_k^2+\bar{\gamma}_1||h_k\mathrm{div}(p-p_k)||^2_{L^2(\Omega)}+\bar{\gamma}_2\eta_k^2\bigg).
\end{aligned}
$$

We finally choose $\gamma_1=\bar{\gamma}_1/\gamma_3, \gamma_2=\bar{\gamma}_2/\gamma_3$, so as to obtain

$$
\mathcal{E}_{k+1}^2+\gamma_1||h_{k+1}\mathrm{div}(p-p_{k+1})||^2_{L^2(\Omega)}+\gamma_2\eta_{k+1}^2\leq\ \beta^2\bigg(\mathcal{E}_k^2+\gamma_1||h_k\mathrm{div}(p-p_k)||^2_{L^2(\Omega)}+\gamma_2\eta_k^2\bigg),
$$

which implies the desired result (5.3). □

6. Optimality of AMFEM

6.1. Auxiliary results

In this subsection, we aim at the discrete upper bound, which is one key to the proof for the quasi-optimal convergence rate. Simultaneously, we shall quote a counting result for the overlay of two conforming meshes.

Theorem 6.1. (Discrete upper bound) *Let $\mathcal{T}_h$ and $\mathcal{T}_H$ be two nested conforming triangulations, $p_h\in M_h$ and $p_H\in M_H$ be the discrete solutions with respect to the meshes $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively, and $\mathcal{F}_H:=\{T\in\mathcal{T}_H: T$ is not included in $\mathcal{T}_h\}$. Then there exists a positive constant C_3 depending only on the shape regularity of $\mathcal{T}_H$ such that*

$$
||A^{-1/2}(p_h-p_H)||^2_{L^2(\Omega)}\leq C_3\eta^2_{\mathcal{T}_H}(p_H,\mathcal{F}_H), \tag{6.1}
$$

$$
\#\mathcal{F}_H\leq\#\mathcal{T}_h-\#\mathcal{T}_H. \tag{6.2}
$$

Proof. The second inequality, i.e., (6.2), follows from the definition of $\mathcal{F}_H$. To prove the first one, we introduce the solution $(\widetilde{p}_h,\widetilde{u}_h)\in M_h\times L_h$ to the problem (4.2). From (2.3) and (4.2), we obtain $\mathrm{div}(\widetilde{p}_h-p_H)=0$, which implies

$$
\int_{\partial\Omega}(\widetilde{p}_h-p_H)\cdot\nu ds=0, \tag{6.3}
$$

where ν is the outward unit normal vector along $\partial\Omega$. Thus $\widetilde{p}_h-p_H$ satisfies the conditions of Theorem 3.1 in [19] on the polygonal domain Ω, namely it is divergence-free and fulfills (6.3). As a result, there exists $\psi_h\in H^1(\Omega)$ with Curl $\psi_h=\widetilde{p}_h-p_H$. Since $\widetilde{p}_h-p_H\in M_h$, this leads to

$$
\psi_h\in S_h^{l+2}:=\bigg\{\psi_h\in C(\overline{\Omega}):\psi_h|_T\in P_{l+2}(T)\ \text{ for all }\ T\in\mathcal{T}_h\bigg\}.
$$

From (4.2) with $q_h = \widetilde{p}_h - p_H$, we get

$$\begin{aligned} ||A^{-1/2}(\widetilde{p}_h - p_H)||^2_{L^2(\Omega)} &= (A^{-1}(\widetilde{p}_h - p_H), \widetilde{p}_h - p_H)_{0,\Omega} \\ &= (A^{-1}\widetilde{p}_h, \widetilde{p}_h - p_H)_{0,\Omega} - (A^{-1}p_H, \widetilde{p}_h - p_H)_{0,\Omega} \\ &= -(A^{-1}p_H, \widetilde{p}_h - p_H)_{0,\Omega} = -(A^{-1}p_H, \mathrm{Curl}\ \psi_h)_{0,\Omega}. \end{aligned} \tag{6.4}$$

Since $\mathrm{div}(\mathrm{Curl}\ \psi_H) = 0$ for any $\psi_H \in S_H^{l+2}$, from (2.3) with $q_H = \mathrm{Curl}\ \psi_H$, we have

$$(A^{-1}p_H, \mathrm{Curl}\ \psi_H)_{0,\Omega} = -(\mathrm{div\ Curl}\ \psi_H, u_H)_{0,\Omega} = 0 \quad \text{for all} \quad \psi_H \in S_H^{l+2}. \tag{6.5}$$

Here the definition of S_H^{l+2} is analogous to S_h^{l+2}.

