Journal of Computational Mathematics
Vol.30, No.1, 2012, 24–33.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1110-m11si09

LINEAR CONVERGENCE OF THE LZI ALGORITHM FOR WEAKLY POSITIVE TENSORS*

Liping Zhang
Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
Email: lzhang@math.tsinghua.edu.cn
Liqun Qi and Yi Xu
Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong
Email: maqilq@polyu.edu.hk yi.xu1983@gmail.com

Abstract

We define weakly positive tensors and study the relations among essentially positive tensors, weakly positive tensors, and primitive tensors. In particular, an explicit linear convergence rate of the Liu-Zhou-Ibrahim(LZI) algorithm for finding the largest eigenvalue of an irreducible nonnegative tensor, is established for weakly positive tensors. Numerical results are given to demonstrate linear convergence of the LZI algorithm for weakly positive tensors.

Mathematics subject classification: 74B99, 15A18, 15A69.
Key words: Irreducible nonnegative tensor, Weakly positive tensor, Largest eigenvalue, Linear convergence.

1. Introduction

Consider an m-order n-dimensional square tensor $\mathcal{A}$ consisting of n^m entries in the real field R:

$$\mathcal{A} = (A_{i_1 \cdots i_m}), \quad A_{i_1 \cdots i_m} \in R, \quad 1 \le i_1, \cdots, i_m \le n.$$

Tensors play an important role in physics, engineering, and mathematics. There are many application domains of tensors such as data analysis and mining, information science, image processing, and computational biology [16].

In 2005, Qi [12] introduced the notion of eigenvalues of higher-order tensors, and studied the existence of both complex and real eigenvalues and eigenvectors. Independently, in the same year, Lim [7] also defined eigenvalues and eigenvectors but restricted them to be real. Unlike matrices, eigenvalue problems for tensors are nonlinear. Nevertheless, eigenvalue problems of higher-order tensors have become an important part of a new applied mathematics branch, numerical multilinear algebra, and found a wide range of practical applications, for more references, see [7, 13–15]. The following definition was first introduced by Qi [12] when m is even and $\mathcal{A}$ is symmetric. Chang, Pearson, and Zhang [3] extended it to general square tensors.

Definition 1.1. *Let C be the complex field. A pair $(\lambda, x) \in C \times (C^n \backslash \{0\})$ is called an eigenvalue-eigenvector pair of $\mathcal{A}$, if they satisfy:*

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{1.1}$$

* Received February 14, 2011 / Revised version received June 9, 2011 / Accepted June 19, 2011 / Published online January 9, 2012 /

where n-dimensional column vectors $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are defined as

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2,\ldots,i_m=1}^{n} A_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1\le i\le n} \qquad \text{and} \qquad x^{[m-1]} := \left(x_i^{m-1} \right)_{1\le i\le n},$$

respectively.

Recently, the largest eigenvalue problem for nonnegative tensors attracted much attention. Chang, Pearson, and Zhang [2] generalized the Perron-Frobenius theorem from nonnegative matrices to irreducible nonnegative tensors. It has numerous applications include multilinear pagerank [7], spectral hypergraph theory [1], and higher-order Markov chains [10]. Pearson [11] introduced the notion of essentially positive tensors and proved that the unique positive eigenvalue is real geometrically simple when the tensor is essentially positive with even order. Here, "real geometrically simple" means that the corresponding real eigenvector is unique up to a scaling constant. Ng, Qi, and Zhou [10] proposed an iterative method for finding the largest eigenvalue of an irreducible nonnegative tensor. The NQZ method in [10] is efficient but it is not always convergent for irreducible nonnegative tensors. Chang, Pearson and Zhang [4] introduced primitive tensors. An essentially positive tensor is a primitive tensor, and a primitive tensor is an irreducible nonnegative tensor, but not vice versa. They established convergence of the NQZ method for primitive tensors. Liu, Zhou, and Ibrahim [9] modified the NQZ method such that the modified algorithm is always convergent for finding the largest eigenvalue of an irreducible nonnegative tensor. Yang and Yang [17] generalized the weak Perron-Frobenius theorem to general nonnegative tensors. Friedland, Gaubert and Han [5] pointed out that the Perron-Frobenius theorem for nonnegative tensors has a very close link with the Perron-Frobenius theorem for homogeneous monotone maps. They introduced weakly irreducible nonnegative tensors and established the Perron-Frobenius theorem for such tensors.

The main contributions of this paper are to introduce the notion of weakly positive tensors, to give the relations among essentially positive tensors, weakly positive tensors, and primitive tensors, and to establish an explicit linear convergence rate of the LZI algorithm in [9] for weakly positive tensors. The linear convergence result is significant for the theory of nonnegative tensors as algorithms for general tensors cannot be so efficient [6, 8].

