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GENERALIZED PRECONDITIONED HERMITIAN AND SKEW-HERMITIAN SPLITTING METHODS FOR NON-HERMITIAN POSITIVE-DEFINITE LINEAR SYSTEMS*

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Abstract

In this paper, a generalized preconditioned Hermitian and skew-Hermitian splitting (GPHSS) iteration method for a non-Hermitian positive-definite matrix is studied, which covers standard Hermitian and skew-Hermitian splitting (HSS) iteration and also many existing variants. Theoretical analysis gives an upper bound for the spectral radius of the iteration matrix. From practical point of view, we have analyzed and implemented inexact generalized preconditioned Hermitian and skew-Hermitian splitting (IGPHSS) iteration, which employs Krylov subspace methods as its inner processes. Numerical experiments from three-dimensional convection-diffusion equation show that the GPHSS and IGPHSS iterations are efficient and competitive with standard HSS iteration and AHSS iteration.

Mathematics subject classification: 65F10, 65F20, 65F30, 65F50.
Key words: Hermitian and skew-Hermitian splitting, Iteration method, Inner iteration.

1. Introduction

We consider the solution of large sparse system of linear equations

$$Ax = b, \qquad A \in \mathbb{C}^{n\times n}, \qquad x, b \in \mathbb{C}^n, \tag{1.1}$$

where A is a non-Hermitian and positive definite matrix. Bai, et al. [10] first presented the Hermitian and skew-Hermitian splitting (HSS) iteration method, which is based on the Hermitian and skew-Hermitian splitting

$$A = H + S, \tag{1.2}$$

where

$$H = \frac{1}{2}(A + A^*), \quad S = \frac{1}{2}(A - A^*) \tag{1.3}$$

is the Hermitian and the skew-Hermitian parts of A, respectively.

Let α be a positive constant. The standard HSS iterative scheme works as follows: Given an arbitrary initial guess $x^{(0)}$, for $k = 0, 1, 2, \cdots$ until $x^{(k)}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{1.4}$$

where α is a given positive constant.

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When H is positive definite, the HSS iteration method is unconditionally convergent for all $\alpha > 0$ and for any choice of $x^{(0)}$. Moreover, an upper bound on the spectral radius of the iteration matrix is given, which can be minimized by choosing $\alpha = \sqrt{\lambda_1 \lambda_n}$, where λ_1 and λ_n are the minimal and the maximal eigenvalues of H, respectively. In order to save the computational cost, the authors considered inexact HSS method with Krylov solvers in the inner iterations, see also [10, 12]. Further, Bai, Golub and Ng in [10] carefully studied the asymptotic convergence rates and the optimal choices of the inner iteration steps for two specific kinds of inexact HSS iteration.

HSS iteration method immediately attracted considerable attention and resulted in many papers devoted to various aspects of the new algorithms. For instance, generalized Hermitian and skew-Hermitian splitting (GHSS) iteration in [14]; preconditioned Hermitian and skew-Hermitian splitting (PHSS) iteration in [8, 13, 14]; lopsided Hermitian and skew-Hermitian splitting (LHSS) iteration in [22]; asymmetric Hermitian and skew-Hermitian splitting (AHSS) iteration in [21]; positive-definite and skew-Hermitian splitting (PSS) iteration and block triangular and skew-Hermitian splitting (BTSS) iterations in [9].

On the other hand, HSS iteration method was successfully extended to the solution of saddle point problem in [16] and preconditioning techniques for Krylov subspace methods [17, 19]; see also [6, 20, 24]. It was noted in [17] that the method can unconditionally convergent when the Hermitian part H is positive semidefinite for the special case of (generalized) saddle point problems. Other developments including studies on the optimal selection of iteration parameters, successive overrelaxation acceleration, extension to certain singular systems and applications of the HSS preconditioner have been well established in [4,5,7,8,11,15,18] and the references therein.

In this paper, based on the splitting (1.2)-(1.3), we generalize the HSS iteration scheme into a new approach, called generalized preconditioned HSS (GPHSS) iteration. By introducing two symmetric positive definite matrices P_1 and P_2, the GPHSS iterative scheme works as follows.

Method 1.1. The GPHSS iteration method. *Given an arbitrary initial guess $x^{(0)}$, for $k = 0, 1, 2, \cdots$ until $x^{(k)}$ converges, compute*

$$\begin{cases} (\alpha P_1 + H)x^{(k+\frac{1}{2})} = (\alpha P_1 - S)x^{(k)} + b, \\ (\beta P_2 + S)x^{(k+1)} = (\beta P_2 - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{1.5}$$

where α and β are given positive constants.

Note that the GPHSS iteration method can cover many existing variants of the standard HSS iteration. For instance, when $\alpha = \beta$ and $P_1 = P_2 = I$, the GPHSS iteration method is equivalent to the standard HSS iteration method in [10]; when $\alpha = 0$ and $P_2 = I$, it leads to the LHSS iteration method in [22]; when $\alpha \neq \beta$ and $P_1 = P_2 = I$, it results in AHSS iteration method in [21] and when $P_1 = P_2$, it is PHSS iteration method in [8, 13].

Theoretical analysis gives an upper bound about the contraction factor of GPHSS iteration method, which shows the relations among GPHSS, HSS and other existing variants. From practical point of view, we also analyze the convergence theory of inexact variants of the GPHSS iteration method and their implementation. A number of numerical experiments from discrete three-dimensional convection-diffusion equation are presented to illustrate the advantages of the GPHSS iteration method.

This paper is organized as follow. We study the convergence properties of the GPHSS iteration method in Section 2. In Section 3, we discuss in detail the implementation of GPHSS iteration method and inexact GPHSS iteration method, and study their convergence property. Numerical results about the GPHSS iteration method and the inexact GPHSS for the application from discrete three-dimensional convection-diffusion equation are reported in Section 4 and, finally we end the paper with concluding remarks in Section 5.

2. Convergence Analysis of the GPHSS Iteration Method

We study the convergence of the GPHSS iteration method and derive an upper bound for the contraction factor in this section. The GPHSS iteration method can be generalized into a two-step splitting iteration framework, see also [2, 3, 10]. We firstly review the convergence criterion for two-step splitting iteration form in [10].

