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SOME RESIDUAL BOUNDS FOR APPROXIMATE EIGENVALUES AND APPROXIMATE EIGENSPACES*

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Abstract

In this paper we consider approximate eigenvalues and approximate eigenspaces for the generalized Rayleigh quotient, and present some residual bounds. Our obtained bounds will improve the existing ones.

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1. Introduction

By $\mathcal{C}^{m\times n}$ we denote the set of $m\times n$ complex matrices, by A^* we denote the conjugate transpose, and by I we denote the identity matrix. The Frobenius norm and the spectral norm of a matrix $\cdot$ are denoted by $\|\cdot\|_F$ and $\|\cdot\|_2$, respectively.

Let A and H be diagonalizable matrices with the following decompositions:

$$A = X\Lambda X^{-1} \equiv \begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} Y_1^* \\ Y_2^* \end{pmatrix} \text{and } H = Z\widetilde{\Lambda}Z^{-1}, \tag{1.1}$$

respectively, where $X\in\mathcal{C}^{n\times n}$, $Z\in\mathcal{C}^{m\times m}$, $X_1\in\mathcal{C}^{n\times m}$ $(m\le n)$,

$$\Lambda_1 = \mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_m), \quad \Lambda_2 = \mathrm{diag}(\lambda_{m+1},\lambda_{m+2},\cdots,\lambda_n),$$
$$\widetilde{\Lambda} = \mathrm{diag}(\widetilde{\lambda}_1,\widetilde{\lambda}_2,\cdots,\widetilde{\lambda}_m).$$

Let A and H have the decomposition (1.1). Then δ_i is denoted by

$$\delta_i = \min_{\lambda\in\lambda(\Lambda_i),\widetilde{\lambda}\in\lambda(\widetilde{\Lambda})} |\lambda-\widetilde{\lambda}|, \quad i=1,2. \tag{1.2}$$

Notice that the decomposition (1.1) implies that

$$X^{-1} = \begin{pmatrix} Y_1^* \\ Y_2^* \end{pmatrix}. \tag{1.3}$$

Let

$$R = AQ_1 - Q_1H \tag{1.4}$$

be the residual matrix of A with Q_1, where $A\in\mathcal{C}^{n\times n}$, $H\in\mathcal{C}^{m\times m}$ and $Q_1\in\mathcal{C}^{n\times m}$ $(m\le n)$, $\mathrm{rank}(Q_1)=m$. The spectrum of H is denoted by $\sigma(H)=\{\widetilde{\lambda}_1,\widetilde{\lambda}_2,\cdots,\widetilde{\lambda}_m\}$.

The quantity $||R||$ can be used to measure the difference between the spectrum $\sigma(H)$ and the spectrum $\sigma(\Lambda_1)$, and between the subspace $\Re(Q_1)$ and the approximate subspace $\Re(X_1)$. Some classical results in this topic are listed below:

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1.1. Approximate eigenvalues

If A and H are Hermitian matrices and Q_1 has orthonormal columns, Kahan proved that there exists a permutation τ of $\langle m\rangle$ such that the following bound

$$\sum_{i=1}^{m}|\lambda_{\tau(i)}-\widetilde{\lambda}_i|^2\leq 2||R||_F^2 \tag{1.5}$$

holds (e.g., see [17]), where $\langle m\rangle=\{1,2,...,m\}$.

If A is Hermitian and Q_1 has the orthonormal columns, $H=Q_1^*AQ_1$ is the Rayleigh quotient matrix, then it holds that [15]

$$\sum_{i=1}^{m}|\lambda_i-\widetilde{\lambda}_i|^2\leq \frac{\|\sin\Theta(Q_1,X_1)\|_2^2}{1-\|\sin\Theta(Q_1,X_1)\|_2^2}||R||_F^2, \tag{1.6}$$

where the angle matrix $\Theta(Y,\widetilde{Y})$ between subspaces $\Re(Y)$ and $\Re(\widetilde{Y})$ is defined by

$$\Theta(Y,\widetilde{Y})=\arccos((Y^*Y)^{-\frac{1}{2}}Y^*\ \widetilde{Y}\ (\widetilde{Y}^*\widetilde{Y})^{-1}\ \widetilde{Y}^*\ Y(Y^*Y)^{-\frac{1}{2}})^{\frac{1}{2}}\ ,$$

Y and $\widetilde{Y}\in\mathcal{C}^{n\times k}(n>k)$ are full column rank matrices. In particular, if Y and $\widetilde{Y}\in\mathcal{C}^{n\times k}(n>k)$ have orthonormal columns, then for any unitarily invariant norm $||\cdot||$ we have

$$||\sin\Theta(Y,\widetilde{Y})||=||(\widetilde{Y}_c)^*Y||, \tag{1.7}$$

where $(\widetilde{Y},\widetilde{Y}_c)$ is an $n\times n$ unitary matrix (e.g., see [13]).

