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AN ANISOTROPIC LOCKING-FREE NONCONFORMING TRIANGULAR FINITE ELEMENT METHOD FOR PLANAR LINEAR ELASTICITY PROBLEM*

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Abstract

The main aim of this paper is to study the nonconforming linear triangular Crouzeix-Raviart type finite element approximation of planar linear elasticity problem with the pure displacement boundary value on anisotropic general triangular meshes satisfying the maximal angle condition and coordinate system condition. The optimal order error estimates of energy norm and L^2-norm are obtained, which are independent of lamé parameter λ. Numerical results are given to demonstrate the validity of our theoretical analysis.

Mathematics subject classification: 65N30, 65N15.
Key words: Planar elasticity, Nonconforming element, Locking-free, Anisotropic meshes.

1. Introduction

We consider the planar linear elasticity problem with the pure displacement boundary value

$$\begin{cases} -\mu\Delta u-(\mu+\lambda)\text{grad}(\text{div}u)=f, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ,μ are Lamé constants, $\lambda\in(0,+\infty)$, $\mu\in[\mu_1,\mu_2]$, $0<\mu_1<\mu_2$. An equivalent variational formulation to problem (1.1) is

$$\begin{cases} \text{find } u\in V \quad \text{such that} \\ a(u,v)=(f,v) \quad \forall v\in V, \end{cases} \tag{1.2}$$

where $V\subset(H_0^1(\Omega))^2$, $u=(u_1,u_2)$, $f=(f_1,f_2)\in(L^2(\Omega))^2$,

$$a(u,v)=\int_\Omega\{\mu\bigtriangledown u\cdot\bigtriangledown v+(\mu+\lambda)(\text{div}u)(\text{div}v)\}dxdy, \qquad (f,v)=\int_\Omega f\cdot vdxdy.$$

It is well-known that if problem (1.1) is approximated by using standard conforming finite elements as the material becomes nearly incompressible, the numerical solutions converge slowly. Such phenomena have been known as numerical locking. The reason for this lies in that the coefficient of the finite element error estimates is dependent on λ, which will extend to ∞ if $\lambda\to\infty$. More detailed explanation of numerical locking can be found in [1–3].

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In order to overcome the locking phenomena, the special finite element methods were used. One direct approach is to use the mixed formulation, which can be found in [4–7]. The other method is to use the nonconforming finite elements approximation of the pure displacement problem. Based on standard finite element methods, [1] and [2] proved that the linear triangular Crouzeix-Raviart nonconforming element is locking-free. [2] and [8] used the so-called reduced integration methods to take account of a class of triangular and quadrilateral elements. [9] also provided a new method to construct locking-free element, and gave a useful nonconforming incomplete biquadratic rectangular element. However, all the above studies rely on the regularity assumption $h_K/\rho_K \le C$ or quasi-uniform assumption $h/\tilde{h} \le C$ [10] of the meshes, where h_K, ρ_K denote the diameter and the radius of inscribed circle of the element K respectively, $h = \max_K h_K$, $\tilde{h} = \min_K h_K$, C is a positive constant independent of h. However, in some cases, the solutions of some elliptic problems may have anisotropic behavior in some parts of the solution domain. An obvious idea to reflect this anisotropy is to employ anisotropic meshes with a finer mesh size in the direction of the rapid variation of the solution and a coarser mesh size in the perpendicular direction. The above assumptions are no longer valid in the case of anisotropic meshes, because the anisotropic elements K are characterized by $h_K/\rho_K \to \infty$, when the limit is considered as $h \to 0$. For the anisotropic elements, the well-known Bramble-Hilbert lemma can not be used directly in estimating the interpolation error. At the same time, the consistency error estimate, the key of the nonconforming finite element analysis, will become very difficult to be dealt with. In recent years, many works have been done to analyze the properties of anisotropic finite elements, especially for the nonconforming finite elements [11–23]. Though [14–18] used the rectangular nonconforming elements to solve the different problems on anisotropic meshes and the Quasi-Wilson element for narrow quadrilateral meshes was discussed in [13], it is difficult to apply these elements to problem (1.1) directly. On the other hand, [12] only discussed the convergence properties for second-order elliptic problem with the nonconforming linear triangular Crouzeix-Raviart type element on anisotropic three-directional meshes. How to extend this element to anisotropic general triangular meshes is still an open problem.

In this paper, we will use the nonconforming linear triangular Crouzeix-Raviart type finite element to approximate problem (1.1) for anisotropic general triangular meshes satisfying the maximal angle condition and coordinate system condition [11]. The optimal order error estimates of energy norm and L^2-norm are obtained by introducing a auxiliary finite element space similar to [12], which are independent of lamé parameter λ. But the analysis is more difficult, and needs more techniques than [12].

The organization of the paper is as follows. In Section 2, we introduce some preliminaries and lemmas. The optimal energy norm and L^2-norm are obtained in Section 3. At last, a numerical example is given to confirm our theoretical analysis in Section 4.

2. Construction of the Nonconforming Anisotropic Element

For the sake of simplicity, we assume that $\Omega \subset R^2$ is a convex polygon composed by a family of triangular meshes J_h, $\Omega = \bigcup_{K \in J_h} \bar{K}$, J_h satisfies the following conditions **(a)** and **(b)** (see Fig. 2.1.), but does not need to satisfy the regularity assumption or quasi-uniform assumption.

(a) Maximal angle condition: There is a constant $\gamma^* < \pi$ (independent of h and $K \in J_h$) such that the maximal interior angle γ of any element K is bounded by γ^*, $\gamma \le \gamma^*$.

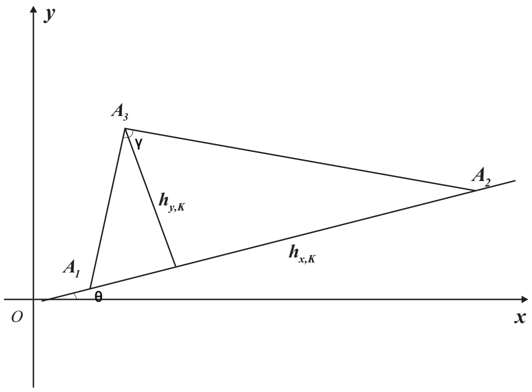

Fig. 2.1. Notation and illustration of K.

(b) Coordinate system condition: The angle θ between the longest side and the $x-$axis is bounded by $|\sin\theta| \le Ch_{y,K}/h_{x,K}$ (where $h_{x,K}$ is the length of the longest edge, $h_{y,K}$ is the length of the high corresponding to the longest edge and C is a positive constant).

