Journal of Computational Mathematics
Vol.30, No.1, 2012, 2–13. http://www.global-sci.org/jcm
doi:10.4208/jcm.1110-m11si07

STEP-LIKE CONTRAST STRUCTURE OF SINGULARLY PERTURBED OPTIMAL CONTROL PROBLEM*

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Abstract

The existence of step-like contrast structure for a class of singularly perturbed optimal control problem is presented by contrast structure theory. By means of direct scheme of boundary function method, we construct the uniformly valid asymptotic solution for the singularly perturbed optimal control problem. As an application, an example is given to illustrate the main result in this paper.

Mathematics subject classification: 34B15, 34E15.
Key words: Singular perturbation, Optimal control problem, Contrast structure.

1. Introduction

The problem of contrast structure is a singularly perturbed problem whose solutions with both internal transition layers and boundary layers (see, e.g., [1-3]). The significant feature of the solution is that it will vary rapidly in the thin internal layer. The contrast structure has a strong application background. For example, in the study of physics , there are cases that their solutions vary rapidly in the interior of domain. In recent years, the study of contrast structure is one of the hot research topics in the study of singular perturbation theory. More and more scholars begin to pay attention to the contrast structure of variational problem. In [4], [5], the authors consider the contrast structures for the simplest vector variational problem and scalar variational problem. One of the basic difficulties for such a problem is unknown of where an internal transition layer is in advance.

Currently, there are mainly two ways to solve this problem. The first way is through the boundary function method [6]. Usually, this method is applied to necessary or sufficient optimality conditions. The second alternative is through direct scheme of boundary function method, which consists in a direct expansion of the optimal control problem. we will apply the direct scheme to the singularly perturbed optimal control problem. As a result of the scheme, we get a minimizing control sequence, each new control approximation decreases the performance index of the given problem. It should be noted that the direct scheme not only make it easy to obtain the relations for the high-order approximations, but also show the nature of the optimal control problem.

In this present paper, we not only prove the existence of step-like contrast structure for the singularly perturbed optimal control problem, but also construct asymptotic solution to the optimal controller and optimal trajectory.

* Received February 23, 2011 / Revised version received May 4, 2011 / Accepted May 24, 2011 / Published online January 9, 2012 /

2. Problem Formulation

Consider the singularly perturbed optimal control problem

$$\begin{cases} J[u] = \int_0^T f(y,u,t)\,\mathrm{d}t \to \min\limits_u, \\ \mu \dfrac{dy}{dt} = a(t)y + b(t)u, \\ y(0,\mu) = y^0\,, \quad y(T,\mu) = y^T. \end{cases} \tag{2.1}$$

where $\mu > 0$ is a small parameter. The following assumptions are fundamental in the theory for the problem in question.

A_1. Suppose that the function $f(y,u,t)$ is sufficiently smooth on the domain $D = \{(y,u,t)|\ |\ y\ | < A, u \in R, 0 \le t \le T\}$, where A is positive constant.
A_2. Suppose that $f_{uu}(y,u,t) > 0$ on the domain D.

Formally setting $\mu = 0$ in (2.1), we obtain the reduced problem

$$J[\bar{u}] = \int_0^T f(\bar{y},\bar{u},t)\,\mathrm{d}t \to \min_{\bar{u}}, \quad \bar{u} = -b^{-1}(t)a(t)\bar{y}. \tag{2.2}$$

For convenience, problem (2.2) can be written in the following equivalent form

$$J[\bar{u}] = \int_0^T F(\bar{y},t)\mathrm{d}t \to \min_{\bar{y}},$$

where $F(\bar{y},t) = f(\bar{y}, -b^{-1}(t)a(t)\bar{y}, t)$.