If A and H are diagonalizable matrices with the decomposition (1.1), and Q_1 has full column rank, then Liu [11] obtained a result as follows: There exists a permutation τ of $\langle m\rangle$ such that

$$\sigma_{\min}^2(Q_1)\sum_{i=1}^{m}|\lambda_{\tau(i)}-\widetilde{\lambda}_i|^2\leq\kappa^2(X)\kappa^2(Z)||R||_F^2\ , \tag{1.8}$$

where $\sigma_{\min}(Q_1)$ denotes the smallest singular value of Q_1. In particular, if A and H are Hermitian matrices, then

$$\sigma_{\min}^2(Q_1)\sum_{i=1}^{m}|\lambda_{\tau(i)}-\widetilde{\lambda}_i|^2\leq||R||_F^2\ . \tag{1.9}$$

It is easy to see that the bound (1.9) generalizes the one in (1.5).

1.2. Approximate eigenspaces

If A and H are Hermitian matrices and Q_1 has orthonormal columns, Kahan and Davis [1] obtained a well-known result, i.e., $\sin\Theta$ Theorem:

$$\|\sin\Theta(Q_1,X_1)\|_F\leq\frac{\|R\|_F}{\delta_2} \tag{1.10}$$

provided $\delta_2>0$, where δ_2 is given by (1.2). If A and H are Hermitian matrices, and Q_1 is a full column rank matrix, then (see, e.g., [13])

$$\sigma_{\min}(Q_1)\|\sin\Theta(Q_1,X_1)\|_F\leq\frac{\|R\|_F}{\delta_2} \tag{1.11}$$

provided $\delta_2 > 0$. Notice that if Q_1 has orthonormal columns, then $\sigma_{\min}(Q_1) = 1$, and hence the bound in (1.11) reduces to the one in (1.10).

Now let A be a diagonalizable matrix with the decomposition (1.1), and let $Q_1 \in \mathcal{C}^{n\times m}$ be any full column rank matrix. Setting $Q_2 \in \mathcal{C}^{n\times(n-m)}$ such that $Q = (Q_1, Q_2)$ is nonsingular, and

$$Q^{-1} \equiv \begin{pmatrix} \widetilde{Q}_1^* \\ \widetilde{Q}_2^* \end{pmatrix}. \tag{1.12}$$

The matrix $H \equiv \widetilde{Q}_1^* A Q_1$ is called the generalized Rayleigh quotient [13].

In some applications, the column of the matrix Q_1 in (1.4) may not be orthonormal and the matrix H in (1.4) may be more general. By this motivation, our work in this paper is to generalize the bound for approximate eigenvalues and eigenvectors, from which one can understand the accuracy of approximate eigenpairs computed by numerical approaches. As we know, the residual bounds for approximate eigenpairs in the following cases have not been studied:

Case 1 A and H are diagonalizable, H is a generalized Rayleigh quotient, and Q_1 is any full rank matrix.

Case 2 A and H are diagonalizable, and Q_1 is any full rank matrix.

Case 3 A is Hermitian and Q_1 has orthonormal columns, but H is an arbitrary matrix.

Hence, in this paper we will consider to bound approximate eigenvalues and approximate eigenspaces in the cases 1-3.

The rest of this paper is organized as follows. In second 2 we consider the case 1, and give bounds for approximate eigenvalues and approximate eigenspaces, which extend some existing bounds; see Theorem 2.1 and Corollary 2.1. In second 3, the residual bounds for approximate eigenvalues and eigenspaces in the case 2 are also provided; see Theorems 3.1 and 3.2 and Corollary 3.1. When H is any matrix, the approximate eigenvalue bound has not been discussed so far, in section 4 we consider the case 3, and obtain some residual bounds for approximate eigenvalues; see Theorem 4.1.

2. Approximate Eigenvalue and Eigenspace Bounds for the Generalized Rayleigh Quotient Case

In this section we consider the case 1 and give bounds for approximate eigenvalues and eigenspaces.

Now let X_1 and $\widetilde{X}_1 \in \mathcal{C}^{n\times k} (n \geq k)$ have full column rank, and let P and $\widetilde{P}$ be the orthonormal projector on to $\Re(X_1)^\perp$ and $\Re(\widetilde{X}_1)$, respectively. Then (see, e.g., [5]) for any unitarily invariant norm $||\cdot||$,

$$||\sin\Theta(X_1, \widetilde{X}_1)|| = ||P\widetilde{P}||, \tag{2.1}$$

where $\Re(X_1)$ and $\Re(\widetilde{X}_1)$ are defined the subspaces spanned by the column vectors of X_1 and $\widetilde{X}_1$, respectively.

Let X_2 be an $n \times (n-m)$ matrix such that $X = (X_1, X_2)$ is nonsingular, and let X^{-1} have the block form (1.3).

The following lemma is useful to prove the main result.