To connect S_h^{l+2} with S_H^{l+2}, we denote the Scott-Zhang interpolation operator (see [27]) by $\mathcal{I}_H : S_h^{l+2} \to S_H^{l+2}$, and recall that $\mathcal{I}_H$ is local in the sense that if $T \in \mathcal{T}_h \cap \mathcal{T}_H$ or $E \in \varepsilon_h \cap \varepsilon_H$ (i.e., T or E is not refined), then $(\psi_h - \mathcal{I}_H\psi_h)|_T = 0$ or $(\psi_h - \mathcal{I}_H\psi_h)|_E = 0$. Consequently, the following estimates hold:

$$||\psi_h - \mathcal{I}_H\psi_h||_{L^2(E)} \lesssim H_E^{1/2}|\psi_h|_{H^1(\omega_E)} \quad \text{for all} \quad E \in \varepsilon_H, \tag{6.6}$$

$$||\psi_h - \mathcal{I}_H\psi_h||_{L^2(T)} \lesssim H_T|\psi_h|_{H^1(\omega_T)} \quad \text{for all} \quad T \in \mathcal{T}_H. \tag{6.7}$$

By taking $\psi_H = \mathcal{I}_H\psi_h$ and using integration by parts, a combination of (6.4)- (6.7) yields

$$\begin{aligned} &||A^{-1/2}(\widetilde{p}_h - p_H)||^2_{L^2(\Omega)} = \sum_{T\in\mathcal{T}_H} -\int_T A^{-1}p_H \cdot \mathrm{Curl}(\psi_h - \psi_H) \\ &= \sum_{T\in\mathcal{T}_H} \int_T \mathrm{curl}(A^{-1}p_H)(\psi_h - \psi_H) - \sum_{E\in\varepsilon_H} \int_E J(A^{-1}p_H \cdot \tau)(\psi_h - \psi_H) \\ &\lesssim \eta_{\mathcal{T}_H}(p_H, \mathcal{F}_H)|\psi_h|_{H^1(\Omega)} \le C_4^{1/2}\eta_{\mathcal{T}_H}(p_H, \mathcal{F}_H)||A^{-1/2}(\widetilde{p}_h - p_H)||_{L^2(\Omega)}, \end{aligned}$$

which implies

$$||A^{-1/2}(\widetilde{p}_h - p_H)||_{L^2(\Omega)} \le C_4^{1/2}\eta_{\mathcal{T}_H}(p_H, \mathcal{F}_H). \tag{6.8}$$

On the other hand, from $\widetilde{p}_h - p_H \in M_h$ we have

$$(A^{-1}(p_h - \widetilde{p}_h), \widetilde{p}_h - p_H)_{0,\Omega} = -(\mathrm{div}(\widetilde{p}_h - p_H), u_h - \widetilde{u}_h)_{0,\Omega} = 0. \tag{6.9}$$

Then a combination of (6.8), (6.9) and Lemma 4.2 shows

$$\begin{aligned} ||A^{-1/2}(p_h - p_H)||^2_{L^2(\Omega)} &= (A^{-1}(p_h - p_H), p_h - p_H)_{0,\Omega} \\ &= (A^{-1}(p_h - \widetilde{p}_h + \widetilde{p}_h - p_H), p_h - \widetilde{p}_h + \widetilde{p}_h - p_H)_{0,\Omega} \\ &= ||A^{-1/2}(p_h - \widetilde{p}_h)||^2_{L^2(\Omega)} + ||A^{-1/2}(\widetilde{p}_h - p_H)||^2_{L^2(\Omega)} \\ &\le C_4\eta^2_{\mathcal{T}_H}(p_H, \mathcal{F}_H) + C_0\mathrm{osc}^2(f_h, \mathcal{T}_H). \end{aligned} \tag{6.10}$$

We note that it holds

$$\begin{aligned} \mathrm{osc}^2(f_h, \mathcal{T}_H) &= \sum_{T\in\mathcal{T}_H} H_T^2||f_h - \Pi_{L_H}f_h||^2_{L^2(T)} = \sum_{T\in\mathcal{F}_H} H_T^2||f_h - \Pi_{L_H}f_h||^2_{L^2(T)} \\ &\le \sum_{T\in\mathcal{F}_H} H_T^2||f_h - f_H||^2_{L^2(T)} = \sum_{T\in\mathcal{F}_H} H_T^2||\Pi_{L_h}(f - \Pi_{L_H}f)||^2_{L^2(T)} \\ &\le \sum_{T\in\mathcal{F}_H} H_T^2||f - \Pi_{L_H}f||^2_{L^2(T)} \le ||H(f - f_H)||^2_{L^2(\mathcal{F}_H)}, \end{aligned} \tag{6.11}$$

where Π_{L_h} and Π_{L_H} are the L^2-projection operator onto L_h and L_H, respectively. As a result, the desired result (6.1) follows from (6.10)–(6.11) with $C_3 := C_4 + C_0$. □

In what follows, we shall quote a counting conclusion from [28] for the overlay $\mathcal{T} := \mathcal{T}_1 \oplus \mathcal{T}_2$ of two conforming triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$, which shows $\mathcal{T}$ is the smallest conforming triangulation for the triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$.