In Section 3, we introduce weakly positive tensors, and give the relations among essentially positive tensors, weakly positive tensors, and primitive tensors. These tensors are all irreducible nonnegative tensors. An essentially positive tensor is both a primitive tensor and a weakly positive tensor, but not vice versa. A primitive tensor may not be a weakly positive tensor. A weakly positive tensor may also not be a primitive tensor. We give a figure to describe their relationships.

We then establish an explicit linear convergence rate of the LZI algorithm for weakly positive tensors in Section 4. We also show that the LZI algorithm terminates after at most K iterations to produce an ε-approximation of the largest eigenvalue of a weakly positive tensor, where K is a constant related to the fixed accurate tolerance ε. Numerical results are given in Section 5 to demonstrate linear convergence of the LZI algorithm for weakly positive tensors.

2. Preliminaries

Let us first recall some definitions on tensors. An m-order n-dimensional tensor $\mathcal{A}$ is called nonnegative if $A_{i_1\cdots i_m} \ge 0$. We call an m-order n-dimensional tensor the unit tensor, denoted by $\mathcal{I}$, if its entries are $\delta_{i_1\dots i_m}$ with $\delta_{i_1\dots i_m}=1$ if and only if $i_1=\cdots=i_m$ and the others are zero. A tensor $\mathcal{A}$ is called reducible, if there exists a nonempty proper index subset $I\subset\{1,2,\cdots,n\}$ such that

$$A_{i_1\cdots i_m}=0,\quad \forall i_1\in I,\quad \forall i_2,\cdots,i_m\notin I.$$

If $\mathcal{A}$ is not reducible, then we say that $\mathcal{A}$ is irreducible.

We now recall some preliminary results. In the following, we state the Perron-Frobenius theorem for nonnegative tensors given in [2, Theorem 1.4].

Theorem 2.1. *If $\mathcal{A}$ is an irreducible nonnegative tensor of order m and dimension n, then there exist $\lambda_0>0$ and $x_0\in R^n, x_0>0$ such that*

$$\mathcal{A}x_0^{m-1}=\lambda_0 x_0^{[m-1]}.$$

Moreover, if λ is an eigenvalue with a nonnegative eigenvector of $\mathcal{A}$, then $\lambda=\lambda_0$. If λ is an eigenvalue of $\mathcal{A}$, then $|\lambda|\le\lambda_0$.

Let $P=\{x\in \mathrm{R}^n:\, x_i\ge 0,\, 1\le i\le n\}$ and $\mathrm{int}(P)=\{x\in\mathrm{R}^n:\, x_i>0,\, 1\le i\le n\}$. The minimax theorem was given in [2] for irreducible nonnegative tensors as follows.

Theorem 2.2. *Assume that $\mathcal{A}$ is an irreducible nonnegative tensor of order m and dimension n. Then*

$$\min_{x\in\mathrm{int}(P)}\max_{1\le i\le n}\frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}=\lambda_0=\max_{x\in\mathrm{int}(P)}\min_{1\le i\le n}\frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}},$$

where λ_0 is the unique positive eigenvalue corresponding to the positive eigenvector.

Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor. Its associated nonlinear map $T_{\mathcal{A}}:P\to P$ was defined in [4] as

$$T_{\mathcal{A}}x=\left(\mathcal{A}x^{m-1}\right)^{[\frac{1}{m-1}]}.$$

The tensor $\mathcal{A}$ is called an *essentially positive tensor* [11] if

$$A_{i\,j\cdots j}>0,\quad i,j\in\{1,2,\cdots,n\}.$$

$\mathcal{A}$ is called a *primitive tensor* [4] if there exists a positive integer ℓ such that

$$T_{\mathcal{A}}^{\ell}(P\backslash\{0\})\subset\mathrm{int}(P).$$

It was proved [4, 11] that an essentially positive tensor is a primitive tensor, and a primitive tensor is an irreducible nonnegative tensor, but not vice versa.

Let $\mathcal{A}$ be an irreducible nonnegative tensor of order m and dimension n. Based on Theorems 2.1 and 2.2, the LZI algorithm presented in [9] works as follows: Choose $x^{(0)}\in\mathrm{int}(P)$ and $\rho>0$. Let $\mathcal{B}=\mathcal{A}+\rho\mathcal{I}$ and let $y^{(0)}=\mathcal{B}\left(x^{(0)}\right)^{m-1}$. For $k=1,2,\cdots$, compute

$$x^{(k)}=\frac{\left(y^{(k-1)}\right)^{[\frac{1}{m-1}]}}{\left\|\left(y^{(k-1)}\right)^{[\frac{1}{m-1}]}\right\|},\quad y^{(k)}=\mathcal{B}\left(x^{(k)}\right)^{m-1},\tag{2.1a}$$

$$\underline{\lambda}_k=\min_{x_i^{(k)}>0}\frac{\left(y^{(k)}\right)_i}{\left(x_i^{(k)}\right)^{m-1}},\quad \overline{\lambda}_k=\max_{x_i^{(k)}>0}\frac{\left(y^{(k)}\right)_i}{\left(x_i^{(k)}\right)^{m-1}}.\tag{2.1b}$$

It is shown in [9, Theorem 2.5] that the obtained sequences $\{\underline{\lambda}_k\}$ and $\{\overline{\lambda}_k\}$ converge to λ, where λ is the largest eigenvalue of $\mathcal{B}$. Furthermore, $\lambda - \rho$ is the largest eigenvalue of $\mathcal{A}$.