A_3. Suppose that there exist two isolated functions $\bar{y} = \varphi_1(t)$, $\bar{y} = \varphi_2(t)$ such that

$$\min_{\bar{y}} F(\bar{y},t) = \begin{cases} F(\varphi_1(t),t) & 0 \le t \le t_0, \\ F(\varphi_2(t),t), & t_0 \le t \le T, \end{cases} \tag{2.3}$$

$$\lim_{t \to t_0^-} \varphi_1(t) \neq \lim_{t \to t_0^+} \varphi_2(t).$$

A_4. Suppose that the transition point t_0 is determined by the following equation

$$F(\varphi_1(t_0),t_0) = F(\varphi_2(t_0),t_0),$$

and satisfies the condition

$$\frac{d}{dt}F(\varphi_1(t_0),t_0) \neq \frac{d}{dt}F(\varphi_2(t_0),t_0).$$

It follows from assumption A_3 that

$$\bar{u}(t) = \begin{cases} \alpha_1(t) = -b^{-1}(t)a(t)\varphi_1(t), & 0 \le t < t_0, \\ \alpha_2(t) = -b^{-1}(t)a(t)\varphi_2(t), & t_0 < t \le T, \end{cases}$$

$$\begin{cases} F_y(\varphi_1(t),t) = 0, & F_{yy}(\varphi_1(t),t) > 0, \quad 0 \le t \le t_0, \\ F_y(\varphi_2(t),t) = 0, & F_{yy}(\varphi_2(t),t) > 0, \quad t_0 \le t \le T. \end{cases} \tag{2.4}$$

Consider the Hamiltonian function

$$H(y,u,\lambda,t)=f(y,u,t)+\lambda\mu^{-1}\big[a(t)y+b(t)u\big],$$

where λ is Lagrange multiplier.

The necessary optimality conditions imply that

$$\begin{cases}\mu y'=a(t)y+b(t)u,\\ \lambda'=-f_y(y,u,t)-\lambda\mu^{-1}a(t),\\ \mu f_u(y,u,t)+\lambda(t)b(t)=0,\\ y(0,\mu)=y^0,\quad y(T,\mu)=y^T.\end{cases} \tag{2.5}$$

From (2.5), we can obtain the following singularly perturbed boundary value problem

$$\begin{cases}\mu y'=a(t)y+b(t)u,\\ \mu u'=g_1(y,u,t)+\mu g_2(y,u,t),\\ y(0,\mu)=y^0,\quad y(T,\mu)=y^T,\end{cases} \tag{2.6}$$

where

$$\begin{aligned}g_1&=b(t)f_{uu}^{-1}f_y-a(t)f_{uu}^{-1}f_u-f_{uu}^{-1}f_{uy}\big(a(t)y+b(t)u\big),\\ g_2&=b^{-1}(t)b'(t)f_{uu}^{-1}f_u-f_{uu}^{-1}f_{ut}.\end{aligned}$$

Nonlinear problem of type (2.6) was considered in [6], in which the existence of solution with step-like contrast structure was shown. By means of the result as described in [6], we show the existence of optimal trajectory with step-like contrast structure.

Now, we state the main result in [6], which we will use in the proofs of our main results.

Theorem 2.1. *Consider the following boundary value problem*

$$\begin{cases}\mu\dfrac{dy}{dt}=F(y,z,t,\mu),\quad \mu\dfrac{dz}{dt}=G(y,z,t,\mu),\\ y(0,\mu)=y^0,\ y(T,\mu)=y^T.\end{cases} \tag{2.7}$$

Suppose that the following assumptions hold:

B_1. *The reduced system*

$$F(\bar y,\bar z,t,0)=0,\quad G(\bar y,\bar z,t,0)=0,$$

has two isolated roots $(\varphi_1(t),\psi_1(t))$ *and* $(\varphi_2(t),\psi_2(t))$.

B_2. *In the phase plane* $(\tilde y,\tilde z)$, *the points* $M_1(\varphi_1(\bar t),\psi_1(\bar t))$ *and* $M_2(\varphi_2(\bar t),\psi_2(\bar t))$ *are stationary saddle points for the associated system*

$$\frac{d\tilde y}{d\tau}=F(\tilde y,\tilde z,\bar t,0),\quad \frac{d\tilde z}{d\tau}=G(\tilde y,\tilde z,\bar t,0), \tag{2.8}$$

where $\bar t$ *is a parameter, and system (2.8) has a first integral* $\Omega_i(\tilde y,\tilde z,\bar t)=\Omega_i(\varphi_i(\bar t),\psi_i(\bar t),\bar t)$, *which passes through* $M_i, i=1,2$.