Lemma 6.1. (Overlay of meshes) *For two conforming triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ the overlay $\mathcal{T} := \mathcal{T}_1 \oplus \mathcal{T}_2$ is conforming, and satisfies*

$$\#\mathcal{T} \leq \#\mathcal{T}_1 + \#\mathcal{T}_2 - \#\mathcal{T}_0.$$

6.2. Quasi-optimal convergence rate

In this subsection, we prove the quasi-optimal convergence rate of the AMFEM with respect to the total error. To this end, we need to introduce a nonlinear approximation class $\mathbb{A}_s$. Let $\mathcal{P}_N$ be the set of all triangulations $\mathcal{T}$ which is refined from $\mathcal{T}_0$ and $\#\mathcal{T} \leq N$. For a given triangulation $\mathcal{T}$ and any constant $\zeta > 0$, which is determined below, denote by $h_{\mathcal{T}}$ the mesh-size function with respect to $\mathcal{T}$, and denote by $p_{\mathcal{T}}$ the approximation solution to the flux with respect to $\mathcal{T}$. For $(q, f, A) \in H(\text{div}, \Omega) \times L^2(\Omega) \times L^\infty(\Omega; \mathbb{R}^{2\times 2})$ and $s > 0$, we define the nonlinear approximation $\mathbb{A}_s$ as

$$\mathbb{A}_s := \{(q, f, A)|\ |(q, f, A)|_s := \sup_{N > N_0 = \#\mathcal{T}_0} N^s \sigma(N; q, f, A) < \infty\},$$

where

$$\sigma(N; q, f, A) := \inf_{\mathcal{T}\in\mathcal{P}_N} \{||A^{-1/2}(q - p_{\mathcal{T}})||^2_{L^2(\Omega)} + \zeta \text{osc}^2_{\mathcal{T}}(p_{\mathcal{T}}, \mathcal{T})\}^{1/2}.$$

We now prove that the approximation p_k generated by the AMFEM concerning the stress variable converges to p in a weighted norm with the same rate $(\#\mathcal{T}_k - \#\mathcal{T}_0)^{-s}$ as the best approximation described by $\mathbb{A}_s$ up to a multiplicative constant. We need to count elements added by handling hanging nodes to keep mesh conformity (see Lemma 2.1), as well as those marked by the estimator (the cardinality of $\mathcal{M}_k$). To this end, we impose more stringent requirements than for convergence of AMFEM. Note that we follow the ideas in [20, 21] to prove the optimality, without using the algorithm that separates the error and the reduction of data oscillation developed in [12].

Assumption 6.1. (Optimality) *We assume the following properties of AMFEM:*
(a) The marking parameter θ satisfies $\theta \in (0, \theta_*)$, where θ_* is determined in Lemma 6.2 *below;*
(b) Procedure MARK selects a set $\mathcal{M}_k$ of marked elements with minimal cardinality;
(c) The distribution of refinement edges on $\mathcal{T}_0$ satisfies condition (b) of Section 4 *in* [29].

The following lemma establishes a link between nonlinear approximation theory and AMFEM through the Dörfler marking strategy. Roughly speaking, we prove that, if an approximation satisfies a suitable total error reduction from $\mathcal{T}_H$ to $\mathcal{T}_h$ ($\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$), the error indicators of the coarser solutions must satisfy a Dörfler property on the set $\mathcal{R}$ of refined elements. In other words, the total error reduction and Dörfler marking are intimately connected.