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}, A = M_i - N_i (i = 1, 2)$ be two splittings of A, and $x^{(0)} \in \mathbb{C}^n$ be a given initial vector. If $x^{(k)}$ is a two-step iteration sequence defined by*

$$\begin{cases} M_1 x^{(k+\frac{1}{2})} = N_1 x^{(k)} + b, \\ M_2 x^{(k+1)} = N_2 x^{(k+\frac{1}{2})} + b, \end{cases} \tag{2.1}$$

$k = 0, 1, \cdots$, then

$$x^{(k+1)} = M_2^{-1} N_2 M_1^{-1} N_1 x^{(k)} + M_2^{-1}(I + N_2 M_1^{-1})b, \quad k = 0, 1, \cdots. \tag{2.2}$$

Moreover, if the spectral radius $\rho(M_2^{-1} N_2 M_1^{-1} N_1) < 1$, then the iterative sequence $\{x^{(k)}\}$ converges to the unique solution $x^* \in \mathbb{C}^n$ of the system of linear equations (1.1) for any initial vector $x^{(0)} \in \mathbb{C}^n$.

This lemma can also be found in [20], which contains a general theory for the so-called alternating iteration method. Applying this lemma to the GPHSS iteration, we obtain the following convergence property.

Theorem 2.1. *Let $P_1 \in \mathbb{C}^{n\times n}$ and $P_2 \in \mathbb{C}^{n\times n}$ be two Hermitian positive definite matrices. Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, respectively, and α be a nonnegative constant and β be a positive constant. Then the iteration matrix $M(\alpha, \beta)$ of GPHSS method is*

$$M(\alpha, \beta) = (\beta P_2 + S)^{-1}(\beta P_2 - H)(\alpha P_1 + H)^{-1}(\alpha P_1 - S). \tag{2.3}$$

Denote $\lambda(\widehat{P}\widehat{P}^*)$ and $\lambda(\widehat{H})$ be the spectral sets of the matrix $\widehat{P}\widehat{P}^*$ and $\widehat{H}$, respectively, and $\sigma(\widehat{S})$ be the singular value set of the matrix $\widehat{S}$, where $\widehat{P} = P_1^{-\frac{1}{2}} P_2^{\frac{1}{2}}$, $\widehat{H} = P_1^{-\frac{1}{2}} H P_1^{-\frac{1}{2}}$ and $\widehat{S} = P_2^{-\frac{1}{2}} S P_2^{-\frac{1}{2}}$. Then,

$$\rho(M(\alpha, \beta)) \le \delta = \kappa(\widehat{P})\delta_1\delta_2, \tag{2.4a}$$

where

$$\delta_1 = (\beta \max_{\gamma_i \in \lambda(\widehat{P}\widehat{P}^*)} |\gamma_i - 1| + |\beta - \alpha|) \max_{\lambda_i \in \lambda(\widehat{H})} \frac{1}{\alpha + \lambda_i} + \max_{\lambda_i \in \lambda(\widehat{H})} \frac{|\lambda_i - \alpha|}{|\lambda_i + \alpha|}, \tag{2.4b}$$

$$\delta_2 = (\alpha \max_{\gamma_i \in \lambda(\widehat{P}\widehat{P}^*)} |\frac{1}{\gamma_i} - 1| + |\alpha - \beta|) \max_{\theta_i \in \sigma(\widehat{S})} \frac{1}{\sqrt{\beta^2 + \theta_i^2}} + 1. \tag{2.4c}$$

Proof. Let

$$M_1 = \alpha P_1 + H, \quad N_1 = \alpha P_1 - S, \quad M_2 = \beta P_2 + S \quad \text{and} \quad N_2 = \beta P_2 - H.$$

Then $\alpha P_1 + H$ and $\beta P_2 + S$ are nonsingular for any nonnegative constants α and positive constants β. So formula (2.3) is valid.

Denote by

$$\widehat{H} = P_1^{-\frac{1}{2}} H P_1^{-\frac{1}{2}}, \quad \widehat{S} = P_2^{-\frac{1}{2}} S P_2^{-\frac{1}{2}}$$

and $\widehat{P} = P_1^{-\frac{1}{2}} P_2^{\frac{1}{2}}$. Then

$$\alpha P_1 + H = P_1^{\frac{1}{2}} (\alpha I + \widehat{H}) P_1^{\frac{1}{2}}, \qquad \beta P_2 - H = P_1^{\frac{1}{2}} (\beta \widehat{P}\widehat{P}^* - \widehat{H}) P_1^{\frac{1}{2}}, \tag{2.5a}$$

$$\beta P_2 + S = P_2^{\frac{1}{2}} (\beta I + \widehat{S}) P_2^{\frac{1}{2}}, \qquad \alpha P_1 - S = P_2^{\frac{1}{2}} (\alpha \widehat{P}^{-1}\widehat{P}^{-*} - \widehat{S}) P_2^{\frac{1}{2}}. \tag{2.5b}$$

Hence,

$$\begin{aligned}
&(\beta P_2 - H)(\alpha P_1 + H)^{-1} \\
=&P_1^{\frac{1}{2}} (\beta \widehat{P}\widehat{P}^* - \widehat{H})(\alpha I + \widehat{H})^{-1} P_1^{-\frac{1}{2}} \\
=&P_1^{\frac{1}{2}} [(\beta \widehat{P}\widehat{P}^* - \alpha I)(\alpha I + \widehat{H})^{-1} + (\alpha I - \widehat{H})(\alpha I + \widehat{H})^{-1}] P_1^{-\frac{1}{2}} \\
=&P_1^{\frac{1}{2}} [\beta(\widehat{P}\widehat{P}^* - I)(\alpha I + \widehat{H})^{-1} + (\beta - \alpha)(\alpha I + \widehat{H})^{-1} + (\alpha I - \widehat{H})(\alpha I + \widehat{H})^{-1}] P_1^{-\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
&(\alpha P_1 - S)(\beta P_2 + S)^{-1} \\
=&P_2^{\frac{1}{2}} (\alpha \widehat{P}^{-1}\widehat{P}^{-*} - \widehat{S})(\beta I + \widehat{S})^{-1} P_2^{-\frac{1}{2}} \\
=&P_2^{\frac{1}{2}} [(\alpha \widehat{P}^{-1}\widehat{P}^{-*} - \beta I)(\beta I + \widehat{S})^{-1} + (\beta I - \widehat{S})(\beta I + \widehat{S})^{-1}] P_2^{-\frac{1}{2}} \\
=&P_2^{\frac{1}{2}} [\alpha(\widehat{P}^{-1}\widehat{P}^{-*} - I)(\beta I + \widehat{S})^{-1} + (\alpha - \beta)(\beta I + \widehat{S})^{-1} + (\beta I - \widehat{H})(\beta I + \widehat{S})^{-1}] P_2^{-\frac{1}{2}},
\end{aligned}$$