For any $K \in J_h$, suppose that the three vertices of K are $A_i(x_i, y_i)$ and the corresponding edges are F_i $(i = 1, 2, 3)$. Let $\hat{K}$ be the reference element on (λ_1, λ_2)-plane with vertices $\hat{d}_1 = (1, 0)$, $\hat{d}_2 = (0, 1)$, and $\hat{d}_3 = (0, 0)$, $\hat{l}_1 = \overline{\hat{d}_2\hat{d}_3}$, $\hat{l}_2 = \overline{\hat{d}_3\hat{d}_1}$, $\hat{l}_3 = \overline{\hat{d}_1\hat{d}_2}$.

The finite element $(\hat{K},\ \hat{P},\ \hat{\sum})$ on $\hat{K}$ is defined by

$$\hat{\sum} = \{\hat{u}^{(1)},\ \hat{u}^{(2)},\ \hat{u}^{(3)}\}, \qquad \hat{P} = \{\ 1,\ \lambda_1,\ \lambda_2\}, \tag{2.1}$$

where

$$\hat{u}^{(i)} = \frac{1}{|\hat{l}_i|}\int_{\hat{l}_i} \hat{u}d\hat{s}, \quad |\hat{l}_i| = \int_{\hat{l}_i} 1d\hat{s}, \qquad i = 1, 2, 3.$$

The interpolation defined above is properly posed and the interpolation function can be expressed as

$$\hat{\Pi}\hat{u} = \hat{u}^{(1)} + \hat{u}^{(2)} - \hat{u}^{(3)} + 2(\hat{u}^{(3)} - \hat{u}^{(1)})\lambda_1 + 2(\hat{u}^{(3)} - \hat{u}^{(2)})\lambda_2. \tag{2.2}$$

Contrary to the Lagrange interpolation (nodal values), this interpolation is defined for $\hat{u} \in W^{1,p}(\hat{K})$, $\forall p \in [1, +\infty)$. Note further that $\hat{\Pi}\hat{\omega} = \hat{\omega}$, $\forall \hat{\omega} \in \hat{P}_1$ ($\hat{P}_1$ is the polynomial set with the order less than or equal to one).

For simplicity, we shall use the abbreviations ∂_i and ∂_{ij} for $\frac{\partial}{\partial i}$ and $\frac{\partial^2}{\partial i \partial j}$, respectively.

Lemma 2.1. *The interpolation operator* $\hat{\Pi}$ *defined by* (2.2) *has the anisotropic interpolation property* [13], *i.e., for any* $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = 1$, *there holds*

$$\|\hat{D}^\alpha(\hat{u} - \hat{\Pi}\hat{u})\|_{0,\hat{K}} \le \hat{C}|\hat{D}^\alpha\hat{u}|_{1,\hat{K}}, \qquad \hat{u} \in H^2(\hat{K}), \tag{2.3}$$

where $\hat{C}$ *is a constant and only relies on reference element* $\hat{K}$.

Proof. Let $\alpha = (1, 0)$. Then

$$\hat{D}^\alpha\hat{\Pi}\hat{u} = \partial_{\lambda_1}(\hat{\Pi}\hat{u}) = 2(\hat{u}^{(3)} - \hat{u}^{(1)}) = |\hat{K}|^{-1}\int_{\hat{K}} \partial_{\lambda_1}\hat{u}d\lambda_1 d\lambda_2 \stackrel{\Delta}{=} F(\hat{D}^\alpha\hat{u}),$$

where $|\hat{K}|$ is the measure of $\hat{K}$. Let $\hat{\omega} = \hat{D}^\alpha\hat{u}$. Then

$$G(\hat{\omega}) = |\hat{K}|^{-1}\int_{\hat{K}} \hat{\omega}d\lambda_1 d\lambda_2 \le \hat{C}\|\hat{\omega}\|_{0,\hat{K}} \le \hat{C}\|\hat{\omega}\|_{1,\hat{K}},$$

apparently G is a continuous linear function. By the anisotropic interpolation theorem [13], we have

$$\|\hat{D}^\alpha(\hat{u}-\hat{\Pi}\hat{u})\|_{0,\hat{K}} \leq \hat{C}|\hat{D}^\alpha\hat{u}|_{1,\hat{K}}.$$

Similarly, (2.3) is valid for $\alpha=(0,1)$, which completes the proof. □

The affine mapping $F_K:\hat{K}\to K$ and the finite element space are defined by

$$\begin{cases} x=(x_1-x_3)\lambda_1+(x_2-x_3)\lambda_2+x_3, \\ y=(y_1-y_3)\lambda_1+(y_2-y_3)\lambda_2+y_3, \end{cases}$$

and

$$V_h=\left\{u_h;\ u_h|_K\circ F_K\in\hat{P}^2,\ \int_F[u_h]ds=0,\ \forall F\subset\partial K, K\in J_h\right\}$$

respectively, where F denotes the edge of K, $[u_h]$ denotes the jumping value of u_h, and if $F\subset\partial\Omega$, then $[u_h]=u_h$.

Let $\Pi_h|_K=\Pi_K=\hat{\Pi}\circ F_K^{-1}$. Then we have the following very important lemma.

Lemma 2.2. $\forall u\in W^{1,p}(K)$ $(p\in[1,+\infty))$, *the interpolation operator* Π_h *satisfies*

$$\text{div}\Pi_K u=M_K(\text{div}u), \tag{2.4}$$

where $M_K v=|K|^{-1}\int_K vdxdy$.

Proof.

$$\begin{aligned}\text{div}\Pi_K u=&|K|^{-1}\int_K div\Pi_K udxdy=|K|^{-1}\int_{\partial K}\Pi_K u\cdot nds=|K|^{-1}\sum_{i=1}^3\int_{F_i}\Pi_K u\cdot n_ids\\ =&|K|^{-1}\sum_{i=1}^3\int_{F_i}u\cdot n_ids=|K|^{-1}\int_K\text{div}udxdy=M_K(\text{div}u).\end{aligned}$$

The proof is completed. □

3. Approximation and Error Estimates

The nonconforming finite element approximation of (1.2) reads as

$$\begin{cases}\text{find } u_h\in V_h \quad \text{such that}\\ a_h(u_h,v_h)=(f,v_h)\quad\forall v_h\in V_h,\end{cases} \tag{3.1}$$

where

$$u_h=(u_{h_1},u_{h_2}),\quad a_h(u_h,v_h)=\sum_{K\in J_h}\int_K\Big\{\mu\bigtriangledown u_h\cdot\bigtriangledown v_h+(\mu+\lambda)(\text{div}u_h)(\text{div}v_h)\Big\}dxdy.$$

The following error estimate can be found in [2].