Lemma 6.2. (Optimality marking) *Assume that the marking parameter θ verifies (a) of Assumption* 6.1. *Let $\mathcal{T}_H$ be an shape regular triangulation of Ω, $p_H \in M_H$ be the approximation* *solution to the flux of* (2.3). *Set* $0 < \mu < 1/2$, *denote* $\zeta \geq 1$ *a constant, which is determined in Lemma* 6.3 *below, and let* $\mathcal{T}_{h_*}$ *be any refinement of* $\mathcal{T}_H$ *such that the approximation solution* $p_{h_*} \in M_{h_*}$ *satisfies*

$$\mathcal{E}_{h_*}^2 + \zeta \text{osc}_{\mathcal{T}_{h_*}}^2(p_{h_*}, \mathcal{T}_{h_*}) \leq \mu\{\mathcal{E}_H^2 + \zeta \text{osc}_{\mathcal{T}_H}^2(p_H, \mathcal{T}_H)\}. \tag{6.12}$$

Then the set $\mathcal{R} := \mathcal{R}_{\mathcal{T}_H \to \mathcal{T}_{h_*}}$ *satisfies the Dörfler property* $\eta_{\mathcal{T}_H}(p_H, \mathcal{R}) \geq \theta \eta_{\mathcal{T}_H}(p_H, \mathcal{T}_H)$.

Proof. A combination of the lower bound (3.3) and (6.12) yields

$$\begin{aligned}(1-2\mu)C_2\eta_{\mathcal{T}_H}^2(p_H, \mathcal{T}_H) \leq &(1-2\mu)\{\mathcal{E}_H^2 + \text{osc}_{\mathcal{T}_H}^2(p_H, \mathcal{T}_H)\}\\ \leq &(1-2\mu)\{\mathcal{E}_H^2 + \zeta\text{osc}_{\mathcal{T}_H}^2(p_H, \mathcal{T}_H)\}\\ \leq &\mathcal{E}_H^2 + \zeta\text{osc}_{\mathcal{T}_H}^2(p_H, \mathcal{T}_H) - \mathcal{E}_{h_*}^2 - 2\zeta\text{osc}_{\mathcal{T}_{h_*}}^2(p_{h_*}, \mathcal{T}_{h_*}).\end{aligned} \tag{6.13}$$

For any $\delta_0 > 0$ which is to be determined below, according to the quasi-orthogonality (4.4), (6.11), and Young inequality, we get

$$\mathcal{E}_H^2 - \mathcal{E}_{h_*}^2 \leq ||A^{-1/2}(p_{h_*} - p_H)||_{L^2(\Omega)}^2 + \delta_0\mathcal{E}_{h_*}^2 + \delta_0^{-1}C_0||H(f-f_H)||_{L^2(\mathcal{R})}^2. \tag{6.14}$$

For the oscillation term we argue according to whether or not an element $T \in \mathcal{T}_H$ belongs to the set of refined elements $\mathcal{R}$. For $T \in \mathcal{R}$ we use the dominance $\text{osc}_{\mathcal{T}_H}^2(p_H, T) \leq \eta_{\mathcal{T}_H}^2(p_H, T)$ (see Remark 2.1). For $T \in \mathcal{T}_H \cap \mathcal{T}_{h_*}$, Corollary 4.2 (Perturbation of oscillation), together with $\sigma_H = p_H$ and $\sigma_h = p_{h_*}$, yields

$$\begin{aligned}&\text{osc}_{\mathcal{T}_H}^2(p_H, \mathcal{T}_H \cap \mathcal{T}_{h_*}) - 2\text{osc}_{\mathcal{T}_{h_*}}^2(p_{h_*}, \mathcal{T}_H \cap \mathcal{T}_{h_*})\\ \leq &2\Lambda_1\text{osc}_{\mathcal{T}_0}^2(A^{-1}, \mathcal{T}_0)||A^{-1/2}(p_{h_*} - p_H)||_{L^2(\Omega)}^2.\end{aligned} \tag{6.15}$$

Combining (6.1) and (6.15), we have

$$\begin{aligned}&\text{osc}_{\mathcal{T}_H}^2(p_H, \mathcal{T}_H) - 2\text{osc}_{\mathcal{T}_{h_*}}^2(p_{h_*}, \mathcal{T}_{h_*})\\ &= \text{osc}_{\mathcal{T}_H}^2(p_H, \mathcal{R}) + \text{osc}_{\mathcal{T}_H}^2(p_H, \mathcal{T}_H \cap \mathcal{T}_{h_*}) - 2\text{osc}_{\mathcal{T}_{h_*}}^2(p_{h_*}, \mathcal{T}_H \cap \mathcal{T}_{h_*}) - 2\text{osc}_{\mathcal{T}_{h_*}}^2(p_{h_*}, \mathcal{T}_{h_*} \setminus \mathcal{T}_H)\\ &\leq \text{osc}_{\mathcal{T}_H}^2(p_H, \mathcal{R}) + 2\Lambda_1\text{osc}_{\mathcal{T}_0}^2(A^{-1}, \mathcal{T}_0)||A^{-1/2}(p_{h_*} - p_H)||_{L^2(\Omega)}^2\\ &\leq (1 + 2C_3\Lambda_1\text{osc}_{\mathcal{T}_0}^2(A^{-1}, \mathcal{T}_0))\eta_{\mathcal{T}_H}^2(p_H, \mathcal{R}).\end{aligned} \tag{6.16}$$