which lead to

$$\begin{aligned}
&\rho(M(\alpha, \beta)) \\
=&\rho((\beta P_2 - H)(\alpha P_1 + H)^{-1}(\alpha P_1 - S)(\beta P_2 + S)^{-1}) \\
=&\rho(P_1^{\frac{1}{2}} (\beta \widehat{P}\widehat{P}^* - \widehat{H})(\alpha I + \widehat{H})^{-1} \widehat{P} (\alpha \widehat{P}^{-1}\widehat{P}^{-*} - \widehat{S})(\beta I + \widehat{S})^{-1} P_2^{-\frac{1}{2}}) \\
=&\rho((\beta \widehat{P}\widehat{P}^* - \widehat{H})(\alpha I + \widehat{H})^{-1} \widehat{P} (\alpha \widehat{P}^{-1}\widehat{P}^{-*} - \widehat{S})(\beta I + \widehat{S})^{-1} \widehat{P}^{-1}) \\
\le&\|(\beta \widehat{P}\widehat{P}^* - \widehat{H})(\alpha I + \widehat{H})^{-1} \widehat{P} (\alpha \widehat{P}^{-1}\widehat{P}^{-*} - \widehat{S})(\beta I + \widehat{S})^{-1} \widehat{P}^{-1}\|_2 \\
\le&\kappa(\widehat{P}) \left[\beta \|\widehat{P}\widehat{P}^* - I\|_2 \|(\alpha I + \widehat{H})^{-1}\|_2 + |\beta - \alpha| \|(\alpha I + \widehat{H})^{-1}\|_2 + \|(\alpha I - \widehat{H})(\alpha I + \widehat{H})^{-1}\|_2\right] \\
&\cdot \left[\alpha \|\widehat{P}^{-1}\widehat{P}^{-*} - I\|_2 \|(\beta I + \widehat{S})^{-1}\|_2 + |\alpha - \beta| \|(\beta I + \widehat{S})^{-1}\|_2 + \|(\beta I - \widehat{S})(\beta I + \widehat{S})^{-1}\|_2\right].
\end{aligned}$$

Denote by $Q(\beta) = (\beta I - \widehat{S})(\beta I + \widehat{S})^{-1}$. It is clear that $\widehat{S}^* = -\widehat{S}$ and

$$\begin{aligned}
Q(\beta)^* Q(\beta) =&(\beta I - \widehat{S})^{-1}(\beta I + \widehat{S})(\beta I - \widehat{S})(\beta I + \widehat{S})^{-1} \\
=&(\beta I - \widehat{S})^{-1}(\beta I - \widehat{S})(\beta I + \widehat{S})(\beta I + \widehat{S})^{-1} = I,
\end{aligned} \tag{2.6}$$

so that $Q(\beta)$ is a unitary matrix, and therefore $\|Q(\beta)\|_2 = 1$.

Note that both $\widehat{P}\widehat{P}^*$ and $\widehat{H}$ are Hermitian positive definite matrices. Denote by $\lambda_i, (i = 1, \cdots, n)$ the eigenvalues of $\widehat{H}$. It is seen that

$$\|(\alpha I - \widehat{H})(\alpha I + \widehat{H})^{-1}\|_2 = \max_{\lambda_i \in \lambda(\widehat{H})} \left|\frac{\alpha - \lambda_i}{\alpha + \lambda_i}\right| < 1 \tag{2.7}$$

is always valid. Let λ_n be the smallest eigenvalue of $\widehat{H}$. Then

$$\|(\alpha I + \widehat{H})^{-1}\|_2 = \max_{\lambda_i \in \lambda(\widehat{H})} \frac{1}{\alpha + \lambda_i} \leq \frac{1}{\alpha + \lambda_n} < \frac{1}{\alpha}. \tag{2.8}$$

Denote by $\gamma_i, (i = 1, \cdots, n)$ the eigenvalues of $\widehat{P}\widehat{P}^*$. Then

$$\|\widehat{P}\widehat{P}^* - I\|_2 \leq \max_{1 \leq i \leq n} |\gamma_i - 1|, \tag{2.9a}$$

$$\|\widehat{P}^{-1}\widehat{P}^{-*} - I\|_2 \leq \max_{1 \leq i \leq n} |\frac{1}{\gamma_i} - 1|. \tag{2.9b}$$

The above inequalities then give the upper bound for $\rho(M(\alpha, \beta))$ in (2.4). □

We give some remarks on the upper bound before going on. First, when $P_1 = P_2$, GPHSS leads to standard preconditioned HSS method where

$$\widehat{P} = P_1^{-\frac{1}{2}} P_2^{\frac{1}{2}} = I, \quad \kappa(\widehat{P}) = 1. \tag{2.10}$$