Lemma 2.1. ([2]) *Let* $T \in \mathcal{C}^{n\times n}$ *and* $\Lambda_i = diag(\lambda_1^{(i)}, ..., \lambda_n^{(i)}) \in \mathcal{C}^{n\times n},\ i = 1, 2, 3, 4.$ *Then there exists a permutation* τ *of* $\langle n\rangle$ *such that*

$$\sigma_{\min}^2(T) \sum \left|\lambda_i^{(1)}\lambda_{\tau(i)}^{(2)} - \lambda_i^{(3)}\lambda_{\tau(i)}^{(4)}\right|^2 \le ||\Lambda_1 T\Lambda_2 - \Lambda_3 T\Lambda_4||_F^2\,,$$

where $\sigma_{\min}(T)$ *is the smallest singular value of* T*.*

Next we consider to bound approximate eigenvalues and approximate eigenspaces, respectively. Let $A \in \mathcal{C}^{\ n\times n}$ and $H \in \mathcal{C}^{\ m\times m}$ be both diagonalizable matrices with the decompositions (1.1). Let Q_1 have full column rank, R, Y_1 and Y_2 be given by (1.3) and (1.4), respectively.

Left-multiplying by X^{-1} on both sides of (1.4) gives

$$X^{-1}R = \Lambda X^{-1}Q_1 - X^{-1}Q_1 H.$$

Substituting $H = Z\widetilde{\Lambda}Z^{-1}$ into the above equation reveals that

$$\Lambda X^{-1}Q_1 Z - X^{-1}Q_1 Z\widetilde{\Lambda} = X^{-1}RZ.$$

Since X^{-1} has the block form (1.3), the above equation can be rewritten as in the block form:

$$\begin{pmatrix} \Lambda_1 Y_1^* Q_1 Z - Y_1^* Q_1 Z\widetilde{\Lambda} \\ \Lambda_2 Y_2^* Q_1 Z - Y_2^* Q_1 Z\widetilde{\Lambda} \end{pmatrix} = X^{-1}RZ. \tag{2.2}$$

Set

$$S_1 = \Lambda_1 Y_1^* Q_1 Z - Y_1^* Q_1 Z\widetilde{\Lambda},\ S_2 = \Lambda_2 Y_2^* Q_1 Z - Y_2^* Q_1 Z\widetilde{\Lambda}.$$

By (2.2), we have

$$X^{-1}RZ = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}.$$

Consequently,

$$||X^{-1}RZ||_F^2 = \|S_1\|_F^2 + \|S_2\|_F^2. \tag{2.3}$$

It can be verified that,

$$\widetilde{Q}_1^* R = \widetilde{Q}_1^* A Q_1 - \widetilde{Q}_1^* Q_1 H = \widetilde{Q}_1^* A Q_1 - H = 0.$$

By the block form of X, we have

$$\widetilde{Q}_1^* X = \begin{pmatrix} \widetilde{Q}_1^* X_1 & \widetilde{Q}_1^* X_2 \end{pmatrix},$$

and hence $\widetilde{Q}_1^* X X^{-1} RZ = 0$, which leads to

$$\widetilde{Q}_1^* X_1 S_1 + \widetilde{Q}_1^* X_2 S_2 = 0.$$

Multiplying by $||\widetilde{Q}_1^* X_2||_2^2$ on both sides of (2.3) and using the inequality

$$\sigma_{\min}(M)\|L\|_F \le ||ML||_F \le \|M\|_2\|L\|_F$$

gives

$$\begin{aligned}||\widetilde{Q}_1^* X_2||_2^2 ||X^{-1}RZ||_F^2 &\geq ||\widetilde{Q}_1^* X_2||_2^2 \|S_1\|_F^2 + \|\widetilde{Q}_1^* X_2 S_2\|_F^2 \\ &= ||\widetilde{Q}_1^* X_2||_2^2 \|S_1\|_F^2 + \|\widetilde{Q}_1^* X_1 S_1\|_F^2 \\ &\geq (||\widetilde{Q}_1^* X_2||_2^2 + \sigma_{\min}^2(\widetilde{Q}_1^* X_1))||S_1\|_F^2,\end{aligned}$$

which implies that

$$||S_1\|_F^2 \leq \frac{||\widetilde{Q}_1^* X_2||_2^2}{||\widetilde{Q}_1^* X_2||_2^2 + \sigma_{\min}^2(\widetilde{Q}_1^* X_1)} ||X^{-1}RZ||_F^2. \tag{2.4}$$

It follows from Lemma 2.1 that

$$\sigma_{\min}^2(Y_1^* Q_1 Z) \sum_{i=1}^{m} |\lambda_{\tau(i)} - \widetilde{\lambda}_i|^2 \leq \|S_1\|_F^2,$$

which, together with (2.4), gives

$$\sigma_{\min}^2(Y_1^* Q_1) \sum_{i=1}^{m} |\lambda_{\tau(i)} - \widetilde{\lambda}_i|^2 \leq \frac{||\widetilde{Q}_1^* X_2||_2^2}{||\widetilde{Q}_1^* X_2||_2^2 + \sigma_{\min}^2(\widetilde{Q}_1^* X_1)} \kappa^2(Z) ||X^{-1}R||_F^2.$$

Then we have the following theorem:

Theorem 2.1. *Let $Q_1 \in \mathcal{C}^{n\times m}$ have full column rank, and let A and $H = \widetilde{Q}_1^* A Q_1$ be diagonalizable matrices with the decomposition* (1.1). *Then there exists a permutation τ of $\langle m \rangle$ such that*

$$\sigma_{\min}^2(Y_1^* Q_1) \sum_{i=1}^{m} |\lambda_{\tau(i)} - \widetilde{\lambda}_i|^2 \leq \frac{||\widetilde{Q}_1^* X_2||_2^2}{||\widetilde{Q}_1^* X_2||_2^2 + \sigma_{\min}^2(\widetilde{Q}_1^* X_1)} \kappa^2(Z) ||X^{-1}R||_F^2. \tag{2.5}$$

Remark 2.1. If A is Hermitian and Q_1 has orthonormal columns, then $Q_1^* = \widetilde{Q}_1^*$ and $H = Q_1^* A Q_1$. Hence X and Z are unitary, and thus $X_1^* = Y_1^*$. So we have

$$||\widetilde{Q}_1^* X_2||_2 = \|\sin\Theta(Q_1, X_1)\|_2$$

and

$$\sigma_{\min}^2(Y_1^* Q_1) = \sigma_{\min}^2(\widetilde{Q}_1^* X_1) = 1 - \|\sin\Theta(Q_1, X_1)\|_2^2.$$

Then the bound (2.5) may reduce to a simple form:

$$\sum_{i=1}^{m} |\lambda_{\tau(i)} - \widetilde{\lambda}_i|^2 \leq \frac{\|\sin\Theta(Q_1, X_1)\|_2^2}{1 - \|\sin\Theta(Q_1, X_1)\|_2^2} ||R||_F^2,$$

which is the Sun's bound (1.6).

Note that the function $f(x) = \frac{x}{1-x}$ is an increased function for $x \in (0, 1)$ and

$$\|\sin\Theta(Q_1, X_1)\|_2 \leq \|\sin\Theta(Q_1, X_1)\|_F \leq \frac{||R||_F}{\delta_2} = \rho_F.$$

It follows from the above result that

$$\sum_{i=1}^{m} |\lambda_{\tau(i)} - \widetilde{\lambda}_i|^2 \leq \frac{1}{1 - \rho_F^2} \frac{||R||_F^2}{\delta_2^2},$$

and from (1.6) we have $\rho_F = ||R||_F / \delta_2 < 1$ (see [15]).

Lemma 2.2. *Suppose that* $X = (X_1, X_2) \in \mathcal{C}^{n\times n}$ *is a nonsingular matrix, where* $X_1 \in \mathcal{C}^{n\times m}$, *and its inverse has the block form* (1.3). *Then for 2-or F-norm* $||\cdot||$ *and any full column matrix* $\widetilde{X}_1 \in \mathcal{C}^{n\times m}$,

$$\|\sin\Theta(X_1,\widetilde{X}_1)\| \le \|Y_2^\dagger\|_2\|\widetilde{X}_1^\dagger\|_2\|Y_2^*\widetilde{X}_1\|, \tag{2.6}$$

where by $M^\dagger$ *we denote the Moore-Penrose inverse of a matrix* M.

Proof. Let $Y_2 = Q_1R_1$ and $\widetilde{X}_1 = \widetilde{Q}_1\widetilde{R}_1$ be QR decompositions, where Q_1 and $\widetilde{Q}_1$ have orthonormal columns, and R_1 and $\widetilde{R}_1$ are nonsingular. Then $P = Q_1Q_1^*$ and $\widetilde{P} = \widetilde{Q}_1\widetilde{Q}_1^*$ are a orthogonal projectors on to $\Re(X_1)^\perp$ and $\Re(\widetilde{X}_1)$, respectively. By (2.1), we have

$$\begin{aligned}\|\sin\Theta(X_1,\widetilde{X}_1)\| &= \|Q_1Q_1^*\widetilde{Q}_1\widetilde{Q}_1^*\| = \|Q_1^*\widetilde{Q}_1\|\\ &= \|R_1^{-*}Y_2^*\widetilde{X}_1\widetilde{R}_1^{-1}\|\\ &\le \|R_1^{-1}\|_2\|\widetilde{R}_1^{-1}\|_2\|Y_2^*\widetilde{X}_1\|\\ &= \|Y_2^\dagger\|_2\|\widetilde{X}_1^\dagger\|_2\|Y_2^*\widetilde{X}_1\|,\end{aligned}$$

which completes the proof. □

By Lemma 2.2 and an analogical proof as Theorem 2.1 we have the following result.

Corollary 2.1. *In the notation of Theorem* 2.1. *If* $\delta_2 > 0$, *then we have*

$$\|\sin\Theta(Q_1,X_1)\|_F^2 \le \|Y_2^\dagger\|_2^2\|Q_1^\dagger\|_2^2\kappa^2(Z)\frac{||\widetilde{Q}_1^*X_2||_2^2}{||\widetilde{Q}_1^*X_2||_2^2+\sigma_{\min}^2(\widetilde{Q}_1^*X_1)}\frac{||X^{-1}R||_F^2}{\delta_2^2}, \tag{2.7}$$

where δ_2 *is given in* (1.2).