Lemma 3.1. *The variational problem* (1.2) *has a unique solution* $u\in(H_0^1(\Omega)\bigcap H^2(\Omega))^2$. *Furthermore, the following elliptic regularity estimate holds*

$$\|u\|_{2,\Omega}+\lambda|\text{div}u|_{1,\Omega}\leq C\|f\|_{0,\Omega}, \tag{3.2}$$

where $C>0$ *is a constant independent of* λ *and* μ.

$\forall v_h \in V_h$, let $\|v_h\|_h = a_h(v_h, v_h)^{\frac{1}{2}}$, then $\|\cdot\|_h$ is a norm over V_h. Now we state the main result of this paper.

Theorem 3.1. *Assume that $u \in (H^2(\Omega) \cap H_0^1(\Omega))^2$ and $u_h \in V_h$ are the solutions of* (1.2) *and* (3.1)*, respectively. Then on anisotropic meshes, we have*

$$\|u - u_h\|_h \le Ch\|f\|_{0,\Omega}. \tag{3.3}$$

Here and later, $C > 0$ is a constant independent of λ and μ, and may be of different value at each occurrence.

Proof. By Strang lemma [10], we have

$$\|u - u_h\|_h \le C\left\{\inf_{v_h\in V_h} \|u - v_h\|_h + \sup_{\omega_h\in V_h\setminus\{0\}} \frac{|E_h(u, \omega_h)|}{\|\omega_h\|_h}\right\}, \tag{3.4}$$

where $E_h(u, \omega_h) = a_h(u, \omega_h) - f(\omega_h)$.

We now begin with estimating the first term on the right hand of (3.4). We know that

$$\begin{aligned}\sum_{K\in J_h} |u - \Pi_h u|_{1,K}^2 &= \sum_{K\in J_h} \int_K [(\partial_x(u - \Pi_h u))^2 + (\partial_y(u - \Pi_h u))^2]dxdy\\ &= \sum_{K\in J_h} (D_1 + D_2),\end{aligned}$$

where $D_1 = \int_K (\partial_x(u - \Pi_h u))^2 dxdy,\ D_2 = \int_K (\partial_y(u - \Pi_h u))^2 dxdy.$

An application of Lemma 2.1 gives

$$\begin{aligned}D_2 =& \int_K (\partial_y(u - \Pi_h u))^2 dxdy = \int_{\hat K} (\partial_y(\hat u - \hat\Pi\hat u))^2 \cdot 2|K|d\lambda_1 d\lambda_2\\
=& \int_{\hat K} [\partial_{\lambda_1}(\hat u - \hat\Pi\hat u)\partial_y\lambda_1 + \partial_{\lambda_2}(\hat u - \hat\Pi\hat u)\partial_y\lambda_2]^2 \cdot 2|K|d\lambda_1 d\lambda_2\\
\le& 2\int_{\hat K} \left\{[\partial_{\lambda_1}(\hat u - \hat\Pi\hat u)\partial_y\lambda_1]^2 + [\partial_{\lambda_2}(\hat u - \hat\Pi\hat u)\partial_y\lambda_2]^2\right\} \cdot 2|K|d\lambda_1 d\lambda_2\\
\le& C \cdot 2|K| \cdot [(\frac{x_3 - x_2}{2|K|})^2|\partial_{\lambda_1}\hat u|_{1,\hat K}^2 + (\frac{x_1 - x_3}{2|K|})^2|\partial_{\lambda_2}\hat u|_{1,\hat K}^2]\\
\le& C \cdot 2|K| \cdot \Big\{(\frac{x_3 - x_2}{2|K|})^2 \int_{\hat K} [(\partial_{\lambda_1\lambda_1}\hat u)^2 + (\partial_{\lambda_1\lambda_2}\hat u)^2]d\lambda_1 d\lambda_2\\
&+ (\frac{x_1 - x_3}{2|K|})^2 \int_{\hat K} [(\partial_{\lambda_2\lambda_1}\hat u)^2 + (\partial_{\lambda_2\lambda_2}\hat u)^2]d\lambda_1 d\lambda_2\Big\}\\
\le& C\Big\{(\frac{x_3 - x_2}{2|K|})^2 \int_K [(\partial_{xx}u)^2(x_1 - x_3)^4 + (\partial_{yx}u)^2(y_1 - y_3)^2(x_1 - x_3)^2\\
&\quad + (\partial_{xy}u)^2(x_1 - x_3)^2(y_1 - y_3)^2 + (\partial_{yy}u)^2(y_1 - y_3)^4\\
&\quad + (\partial_{xx}u)^2(x_1 - x_3)^2(x_2 - x_3)^2 + (\partial_{yx}u)^2(y_1 - y_3)^2(x_2 - x_3)^2\\
&\quad + (\partial_{xy}u)^2(x_1 - x_3)^2(y_2 - y_3)^2 + (\partial_{yy}u)^2(y_1 - y_3)^2(y_2 - y_3)^2]dxdy\\
&\quad + (\frac{x_1 - x_3}{2|K|})^2 \int_K [(\partial_{xx}u)^2(x_2 - x_3)^2(x_1 - x_3)^2 + (\partial_{yx}u)^2(y_2 - y_3)^2(x_1 - x_3)^2\\
&\quad + (\partial_{xy}u)^2(x_2 - x_3)^2(y_1 - y_3)^2 + (\partial_{yy}u)^2(y_2 - y_3)^2(y_1 - y_3)^2\\
&\quad + (\partial_{xx}u)^2(x_2 - x_3)^4 + (\partial_{yx}u)^2(y_2 - y_3)^2(x_2 - x_3)^2\\
&\quad + (\partial_{xy}u)^2(x_2 - x_3)^2(y_2 - y_3)^2 + (\partial_{yy}u)^2(y_2 - y_3)^4]dxdy\Big\}.\end{aligned} \tag{3.5}$$

Obviously, we can derive that

$$\frac{(y_1-y_3)^2}{(2|K|)^2} \le \frac{C}{h_{x,K}^2}, \quad \frac{(y_2-y_3)^2}{(2|K|)^2} \le \frac{C}{h_{x,K}^2}.$$

If the maximal interior angle γ of K is an obtuse angle, then from condition **(a)**, there holds $1/\sin\gamma^* \le C$. So we have

$$\frac{(x_3-x_2)^2(x_1-x_3)^2}{(2|K|)^2} = \frac{(x_3-x_2)^2(x_1-x_3)^2}{(|\overline{A_1A_3}||\overline{A_2A_3}|\sin\gamma)^2} = \frac{1}{\sin^2\gamma^*} \le C, \tag{3.6}$$

and

$$\frac{(x_3-x_2)^2(x_1-x_3)^4}{(2|K|)^2} \le Ch^2, \quad \frac{(x_3-x_2)^4(x_1-x_3)^2}{(2|K|)^2} \le Ch^2. \tag{3.7}$$