In this paper we intend to establish an explicit linear convergence rate of the LZI algorithm. For this purpose, we introduce a new class of nonnegative tensors, i.e., weakly positive tensors, in the next section.

3. Weakly Positive Tensors

In this section we introduce the notion of weakly positive tensors, and give the relations among essentially positive tensors, weakly positive tensors, and primitive tensors.

Definition 3.1. *Let $\mathcal{A}$ be a nonnegative tensor of order m and dimension n. $\mathcal{A}$ is weakly positive if*

$$A_{i\,j\ldots j} > 0 \quad \text{for } i \neq j \text{ and } i, j \in \{1, 2, \cdots, n\}.$$

Clearly, essentially positive tensors are weakly positive, but not vice versa..

Theorem 3.1. *If a nonnegative tensor $\mathcal{A}$ is weakly positive, then it is irreducible.*

Proof. Suppose $\mathcal{A}$ is a reducible tensor, then there exists a nonempty proper index subset $I \subset \{1, 2, \cdots, n\}$ such that

$$A_{i_1 \cdots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \cdots, i_m \notin I.$$

In particular, for $i \in I$ and $j \notin I$ we have $A_{i\,j\ldots j} = 0$. This is a contradiction since $\mathcal{A}$ is weakly positive.

The following example shows the converse of Theorem 3.1 is false.

Example 3.1. Let a 3-order 3-dimensional tensor $\mathcal{A}$ be defined by $A_{122} = A_{133} = A_{211} = A_{311} = 1$ and zero otherwise.

$\mathcal{A}$ is irreducible but not primitive [4, Example 3.6]. Moreover, it is not weakly positive since $A_{322} = A_{233} = 0$.

By Theorem 3.1, [11, Theorem 3.2] and [4, Theorem 2.7], essentially positive tensors, primitive tensors, and weakly positive tensors are irreducible, but not vice versa.

The following example [4, Example 3.3] shows that there are some irreducible tensors which are both weakly positive and primitive, but not essentially positive.

Example 3.2. Consider the 3-order 3-dimensional tensor $\mathcal{A}$ defined by $A_{122} = A_{133} = A_{211} = A_{233} = A_{311} = A_{322} = 1$ and zero otherwise.

Clearly, $\mathcal{A}$ is weakly positive. It was shown in [4] that $\mathcal{A}$ is primitive but not essentially positive.

Furthermore, there exist some primitive tensors which are not weakly positive, and some weakly positive tensors which are not primitive. These can be seen from the following examples.

Example 3.3. Consider the 3-order 3-dimensional tensor $\mathcal{A}$ defined by $A_{133} = A_{211} = A_{311} = A_{322} = 1$ and zero otherwise.

$\mathcal{A}$ is primitive [4, Example 3.4] but not weakly positive since $A_{122} = A_{233} = 0$.

Example 3.4. Consider the 3-order 2-dimensional tensor $\mathcal{A}$ defined by $A_{122} = A_{211} = 1$ and zero elsewhere.

Clearly, $\mathcal{A}$ is weakly positive but not primitive [9, Example 2.1].

We illustrate the relations of notions of irreducible, weakly positive, primitive, essentially positive on higher-order tensors in Fig. 3.1.

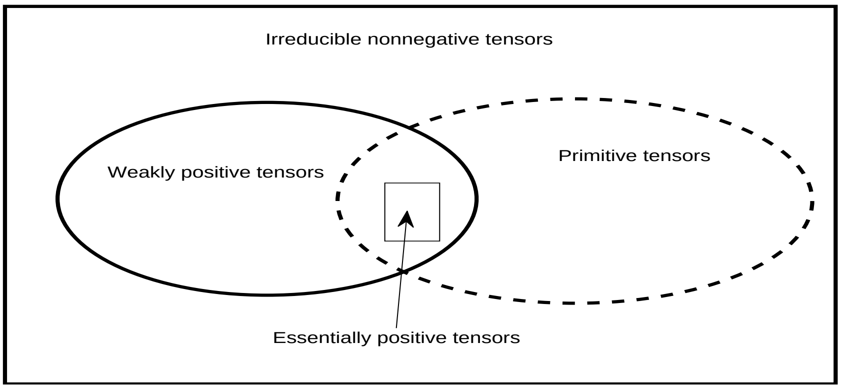

Fig. 3.1. Relations of essentially positive, weakly positive, and primitive tensors

4. Linear Convergence for Weakly Positive Tensors

Clearly, the LZI algorithm is convergent for weakly positive tensors by Theorem 2.5 in [9] and Theorem 3.1. We now establish linear convergence of the LZI algorithm under the weakly positive assumption. We first prove the following lemma.