In this case, the upper bound in (2.4) results in

$$\delta = \left(\max_{\lambda_i \in \lambda(\widehat{H})} \frac{|\beta - \alpha|}{\alpha + \lambda_i} + \max_{\lambda_i \in \lambda(\widehat{H})} \frac{|\lambda_i - \alpha|}{|\lambda_i + \alpha|}\right)\left(\max_{\theta_i \in \sigma(\widehat{S})} \frac{|\alpha - \beta|}{\sqrt{\beta^2 + \theta_i^2}} + 1\right), \tag{2.11}$$

which is a bit looser than the upper bound in [21], but also includes the special case in [10]:

$$\delta = \max_{\lambda_i \in \lambda(\widehat{H})} \frac{|\lambda_i - \alpha|}{|\lambda_i + \alpha|}, \tag{2.12}$$

when $\alpha = \beta$. On the other hand, when $P_1 \neq P_2$ and $\alpha = \beta$, the upper bound in (2.4) leads in

$$\left(\max_{\gamma_i \in \lambda(\widehat{P}\widehat{P}^*)} |\gamma_i - 1| \max_{\lambda_i \in \lambda(\widehat{H})} \frac{\alpha}{\alpha + \lambda_i} + \max_{\lambda_i \in \lambda(\widehat{H})} \frac{|\lambda_i - \alpha|}{|\lambda_i + \alpha|}\right)\left(\max_{\gamma_i \in \lambda(\widehat{P}\widehat{P}^*)} |\frac{1}{\gamma_i} - 1| \max_{\theta_i \in \sigma(\widehat{S})} \frac{\alpha}{\sqrt{\alpha^2 + \theta_i^2}} + 1\right).$$

In this case, it is observed that the upper bound could be minimized to

$$\max_{\lambda_i \in \lambda(\widehat{H})} \frac{|\lambda_i - \alpha|}{|\lambda_i + \alpha|},$$

if and only if $\widehat{P}$ is orthogonal.

The approach to minimize the the upper bound is very important in theoretical viewpoint. However, it is not practical since the corresponding spectral radius of the iteration matrix $M(\alpha, \beta)$ is not optimal. How to choose the suitable preconditioners and parameters for practical problem is still a great challenge.

3. The Inexact GPHSS Iteration Method and Its Convergence

From the process of GPHSS iteration, it is required to solve two systems of linear equations whose coefficient matrices are $\alpha P_1 + H$ and $\beta P_2 + S$, respectively, which is costly and even impractical in actual implementation. To improve computing efficiency of the GPHSS iteration method, we propose to solve the two subproblems iteratively [1,10,12], which leads to IGPHSS iteration scheme.

Since $\alpha P_1 + H$ is Hermitian positive definite, one can solve this system of linear equations by CG; and for the solution of the system of linear equations with coefficient matrix $\beta P_2 + S$, some Krylov subspace method can be considered, e.g., CGNR [23], Lanczos [25] and GMRES [23]. Here, taking CG and CGNR for example, we describe the IGPHSS iteration scheme in the following algorithm.

Algorithm 3.1. The IGPHSS(CG, CGNR) iteration method.
Input an initial guess $x^{(0)}$, the stopping tolerance ϵ for the outer iteration method, the largest admissible number $k_{\max}$ of the outer iteration steps, two stopping tolerances $\{\eta_k\}$ and $\{\tau_k\}$ for the inner CG and the inner CGNR iteration methods, respectively.

1. Set $k := 0$;
2. Compute $r^{(0)} = b - Ax^{(0)}$ *and* $\rho^{(0)} = \|r^{(0)}\|_2$;
3. If $\rho^{(0)} \leq \|b\|_2$, then stop;
4. For $k = 0, 1, 2, \cdots, k_{\max}$ *and* $\rho_k \leq \|b\|_2$, Do
5. Solve $(\alpha P_1 + H)z^{(k)} = r^{(k)}$ *by* CG until the residual $p^{(k)} = r^{(k)} - (\alpha P_1 + H)z^{(k)}$ satisfies $\|p^{(k)}\| \leq \eta_k \|r^{(k)}\|$;
6. Compute $x^{(k+1/2)} = x^{(k)} + z^{(k)}$;
7. Compute $r^{(k+1/2)} = b - Ax^{(k+1/2)}$;
8. Solve $(\beta P_2 + S)z^{(k+1/2)} = r^{(k+1/2)}$ by CGNR until the residual $q^{(k+1/2)} = r^{(k+1/2)} - (\beta P_2 + S)z^{(k+1/2)}$ satisfies $\|q^{(k+1/2)}\| \leq \tau_k \|r^{(k+1/2)}\|$;
9. Compute $x^{(k+1)} = x^{(k+1/2)} + z^{(k+1/2)}$;
10. Compute $r^{(k+1)} = b - Ax^{(k+1)}$;
11. Compute $\rho^{(k+1)} = \|r^{(k+1)}\|_2$;
12. EndFor

We should remark that the convergence criterion ϵ for the outer iteration is a small constant to make the approximate solution accurate enough; while the stopping criterion $\{\eta_k\}$ and $\{\tau_k\}$ for the inner iteration is relatively large for rough approximate solution in the half steps. Usually, $\{\eta_k\}$ and $\{\tau_k\}$ are chosen to be larger than ϵ; for details, refer to [10,12,22]. If both tolerances $\{\eta_k\}$ and $\{\tau_k\}$ are all zeros, then the inner systems is solved exactly and the IGPHSS iteration becomes the exact GPHSS iteration method.

To analyze the convergence property of IGPHSS method, we review the following lemma presented in [10], where $|||x|||_{M_2} = \|Mx\|_2$ for all $x \in \mathbb{C}^n$.

Lemma 3.1. *Let $A \in \mathbb{C}^{n\times n}, A = M_i - N_i (i = 1, 2)$ be two splittings of the matrix A. If $\{x^{(k)}\}$ is an iteration sequence defined as follows:*

$$x^{(k+1/2)} = x^{(k)} + z^{(k)}, \qquad \text{with} \quad M_1 z^{(k)} = r^{(k)} + p^{(k)}, \tag{3.1}$$

satisfying $\|p^{(k)}\| \le \eta_k \|r^{(k)}\|$, *where* $r^{(k)} = b - Ax^{(k)}$; *and*

$$x^{(k+1)} = x^{(k+1/2)} + z^{(k+1/2)}, \qquad \text{with} \quad M_2 z^{(k+1/2)} = r^{(k+1/2)} + q^{(k+1/2)}, \tag{3.2}$$

satisfying $\|q^{(k+1/2)}\| \le \tau_k \|r^{(k+1/2)}\|$, *where* $r^{(k+1/2)} = b - Ax^{(k+1/2)}$, *then* $\{x^{(k)}\}$ *is of the form*

$$x^{(k+1)} = \quad M_2^{-1} N_2 M_1^{-1} N_1 x^{(k)} + M_2^{-1}(I + N_2 M_1^{-1})b + M_2^{-1}(N_2 M_1^{-1} p^{(k)} + q^{(k+1/2)}). \tag{3.3}$$