Remark 2.2. If A is Hermitian and Q_1 has orthonormal columns, then (2.7) reduces to

$$\|\sin\Theta(Q_1,X_1)\|_F \le \frac{\|\cos\Theta(Q_1,X_1)\|_2}{\delta_2}||R||_F. \tag{2.8}$$

In particular, let $\mu = \rho(y) = y^*Ay$, $r(y) = Ay - y\mu$ and $\delta = \min\{|\mu-\lambda_i|,\ i = 2,...,n\}$. Then by (2.8) we have

$$\delta^2\sin^2\Theta(x,y) \le (1-\sin^2\Theta(x,y))||r(y)||_F^2.$$

Hence

$$\sin\Theta(x,y) \le \frac{||r(y)||_F}{\sqrt{\delta^2+||r(y)||_F^2}},$$

which is always sharper than the one in [1] (see also [12,17]), i.e.,

$$\sin\Theta(x,y) \le \frac{||r(y)||_F}{\delta}.$$

3. Bounds for the Diagonalizable Matrix Case

In this section we deal with the case that H is a diagonalizable matrix. As we know, Liu gave the bound (1.8) for approximate eigenvalues, however no approximate eigenspace bound has been given. Here we will provide some alternative bounds for approximate eigenvalues and approximate eigenspaces, respectively.

Our first result is the residual bound for approximate eigenspaces.

Theorem 3.1. *Let $A \in \mathcal{C}^{n\times n}$ and $H \in \mathcal{C}^{m\times m}$ be both diagonalizable matrices with the decomposition* (1.1)*, and let Q_1 have full column rank. If $\delta_2 > 0$, then*

$$\|\sin\Theta(Q_1, X_1)\|_F \leq \|Q_1^\dagger\|_2 \kappa(Z)\kappa(Y_2)\frac{||R||_F}{\delta_2},$$

where $\kappa(Y_2) = \|Y_2\|_2\|Y_2^\dagger\|_2$. Moreover, if Q_1 has orthonormal columns, then

$$\|\sin\Theta(Q_1, X_1)\|_F \leq \kappa(Z)\kappa(Y_2)\frac{||R||_F}{\delta_2},$$

Proof. By (2.2) we have

$$\|\Lambda_1 Y_1^* Q_1 Z - Y_1^* Q_1 Z\widetilde{\Lambda}\|_F^2 + \|\Lambda_2 Y_2^* Q_1 Z - Y_2^* Q_1 Z\widetilde{\Lambda}\|_F^2 = ||X^{-1}RZ||_F^2. \tag{3.1}$$

Thus

$$\sum_{i\in<n-m>, j\in<m>} |\left(\Lambda_2 Y_2^* Q_1 Z - Y_2^* Q_1 Z\widetilde{\Lambda}\right)_{ij}|^2 \geq \delta_2^2 \sum_{i\in<n-m>, j\in<m>} |(Y_2^* Q_1 Z)_{ij}|^2,$$

which implies that

$$\delta_2^2||Y_2^* Q_1||_F^2 \leq ||Z^{-1}||_2^2\|\Lambda_2 Y_2^* Q_1 Z - Y_2^* Q_1 Z\widetilde{\Lambda}\|_F^2. \tag{3.2}$$

By Lemma 2.2, we have

$$\|\sin\Theta(Q_1, X_1)\|_F \leq \|Y_2^\dagger\|_2\|Q_1^\dagger\|_2||Z^{-1}||_2\|Y_2^* Q_1\|_F.$$

which together with (3.2) gives that

$$\frac{\delta_2^2}{\|Y_2^\dagger\|_2^2\|Q_1^\dagger\|_2^2||Z^{-1}||_2^2}\|\sin\Theta(Q_1, X_1)\|_F^2 \leq \|\Lambda_2 Y_2^* Q_1 Z - Y_2^* Q_1 Z\widetilde{\Lambda}\|_F^2. \tag{3.3}$$

By (1.3) and (2.2) we have

$$\Lambda_2 Y_2^* Q_1 Z - Y_2^* Q_1 Z\widetilde{\Lambda} = Y_2^* RZ,$$

combining with (3.3) gives

$$\frac{\delta_2^2}{\|Y_2^\dagger\|_2^2\|Q_1^\dagger\|_2^2||Z^{-1}||_2^2}\|\sin\Theta(Q_1, X_1)\|_F^2 \leq \|Y_2^* RZ\|_F^2,$$

from which one may deduce the desired bound. □

Remark 3.1. Let A and its perturbed matrix $\widetilde{A}$ be diagonalizable, and let

$$Y_2^* A = \Lambda_1 Y_2^*, \qquad \widetilde{A}\widetilde{X}_1 = \widetilde{X}_1\widetilde{\Lambda}_1. \tag{3.4}$$

By Theorem 3.1 we may obtain the bound for $\|\sin\Theta(X_1, \widetilde{X}_1)\|_F$. In fact, by (3.4) we have

$$(A - \widetilde{A})\widetilde{X}_1 = A\widetilde{X}_1 - \widetilde{X}_1\widetilde{\Lambda}_1.$$