If the maximal interior angle γ of K is an acute angle or a right angle, then

$$\frac{(x_3-x_2)^2(x_1-x_3)^2}{(2|K|)^2} \le C,$$

(3.7) is still valid. Hence

$$D_2 = \int_K (\partial_y(u-\Pi_h u))^2 dxdy \le Ch^2|u|_{2,K}^2. \tag{3.8}$$

Similarly, we have

$$D_1 = \int_K (\partial_x(u-\Pi_h u))^2 dxdy \le Ch^2|u|_{2,K}^2. \tag{3.9}$$

Combining with these two estimates (3.8) and (3.9), yields

$$\sum_{K\in J_h} |u-\Pi_h u|_{1,K}^2 \le Ch^2 \sum_{K\in J_h} |u|_{2,K}^2 = Ch^2|u|_{2,\Omega}^2. \tag{3.10}$$

It follows from Lemma 2.2, inequality (3.10) and Lemma 3.1 that

$$\begin{aligned}
\inf_{v_h\in V_h} \|u-v_h\|_h^2 \le& \|u-\Pi_h u\|_h^2 = a_h(u-\Pi_h u, u-\Pi_h u)\\
=&\mu\sum_{K\in J_h}|u-\Pi_h u|_{1,K}^2 + (\mu+\lambda)\sum_{K\in J_h}\|\mathrm{div}u - \mathrm{div}\Pi_h u\|_{0,K}^2\\
=&\mu\sum_{K\in J_h}|u-\Pi_h u|_{1,K}^2 + (\mu+\lambda)\sum_{K\in J_h}\|\mathrm{div}u - M_K\mathrm{div}u\|_{0,K}^2\\
\le& Ch^2\sum_{K\in J_h}|u|_{2,K}^2 + C(\mu+\lambda)h^2\sum_{K\in J_h}|\mathrm{div}u|_{1,K}^2\\
\le& Ch^2\left(\sum_{K\in J_h}|u|_{2,K}^2 + \lambda\sum_{K\in J_h}|\mathrm{div}u|_{1,K}^2\right)\\
\le& Ch^2(|u|_{2,\Omega}^2 + \lambda|\mathrm{div}u|_{1,\Omega}^2) \le Ch^2\|f\|_{0,\Omega}^2.
\end{aligned}$$

Thus

$$\inf_{v_h\in V_h}\|u-v_h\|_h \le Ch\|f\|_{0,\Omega}. \tag{3.11}$$

Next, we turn to estimate the consistency error, the second term on the right hand of (3.4). In order to do this, we introduce the auxiliary finite element space $\tilde{V}_h$, which can be defined by

$$\tilde{V}_h = \left\{\tilde{u}_h \in (L^2(\Omega))^2;\ \ \tilde{u}_h|_K \in (\mathrm{span}\{1,y\})^2,\ \forall K, \int_{F_L}[\tilde{u}_h]ds = 0\right\},$$

where F_L are the two longer edges of K.

For an arbitrary but fixed $u_h \in V_h$, we define $\tilde{u}_h \in \tilde{V}_h$ such that

$$\int_{F_L} u_h ds = \int_{F_L} \tilde{u}_h ds, \qquad \forall F_L \in \partial K. \tag{3.12}$$

Since triangles have exactly two longer edges F_L, this definition is meaningful for the above element. Owing to $\partial u_h/\partial y$ and $\partial \tilde{u}_h/\partial y$ are constants, by means of Green's formula and (3.12), we have

$$\partial_y(u_h - \tilde{u}_h) = |K|^{-1}\int_K \partial_y(u_h - \tilde{u}_h)dxdy = |K|^{-1}\sum_{F_L\in\partial K}\int_{F_L}(u_h - \tilde{u}_h)\cdot n_y ds = 0,$$

where $n = (n_x, n_y)$ is outward unit normal vector to ∂K, so we get

$$\partial_y u_h = \partial_y \tilde{u}_h, \qquad \partial_x \tilde{u}_h = 0. \tag{3.13}$$

Using (3.12) and (3.13), we can derive

$$\|\hat{u}_h - \tilde{\hat{u}}_h\|_{0,\hat{K}} \le \hat{C}|\hat{u}_h - \tilde{\hat{u}}_h|_{1,\hat{K}} \tag{3.14}$$

and

$$\begin{aligned}\|u_h - \tilde{u}_h\|_{0,K} &= |K|^{\frac{1}{2}}\cdot|\hat{K}|^{-\frac{1}{2}}\|\hat{u}_h - \tilde{\hat{u}}_h\|_{0,\hat{K}} \\ &\le C|K|^{\frac{1}{2}}\cdot|\hat{K}|^{-\frac{1}{2}}|\hat{u}_h - \tilde{\hat{u}}_h|_{1,\hat{K}} \le Ch_{x,K}\|\partial_x u_h\|_{0,K}.\end{aligned} \tag{3.15}$$

Let $M_F u_i = \frac{1}{|F|}\int_F u_i ds$ $(i = 1, 2)$, then we have

$$\begin{aligned}
E_h(u,\omega_h) =& \sum_{K\in J_h}\int_K \Big[\mu\nabla u\cdot\nabla\omega_h + (\mu+\lambda)(\mathrm{div}u)(\mathrm{div}\omega_h)\Big]dxdy - \int_\Omega f\cdot\omega_h dxdy \\
=& \sum_{K\in J_h}\int_K \Big[\mu(\partial_x u_1\partial_x\omega_{h_1} + \partial_y u_1\partial_y\omega_{h_1} + \partial_x u_2\partial_x\omega_{h_2} + \partial_y u_2\partial_y\omega_{h_2}) \\
&+ (\mu+\lambda)(\mathrm{div}u\partial_x\omega_{h_1} + \mathrm{div}u\partial_y\omega_{h_2})\Big]dxdy - \sum_{K\in J_h}\int_K (f_1\omega_{h_1} + f_2\omega_{h_2})dxdy \\
=& \sum_{K\in J_h}\int_K \Big[\mu(\partial_x u_1\partial_x\omega_{h_1} + \partial_y u_1\partial_y\tilde{\omega}_{h_1} + \partial_x u_2\partial_x\omega_{h_2} + \partial_y u_2\partial_y\tilde{\omega}_{h_2}) \\
&+ (\mu+\lambda)(\mathrm{div}u\partial_x\omega_{h_1} + \mathrm{div}u\partial_y\tilde{\omega}_{h_2})\Big]dxdy - \sum_{K\in J_h}\int_K (f_1\omega_{h_1} + f_2\omega_{h_2})dxdy \\
=& -\sum_{K\in J_h}\int_K \Big\{\mu(\partial_{xx}u_1\omega_{h_1} + \partial_{yy}u_1\tilde{\omega}_{h_1} + \partial_{xx}u_2\omega_{h_2} + \partial_{yy}u_2\tilde{\omega}_{h_2}) \\
&+ (\mu+\lambda)[\partial_x(\mathrm{div}u)\omega_{h_1} + \partial_y(\mathrm{div}u)\tilde{\omega}_{h_2}] + f_1\omega_{h_1} + f_2\omega_{h_2}\Big\}dxdy \\
&+ \sum_{K\in J_h}\int_{\partial K}\Big\{\mu(\partial_x u_1\omega_{h_1}n_x + \partial_y u_1\tilde{\omega}_{h_1}n_y + \partial_x u_2\omega_{h_2}n_x + \partial_y u_2\tilde{\omega}_{h_2}n_y) \\
&+ (\mu+\lambda)[(\mathrm{div}u)\omega_{h_1}n_x + (\mathrm{div}u)\tilde{\omega}_{h_2}n_y]\Big\}ds
\end{aligned}$$