Moreover, if $x^* \in \mathbb{C}^n$ *is the exact solution of the system of linear equation* (1.1), *then we have*

$$|||x^{(k+1)} - x^*|||_{M_2} \le (\zeta + \mu\theta\eta_k + \theta(\rho + \theta\nu\eta_k)\tau_k)|||x^{(k+1)} - x^*|||_{M_2}, \qquad k = 0, 1, \cdots, \tag{3.4a}$$

where

$$\zeta = \|N_2 M_1^{-1} N_1 M_2^{-1}\|, \quad \rho = \|M_2 M_1^{-1} N_1 M_2^{-1}\|, \tag{3.4b}$$

$$\mu = \|N_2 M_1^{-1}\|, \quad \theta = \|A M_2^{-1}\|, \quad \nu = \|M_2 M_1^{-1}\|. \tag{3.4c}$$

In particular, if

$$\zeta + \mu\theta\eta_{\max} + \theta(\rho + \theta\nu\eta_{\max})\tau_{\max} < 1, \tag{3.5}$$

then the iteration sequence $\{x^{(k)}\}$ *converges to* $x^* \in \mathbb{C}^n$, *where* $\eta_{\max} = \max_k\{\eta_k\}$ *and* $\tau_{\max} = \max_k\{\tau_k\}$.

According to this lemma, we derive the convergence theorem for IGPHSS iteration method.

Theorem 3.2. *Let* $P_1, P_2 \in \mathbb{C}^{n\times n}$ *be two Hermitian positive definite matrices. Let* $A \in \mathbb{C}^{n\times n}$ *be a positive definite matrix,* H *and* S *be its Hermitian and skew-Hermitian parts,* α *be a nonnegative constant and* β *be a positive constant. If* $\{x^{(k)}\}$ *is an iteration sequence generated by the IGPHSS iteration method and if* $x^* \in \mathbb{C}^n$ *is the exact solution of the system of linear equation* (1.1), *then it holds that*

$$|||x^{(k+1)} - x^*||| \le (\delta(\alpha,\beta) + \mu\theta\eta_k + \theta(\rho + \theta\nu\eta_k)\tau_k)|||x^{(k)} - x^*|||, \qquad k = 0, 1, \cdots, \tag{3.6a}$$

where

$$\rho = \|(\beta P_2 + S)(\alpha P_1 + H)^{-1}(\alpha P_1 - S)(\beta P_2 + S)^{-1}\|_2, \qquad \theta = \|A(\beta P_2 + S)^{-1}\|_2, \tag{3.6b}$$

$$\mu = \|(\beta P_2 - H)(\alpha P_1 + H)^{-1}\|_2, \qquad \nu = \|(\beta P_2 + S)(\alpha P_1 + H)^{-1}\|_2. \tag{3.6c}$$

In particular, when

$$\delta(\alpha,\beta) + \eta_{\max}\rho + \tau_{\max}\rho(\omega + \eta_{\max}\theta\rho) < 1, \tag{3.7}$$

the iteration sequence $\{x^{(k)}\}$ *converges to* x^*, *where* $\eta_{\max} = \max_k\{\eta_k\}$ *and* $\tau_{\max} = \max_k\{\tau_k\}$.

Replacing M_i and $N_i (i = 1, 2)$ in Lemma 3.1 with

$$M_1 = \alpha P_1 + H, \qquad N_1 = \alpha P_1 - S, \tag{3.8a}$$

$$M_2 = \beta P_2 + S, \qquad N_2 = \beta P_2 - H, \tag{3.8b}$$

we straightforwardly obtain the proof of Theorem 3.2.

We remark that Theorem 3.2 gives the choices of the tolerances $\{\eta_k\}$ and $\{\tau_k\}$ for convergence. It is clear that there is a trade-off between inner and outer iterations, which depends on the choices of $\{\eta_k\}$ and $\{\tau_k\}$. However, the theoretical optimal tolerances $\{\eta_k\}$ and $\{\tau_k\}$ are difficult to be analyzed.

Since the convergence rate of Krylov subspace iteration depends on large extent on the size, shape, and location of the entire spectrum of the coefficient matrix, the convergence rate of IGPHSS method combined with Krylov subspace iteration is not only determined by the spectral radius of the iteration matrix. In practice, the optimal parameters α and β should be chosen to minimize the number of iteration, or the total CPU time.

4. Numerical Results

In this section, we illustrate the efficiency of GPHSS iteration method by a number of numerical experiments on the application from discrete convection-diffusion problem. Numerical comparisons with the standard HSS method and the AHSS method are also presented to show the advantage of GPHSS iteration method.

Consider the discrete convection-diffusion equation of the form

$$-\Delta u + q\nabla u = f, \qquad u \in \Omega, \tag{4.1}$$

with the homogeneous Dirichlet boundary conditions, where q is a constant number and f is a given function. Here, we discretize the problem with seven-point finite-difference with the same numbers (m) of grid points in all three directions, which results in a positive-definite linear system with the number $n = m^3$ of unknowns.

If we choose different difference scheme (central or upwind difference scheme) for convection term, the above problem will lead to different system of linear equations. The spectral properties of these problems were carefully studied in [10] and analyzed in [22]. Thus we take these problems as examples to illustrate the efficiency of our methods.

In our experiments, we take $P_1 = I$ and $P_2 = \text{tridiag}(H)$ in GPHSS iteration method, where I is a identity matrix and $\text{tridiag}(H)$ is a tridiagonal matrix constructed by the diagonal, the upper diagonal and the lower diagonal of the matrix H.

4.1. Spectral radius

In this section, we will show how to choose optimal parameters α and β for P_1 and P_2, respectively, to minimize the spectral radius of the iteration matrix of GPHSS iteration method with different values of q and difference schemes. In our experiments, the size of the tested matrix is 512×512.