Lemma 4.1. *Choose $x^{(0)}$ as the vector of ones in the LZI algorithm, then it generates two sequences $\{\underline{\lambda}_k\}$ and $\{\overline{\lambda}_k\}$. Let $\mathcal{B}^{(0)} = \mathcal{B}$ and*

$$S_i^{(0)} = \sum_{i_2,\ldots,i_m=1}^{n} B_{ii_2\ldots i_m}, \quad i = 1, \cdots, n.$$

For $k = 1, \cdots,$ and $i = 1, \cdots, n$, define

$$S_i^{(k)} = \sum_{i_2,\ldots,i_m=1}^{n} B_{ii_2\cdots i_m}^{(k)}$$

where

$$B_{ii_2\cdots i_m}^{(k)} = B_{ii_2\ldots i_m}^{(k-1)} \frac{\left(S_{i_2}^{(k-1)}\right)^{\frac{1}{m-1}} \cdots \left(S_{i_m}^{(k-1)}\right)^{\frac{1}{m-1}}}{S_i^{(k-1)}}. \tag{4.1}$$

Then, for $k = 1, 2, \cdots,$ we have

$$\underline{\lambda}_k = \min_{1\le i\le n} S_i^{(k)}, \qquad \overline{\lambda}_k = \max_{1\le i\le n} S_i^{(k)}. \tag{4.2}$$

Proof. For $k = 0, 1, \cdots,$ denote

$$M^{(k)} = \left\| \left(y^{(k)}\right)^{[\frac{1}{m-1}]} \right\|^{m-1}.$$

Obviously, $y_i^{(0)} = S_i^{(0)}$, $1 \le i \le n$. Thus, it follows the LZI algorithm that

$$x_i^{(1)} = \frac{\left(S_i^{(0)}\right)^{\frac{1}{m-1}}}{\left(M^{(0)}\right)^{\frac{1}{m-1}}}, \quad y_i^{(1)} = \frac{S_i^{(0)} S_i^{(1)}}{M^{(0)}}, \quad x_i^{(2)} = \frac{\left(S_i^{(0)} S_i^{(1)}\right)^{\frac{1}{m-1}}}{\left(M^{(0)} M^{(1)}\right)^{\frac{1}{m-1}}}.$$

According to the framework of the LZI algorithm and (4.1), by induction, we have for $1 \le i \le n$ and $k = 1, 2, \cdots$,

$$x_i^{(k)} = \frac{\left(\prod_{j=0}^{k-1} S_i^{(j)}\right)^{\frac{1}{m-1}}}{\left(\prod_{j=0}^{k-1} M^{(j)}\right)^{\frac{1}{m-1}}},$$

and hence

$$\begin{aligned}
\frac{(\mathcal{B}(x^{(k)})^{m-1})_i}{(x_i^{(k)})^{m-1}} &= \sum_{i_2,\ldots,i_m=1}^{n} B_{ii_2\ldots i_m}^{(0)} \frac{\left(\prod_{j=0}^{k-1} S_{i_2}^{(j)}\right)^{\frac{1}{m-1}} \cdots \left(\prod_{j=0}^{k-1} S_{i_m}^{(j)}\right)^{\frac{1}{m-1}}}{\prod_{j=0}^{k-1} S_i^{(j)}} \\
&= \sum_{i_2,\ldots,i_m=1}^{n} B_{ii_2\ldots i_m}^{(1)} \frac{\left(\prod_{j=1}^{k-1} S_{i_2}^{(j)}\right)^{\frac{1}{m-1}} \cdots \left(\prod_{j=1}^{k-1} S_{i_m}^{(j)}\right)^{\frac{1}{m-1}}}{\prod_{j=1}^{k-1} S_i^{(j)}} \\
&= \cdots = \sum_{i_2,\ldots,i_m=1}^{n} B_{ii_2\ldots i_m}^{(k-1)} \frac{\left(S_{i_2}^{(k-1)}\right)^{\frac{1}{m-1}} \cdots \left(S_{i_m}^{(k-1)}\right)^{\frac{1}{m-1}}}{S_i^{(k-1)}} \\
&= S_i^{(k)},
\end{aligned}$$

which, together with (2.1), implies that (4.2) holds. □

By Theorem 2.5 in [9], we have

$$\lim_{k\to\infty} \left(\overline{\lambda}_k - \underline{\lambda}_k\right) = 0.$$

We will show that $\{\overline{\lambda}_k - \underline{\lambda}_k\}$ linearly converges to zero with an explicit convergence rate when $\mathcal{A}$ is a weakly positive tensor. We need the following lemma.