$$
\begin{aligned}
=&-\sum_{K\in J_h}\int_K\Big\{[\mu\partial_{xx}u_1+(\mu+\lambda)\partial_x(\mathrm{div}u)+f_1](\omega_{h_1}-\tilde{\omega}_{h_1})\\
&+(\mu\partial_{xx}u_2+f_2)(\omega_{h_2}-\tilde{\omega}_{h_2})\Big\}dxdy\\
&+\sum_{K\in J_h}\int_{\partial K}\Big\{\mu(\partial_x u_1\omega_{h_1}n_x+\partial_y u_1\tilde{\omega}_{h_1}n_y+\partial_x u_2\omega_{h_2}n_x+\partial_y u_2\tilde{\omega}_{h_2}n_y)\\
&+(\mu+\lambda)[(\mathrm{div}u)\omega_{h_1}n_x+(\mathrm{div}u)\tilde{\omega}_{h_2}n_y]\Big\}ds\\
=&-\sum_{K\in J_h}\int_K[\mu\partial_{xx}u_1+(\mu+\lambda)\partial_x(\mathrm{div}u)+f_1](\omega_{h_1}-\tilde{\omega}_{h_1})dxdy\\
&-\sum_{K\in J_h}\int_K(\mu\partial_{xx}u_2+f_2)(\omega_{h_2}-\tilde{\omega}_{h_2})dxdy\\
&+\sum_{K\in J_h}\int_{\partial K}[\mu(\partial_x u_1\omega_{h_1}n_x+\partial_x u_2\omega_{h_2}n_x)+(\mu+\lambda)(\mathrm{div}u)\omega_{h_1}n_x]ds\\
&+\sum_{K\in J_h}\int_{\partial K}\Big[\mu(\partial_y u_1\tilde{\omega}_{h_1}n_y+\partial_y u_2\tilde{\omega}_{h_2}n_y)+(\mu+\lambda)(\mathrm{div}u)\tilde{\omega}_{h_2}n_y\Big]ds\\
=&E_1+E_2+E_3+E_4.
\end{aligned}\tag{3.16}
$$

We take into account the first term of (3.16), i.e., the integral on K. From (3.2) and (3.15), we have

$$
\begin{aligned}
|E_1|\le&\sum_{K\in J_h}h_{x,K}(|u|_{2,K}+\lambda|\mathrm{div}u|_{1,K}+\|f_1\|_{0,K})\|\partial_x\omega_{h_1}\|_{0,K}\\
\le&Ch\|f\|_{0,\Omega}\|\omega_h\|_h.
\end{aligned}\tag{3.17}
$$

Similarly,

$$
|E_2|\le\sum_{K\in J_h}\|\mu\partial_{xx}u_2+f_2\|_{0,K}\|\omega_{h_2}-\tilde{\omega}_{h_2}\|_{0,K}\le Ch\|f\|_{0,\Omega}\|\omega_h\|_h.\tag{3.18}
$$

As to the third term of (3.16), i.e., the integral along ∂K, we have

$$
\begin{aligned}
|E_3|=&\left|\sum_{K\in J_h}\sum_{F\subset\partial K}\int_F[\mu\left(\partial_x u_1\omega_{h_1}n_x+\partial_x u_2\omega_{h_2}n_x\right)+(\mu+\lambda)\left(\mathrm{div}u\right)\omega_{h_1}n_x]ds\right|\\
=&\Bigg|\sum_{K\in J_h}\sum_{F\subset\partial K}n_x\int_F\Big\{\mu[(\partial_x u_1-M_F\partial_x u_1)\left(\omega_{h_1}-M_F\omega_{h_1}\right)\\
&+\left(\partial_x u_2-M_F\partial_x u_2\right)\left(\omega_{h_2}-M_F\omega_{h_2}\right)]\\
&+(\mu+\lambda)\left[(\mathrm{div}u)-M_F\left(\mathrm{div}u\right)\right]\left(\omega_{h_1}-M_F\omega_{h_1}\right)\Big\}ds\Bigg|\\
\le&\sum_{K\in J_h}\sum_{F\subset\partial K}\frac{|F||n_x|}{2|K|}\Bigg[\mu\Bigg(\sum_{q\in\{x,y\}}h_{q,K}^2\|\partial_{xq}u_1\|_{0,K}^2\Bigg)^{\frac12}\Bigg(\sum_{q\in\{x,y\}}h_{q,K}^2\|\partial_q\omega_{h_1}\|_{0,K}^2\Bigg)^{\frac12}
\end{aligned}
$$

$$+\mu\Big(\sum_{q\in\{x,y\}} h_{q,K}^2\|\partial_{xq}u_2\|_{0,K}^2\Big)^{\frac{1}{2}}\Big(\sum_{q\in\{x,y\}} h_{q,K}^2\|\partial_q\omega_{h_2}\|_{0,K}^2\Big)^{\frac{1}{2}}$$
$$+(\mu+\lambda)\Big(\sum_{q\in\{x,y\}} h_{q,K}^2|\mathrm{div}u|_{1,K}^2\Big)^{\frac{1}{2}}\Big(\sum_{q\in\{x,y\}} h_{q,K}^2\|\partial_q\omega_{h_1}\|_{0,K}^2\Big)^{\frac{1}{2}}\Big]. \tag{3.19}$$