B_3. *The equations* $\Omega_i(\tilde y,\tilde z,\bar t)=\Omega_i(\varphi_i(\bar t),\psi_i(\bar t),\bar t)$ *are solvable with respect to* $\tilde z$:

$$S_{M_1}:\quad \tilde z^{(-)}=V(\tilde y,\varphi_1(\bar t),\psi_1(\bar t),\bar t),$$

$$S_{M_2}: \quad \tilde{z}^{(+)} = V(\tilde{y}, \varphi_2(\bar{t}), \psi_2(\bar{t}), \bar{t}).$$

B_4. *The equation* $H(\bar{t}) = \tilde{z}^{(+)} - \tilde{z}^{(-)}$ *has a solution* $\bar{t} = t_0 \in (0, T)$*, such that* $\frac{d}{dt}H(t_0) \neq 0$. *Then the boundary value problem* (2.7) *has a step-like contrast structure solution satisfying the limiting relations*

$$\lim_{\mu\to 0} y(t,\mu) = \begin{cases} \varphi_1(t), & t < t_0, \\ \varphi_2(t), & t > t_0, \end{cases} \quad \lim_{\mu\to 0} z(t,\mu) = \begin{cases} \psi_1(t), & t < t_0, \\ \psi_2(t), & t > t_0. \end{cases}$$

3. Existence of Step-Like Contrast Structure

As mentioned above, problem (2.6) is a special case of the more general problem (2.7). Therefore, under suitable conditions, the extremal trajectory (the solution to the system of Euler equations (2.6)) contains a step-like contrast structure.

It is easy to see that the associated system for (2.6) can be written as

$$\begin{cases} \dfrac{du}{d\tau} = b(\bar{t}) f_{uu}^{-1} f_y - a(\bar{t}) f_{uu}^{-1} f_u - f_{uu}^{-1} f_{uy}\big(a(\bar{t})y + b(\bar{t})u\big), \\ \dfrac{dy}{d\tau} = a(\bar{t})y + b(\bar{t})u, \end{cases} \tag{3.1}$$

where $\bar{t} \in [0, T]$ is a parameter.

Now we will state and prove some useful lemmas, which will be used to prove our main results. We begin with the following lemma.

Lemma 3.1. *Suppose that* A_1 -A_4 *hold. Then associated system* (3.1) *has two equilibria* $M_i(\varphi_i(\bar{t}), \alpha_i(\bar{t}))$, $i = 1, 2$ *, which are both saddle points.*

Proof. Let

$$\begin{aligned} H(y,u,\bar{t}) &= b(\bar{t}) f_{uu}^{-1} f_y - a(\bar{t}) f_{uu}^{-1} f_u - f_{uu}^{-1} f_{uy}\big(a(\bar{t})y + b(\bar{t})u\big), \\ G(y,u,\bar{t}) &= a(\bar{t})y + b(\bar{t})u. \end{aligned}$$

Obviously, $M_i\big(\varphi_i(\bar{t}), \alpha_i(\bar{t})\big)$, $i = 1, 2$ are two isolated solutions of the reduced system

$$H(y,u,\bar{t}) = 0, \quad G(y,u,\bar{t}) = 0.$$

Moreover, the characteristic equation of the system (3.1) is given by

$$\lambda^2 - a^2(\bar{t}) - b^2(\bar{t})\left(\bar{f}_{uu}^{-1}\bar{f}_{yy} - 2b^{-1}(\bar{t})a(\bar{t})\bar{f}_{uu}^{-1}\bar{f}_{uy}\right) = 0,$$

where $\bar{f}_{uu}^{-1}$, $\bar{f}_{yy}$, $\bar{f}_{uy}$ are calculated in $\big(\varphi_i(\bar{t}), \alpha_i(\bar{t}), \bar{t}\big)$, $i = 1, 2$. Using assumption (2.4), we obtain

$$\lambda^2 = a^2(\bar{t}) + b^2(\bar{t})\left(\bar{f}_{uu}^{-1}\bar{f}_{yy} - 2b^{-1}(\bar{t})a(\bar{t})\bar{f}_{uu}^{-1}\bar{f}_{uy}\right) > 0.$$

Hence, in the phase plane (y, u), $M_i\big(\varphi_i(\bar{t}), \alpha_i(\bar{t})\big)$, $i = 1, 2$ are both saddle points. □

Lemma 3.2. *For fixed $\bar{t} \in [0,T]$, associated system* (3.1) *has a first integral*

$$\Big(a(\bar{t})y + b(\bar{t})u\Big) f_u(y,u,\bar{t}) - b(\bar{t}) f(y,u,\bar{t}) = C, \tag{3.2}$$

where C is a constant.