Lemma 4.2. *For $k = 1, 2, \cdots$,*

$$\begin{aligned}
&B_{i\,i\ldots i}^{(k)} = A_{i\,i\ldots i} + \rho, && i = 1, \cdots, n. \\
&B_{i\,j\ldots j}^{(k)} B_{j\,i\ldots i}^{(k)} = A_{i\,j\ldots j} A_{j\,i\ldots i}, && i \ne j,\ i, j \in \{1, \cdots, n\}.
\end{aligned}$$

Proof. For $i = 1, 2, \cdots, n$, it follows from (4.1) that

$$\begin{aligned}
B_{i\,i\ldots i}^{(k)} &= B_{i\,i\ldots i}^{(k-1)} \frac{\left(S_i^{(k-1)}\right)^{\frac{1}{m-1}} \cdots \left(S_i^{(k-1)}\right)^{\frac{1}{m-1}}}{S_i^{(k-1)}} \\
&= B_{i\,i\ldots i}^{(k-1)} = \cdots = B_{i\,i\ldots i} = A_{i\,i\ldots i} + 1.
\end{aligned}$$

For $i, j \in \{1, \cdots, n\}$ and $i \ne j$, according to (4.1), by a direct computation,

$$\begin{aligned}
B_{i\,j\ldots j}^{(k)} B_{j\,i\ldots i}^{(k)} &= B_{i\,j\ldots j}^{(k-1)} \frac{S_j^{(k-1)}}{S_i^{(k-1)}}\, B_{j\,i\ldots i}^{(k-1)} \frac{S_i^{(k-1)}}{S_j^{(k-1)}} \\
&= B_{i\,j\ldots j}^{(k-1)} B_{j\,i\ldots i}^{(k-1)} = \cdots = B_{i\,j\ldots j} B_{j\,i\ldots i} = A_{i\,j\ldots j} A_{j\,i\ldots i}.
\end{aligned}$$

The following theorem shows that the LZI algorithm has an explicit linear convergence rate under the weakly positive assumption.

Theorem 4.1. *Let $\mathcal{A}$ be a nonnegative tensor of order m and dimension n. Choose $x^{(0)}$ as the vector of ones in the LZI algorithm. Let $\{\underline{\lambda}_k\}$ and $\{\overline{\lambda}_k\}$ be the sequences generated by the LZI algorithm. If $\mathcal{A}$ is weakly positive, then*

$$\overline{\lambda}_{k+1} - \underline{\lambda}_{k+1} \le \alpha \left(\overline{\lambda}_k - \underline{\lambda}_k\right), \quad k = 1, 2, \cdots, \tag{4.3}$$

where

$$\alpha = 1 - \frac{\beta}{\overline{\mu}} \in (0,1), \qquad \overline{\mu} = \rho + \max_{1\le i\le n} \mu_i, \tag{4.4a}$$

$$\beta = \min\left\{ \min_{i,j\in\{1,2,\ldots,n\},\, i\neq j} A_{i\,j\ldots j}, \quad \rho + \min_{1\le i\le n} A_{i\,i\ldots i} \right\}, \tag{4.4b}$$

$$\mu_i = \sum_{i_2,\ldots,i_m=1}^{n} A_{ii_2\ldots i_m}. \tag{4.4c}$$

Proof. By Theorem 3.1 and Theorem 2.4 in [9], $\mathcal{B}$ is irreducible.

By (4.2), without loss of generality, assume that $\overline{\lambda}_{k+1} = S_p^{(k+1)}$ and $\underline{\lambda}_{k+1} = S_q^{(k+1)}$. Then we have

$$\overline{\lambda}_{k+1} - \underline{\lambda}_{k+1} = \sum_{i_2,\ldots,i_m=1}^{n} \left(\frac{B^{(k)}_{pi_2\ldots i_m}}{S_p^{(k)}} - \frac{B^{(k)}_{qi_2\ldots i_m}}{S_q^{(k)}} \right) \left(S_{i_2}^{(k)}\right)^{\frac{1}{m-1}} \cdots \left(S_{i_m}^{(k)}\right)^{\frac{1}{m-1}}. \tag{4.5}$$

Define

$$I = \{(i_2,\ldots,i_m) \,|\, i_2,\ldots,i_m \in \{1,\ldots,n\}\},$$
$$I(k) = \left\{ (i_2,\ldots,i_m) \in I \;\middle|\; \frac{B^{(k)}_{pi_2\ldots i_m}}{S_p^{(k)}} \ge \frac{B^{(k)}_{qi_2\ldots i_m}}{S_q^{(k)}} \right\}.$$