If F is the shortest edge, we obviously have

$$\frac{|F||n_x|}{2|K|}\le\frac{C}{h_{x,K}}. \tag{3.20}$$

If F is the longest edge, using the coordinate system condition **(b)**, we can derive that

$$\frac{|F||n_x|}{2|K|}=\frac{|F||\cos\alpha|}{2|K|}=\frac{|F||\sin\theta|}{2|K|}\le C\frac{|F|}{2|K|}\frac{h_{y,K}}{h_{x,K}}\le\frac{C}{h_{x,K}}. \tag{3.21}$$

If F is the remainder edge of the element K, owing to $\beta\le\angle A_2$ or $\beta\le\theta$ (see Fig. 3.1), and

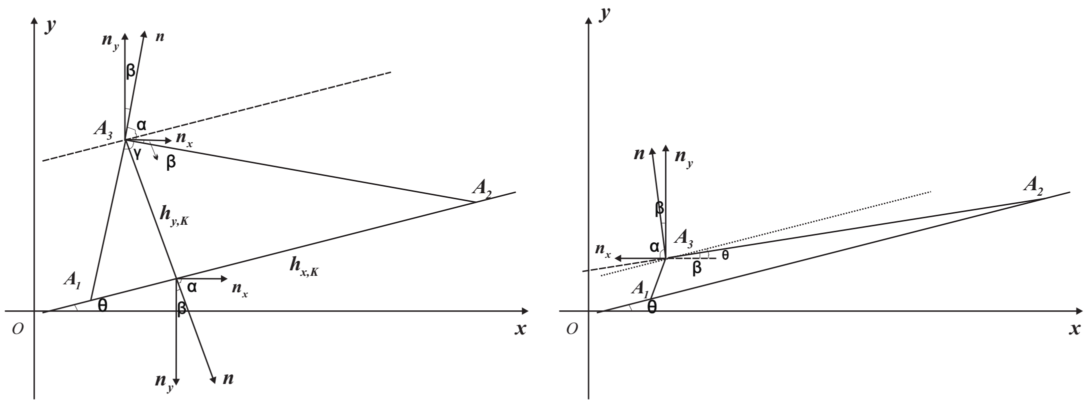

Fig. 3.1. The outward unit normal vectors to two longer edges (two cases).

$C_1h_{x,K}\le|A_2A_3|\le C_2h_{x,K}$ [11] (C_1 and C_2 are constants), we also have

$$\frac{|F||n_x|}{2|K|}=\frac{|F||\cos\alpha|}{2|K|}=\frac{|F||\sin\beta|}{2|K|}\le C\frac{|F|}{2|K|}\frac{h_{y,K}}{h_{x,K}}\le\frac{C}{h_{x,K}}. \tag{3.22}$$

It follows from the inequalities (3.19)–(3.22) and Lemma 3.1 that

$$|E_3|\le Ch(\|u\|_{2,\Omega}+\lambda|\mathrm{div}u|_{1,\Omega})\|\omega_h\|_h\le Ch\|f\|_{0,\Omega}\|\omega_h\|_h. \tag{3.23}$$

Finally, for the last term E_4, since $\frac{\partial \tilde{\omega}_h}{\partial x} = 0$ and $\|\tilde{\omega}_h - M_F\tilde{\omega}_h\|_{0,F} \le h_{y,K}\|\frac{\partial \tilde{\omega}_h}{\partial y}\|_{0,K}$, we obtain

$$\begin{aligned}
|E_4| =& \Big| \sum_{K\in J_h}\sum_{F\subset\partial K} n_y \int_F \Big\{ \mu\Big[(\partial_y u_1 - M_F\partial_y u_1)(\tilde{\omega}_{h_1} - M_F\tilde{\omega}_{h_1}) \\
& \quad + (\partial_y u_2 - M_F\partial_y u_2)(\tilde{\omega}_{h_2} - M_F\tilde{\omega}_{h_2})\Big] \\
& \quad + (\mu+\lambda)[(\mathrm{div} u) - M_F(\mathrm{div} u)](\tilde{\omega}_{h_2} - M_F\tilde{\omega}_{h_2})\Big\} ds \Big| \\
\le & C \sum_{K\in J_h}\sum_{F\subset\partial K} (h_{y,K})^{-1}\Big[\Big(\sum_{q\in\{x,y\}} h_{q,K}^2 \|\partial_{yq} u_1\|_{0,K}^2 \Big)^{\frac{1}{2}} h_{y,K}\|\partial_y \tilde{\omega}_{h_1}\|_{0,K} \\
& \quad + \Big(\sum_{q\in\{x,y\}} h_{q,K}^2 \|\partial_{yq} u_2\|_{0,K}^2 \Big)^{\frac{1}{2}} h_{y,K}\|\partial_y \tilde{\omega}_{h_2}\|_{0,K} \\
& \quad + (\mu+\lambda)\Big(\sum_{q\in\{x,y\}} h_{q,K}^2 |\mathrm{div} u|_{1,K}^2 \Big)^{\frac{1}{2}} h_{y,K}\|\partial_y \tilde{\omega}_{h_2}\|_{0,K}\Big] \\
\le & Ch(\|u\|_{2,\Omega} + \lambda|\mathrm{div} u|_{1,\Omega})\|\omega_h\|_h \\
\le & Ch\|f\|_{0,\Omega}\|\omega_h\|_h. \qquad (3.24)
\end{aligned}$$

Combining the results (3.16)–(3.18), (3.23) and (3.24), we have

$$|E_h(u,\omega_h)| \le Ch\|f\|_{0,\Omega}\|\omega_h\|_h. \qquad (3.25)$$

Thus the desired result (3.3) follows from (3.11) and (3.25). □

Now we start to derive the following optimal L^2-norm error estimate by using Aubin-Nitsche technique.

Theorem 3.2. *Under the assumptions of Theorem* 3.1*, we have*

$$\|u - u_h\|_{0,\Omega} \le Ch^2\|f\|_{0,\Omega}. \qquad (3.26)$$

Proof. Let $g \in (H^2(\Omega)\cap H_0^1(\Omega))^2$ satisfy

$$\begin{cases} -\mu\Delta g - (\mu+\lambda)\mathrm{grad}(\mathrm{div} g) = u - u_h & \text{in } \Omega, \\ g = 0 & \text{on } \partial\Omega. \end{cases} \qquad (3.27)$$

Then, corresponding to (3.2), the following estimate holds

$$\|g\|_{2,\Omega} + \lambda|\mathrm{div} g|_{1,\Omega} \le C\|u - u_h\|_{0,\Omega}. \qquad (3.28)$$