Now let $H = \widetilde{\Lambda}_1$, $Q_1 = \widetilde{X}_1$ and $R = (A - \widetilde{A})\widetilde{X}_1$. Then $Z = I$. It follows from Theorem 3.1 that

$$\begin{aligned}\delta_2\|\sin\Theta(X_1, \widetilde{X}_1)\|_F &\leq \|\widetilde{X}_1^\dagger\|_2\kappa(Y_2)||R||_F \\ &= \|\widetilde{X}_1^\dagger\|_2\|\widetilde{X}_1\|_2\kappa(Y_2)||A - \widetilde{A}||_F = \kappa(\widetilde{X}_1)\kappa(Y_2)||E||_F,\end{aligned}$$

where $E = A - \widetilde{A}$ and $\kappa(\widetilde{X}_1) = \|\widetilde{X}_1^\dagger\|_2\|\widetilde{X}_1\|_2$, which is a result of [5].

Theorem 3.2. *In the notation of Theorem* 3.1. *Then there exists a permutation* τ *of* $\langle m\rangle$ *such that*

$$\sigma_{\min}^2(Y_1^*Q_1)\sum_{i=1}^m|\lambda_i-\widetilde{\lambda}_{\tau(i)}|^2\le\kappa^2(Z)||Y_1^*R||_F^2, \tag{3.5}$$

where Y_1 *is given by* (1.3).

Proof. By (2.2) we have

$$\Lambda_1Y_1^*Q_1Z-Y_1^*Q_1Z\widetilde{\Lambda}=Y_1^*RZ.$$

It follows from Lemma 2.1 that there exists a permutation τ of $\langle m\rangle$ such that

$$\sigma_{\min}^2(Y_1^*Q_1Z)\sum_{i=1}^m|\lambda_i-\widetilde{\lambda}_{\tau(i)}|^2\le\|Y_1^*RZ\|_F^2\,,$$

which together with the fact that

$$\sigma_{\min}(Y_1^*Q_1Z)\ge||Z^{-1}||_2^{-1}\sigma_{\min}(Y_1^*Q_1)$$

gives the bound (3.5). This completes the proof of the theorem. □

The bound (3.5) can be expressed as follows:

Corollary 3.1. *In the notation of Theorem* 3.1. *Then*

$$\sqrt{\sum_{i=1}^m|\lambda_i-\widetilde{\lambda}_{\tau(i)}|^2}\le\frac{\|Q_1^\dagger\|_2\kappa(Y_1)\kappa(Z)}{\sqrt{1-\|\sin\Theta(Q_1,Y_2)\|_2^2}}||R||_F. \tag{3.6}$$

Proof. Let $Q_1=\widetilde{Q}_1\widetilde{R}_1$ and $Y_1=\widehat{Q}_1\widehat{R}_1$ be the QR decompositions. Then

$$\begin{aligned}\sigma_{\min}(Y_1^*Q_1)&=\sigma_{\min}(\widehat{R}_1^*\widehat{Q}_1^*\widetilde{Q}_1\widetilde{R}_1)\\&\ge\sigma_{\min}(\widehat{Q}_1^*\widetilde{Q}_1)||\widetilde{R}_1^{-1}||_2^{-1}||\widetilde{R}_1^{-1}||_2^{-1}\\&=\frac{\sigma_{\min}(\widehat{Q}_1^*\widetilde{Q}_1)}{\|Y_1^\dagger\|_2\|Q_1^\dagger\|_2}.\end{aligned}$$

Note that $\sigma_{\min}^2(\widehat{Q}_1^*\widetilde{Q}_1)=1-\|\sin\Theta(Q_1,Y_2)\|_2^2$. Then by (3.7) we have

$$\sigma_{\min}(Y_1^*Q_1)\ge\frac{\sqrt{1-\|\sin\Theta(Q_1,Y_1)\|_2^2}}{\|Y_2^\dagger\|_2\|Q_1^\dagger\|_2},$$

which, together with (3.5), gives the desired bound (3.6). □

Remark 3.2. Let $X_1=X$ and $H=\widetilde{A}$, where $\widetilde{A}=A+E$ is the perturbed matrix of A, and $Q_1=I$. Then $R=A-\widetilde{A}=E$. Hence the bounds (3.5) and (3.6) reduce to

$$\sqrt{\sum_{i=1}^n|\lambda_i-\widetilde{\lambda}_{\tau(i)}|^2}\le\kappa(X)\kappa(Z)||E||_F. \tag{3.7}$$

The perturbation bound (3.7) was obtained by Sun [14] and Zhang [18], which is a generalization of the Hoffman and Wielandt bound.