The convergence result and upper bound imply

$$\begin{aligned}\mathcal{E}_{h_*}^2 \leq &\beta^2\{\mathcal{E}_H^2 + \gamma_1||H\text{div}(p - p_H)||_{L^2(\Omega)}^2 + \gamma_2\eta_{\mathcal{T}_H}^2(P_H, \mathcal{T}_H)\}\\ \leq &\beta^2(C_1 + \gamma_1 + \gamma_2)\eta_{\mathcal{T}_H}^2(P_H, \mathcal{T}_H).\end{aligned} \tag{6.17}$$

Applying the discrete upper bound (6.1) to (6.14), and combing (6.13)-(6.14), (6.16)-(6.17), we arrive at

$$\Big((1-2\mu)C_2 - \beta^2(C_1 + \gamma_1 + \gamma_2)\delta_0\Big)\eta_{\mathcal{T}_H}^2(P_H, \mathcal{T}_H) \leq g(\delta_0)\eta_{\mathcal{T}_H}^2(P_H, \mathcal{R}),$$

where $g(\delta_0)$ is the value of the function $g(\delta) := C_3 + \delta^{-1}C_0 + (1 + 2C_3\Lambda_1\text{osc}_{\mathcal{T}_0}^2(A^{-1}, \mathcal{T}_0))\zeta$ at the point δ_0. This inequality means

$$\eta_{\mathcal{T}_H}^2(P_H, \mathcal{R}) \geq \theta_*^2\eta_{\mathcal{T}_H}^2(P_H, \mathcal{T}_H) \geq \theta^2\eta_{\mathcal{T}_H}^2(P_H, \mathcal{T}_H),$$

where $\theta_*^2 = ((1-2\mu)C_2 - \beta^2(C_1+\gamma_1+\gamma_2)\delta_0)/g(\delta_0)$. This implies that θ_*^2 is chosen as the maximum value of the function

$$\theta_*^2(\delta) = ((1-2\mu)C_2 - \beta^2(C_1+\gamma_1+\gamma_2)\delta)/g(\delta)$$

on the interval $(0,(1-2\mu)C_2/(\beta^2(C_1+\gamma_1+\gamma_2)))$, and that δ_0 is the corresponding the maximum value point. This completes the proof. □

The fact that procedure MARK selects the set of marked elements $\mathcal{M}_k$ with minimal cardinality, establishes a link between the best mesh and triangulations generated by AMFEM, and forms crucial idea of AFEM (see [28]). In what follows we shall use this fact.

Lemma 6.3. (Cardinality of $\mathcal{M}_k$) *Assume that the marking parameter θ verifies (a) of Assumption* 6.1*, and procedure MARK satisfies (b) of Assumption* 6.1*. Let p solve the problem* (2.1)*, and let $\{\mathcal{T}_k;(M_k,L_k);p_k;\mathcal{E}_k\}_{k\geq 0}$ be the sequence of meshes, finite element spaces, the discrete solution produced by AMFEM, and the stress variable error. If $(p,f,A)\in\mathbb{A}_s$, then the following estimate is valid:*

$$\#\mathcal{M}_k \lesssim \mu^{-1/2s}|(p,f,A)|_s^{1/s} C_A^{1/2s}\{\mathcal{E}_k^2 + \zeta \mathrm{osc}_{\mathcal{T}_k}^2(p_k,\mathcal{T}_k)\}^{-1/2s}. \tag{6.18}$$

Proof. Let μ be the same as defined in Lemma 6.2, for any positive constant ζ, which is determined below, we set

$$\varepsilon^2 := 4^{-1}C_A^{-1}\mu(\mathcal{E}_k^2 + \zeta\mathrm{osc}_{\mathcal{T}_k}^2(p_k,\mathcal{T}_k)),$$

where the constant C_A is defined by

$$C_A := \max\{(1-\delta_4)^{-1}, 2+2\Lambda_1^2\mathrm{osc}_{\mathcal{T}_0}^2(A^{-1},\mathcal{T}_0)C_0\delta_4^{-1}\}$$

with any $0<\delta_4<1$. Let $[\varepsilon^{-1/s}|(p,f,A)|_s^{1/s}]$ denote the integer component of $\varepsilon^{-1/s}|(p,f,A)|_s^{1/s}$, set $N_\varepsilon := [\varepsilon^{-1/s}|\ (p,f,A)|_s^{1/s}]+1$. Recall

$$\sigma(N_\varepsilon + \#\mathcal{T}_0 - 1; p,f,A) := \inf_{\mathcal{T}\in\mathcal{P}_{N_\varepsilon+\#\mathcal{T}_0-1}}\{\mathcal{E}_\mathcal{T}^2 + \zeta\mathrm{osc}_\mathcal{T}^2(p_\mathcal{T},\mathcal{T})\}^{1/2}.$$