Let $g_h \in V_h$ be the solution of the following weak formulation

$$a_h(g_h, v_h) = (u - u_h, v_h), \qquad \forall v_h \in V_h. \qquad (3.29)$$

According to Theorem 3.1,

$$\|g - g_h\|_h \le Ch\|u - u_h\|_{0,\Omega}. \qquad (3.30)$$

Taking the L^2-norm product of (3.27) with $u-u_h$ and then employing the Green's formula piecewise, we have

$$\begin{aligned}
&\|u-u_h\|_{0,\Omega}^2\\
=&a_h(g,u-u_h)-(\mu+\lambda)\sum_{K\in J_h}\int_{\partial K}\mathrm{div}g(u-u_h)\cdot nds-\mu\sum_{K\in J_h}\int_{\partial K}\partial_n g(u-u_h)ds\\
=&a_h(g-g_h,u-u_h)-(\mu+\lambda)\sum_{K\in J_h}\int_{\partial K}\mathrm{div}g(u-u_h)\cdot nds-\mu\sum_{K\in J_h}\int_{\partial K}\partial_n g(u-u_h)ds\\
&\quad+(\mu+\lambda)\sum_{K\in J_h}\int_{\partial K}\mathrm{div}ug_h\cdot nds+\mu\sum_{K\in J_h}\int_{\partial K}\partial_n ug_hds\\
=&R_1+(\mu+\lambda)R_2+\mu R_3, \qquad (3.31)
\end{aligned}$$

where

$$\begin{aligned}
R_1&=a_h(g-g_h,u-u_h),\\
R_2&=-\sum_{K\in J_h}\int_{\partial K}\mathrm{div}g(u-u_h)\cdot nds+\sum_{K\in J_h}\int_{\partial K}\mathrm{div}ug_h\cdot nds,\\
R_3&=-\sum_{K\in J_h}\int_{\partial K}\partial_n g(u-u_h)ds+\sum_{K\in J_h}\int_{\partial K}\partial_n ug_hds. \qquad (3.32)
\end{aligned}$$

Now we estimate $|R_i|$ (i=1,2,3) one by one. By Theorem 3.1 and (3.30), the following estimate holds

$$|R_1|\le C\|g-g_h\|_h\|u-u_h\|_h\le Ch^2\|f\|_{0,\Omega}\|u-u_h\|_{0,\Omega}. \qquad (3.33)$$

By the similar error estimate technique of (3.25), and combining with Theorem 3.1 and (3.30), yields

$$\begin{aligned}
|R_2|\le&\Big|\sum_{K\in J_h}\int_{\partial K}\mathrm{div}g(u-u_h)\cdot nds\Big|+\Big|\sum_{K\in J_h}\int_{\partial K}\mathrm{div}ug_h\cdot nds\Big|\\
\le&Ch\|u-u_h\|_h|\mathrm{div}g|_{1,\Omega}+Ch|\mathrm{div}u|_{1,\Omega}\|g-g_h\|_h\\
\le&Ch^2\|f\|_{0,\Omega}|\mathrm{div}g|_{1,\Omega}+Ch^2|\mathrm{div}u|_{1,\Omega}\|u-u_h\|_{0,\Omega} \qquad (3.34)
\end{aligned}$$

and

$$\begin{aligned}
|R_3|=&\Big|\sum_{K\in J_h}\int_{\partial K}\partial_n g(u-u_h)ds+\sum_{K\in J_h}\int_{\partial K}\partial_n u(g-g_h)ds\Big|\\
\le&Ch\|u-u_h\|_h|g|_{2,\Omega}+Ch|u|_{2,\Omega}\|g-g_h\|_h\\
\le&Ch^2\|f\|_{0,\Omega}\|u-u_h\|_{0,\Omega}. \qquad (3.35)
\end{aligned}$$

Collecting the estimates (3.33) to (3.35), together with (3.2) and (3.28), we have

$$\begin{aligned}
\|u-u\|_{0,\Omega}^2\le&Ch^2\Big[\|f\|_{0,\Omega}\|u-u_h\|_{0,\Omega}+\mu\|f\|_{0,\Omega}\|u-u_h\|_{0,\Omega}\\
&+(\mu+\lambda)(\|f\|_{0,\Omega}|\mathrm{div}g|_{1,\Omega}+|\mathrm{div}u|_{1,\Omega}\|u-u_h\|_{0,\Omega})\Big]\\
\le&Ch^2\|f\|_{0,\Omega}\|u-u_h\|_{0,\Omega}. \qquad (3.36)
\end{aligned}$$

Therefore,

$$\|u-u_h\|_{0,\Omega}\le Ch^2\|f\|_{0,\Omega}.$$

The proof is completed. □

Table 4.1: The error $\|u - u_h\|_h$.

$m_1 \times m_2$	10×80	20×160	40×320	80×640	α
$\lambda = 9$	10.29813758	5.10505902	2.54804779	1.27360420	1.0051
$\lambda = 99$	10.14780095	5.04018850	2.51678398	1.25778677	1.0040
$\lambda = 999$	10.14517060	5.03512340	2.51295416	1.25604985	1.0046
$\lambda = 9999$	10.13446599	5.03533038	2.51349808	1.25603180	1.0041

Table 4.2: The error $\|u - u_h\|_{0,\Omega}$.

$m_1 \times m_2$	10×80	20×160	40×320	80×640	α
$\lambda = 9$	0.55616049	0.13756083	0.03435003	0.00858766	2.0056
$\lambda = 99$	0.55960322	0.13887440	0.03469492	0.00867049	2.0040
$\lambda = 999$	0.56212467	0.13928850	0.03476480	0.00868709	2.0052
$\lambda = 9999$	0.56076504	0.13927853	0.03478685	0.00869010	2.0039

4. Numerical Example

In order to check the convergence behavior of the nonconforming linear triangular Crouzeix-Raviart type finite element on anisotropic meshes as $\lambda \to \infty$, we carry out a numerical example in this section. We consider problem (1.1) with $\Omega = [0,2]^2$ and $f = (f_1, f_2) \in L^2(\Omega)$, where

$$\begin{aligned} f_1 =& -(x^4 - 4x^3 + 4x^2)\Big[n(n+1)(n+2)y^{n-1} - 4n(n-1)(n+1)y^{n-2} + 4n(n-1)(n-2)y^{n-3}\Big] \\ & - (12x^2 - 24x + 8)\Big[(n+2)y^{n+1} - 4(n+1)y^n + 4ny^{n-1}\Big] + \frac{\mu}{1+\lambda}\sin\pi x \sin\pi y, \\ f_2 =& (24x - 24)\left[y^n(y-2)^2\right] + \frac{\mu}{1+\lambda}\sin\pi x\sin\pi y \\ & + (4x^3 - 12x^2 + 8x)\Big[(n+1)(n+2)y^n - 4n(n+1)y^{n-1} + 4n(n-1)y^{n-2}\Big]. \end{aligned}$$