By the definition of $S_i^{(k)}$, for $i = 1, \cdots, n$, we have

$$\sum_{(i_2,\ldots,i_m)\in I(k)} \frac{B^{(k)}_{ii_2\ldots i_m}}{S_i^{(k)}} + \sum_{(i_2,\ldots,i_m)\in I\setminus I(k)} \frac{B^{(k)}_{ii_2\ldots i_m}}{S_i^{(k)}} = 1.$$

Combining the two equalities with $i = p$ and q , we have

$$\sum_{(i_2,\ldots,i_m)\in I(k)} \left(\frac{B^{(k)}_{pi_2\ldots i_m}}{S_p^{(k)}} - \frac{B^{(k)}_{qi_2\ldots i_m}}{S_q^{(k)}} \right) = - \sum_{(i_2,\ldots,i_m)\in I\setminus I(k)} \left(\frac{B^{(k)}_{pi_2\ldots i_m}}{S_p^{(k)}} - \frac{B^{(k)}_{qi_2\ldots i_m}}{S_q^{(k)}} \right). \tag{4.6}$$

Combining (4.2), (4.5) and (4.6), we obtain

$$
\begin{aligned}
&\overline{\lambda}_{k+1} - \underline{\lambda}_{k+1} \\
&= \sum_{(i_2,\ldots,i_m)\in I(k)} \left(\frac{B^{(k)}_{pi_2\ldots i_m}}{S^{(k)}_p} - \frac{B^{(k)}_{qi_2\ldots i_m}}{S^{(k)}_q} \right) \left(S^{(k)}_{i_2}\right)^{\frac{1}{m-1}} \cdots \left(S^{(k)}_{i_m}\right)^{\frac{1}{m-1}} \\
&\qquad + \sum_{(i_2,\ldots,i_m)\in I\setminus I(k)} \left(\frac{B^{(k)}_{pi_2\ldots i_m}}{S^{(k)}_p} - \frac{B^{(k)}_{qi_2\ldots i_m}}{S^{(k)}_q} \right) \left(S^{(k)}_{i_2}\right)^{\frac{1}{m-1}} \cdots \left(S^{(k)}_{i_m}\right)^{\frac{1}{m-1}} \\
&\le \overline{\lambda}_k \sum_{(i_2,\ldots,i_m)\in I(k)} \left(\frac{B^{(k)}_{pi_2\ldots i_m}}{S^{(k)}_p} - \frac{B^{(k)}_{qi_2\ldots i_m}}{S^{(k)}_q} \right) + \underline{\lambda}_k \sum_{(i_2,\ldots,i_m)\in I\setminus I(k)} \left(\frac{B^{(k)}_{pi_2\ldots i_m}}{S^{(k)}_p} - \frac{B^{(k)}_{qi_2\ldots i_m}}{S^{(k)}_q} \right) \\
&= \left(\overline{\lambda}_k - \underline{\lambda}_k\right) \sum_{(i_2,\ldots,i_m)\in I(k)} \left(\frac{B^{(k)}_{pi_2\ldots i_m}}{S^{(k)}_p} - \frac{B^{(k)}_{qi_2\ldots i_m}}{S^{(k)}_q} \right) \\
&\le \left(\overline{\lambda}_k - \underline{\lambda}_k\right) \left(1 - \frac{\Delta_1 + \Delta_2}{\overline{\mu}}\right),
\end{aligned}
\tag{4.7}
$$

where

$$
\Delta_1 = \sum_{(i_2,\ldots,i_m)\in I\setminus I(k)} B^{(k)}_{pi_2\ldots i_m}, \quad \Delta_2 = \sum_{(i_2,\ldots,i_m)\in I(k)} B^{(k)}_{qi_2\ldots i_m}.
$$

Since $\mathcal{A}$ is weakly positive, we have

$$
0 < \beta < \overline{\mu} \quad \Longrightarrow \quad 0 < \alpha < 1. \tag{4.8}
$$

We now consider the subindex array $(p,\ldots,p) \in I$. If $(p,\ldots,p) \in I\setminus I(k)$, then the summation Δ_1 must include $B^{(k)}_{p\,p\ldots p}$. If $(p,\ldots,p) \in I(k)$, there are two possibilities:

(i) $(q,\ldots,q) \in I(k)$. In this case, the summation Δ_2 must include $B^{(k)}_{q\,q\ldots q}$.

(ii) $(q,\ldots,q) \in I\setminus I(k)$. In this case, the summation Δ_1 must include $B^{(k)}_{p\,q\ldots q}$ and the summation Δ_2 must include $B^{(k)}_{q\,p\ldots p}$.