Denote $\tilde\varepsilon := \min\{1,\varepsilon\}$. There exists $\mathcal{T}_\varepsilon \in \mathcal{P}_{N_\varepsilon+\#\mathcal{T}_0-1}$ with $\#\mathcal{T}_\varepsilon \leq N_\varepsilon + \#\mathcal{T}_0 - 1$ such that

$$\{\mathcal{E}_{h_\varepsilon}^2 + \zeta\mathrm{osc}_{\mathcal{T}_\varepsilon}^2(p_\varepsilon,\mathcal{T}_\varepsilon)\}^{1/2} \leq (1+\tilde\varepsilon)\sigma(N_\varepsilon+\#\mathcal{T}_0-1;p,f,A),$$

where p_ε is the discrete flux approximation to $p = A\nabla u$ with respect to the mesh $\mathcal{T}_\varepsilon$, $\mathcal{E}_{h_\varepsilon}^2 := ||A^{-1/2}(p-p_\varepsilon)||_{L^2(\Omega)}^2$. This inequality leads to

$$\begin{aligned}
&N_\varepsilon^s\{\mathcal{E}_{h_\varepsilon}^2 + \zeta\mathrm{osc}_{\mathcal{T}_\varepsilon}^2(p_\varepsilon,\mathcal{T}_\varepsilon)\}^{1/2} \leq (N_\varepsilon+\#\mathcal{T}_0-1)^s\{\mathcal{E}_{h_\varepsilon}^2 + \zeta\mathrm{osc}_{\mathcal{T}_\varepsilon}^2(p_\varepsilon,\mathcal{T}_\varepsilon)\}^{1/2}\\
\leq&(1+\tilde\varepsilon)(N_\varepsilon+\#\mathcal{T}_0-1)^s\sigma(N_\varepsilon+\#\mathcal{T}_0-1;p,f,A)\\
\leq&(1+\tilde\varepsilon)\sup_{N>0}N^s\sigma(N;p,f,A) = (1+\tilde\varepsilon)|(p,f,A)|_s.
\end{aligned}\tag{6.19}$$

From the above inequality (6.19), we obtain

$$\{\mathcal{E}_{h_\varepsilon}^2 + \zeta\mathrm{osc}_{\mathcal{T}_\varepsilon}^2(p_\varepsilon,\mathcal{T}_\varepsilon)\}^{1/2} \leq \frac{(1+\tilde\varepsilon)|(p,f,A)|_s}{N_\varepsilon^s} \leq 2\varepsilon, \tag{6.20}$$

$$\#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \leq N_\varepsilon - 1 \leq \varepsilon^{-1/s}|(p,f,A)|_s^{1/s}. \tag{6.21}$$

Let $\mathcal{T}_* := \mathcal{T}_\varepsilon \oplus \mathcal{T}_k$ be the overlay of $\mathcal{T}_\varepsilon$ and $\mathcal{T}_k$, h_* be the mesh-size function with respect to $\mathcal{T}_*$, and p_* be the approximation solution to the flux onto $\mathcal{T}_*$. We show that there is a reduction with a factor μ of the total error between p_* and p_k. Notice that $\mathcal{T}_*$ is a refinement of $\mathcal{T}_\varepsilon$, and recall that $\mathcal{E}^2_{h_*} := ||A^{-1/2}(p-p_*)||^2_{L^2(\Omega)}$, by the quasi-orthogonality (4.4) and Lemma 4.3, for any $\delta_4 > 0$, we have

$$\mathcal{E}^2_{h_*} \le \mathcal{E}^2_{h_\varepsilon} - ||A^{-1/2}(p_*-p_\varepsilon)||^2_{L^2(\Omega)} + \delta_4 \mathcal{E}^2_{h_*} + C_0\delta_4^{-1}\mathrm{osc}^2_{\mathcal{T}_\varepsilon}(p_\varepsilon, \mathcal{T}_\varepsilon). \tag{6.22}$$