Proof. Let $y' = \dfrac{dy}{d\tau}$, $u' = \dfrac{du}{d\tau}$. Then the first equation in (3.1) can be written as

$$f_{uu}(y,u,\bar{t})u' = b(\bar{t}) f_y(y,u,\bar{t}) - a(\bar{t}) f_u(y,u,\bar{t}) - f_{uy}\big(a(\bar{t})y + b(\bar{t})u\big). \tag{3.3}$$

Using the second equation of (3.1) , we get

$$f_{uu}(y,u,\bar{t})u' - b(\bar{t}) f_y(y,u,\bar{t}) + a(\bar{t}) f_u(y,u,\bar{t}) + f_{uy} y' = 0. \tag{3.4}$$

In view of $y'' = a(\bar{t})y' + b(\bar{t})u'$, we obtain

$$\frac{d}{d\tau}\Big(y' f_u(y,u,\bar{t}) - b(\bar{t}) f(y,u,\bar{t})\Big) = 0.$$

Therefore, the first integral for (3.1) is given by (3.2). □

Lemma 3.3. *Suppose that A_1-A_2 and $u \neq -a(\bar{t})b^{-1}(\bar{t})y$ hold. Then, for fixed $\bar{t} \in [0,T]$, the first integral* (3.2) *is solvable with respect to u.*

Proof. Let

$$g(y,u,\bar{t}) = \Big(a(\bar{t})y + b(\bar{t})u\Big) f_u(y,u,\bar{t}) - b(\bar{t}) f(y,u,\bar{t}) - C.$$

Obviously

$$\begin{aligned} g_u(y,u,\bar{t}) &= b(\bar{t}) f_u(y,u,\bar{t}) + \Big(a(\bar{t})y + b(\bar{t})u\Big) f_{uu}(y,u,\bar{t}) - b(\bar{t}) f_u(y,u,\bar{t}) \\ &= \Big(a(\bar{t})y + b(\bar{t})u\Big) f_{uu}(y,u,\bar{t}) \neq 0. \end{aligned}$$

By the implicit function theorem , the equation $g(y,u,\bar{t}) = 0$ is solvable with respect to u:

$$u = h(y,\bar{t},C), \quad (y,\bar{t}) \in D_1, \tag{3.5}$$

where $D_1 = \{(y,t) | |y| \leq A, 0 \leq \bar{t} \leq T\}$.

Let us continue the verification of the assumptions of Theorem 2.1. Obviously, there exist two separate orbits S_{M_1} and S_{M_2} that pass through the saddle points M_1 and M_2, which satisfy the equations

$$S_{M_1}: \quad (a(\bar{t})y + b(\bar{t})u) f_u(y,u,\bar{t}) - b(\bar{t}) f(y,u,\bar{t}) = -b(\bar{t}) f(\varphi_1(\bar{t}), \alpha_1(\bar{t}), \bar{t}), \tag{3.6a}$$

$$S_{M_2}: \quad (a(\bar{t})y + b(\bar{t})u) f_u(y,u,\bar{t}) - b(\bar{t}) f(y,u,\bar{t}) = -b(\bar{t}) f(\varphi_2(\bar{t}), \alpha_2(\bar{t}), \bar{t}). \tag{3.6b}$$

It follows from Lemma 3.3 that

$$u^{(-)}(\tau,\bar{t}) = h^{(-)}(y^{(-)},\bar{t},\varphi_1(\bar{t})), \quad u^{(+)}(\tau,\bar{t}) = h^{(+)}(y^{(+)},\bar{t},\varphi_2(\bar{t})). \tag{3.7}$$