Remark 3.3. It is difficult to compare the bound (3.5) with (1.8). But the following example illustrates that our bound is sharper. Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$Q_1 = \begin{pmatrix} 1 & -1 \\ 0.5 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 2.1 & 0 \\ 0 & 1.1 \end{pmatrix}.$$

Then

$$R = \begin{pmatrix} -.1 & -.9 \\ 1.95 & 2.9 \\ -1.1 & -.1 \\ -1.1 & -.1 \end{pmatrix}, \quad Z = I$$

A simple calculation gives

$$\frac{||Y_1^* R||_F^2}{\sigma_{\min}^2(Y_1^* Q_1)} = 6.722.$$

However,

$$\frac{\kappa^2(X)||R||_F^2}{\sigma_{\min}^2(Q_1)} = 217.97.$$

4. Approximate Eigenvalue Bounds for General Cases

In this section we consider the third case, i.e., A is Hermitian and Q_1 is column orthonormal, but H is an $m \times m$ matrix. It is known that any matrix can be rewritten as follows:

$$H = U \begin{pmatrix} H_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & H_s \end{pmatrix} U^*, \tag{4.1}$$

where U is unitary, $s \geq 1$.

The following lemma is important to deduce our bound.

Lemma 4.1. ([8]) *Let $\widetilde{H}$ be an $m \times m$ normal matrix and H have the decomposition* (4.1). *Then there exists a permutation τ of $\langle m \rangle$ such that*

$$\sum_{i=1}^{m} |\lambda_{\tau(i)} - \widetilde{\lambda}_i|^2 \leq (m - s + 1)||\widetilde{H} - H||_F^2, \tag{4.2}$$

where $\lambda_i \in \sigma(H)$ and $\widetilde{\lambda}_i \in \sigma(\widetilde{H})$.

Let P_{Q_1} denote the orthonormal projector on to $\Re(Q_1)$. By $P_{Q_1}^{\perp} = I - P_{Q_1}$ we denote the projection complementary to P_{Q_1}.

Theorem 4.1. *Let A be Hermitian and Q_1 have orthonormal columns, and let H be an $m \times m$ matrix with the decomposition* (4.1). *Then there exists a permutation τ of $\langle m \rangle$ such that*

$$\sum_{i=1}^{m} |\lambda_{\tau(i)} - \widetilde{\lambda}_i|^2 \leq (m-s+1)(||(I, P_{Q_1}^{\perp})^T R||_F^2 \tag{4.3}$$

$$\leq 2(m-s+1)||R||_F^2. \tag{4.4}$$

Proof. Let Q_2 be column orthonormal so that $Q = (Q_1, Q_2)$ is unitary, and let $\widetilde{R} = AQ_2 - Q_2(Q_2^* A Q_2)$. Then

$$(R, \widetilde{R}) = AQ - Q \begin{pmatrix} H & 0 \\ 0 & Q_2^* A Q_2 \end{pmatrix}.$$

Left-multiplying by Q on both sides of the above equality gives that

$$Q^*(R, \widetilde{R}) = Q^* A Q - \begin{pmatrix} H & 0 \\ 0 & Q_2^* A Q_2 \end{pmatrix}.$$

Note that $Q_2^* A Q_2$ is Hermitian. Then there exists a unitary matrix V such that

$$Q_2^* A Q_2 = V \begin{pmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{n-m} \end{pmatrix} V^*,$$

where α_i is a real number, $i = 1, ..., n-m$. Let

$$W = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

Since H has the decomposition (4.1), we have

$$\begin{pmatrix} H & 0 \\ 0 & Q_2^* A Q_2 \end{pmatrix} = W \begin{pmatrix} H_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & H_s & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \alpha_1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \alpha_{n-m} \end{pmatrix} W^*.$$

Applying Lemma 4.1 to matrices $Q^* A Q$ and $\text{diag}(H, Q_2^* A Q_2)$ reveals that there exists a permutation τ of $\langle m \rangle$ such that

$$\begin{aligned} \sum_{i=1}^{m} |\lambda_{\tau(i)} - \widetilde{\lambda}_i|^2 &\leq (m-s+1)(||Q^*(R, \widetilde{R})||_F^2 \\ &= (m-s+1)(||Q^* R||_F^2 + ||Q^* \widetilde{R}||_F^2) \\ &= (m-s+1)(||R||_F^2 + ||Q^* \widetilde{R}||_F^2). \end{aligned} \tag{4.5}$$

Note that

$$Q^* \widetilde{R} = \begin{pmatrix} Q_1^* \widetilde{R} \\ Q_2^* \widetilde{R} \end{pmatrix},$$

$Q_2^*\widetilde{R} = Q_2^*(AQ_2 - Q_2Q_2^*AQ_2) = 0$ and $Q_1^*\widetilde{R} = Q_1^*AQ_2 = R^*Q_2$. Then

$$||Q^*\widetilde{R}||_F^2 = ||Q_2^*R||_F^2 = ||P_{Q_1}^{\perp}R||_F^2.$$

Hence

$$\begin{aligned} &||R||_F^2 + ||Q^*\widetilde{R}||_F^2 \\ =& ||R||_F^2 + ||P_{Q_1}^{\perp}R||_F^2 = ||(I, P_{Q_1}^{\perp})^T R||_F^2, \end{aligned}$$

which together with (4.5) gives the bound (4.3).