By the second inequality (4.11) of Lemma 4.5 with $p_\varepsilon, p_* \in M_{h_*}$, for all $T \in \mathcal{T}_*$, we have

$$\mathrm{osc}^2_{\mathcal{T}_*}(p_*, T) \le 2\mathrm{osc}^2_{\mathcal{T}_*}(p_\varepsilon, T) + 2\bar{\Lambda}_1^2 \mathrm{osc}^2_{\mathcal{T}_*}(A^{-1}, T)||A^{-1/2}(p_*-p_\varepsilon)||^2_{L^2(\omega_T)}.$$

Summing on $\mathcal{T}_*$, the monotonicity of the data oscillation (see Remarks 2.1 and 4.1), we get

$$\begin{aligned}\mathrm{osc}^2_{\mathcal{T}_*}(p_*, \mathcal{T}_*) \le& 2\mathrm{osc}^2_{\mathcal{T}_*}(p_\varepsilon, \mathcal{T}_*) + 2\Lambda_1^2\mathrm{osc}^2_{\mathcal{T}_*}(A^{-1}, \mathcal{T}_*)||A^{-1/2}(p_*-p_\varepsilon)||^2_{L^2(\Omega)}\\ \le& 2\mathrm{osc}^2_{\mathcal{T}_\varepsilon}(p_\varepsilon, \mathcal{T}_\varepsilon) + 2\Lambda_1^2\mathrm{osc}^2_{\mathcal{T}_0}(A^{-1}, \mathcal{T}_0)||A^{-1/2}(p_*-p_\varepsilon)||^2_{L^2(\Omega)}.\end{aligned} \tag{6.23}$$

Set $\bar{\zeta} := (2\Lambda_1^2\mathrm{osc}^2_{\mathcal{T}_0}(A^{-1}, \mathcal{T}_0))^{-1}$, a combination of (6.22)-(6.23) yields

$$\mathcal{E}^2_{h_*} + \bar{\zeta}\mathrm{osc}^2_{\mathcal{T}_*}(p_*, \mathcal{T}_*) \le \mathcal{E}^2_{h_\varepsilon} + \delta_4\mathcal{E}^2_{h_*} + (C_0\delta_4^{-1} + 2\bar{\zeta})\mathrm{osc}^2_{\mathcal{T}_\varepsilon}(p_\varepsilon, \mathcal{T}_\varepsilon). \tag{6.24}$$

We then choose $\zeta = \bar{\zeta}(1-\delta_4)^{-1}$, which indicates that δ_4 satisfies $\bar{\zeta}(1-\delta_4)^{-1} \ge 1$. The inequality (6.24) and (6.20) imply

$$\begin{aligned}\mathcal{E}^2_{h_*} + \zeta\mathrm{osc}^2_{\mathcal{T}_*}(p_*, \mathcal{T}_*) \le& C_A\{\mathcal{E}^2_{h_\varepsilon} + \zeta\mathrm{osc}^2_{\mathcal{T}_\varepsilon}(p_\varepsilon, \mathcal{T}_\varepsilon)\}\\ \le& 4\varepsilon^2 C_A = \mu\{\mathcal{E}^2_k + \zeta\mathrm{osc}^2_{\mathcal{T}_k}(p_k, \mathcal{T}_k)\}.\end{aligned} \tag{6.25}$$

Hence, we deduce from optimality marking (Lemma 6.2) that the subset $\mathcal{R} := \mathcal{R}_{\mathcal{T}_k \to \mathcal{T}_*} \subset \mathcal{T}_k$ verifies the Dörfler property (2.5) for $\theta < \theta_*$. The fact that procedure $MARK$ selects a subset $\mathcal{M}_k \subset \mathcal{T}_k$ with minimal cardinality satisfying the same property (2.5), and (6.21) leads to

$$\begin{aligned}&\#\mathcal{M}_k \le \#\mathcal{R} \le \#\mathcal{T}_* - \#\mathcal{T}_k \le \#\mathcal{T}_\varepsilon + \#\mathcal{T}_k - \#\mathcal{T}_0 - \#\mathcal{T}_k\\ =&\#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \le |(p, f, A)|_s^{1/s}\varepsilon^{-1/s}\\ \le&(4C_A)^{1/2s}\mu^{-1/2s}|(p, f, A)|_s^{1/s}\{\mathcal{E}^2_k + \zeta\mathrm{osc}^2_{\mathcal{T}_k}(p_k, \mathcal{T}_k)\}^{-1/2s},\end{aligned}$$

which implies the desired result (6.18). In the third step above, we have used the overlay of two meshes (Lemma 6.1). □