Let

$$H(\bar{t}) = u^{(-)}(0,\bar{t}) - u^{(+)}(0,\bar{t}) = h^{(-)}(y^{(-)}(0),\bar{t},\varphi_1(\bar{t})) - h^{(+)}(y^{(+)}(0),\bar{t},\varphi_2(\bar{t})),$$

where

$$y^{(-)}(0) = y^{(+)}(0) = \frac{1}{2}(\varphi_1(\bar{t}) + \varphi_2(\bar{t})) = \beta(\bar{t}).$$

Lemma 3.4. *Suppose that A_1-A_4 hold. Then, we get*

$$a(\bar{t})+b(\bar{t})h_y(\varphi_i(\bar{t}),\bar{t})=\pm\sqrt{\big(b^2(\bar{t})f_{yy}-2a(\bar{t})b(\bar{t})f_{uy}+a^2(\bar{t})f_{uu}\big)f_{uu}^{-1}},\quad i=1,2. \tag{3.8}$$

where f_{yy}, f_{uy} and f_{uu} are calculated in $(\varphi_i(\bar{t}),\alpha_i(\bar{t}),\bar{t})$, $i=1,2$.

Proof. Differentiating the implicit function, we have

$$h_y(y,\bar{t})=\frac{du}{dy}=\frac{b(\bar{t})f_y-a(\bar{t})f_u-(a(\bar{t})y+b(\bar{t})u(\bar{t}))f_{yu}}{(a(\bar{t})y+b(\bar{t})u(\bar{t}))f_{uu}}.$$

From A_2, A_3, we obtain

$$\Big(b^2(\bar{t})f_{yy}-2a(\bar{t})b(\bar{t})f_{uy}+a^2(\bar{t})f_{uu}\Big)f_{uu}^{-1}>0\quad\text{and}\quad f_{uu}^{-1}>0.$$

Using L'Hospital's rule , in the neighborhood of saddle points, we obtain (3.8). □

Lemma 3.5. *Suppose that A_1-A_4 hold. Then $H(t_0)=0$ if and only if*

$$f(\varphi_1(t_0),\alpha_1(t_0),t_0)=f(\varphi_2(t_0),\alpha_2(t_0),t_0). \tag{3.9}$$

Proof. Setting $\tau=0$, $\bar{t}=t_0$ in (3.6a) and (3.6b), we obtain

$$\begin{aligned}&[a(t_0)\beta(t_0)+b(t_0)h^{(-)}(t_0)]f_u(\beta(t_0),h^{(-)}(t_0),t_0)-b(t_0)f(\beta(t_0),h^{(-)}(t_0),t_0)\\&\qquad=-b(t_0)f(\varphi_1(t_0),\alpha_1(t_0),t_0), &(3.10a)\\&[a(t_0)\beta(t_0)+b(t_0)h^{(+)}(t_0)]f_u(\beta(t_0),h^{(+)}(t_0),t_0)-b(t_0)f(\beta(t_0),h^{(+)}(t_0),t_0)\\&\qquad=-b(t_0)f(\varphi_2(t_0)),\alpha_2(t_0),t_0), &(3.10b)\end{aligned}$$

where

$$h^{(-)}(t_0)=h^{(-)}(\beta(t_0),\varphi_1(t_0),t_0),\quad h^{(+)}(t_0)=h^{(+)}(\beta(t_0),\varphi_2(t_0),t_0), \tag{3.10c}$$

Necessity follows directly from (3.10), and sufficiency follows from (3.5). □

Lemma 3.6. *Suppose that A_1-A_4 hold. Then $\dfrac{d}{dt}H(t_0)\neq 0$ if and only if*

$$\frac{d}{dt}f(\varphi_1(t_0),\alpha_1(t_0),t_0)\neq\frac{d}{dt}f(\varphi_2(t_0),\alpha_2(t_0),t_0). \tag{3.11}$$