From the above discussion, by Lemma 4.2, we obtain

$$
\begin{aligned}
\Delta_1 + \Delta_2 &\ge \min\left\{ B^{(k)}_{p\,p\ldots p},\, B^{(k)}_{q\,q\ldots q},\, B^{(k)}_{p\,q\ldots q} + B^{(k)}_{q\,p\ldots p} \right\} \\
&\ge \min\left\{ A_{p\,p\ldots p} + \rho,\, A_{q\,q\ldots q} + \rho,\, 2\sqrt{A_{p\,q\ldots q}\, A_{q\,p\ldots p}} \right\} \ge \beta.
\end{aligned}
\tag{4.9}
$$

Combining (4.7), (4.8) and (4.9), we obtain the inequality (4.3) for $k = 1, 2, \cdots$. □

Corollary 4.1. *Let α and $\overline{\mu}$ be defined by (4.4), and let $\underline{\mu} = \rho + \min_{1\le i\le n} \mu_i$. Let $\varepsilon > 0$ be a sufficiently small number. If $\mathcal{A}$ is weakly positive, then the LZI algorithm terminates in at most*

$$
K = \left\lceil \frac{\log\left(\frac{\varepsilon}{\overline{\mu} - \underline{\mu}}\right)}{\log(\alpha)} \right\rceil + 1 \tag{4.10}
$$

iterations with $\overline{\lambda}_K - \underline{\lambda}_K < \varepsilon$.

Table 5.1: Output of the LZI Algorithm for $\mathcal{A}1$, $\mathcal{A}2$ and $\mathcal{A}3$ with $m = 3$

Tensor	Dimension	Eig	No. Iter	Ratio
$\mathcal{A}1$	$n = 5$	2	18	0.3333, 0.3333, 0.3333, 0.3333, 0.3333
	$n = 10$	3	30	0.5000, 0.5000, 0.5000, 0.5000, 0.5000
	$n = 20$	4.3589	45	0.6268, 0.6268, 0.6268, 0.6268, 0.6268
	$n = 40$	6.2450	66	0.7239, 0.7239, 0.7239, 0.7239, 0.7239
	$n = 80$	8.8882	97	0.7977, 0.7977, 0.7977, 0.7977, 0.7977
$\mathcal{A}2$	$n = 5$	24.5919	19	0.3300, 0.3297, 0.3297, 0.3297, 0.3297
	$n = 10$	103.6157	15	0.1828, 0.1830, 0.1831, 0.1833, 0.1833
	$n = 20$	423.2622	12	0.1144, 0.1143, 0.1143, 0.1142, 0.1142
	$n = 40$	1709	11	0.0899, 0.0898, 0.0898, 0.0898, 0.0898
	$n = 80$	6866	11	0.0802, 0.0802, 0.0802, 0.0802, 0.0802
$\mathcal{A}3$	$n = 5$	1.6717	22	0.3821, 0.4820, 0.2895, 0.5704, 0.2764
	$n = 10$	2.1663	30	0.6571, 0.3774, 0.6650, 0.4017, 0.6179
	$n = 20$	2.7478	39	0.6728, 0.4971, 0.6540, 0.5713, 0.5520
	$n = 40$	3.4611	50	0.6592, 0.6386, 0.6106, 0.7454, 0.6238
	$n = 80$	4.3507	64	0.6700, 0.6671, 0.7421, 0.7494, 0.7440

Proof. By Theorem 4.1, we have for $k = 1, 2, \cdots$,

$$\overline{\lambda}_k - \underline{\lambda}_k \le \alpha^k(\overline{\mu} - \underline{\mu}). \tag{4.11}$$

It follows from (4.10) and $\alpha \in (0, 1)$ that

$$\log(\alpha^K) = K\log(\alpha) < \log(\alpha)\frac{\log\left(\dfrac{\varepsilon}{\overline{\mu} - \underline{\mu}}\right)}{\log(\alpha)} = \log\left(\frac{\varepsilon}{\overline{\mu} - \underline{\mu}}\right),$$

which yields $\alpha^K < \varepsilon/(\overline{\mu} - \underline{\mu})$. This, together with (4.11), implies

$$\overline{\lambda}_K - \underline{\lambda}_K \le \alpha^K(\overline{\mu} - \underline{\mu}) < \varepsilon.$$

This completes the proof. □

5. Numerical Experiments

To demonstrate the linear convergence for weakly positive tensors, we made numerical experiments for the LZI algorithm [9] on some numerical examples with the stop rule $\varepsilon = 10^{-8}$. We consider the following three classes of nonnegative tensors of order $m = 3$ and dimension n:

$\mathcal{A}1$: $A1_{1ii} = A1_{i11} = 1$ for $i = 2, 3, \cdots, n$, and zero elsewhere.

$\mathcal{A}2$: $A2_{ijj} = i + j$ for $i, j = 1, 2, \cdots, n$ and $i \neq j$, and zero elsewhere.