The exact solution of problem (1.1) is $u = (u_1, u_2)$, where

$$\begin{aligned} u_1 &= (x^4 - 4x^3 + 4x^2)(y^{n+2} - 4y^{n+1} + 4y^n) + \frac{\mu}{1+\lambda}\sin\pi x \sin\pi y, \\ u_2 &= -(4x^3 - 12x^2 + 8x)[y^n(y-2)^2] + \frac{\mu}{1+\lambda}\sin\pi x\sin\pi y. \end{aligned}$$

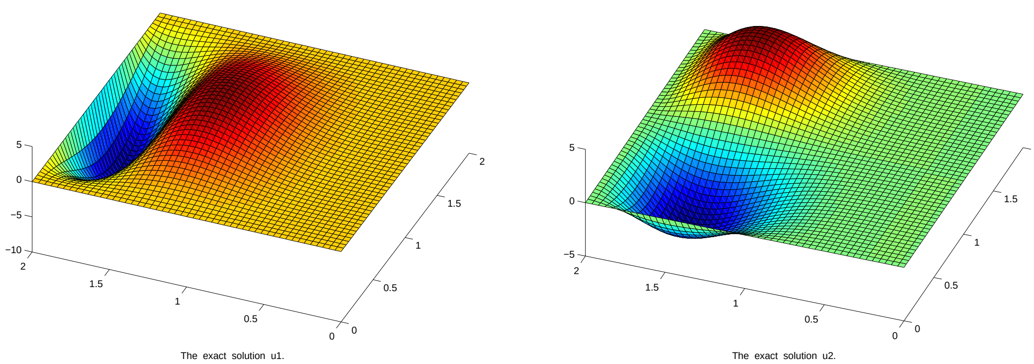

Fig. 4.1. The exact solutions u_1 and u_2 with $n = 6$ and $\lambda = 999$.

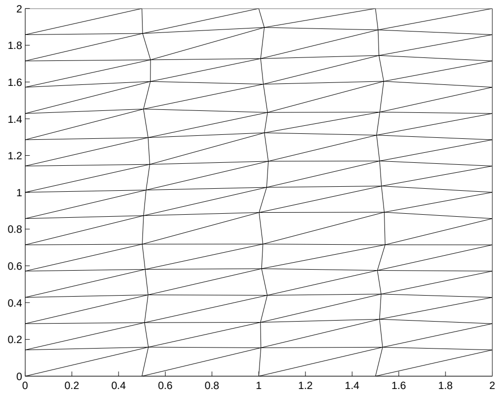

Fig. 4.2. The subdivision meshes of Ω.

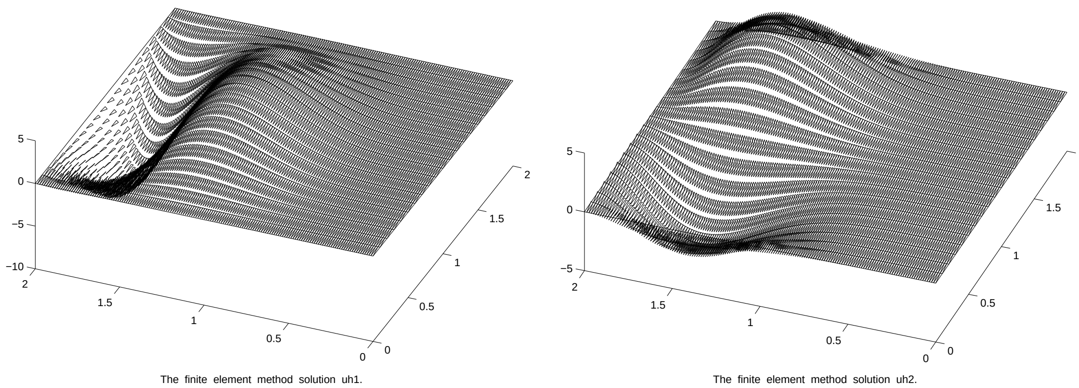

Fig. 4.3. The finite element method solutions u_{h_1} and u_{h_2} with $n = 6$ and $\lambda = 999$.

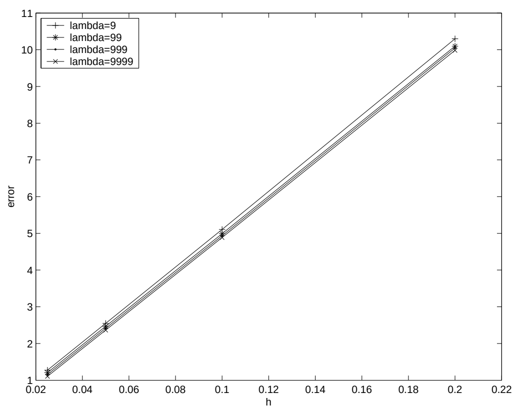

Fig. 4.4. The error $\|u - u_h\|_h$.

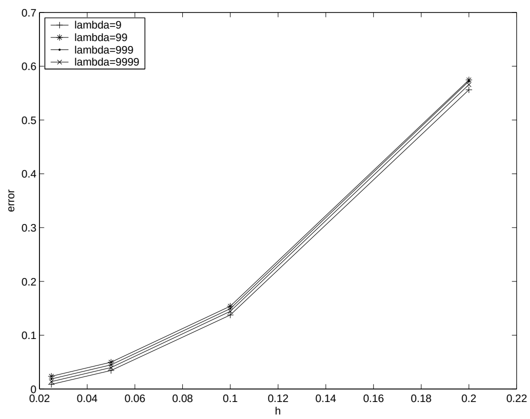

Fig. 4.5. The error $\|u - u_h\|_{0,\Omega}$.

It can be seen that the exact solutions u_1 and u_2 vary significantly in the y-direction when n is a big positive integer (see Fig. 4.1). This anisotropic behavior makes it necessary to use a smaller mesh size in y-direction and a larger mesh size in x-direction. So we first divide the boundary of Ω into m_1 and m_2 equal intervals along x-axis and y-axis, respectively, to get the uniformly right triangle meshes. Then we do some perturbations on interior points (see Fig. 4.2). Obviously, the triangular meshes satisfy the conditions **(a)** and **(b)**. The computation is carried out for $\mu = 1$, $n = 6$ and $m_1 : m_2 = 1 : 8$, and the numerical results are listed in Tables 1–2 , and pictured in Figs. 4.3–4.5, respectively , where α represents the average convergence order.

From the above tables we can see that the optimal convergence orders are obtained in the energy and L^2-norms, which are independent of the Lamé parameter λ for anisotropic meshes. Thus we can devise the better meshes to improve the computing accuracy.

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