Proof. Setting $\tau=0$ in (3.6) yields

$$\begin{aligned}&(a(\bar{t})\beta(\bar{t})+b(\bar{t})h^{(-)}(\bar{t}))f_u(\beta(\bar{t}),h^{(-)}(\bar{t}),\bar{t})-b(\bar{t})f(\beta(\bar{t}),h^{(-)}(\bar{t}),\bar{t})\\&\qquad=-b(\bar{t})f(\varphi_1(\bar{t}),\alpha_1(\bar{t}),\bar{t}), &(3.12a)\\&(a(\bar{t})\beta(\bar{t})+b(\bar{t})h^{(+)}(\bar{t}))f_u(\beta(\bar{t}),h^{(+)}(\bar{t}),\bar{t})-b(\bar{t})f(\beta(\bar{t}),h^{(+)}(\bar{t}),\bar{t})\\&\qquad=-b(\bar{t})f(\varphi_2(\bar{t}),\alpha_2(\bar{t}),\bar{t}), &(3.12b)\end{aligned}$$

where

$$h^{(-)}(\bar{t})=h^{(-)}(\beta(\bar{t}),\varphi_1(\bar{t}),\bar{t}),\quad h^{(+)}(\bar{t})=h^{(+)}(\beta(\bar{t}),\varphi_2(\bar{t}),\bar{t}). \tag{3.12c}$$

Differentiating (3.12) with respect to $\bar{t}$, we obtain

$$\begin{aligned}
&\frac{d}{d\bar{t}}\Big(a(\bar{t})\beta(\bar{t})+b(\bar{t})h^{(-)}(\bar{t})\Big)f_u(\beta(\bar{t}),h^{(-)}(\bar{t}),\bar{t})+\Big(a(\bar{t})\beta(\bar{t})+b(\bar{t})h^{(-)}(\bar{t})\Big)\frac{d}{d\bar{t}}f_u(\beta(\bar{t}),h^{(-)}(\bar{t}),\bar{t})\\
&\qquad-\Big(b'(\bar{t})f(\beta(\bar{t}),h^{(-)}(\bar{t}),\bar{t})+b(\bar{t})\frac{d}{d\bar{t}}f(\beta(\bar{t}),h^{(-)}(\bar{t}),\bar{t})\Big)\\
&\qquad=-\Big(b'(\bar{t})f(\varphi_1(\bar{t}),\alpha_1(\bar{t}),\bar{t})+b(\bar{t})\frac{d}{d\bar{t}}f(\varphi_1(\bar{t}),\alpha_1(\bar{t}),\bar{t})\Big),
\end{aligned}\tag{3.13}$$

$$\begin{aligned}
&\frac{d}{d\bar{t}}\Big(a(\bar{t})\beta(\bar{t})+b(\bar{t})h^{(+)}(\bar{t})\Big)f_u(\beta(\bar{t}),h^{(+)}(\bar{t}),\bar{t})+\Big(a(\bar{t})\beta(\bar{t})+b(\bar{t})h^{(+)}(\bar{t})\Big)\frac{d}{d\bar{t}}f_u(\beta(\bar{t}),h^{(+)}(\bar{t}),\bar{t})\\
&\qquad-\Big(b'(\bar{t})f(\beta(\bar{t}),h^{(+)}(\bar{t}),\bar{t})+b(\bar{t})\frac{d}{d\bar{t}}f(\beta(\bar{t}),h^{(+)}(\bar{t}),\bar{t})\Big)\\
&\qquad=-\Big(b'(\bar{t})f(\varphi_2(\bar{t}),\alpha_2(\bar{t}),\bar{t})+b(\bar{t})\frac{d}{d\bar{t}}f(\varphi_2(\bar{t}),\alpha_2(\bar{t}),\bar{t})\Big).
\end{aligned}\tag{3.14}$$

Letting $\bar{t}=t_0$ yields

$$\begin{aligned}
&\Big(a(t_0)\beta(t_0)+b(\bar{t})h^{(-)}(t_0)\Big)f_{u^2}(\beta(t_0),h(t_0),t_0)\frac{d}{dt}H(t_0)\\
&=-b(t_0)\Big(\frac{d}{dt}f(\varphi_1(t_0),\alpha_1(t_0),t_0)-\frac{d}{dt}f(\varphi_2(t_0),\alpha_2(t_0),t_0)\Big).
\end{aligned}\tag{3.15}$$

Using assumptions A_1 and A_2, and also the fact that different orbits do not intersect with the line $\bar{u}=\alpha_i(t_0), i=1,2$ at the point $y=\beta(t_0)$, we know that $\frac{d}{dt}H(t_0)\neq 0$ if and only if (3.11) holds. □

From Lemmas 3.2 and 3.5, it is easy to obtain the next lemma.