The bound (4.4) follows from (4.3) and the fact that $||(I, P_{Q_1}^{\perp})^T R||_F^2 \leq 2||R||_F^2$. This completes the proof of the theorem. □

Without any restriction on H, then $s \geq 1$, and hence by Theorem 4.1 we have the following corollary:

Corollary 4.1. *Let A be Hermitian and Q_1 have orthonormal columns, and let H be any $m\times m$ matrix. Then there exists a permutation τ of $\langle m\rangle$ such that*

$$\sqrt{\sum_{i=1}^{m} |\lambda_{\tau(i)} - \widetilde{\lambda}_i|^2} \leq \sqrt{m}||(I, P_{Q_1}^{\perp})^T R||_F \leq \sqrt{2m}||R||_F \tag{4.6}$$

Remark 4.1. If H is Hermitian, then H has the spectral decomposition, and thus $s = m$. In this case the bound (4.4) reduces to (1.5). Taking $Q_1 = I$ and $H = \widetilde{A}$, then the bound (4.3) reduces to the bound (4.2), similarly, the first inequality in the bound (4.6) can reduce to the Sun's bound (see [16]):

$$\sqrt{\sum_{i=1}^{n} |\lambda_{\tau(i)} - \widetilde{\lambda}_i|^2} \leq \sqrt{n}||R||_F.$$

If we do some restriction on Q_1, then we have

Corollary 4.2. *In the notation of Theorem 4.2. If $\Re(Q_1)$ is an invariant subspace of A, then there exists a permutation τ of $\langle m\rangle$ such that*

$$\sqrt{\sum_{i=1}^{m} |\lambda_{\tau(i)} - \widetilde{\lambda}_i|^2} \leq \sqrt{(m-s+1)}||R||_F. \tag{4.7}$$

Proof. Let Q_2 be defined as in the proof of Theorem 4.1. Since Q_1 is an invariant subspace of A, we have $Q_2^*AQ_1 = 0$. By the proof of Theorem 4.1 we have $Q_1^*\widetilde{R} = Q_1^*AQ_2 = 0$, and hence $||Q^*\widetilde{R}||_F^2 = 0$, which together with (4.5) gives the bound (4.7). □

Remark 4.2. Many examples illustrate that the bound (4.7) holds without the restriction on Q_1. However, we can not prove it, which remains open.

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References

[1] C. Davis and W. Kahan, The rotation of eigenvectors by a perturbation III, *SIAM J. Numer. Anal.*, **7** (1970), 1-46.

[2] L. Elsner and S. Friedland, Singular values, doubly stochastic matrices and applications, *Linear Algebra Appl.*, **220** (1995), 161-169.

[3] A.J. Hoffman and H.W. Wielandt, The variation of the spectrum of a normal matrix, *Duke Math. J.*, **20** (1953), 37-39.

[4] I.C.F. Ipsen, Relative perturbation results for the matrix eigenvalues and singular values, *Acta Numerica*, 1998, 151-201.

[5] I.C.F. Ipsen, A note on unifying absolute and relative perturbation bounds, *Linear Algebra Appl.*, **358** (2003), 239-253.

[6] R.C. Li, Relative perturbation theory: (I) Eigenvalue and singular value variations, *SIAM J. Matrix Anal. Appl.*, **19** (1998), 956-982.

[7] R.C. Li, Relative perturbation theory: (II) Eigenspace and singular subspace variations, *SIAM J. Matrix Anal. Appl.*, **20** (1999), 471-492.

[8] W. Li and W.W. Sun, The perturbation bounds for eigenvalues of normal matrices, *Numer. Linear Algebra. Appl.*, **12** (2005), 89-94.

[9] W. Li and W.W. Sun, Combined perturbation bounds I: Eigensystems and singular value decomposition, *SIAM J. Matrix Anal. Appl.*, **29** (2007), 643-655.

[10] W. Li and W.W. Sun, Combined perturbation bounds II: Polar decompositions, *Science in China, Series A: Mathematics*, **50** (2007), 1339-1346.

[11] X.G. Liu, The perturbation on matrix eigenvalues associated with invariant subspaces, *J. of Ocean Univ. of Qindao*, **19** (1989), 91-95.

[12] B.N. Pattlet, The Symmetry Eigenvalue Problems, Prentince-Hall, Englewood Cliffs, NJ, 1980.

[13] G. Stewart and J.G. Sun, Matrix Perturbation Theory, Academic Press, Boston, 1990.

[14] J.G. Sun, On the perturbation of the eigenvalues of a normal matrix, *Math. Numer. Sinica*, **6** (1984), 334-336, (in Chinese).

[15] J.G. Sun, Eigenvalues of Rayleigh quotient, *Numer. Math.*, **59** (1991), 603-614.

[16] J.G. Sun, On the variation of the spectrum of a normal matrix, *Linear Algebra Appl.*, **246** (1996), 215-223.

[17] J.G. Sun, Matrix Perturbation Analysis, Science Press, Beijing, 2001, (in Chinese).

[18] Z.Y. Zhang, On the perturbation of the eigenvalues of a non-defective matrix, *Math. Numer. Sinica*, **6** (1986), 106-108, (in Chinese).