In order to compare the optimal spectral radius of GPHSS iteration method to those of HSS and AHSS iteration methods, we plot the curves of the spectral radius of the iteration matrix versus α in Figs. 4.1 and 4.2 with respect to the central and the upwind difference scheme respectively. Here, for every curve of GPHSS iteration method, the parameter β is chosen to minimize the spectral radius of iteration matrix.

From Fig. 4.1, it is clear that the spectral radius of GPHSS iteration matrix is always smaller than AHSS iteration method when $q = 10$. From Fig. 4.2, the spectral radius of iteration matrix of GPHSS iteration method is always the smaller one between these methods when $q = 10$ and 100. Furthermore, the optimal spectral radius of GPHSS iteration method is generally smaller than those of HSS and AHSS iteration methods.

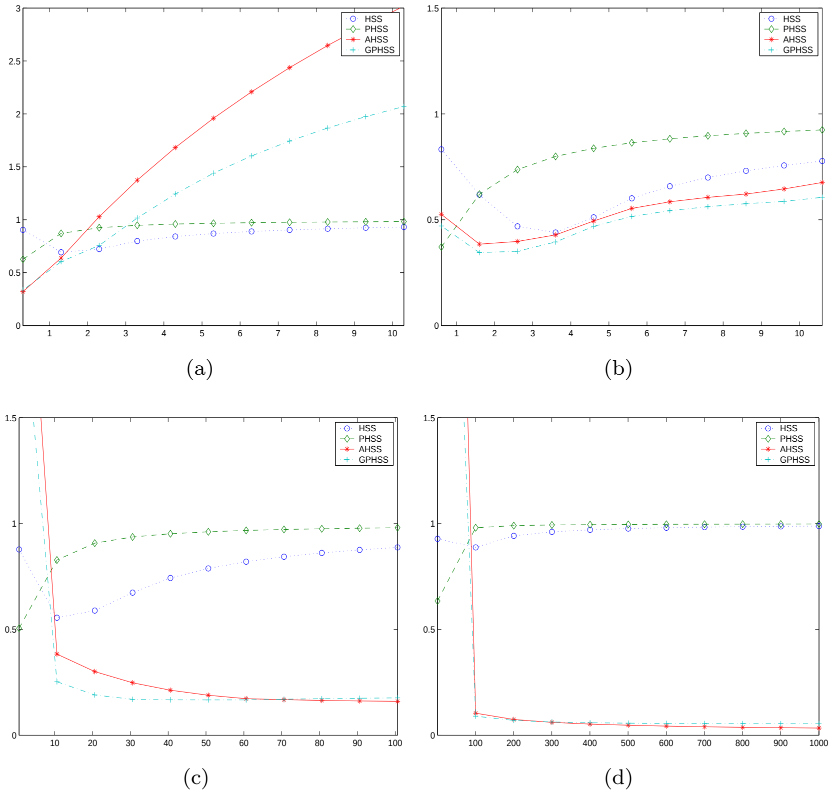

Fig. 4.1. Central difference scheme. The spectral radii of the iteration matrices of HSS, AHSS, GPHSS method versus α: (a) $q = 1$; (b) $q = 10$; (c) $q = 100$; and (d) $q = 1000$.

In Tables 1 and 2, we list the optimal parameters for HSS, AHSS and GPHSS iteration methods, denoted by α_H^*, α_A^* and (α_G^*, β_G^*), as well as the corresponding spectral radii of the iteration matrices for different values of q and different difference schemes, respectively.

From Tables 4.1 and 4.2, we observe that the optimal spectral radius of GPHSS iteration method is smaller than those of HSS and AHSS iteration methods except for the case where convection term is discreted by central difference scheme and $q = 1000$. Also, we see that for GPHSS method, α_G^* is smaller than β_G^* when q is relatively small, e.g., $q = 1$, and bigger

Table 4.1: Central difference scheme. The optimal parameters α^*(or (α^*, β^*)) and the corresponding spectral radii for the iteration matrices of HSS, AHSS and GPHSS iteration methods with respect to different q .

	q	1	10	100	1000
HSS	α_H^*	2.0	3.1	5.0	2.0
	$\rho(M(\alpha_H^*))$	0.70	0.41	0.53	0.69
AHSS	α_A^*	0.1	2.0	200	1000
	β_A^*	1.4	3.1	6.0	6.0
	$\rho(M(\alpha_A^*, \beta_A^*))$	0.18	0.38	0.17	0.03
GPHSS	$\alpha_{G(a)}^*$	0.1	2.0	30	1000
	$\beta_{G(a)}^*$	0.4	0.6	1.0	1.0
	$\rho(M(\alpha_G^*, \beta_G^*))$	0.10	0.34	0.16	0.05

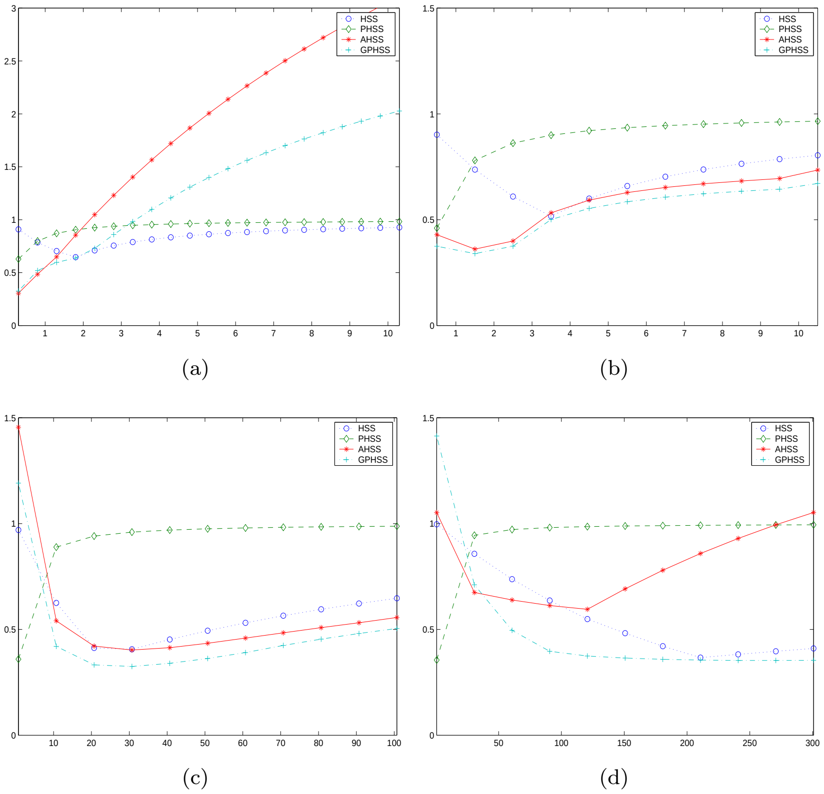

Fig. 4.2. Upwind difference scheme. The spectral radii of the iteration matrices of HSS, AHSS, GPHSS methods versus α: (a) $q = 1$; (b) $q = 10$; (c) $q = 100$; and (d) $q = 1000$.