Lemma 3.7. *Suppose that A_1-A_4 hold. Then there exists $\bar{t}=t_0$ at which associated system* (3.1) *has a heteroclinic orbit connecting saddle points $M_1(\varphi_1(t_0),\alpha_1(t_0))$ and $M_2(\varphi_2(t_0),\alpha_2(t_0))$.*

From the above discussions, we know that the boundary value problem (2.6) satisfies all the assumptions of Theorem 2.1. Then problem (2.1) has an extremal trajectory $y(t,\mu)$ with a step-like contrast structure.

Theorem 3.1. *Suppose that A_1-A_4 hold. Then for sufficiently small $\mu>0$, the optimal control problem* (2.1) *has an extremal trajectory $y(t,\mu)$ with a step-like contrast structure*

$$\lim_{\mu\to 0} y(t,\mu)=\begin{cases}\varphi_1(t), & 0\le t<t_0,\\ \varphi_2(t), & t_0<t\le T.\end{cases}$$

4. Construction of Asymptotic Solution

An asymptotic solution of problem (2.1) is sought in the form

$$\begin{cases} y(t,\mu)=\sum\limits_{k=0}^{\infty}\mu^k(\bar{y}_k(t)+L_k y(\tau_0)+Q_0^{(-)}y(\tau)), & 0\le t<t^*,\\ u(t,\mu)=\sum\limits_{k=0}^{\infty}\mu^k(\bar{u}_k(t)+L_k u(\tau_0)+Q_0^{(-)}u(\tau)), \end{cases}\tag{4.1}$$

$$\begin{cases} y(t,\mu) = \sum\limits_{k=0}^{\infty} \mu^k(\bar{y}_k(t) + Q_0^{(+)}y(\tau) + R_k y(\tau_1)), \quad t^* < t \le T, \\ u(t,\mu) = \sum\limits_{k=0}^{\infty} \mu^k(\bar{u}_k(t) + Q_0^{(+)}u(\tau) + R_k u(\tau_1)), \end{cases} \tag{4.2}$$

where $\tau_0 = t\mu^{-1}$, $\tau = (t - t^*)\mu^{-1}$, $\tau_1 = (t - T)\mu^{-1}$, $L_k y(\tau_0)$ are coefficients of boundary layer terms at $t = 0$, $R_k(\tau_1)$ are coefficients of boundary layer terms at $t = T$, $Q_k^{(\mp)}(\tau)$ are left and right coefficients of internal transition terms at $t = t^*$.

The position of a transition time $t^*(\mu) \in [0, T]$ is unknown in advance. Suppose that t^* has also asymptotic expression of the form

$$t^* = t_0 + \mu t_1 + \cdots + \mu^k t_k + \cdots .$$

The coefficients of the above series are determined during the construction of an asymptotic solution.

From the main results of [4], we obtain

$$\min_u J[u] = \min_{u_0} J(u_0) + \sum_{i=1}^{n} \mu^i \min_{u_i} \tilde{J}_i(u_i) + \cdots ,$$

where

$$\tilde{J}_i(u_i) = J_i(u_i, \tilde{u}_{i-1}, \cdots, \tilde{u}_0), \quad \tilde{u}_k = \arg(\min_{u_k} \tilde{J}_k(u_k)), \quad 0 \le k \le i-1.$$

Substituting (4.1) and (4.2) into (2.1), and equating separately the terms on t, τ_0, τ and τ_1 by the boundary function method, we can obtain a series of variational problems to determine $\{\bar{y}_k(t), \bar{u}_k(t)\}$, $\{L_k y(\tau_0), L_k u(\tau_0)\}$, $\{Q_k^{(\mp)}y(\tau), Q_k^{(\mp)}u(\tau)\}$, $\{R_k y(\tau_1), R_k u(\tau_1)\}$, $k \ge 0$ respectively.