Theorem 6.2. *Let $\{\mathcal{T}_k; \mathcal{E}_k; p_k; \mathrm{osc}_{\mathcal{T}_k}(p_{k,\mathcal{T}_k})\}_{k\ge 0}$ be the sequence of meshes, the stress variable error, the approximation solution to the flux, and the oscillation of data produced by the AMFEM, and let $(p, f, A) \in \mathbb{A}_s$, and set the function $\Theta(s, \theta, \mathbb{A}_s)$ to describe the asymptotics of the AMFEM as $\theta \to \theta_*$ or $s \to 0$, which is determined below. Then it holds*

$$(\mathcal{E}^2_k + \gamma_2\mathrm{osc}^2_{\mathcal{T}_k}(p_k, \mathcal{T}_k))^{1/2} \le \Theta(s, \theta, \mathbb{A}_s)(\#\mathcal{T}_k - \#\mathcal{T}_0)^{-s}. \tag{6.26}$$

Proof. From the complexity of $REFINE$ (Lemma 2.1) and the cardinality (6.18) of $\mathcal{M}_k$, it holds

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \sum_{i=0}^{k-1}\#\mathcal{M}_i \lesssim \rho\sum_{i=0}^{k-1}\left(\mathcal{E}^2_i + \zeta\mathrm{osc}^2_{\mathcal{T}_i}(p_i, \mathcal{T}_i)\right)^{-1/2s}, \tag{6.27}$$

where $\rho := \mu^{-1/2s} C_A^{1/2s} |(p, f, A)|_s^{1/s}$. Set $\gamma_3 := \gamma_1 + \gamma_2$, and denote $\gamma_4 := \max(1 + \gamma_3 C_2^{-1}, \gamma_3 C_2^{-1} \zeta^{-1})$. It follows from the lower bound (3.3) that for $0 \le i \le k$

$$\begin{aligned}&\mathcal{E}_i^2 + \gamma_3 \mathrm{osc}_{\mathcal{T}_i}^2(p_i, \mathcal{T}_i) \le \mathcal{E}_i^2 + \gamma_3 \eta_{\mathcal{T}_i}^2(p_i, \mathcal{T}_i)\\ \le&\mathcal{E}_i^2 + \gamma_3 C_2^{-1}(\mathcal{E}_i^2 + \mathrm{osc}_{\mathcal{T}_i}^2(p_i, \mathcal{T}_i) \le \gamma_4(\mathcal{E}_i^2 + \zeta \mathrm{osc}_{\mathcal{T}_i}^2(p_i, \mathcal{T}_i)).\end{aligned} \tag{6.28}$$

On the other hand, the linear rate $\beta = \beta(\theta) < 1$ of convergence for the quasi error implies that for $0 \le i \le k$

$$\begin{aligned}&\mathcal{E}_k^2 + \gamma_2 \eta_{\mathcal{T}_k}^2(p_k, \mathcal{T}_k) \le \mathcal{E}_k^2 + \gamma_1 ||h_k \mathrm{div}(p - p_k)||_{L^2(\Omega)}^2 + \gamma_2 \eta_{\mathcal{T}_k}^2(p_k, \mathcal{T}_k)\\ \le&\beta^{2(k-i)}\left(\mathcal{E}_i^2 + \gamma_1 ||h_i \mathrm{div}(p - p_i)||_{L^2(\Omega)}^2 + \gamma_2 \eta_{\mathcal{T}_i}^2(p_i, \mathcal{T}_i)\right)\\ \le&\beta^{2(k-i)}\left(\mathcal{E}_i^2 + \gamma_3 \eta_{\mathcal{T}_i}^2(p_i, \mathcal{T}_i)\right).\end{aligned} \tag{6.29}$$

The above three inequalities, (6.27)-(6.29), indicate

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \rho \gamma_4^{1/2s} \{\mathcal{E}_k^2 + \gamma_2 \eta_{\mathcal{T}_k}^2(p_k, \mathcal{T}_k)\}^{-1/2s} \times \sum_{i=0}^k \beta^{i/s}.$$

It follows from the fact $\beta < 1$ that the geometric series is bounded by the constant $s_\theta = 1/(1 - \beta^{1/s})$. Since

$$\mathrm{osc}_{\mathcal{T}_k}^2(p_k, \mathcal{T}_k) \le \eta_{\mathcal{T}_k}^2(p_k, \mathcal{T}_k),$$

it holds

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \le C_4 s_\theta \rho \gamma_4^{1/2s}\left(\mathcal{E}_k^2 + \gamma_2 \mathrm{osc}_{\mathcal{T}_k}^2(p_k, \mathcal{T}_k)\right)^{-1/2s}. \tag{6.30}$$

Finally the desired result (6.26) follows from (6.30) with $\Theta(s, \theta, \mathbb{A}_s) := C_4 s_\theta^s \rho^s \gamma_4^{1/2}$. $\square$

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