than β_G^* when q becomes large, e.g., $q = 10, 100$ and 1000. This further illustrates that the convergence condition in Theorem 2.1 is a sufficient but not a necessary one.

In Figs. 4.3 and 4.4, the eigenvalue distribution of GPHSS iteration matrices with the optimal parameters listed in Tables 4.1 and 4.2 is plotted for upwind and central difference schemes, respectively, when $q = 1, 10, 100, 1000$.

From Figs. 4.3 and 4.4, we note that the eigenvalues of the iteration matrix of GPHSS iteration method are clustered and the corresponding spectral radii are all smaller than 1, which shows the potential that fast convergence performance of GPHSS iteration method could

Table 4.2: Upwind difference scheme. The optimal parameters and the corresponding spectral radii for the iteration matrices of HSS, AHSS and GPHSS iteration methods with respect to different q .

	q	1	10	100	1000
HSS	α_H^*	2.0	3.1	30	200
	$\rho(M(\alpha_H^*))$	0.70	0.51	0.40	0.38
AHSS	α_A^*	0.1	1.1	30	100
	β_A^*	1.4	4.2	30	101
	$\rho(M(\alpha_A^*, \beta_A^*))$	0.18	0.36	0.40	0.61
GPHSS	α_G^*	0.1	1.1	30	100
	β_G^*	0.4	0.5	0.7	0.6
	$\rho(M(\alpha_G^*, \beta_G^*))$	0.10	0.32	0.32	0.38

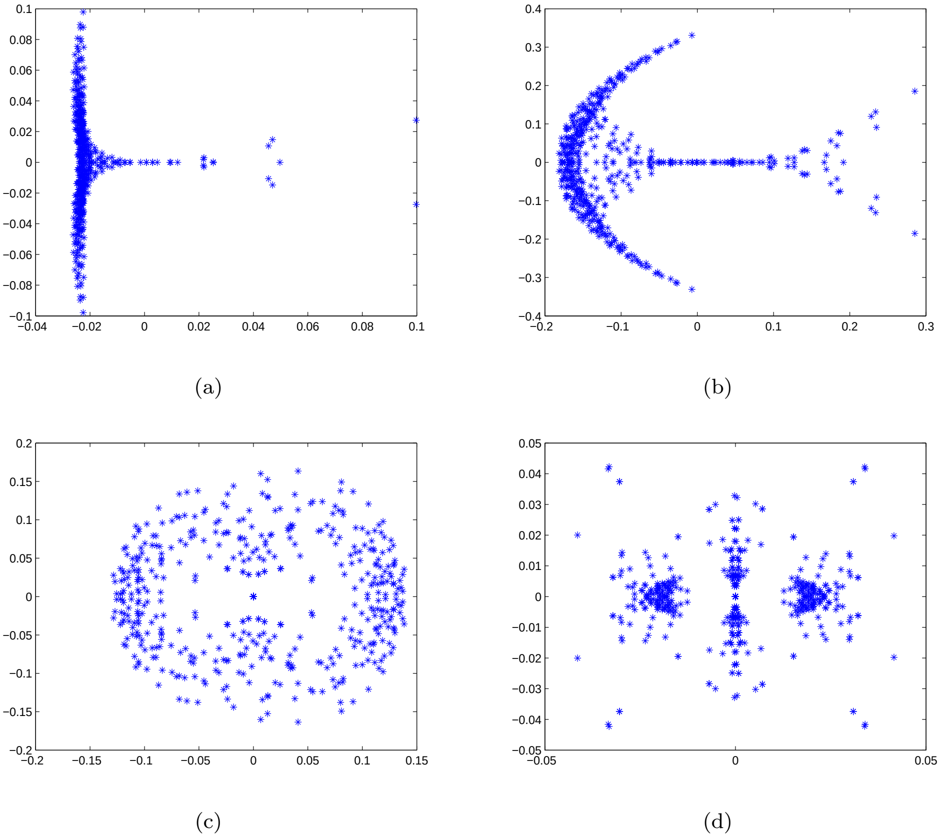

Fig. 4.3. Central difference scheme. Eigenvalue distributions for the iteration matrix of GPHSS iteration method with optimal parameters α and β when $n = 512$: (a) $q = 1$; (b) $q = 10$; (c) $q = 100$; and (d) $q = 1000$.

be expected.

4.2. Results for GPHSS and IGPHSS iteration methods

In this subsection, we focus our attention on the convergence performance of GPHSS and IGPHSS iterations.

The system of linear equations is discretized from convection-diffusion equation 4.1, where the right-hand side vector is chosen such that the true solution is $x^* = (1, 1, \cdots, 1)^T$. All numerical experiments were done in MATLAB with the initial guess $x_0 = (0, 0, \cdots, 0)^T$, and terminated when $\|r_k\|_2/\|r_0\|_2 < 10^{-6}$, where r_k is the residual of the kth iteration.

In Table 4.3, we list the numbers of iteration steps and the computational times for HSS, AHSS and GPHSS iteration methods using the optimal parameters in Tables 4.1 and 4.2 with respect to different difference schemes and different values of q, respectively.

From Table 4.3, we see that GPHSS iteration method is the best among three methods in terms of number of iteration steps and computational time when q varies. This further verifies the spectral analysis in the above section.