$\mathcal{A}3$: $A3_{1nn} = 1$, $A3_{i11} = 1$ for $i = 2, 3, \cdots, n$, $A3_{nii} = 1$ for $i = 1, 2, \cdots, n-1$, and zero elsewhere.

Note that $\mathcal{A}1$ is irreducible, but not primitive and weakly positive. $\mathcal{A}2$ is primitive and weakly positive, but not essentially positive. $\mathcal{A}3$ is primitive, but not weakly positive. We apply the LZI algorithm to find the largest eigenvalues of $\mathcal{A}1$, $\mathcal{A}2$ and $\mathcal{A}3$ with different dimensions. We summarize the numerical results in the following table, where *No.Iter* denotes the number of iterations, *Eig* denotes the largest eigenvalue, and *Ratio* denotes the ratio of $\overline{\lambda}_{k+1} - \underline{\lambda}_{k+1}$ to $\overline{\lambda}_k - \underline{\lambda}_k$ at the last 5 iterations.

From the last column of Table 1, we see that the LZI algorithm for $\mathcal{A}1$ and $\mathcal{A}2$ is linearly convergent and it is not true for $\mathcal{A}3$. This shows that the assumption that $\mathcal{A}$ is weakly positive is a sufficient condition for the inequality (4.3), but may not be a necessary condition. There may exist a weaker condition to guarantee the linear convergence of the LZI algorithm, but we have not been able to identify it at this moment. We put this as an open question for further study.

Acknowledgments. This first author's work was supported by the National Natural Science Foundation of China (Grant No. 10871113). This second author's work was supported by the Hong Kong Research Grant Council.

References

[1] S.R. Bulò and M. Pelillo, New bounds on the clique number of graphs based on spectral hypergraph theory, in: T. Stützle ed., *Learning and Intelligent Optimization*, Springer Verlag, Berlin, (2009), 45-48.

[2] K.C. Chang, K. Pearson and T. Zhang, Perron Frobenius Theorem for nonnegative tensors, *Commu. Math. Sci.*, **6** (2008), 507-520.

[3] K.C. Chang, K. Pearson and T. Zhang, On eigenvalue problems of real symmetric tensors, *J. Math. Anal. Appl.*, **350**(2009), 416-422.

[4] K.C. Chang, K. Pearson and T. Zhang, Primitivity, the convergence of the NZQ method, and the largest eigenvalue for nonnegative tensors, *SIAM J. Matrix Anal. A.*, **32** (2011), 806-819.

[5] S. Friedland, S. Gaubert and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions, to appear in: *Linear Algebra Appl.*

[6] C. Hillar and L.-H. Lim, Most tensor problems are NP hard, arXiv:0911.1393, 10 Nov 2009.

[7] L.-H. Lim, Singular values and eigenvalues of tensors: a variational approach, in *Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Addaptive Processing (CAMSAP'05)*, Vol. 1, IEEE Computer Society Press, Piscataway, NJ, 2005, 129-132.

[8] C. Ling, J. Nie, L. Qi and Y. Ye, Bi-quadratic optimization over unit spheres and semidefinite programming relaxations, *SIAM J. Optimiz.*, **20** (2009), 1286-1310.

[9] Y. Liu, G. Zhou and N.F. Ibrahim, An always convergent algorithm for the largest eigenvalue of an irreducible nonnegative tensor, *J. Comput. Appl. Math.*, **235** (2010), 286-292.

[10] M. Ng, L. Qi and G. Zhou, Finding the largest eigenvalue of a nonnegative tensor, *SIAM J. Matrix Anal. A.*, **31** (2009), 1090-1099.

[11] K. Pearson, Essentially Positive Tensors, *International Journal of Algebra*, **4** (2010), 421-427.

[12] L. Qi, Eigenvalues of a real supersymmetric tensor,*J. Symb. Comput.*, **40** (2005), 1302-1324.

[13] L. Qi, Eigenvalues and invariants of tensor, *J. Math. Anal. Appl.*, **325** (2007), 1363-1377.

[14] L. Qi, W. Sun and Y. Wang, Numerical multilinear algebra and its applications, *Frontiers Math. China*, **2** (2007), 501-526.

[15] L. Qi, Y. Wang and E.X. Wu, D-eigenvalues of diffusion kurtosis tensor, *J. Comput. Appl. Math.*, **221** (2008), 150-157.

[16] C. Van Loan, Future dirctions in tensor-based computation and modeling, Workshop Report in *Arlington, Virginia at National Science Foundation*, February 20-21, 2009. `http://www.cs.cornell.edu/cv/TenWork/Home.htm`.

[17] Y.N. Yang and Q.Z. Yang, Further results for Perron-Frobenius Theorem for nonnegative tensors, *SIAM J. Matrix Anal. A.*, **31** (2010), 2517-2530.