To test the efficiency of IGPHSS(CG, CGNR) iteration method, we enlarge the size of the problem and use the optimal parameters in Tables 4.1 and 4.2 for different difference schemes.

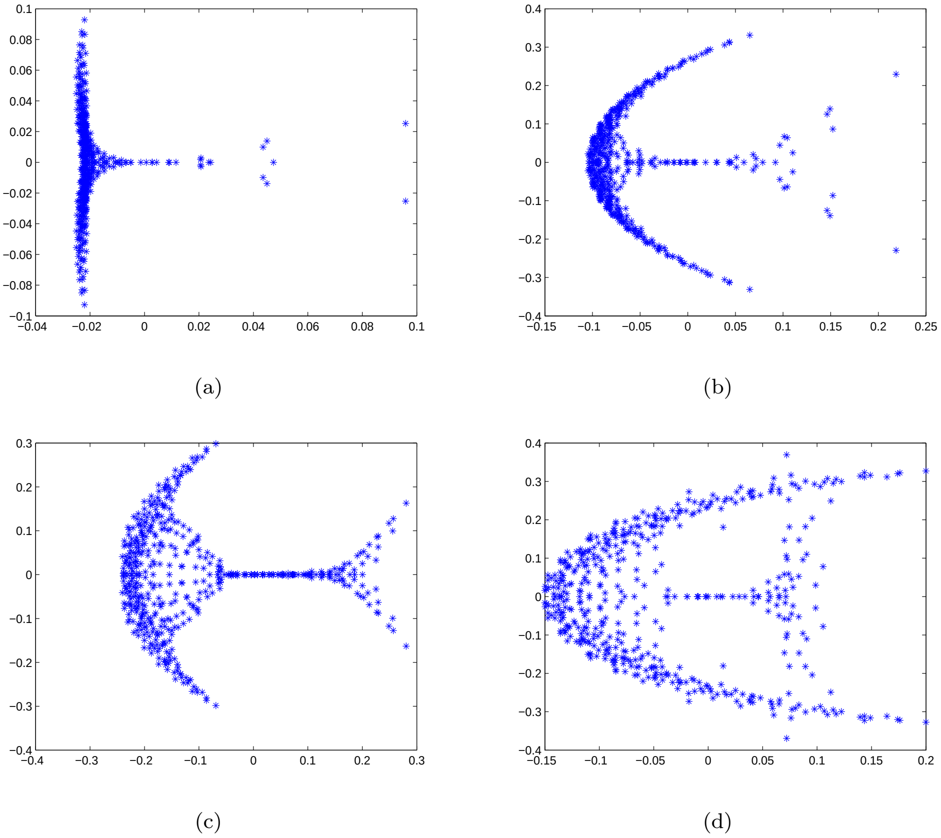

Fig. 4.4. Upwind difference scheme. Eigenvalues distributions of the iteration matrices of GPHSS iteration method with optimal parameters α and β when $n = 512$: (a) $q = 1$; (b) $q = 10$; (c) $q = 100$; and (d) $q = 1000$.

In Tables 4.4 and 4.5, we list the number of iteration steps, average number of CG iteration steps, average number of CGNR iteration steps and CPU time of inexact GPHSS(CG, CGNR) iteration method for central and upwind difference scheme, respectively.

From Tables 4.4 and 4.5, we see that inexact GPHSS(CG, CGNR) iteration method con-

Table 4.3: The numbers of iteration steps and computational times for HSS, AHSS and GPHSS iteration methods.

Difference Scheme	q	HSS		AHSS		GPHSS	
		IT	CPU	IT	CPU	IT	CPU
Central	1	34	7.82	7	2.01	7	1.61
	10	21	5.48	17	4.17	15	3.70
	100	21	4.93	16	3.85	10	2.31
	1000	31	7.33	6	1.44	6	1.44
Upwind	1	33	7.71	7	1.65	7	1.58
	10	24	5.88	14	3.49	13	3.01
	100	21	5.35	21	4.93	16	3.78
	1000	17	4.18	29	6.66	16	3.60

Table 4.4: Central difference scheme: The number of iteration steps and the computational time for IGPHSS(CG, CGNR) iteration method.

q	$n = 512$		$n = 4096$		$n = 32768$	
	IT	CPU	IT	CPU	IT	CPU
1	18(4.00, 0.33)	0.08	21(12.62, 0.67)	2.45	38(34.05, 5.11)	47.77
10	24(5.17, 1.96)	0.13	50(10.78, 9.12)	7.86	177(36.32, 27.90)	808.03
100	18(1.00, 8.94)	0.28	25(1.96, 15.24)	5.82	61(3.90, 16.59)	164.05
1000	11(0.09, 29.09)	0.50	9(0.11, 48.33)	6.22	16(0.44, 54.69)	132.80

Table 4.5: Upwind difference scheme: The number of iteration steps and the computational time for IGPHSS(CG, CGNR) iteration method.

q	$n = 512$		$n = 4096$		$n = 32768$	
	IT	CPU	IT	CPU	IT	CPU
1	19(3.89, 0.32)	0.09	22(12.18, 0.64)	2.81	38(34.26, 5.08)	47.70
10	21(7.00, 1.29)	0.09	33(12.30, 6.39)	4.02	120(35.98, 19.19)	405.46
100	18(3.56, 5.67)	0.20	40(4.72, 9.30)	6.23	109(10.75, 21.11)	371.84
1000	16(5.81, 6.00)	0.19	22(4.95, 7.64)	3.40	48(6.56, 13.19)	104.58

verges when using these optimal parameters. The number of iteration steps and the computational time of IGPHSS(CG, CGNR) iteration method increase as the size of the problem grows up. This confirms that IGPHSS(CG, CGNR) iteration method is efficient for solving large size system of linear equations.

5. Conclusion

In this paper, we have generalized the HSS method to the GPHSS method for solving non-Hermitian positive-definite system of linear equations. Theoretical analysis shows that for any initial guess the GPHSS method converges to the unique solution of the linear system for a wide range of the parameters. Then, an inexact version has been presented and implemented for saving the computational cost. Numerical experiments show that GPHSS method and IGPHSS method are efficient and competitive with the existing HSS and AHSS methods.

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