The variational problem to determine the zero-order coefficients of regular terms $\{\bar{y}_0(t),\ \bar{u}_0(t)\}$ are given by

$$\begin{cases} J_0(\bar{u}_0) = \int_0^T f(\bar{y}_0, \bar{u}_0, t)\, dt \to \min\limits_{\bar{u}_0}, \\ a(t)\bar{y}_0 + b(t)\bar{u}_0 = 0. \end{cases} \tag{4.3}$$

By assumption A_3, we get

$$\bar{y}_0 = \begin{cases} \varphi_1(t) \ , 0 \le t < t_0, \\ \varphi_2(t) \ , t_0 < t \le T, \end{cases} \tag{4.4a}$$

$$\bar{u}_0 = \begin{cases} \alpha_1(t) = -a(t)b^{-1}(t)\varphi_1(t) \ , 0 \le t < t_0, \\ \alpha_2(t) = -a(t)b^{-1}(t)\varphi_2(t) \ , t_0 < t \le T, \end{cases} \tag{4.4b}$$

The following variational problems to determine $\{Q_0^{(\mp)}y(\tau), Q_0^{(\mp)}u(\tau)\}$ are given by

$$\begin{cases} Q_0^{(\mp)}J = \int_{-\infty(0)}^{0(+\infty)} \Delta_0^{(\mp)} f(\varphi_{1,2}(t_0) + Q_0^{(\mp)}y, \alpha_{1,2}(t_0) + Q_0^{(\mp)}u, t_0)\, d\tau \to \min\limits_{Q_0^{(\mp)}u}, \\ \dfrac{d}{d\tau} Q_0^{(\mp)}y = a(t_0)(\varphi_{1,2}(t_0) + Q_0^{\mp}y) + b(t_0)(\alpha_{1,2}(t_0) + Q_0^{(\mp)}u), \\ Q_0^{(\mp)}y(0) = \beta(t_0) - \varphi_{1,2}(t_0), \quad Q_0^{(\mp)}y(\mp\infty) = 0, \end{cases} \tag{4.5a}$$

where

$$\Delta_0^{(\mp)} f = f(\varphi_{1,2}(t_0) + Q_0^{(\mp)}y, \alpha_{1,2}(t_0) + Q_0^{(\mp)}u, t_0) - f(\varphi_{1,2}(t_0), \alpha_{1,2}(t_0), t_0). \tag{4.5b}$$

Making the substitutions

$$\tilde{y}^{(\mp)} = \varphi_{1,2}(t_0) + Q_0^{(\mp)} y(\tau), \quad \tilde{u}^{(\mp)} = \alpha_{1,2}(t_0) + Q_0^{(\mp)} u(\tau),$$

we obtain

$$\begin{cases} Q_0^{(\mp)} J = \int_{-\infty(0)}^{0(+\infty)} \Delta_0^{(\mp)} \tilde{f}(\tilde{y}^{(\mp)}(\tau), \tilde{u}^{(\mp)}(\tau), t_0)\, d\tau \to \min\limits_{\tilde{u}^{(\mp)}(\tilde{y}^{(\mp)})}, \\ \dfrac{d\tilde{y}^{(\mp)}}{d\tau} = a(t_0)\tilde{y}^{(\mp)} + b(t_0)\tilde{u}^{(\mp)}, \\ \tilde{y}^{(\mp)}(0) = \beta(t_0), \quad \tilde{y}^{(\mp)}(\mp\infty) = \varphi_{1,2}(t_0). \end{cases} \tag{4.6}$$

The substitution

$$\frac{d\tilde{y}^{(\mp)}}{a(t_0)\tilde{y}^{(\mp)} + b(t_0)\tilde{u}^{(\mp)}} = d\tau, \tag{4.7}$$

produces the following variational problem, which is explicitly independent of τ

$$Q_0^{(\mp)} J = \int_{\varphi_1(t_0)(\beta(t_0))}^{\beta(t_0)(\varphi_2(t_0))} \frac{\Delta_0 \tilde{f}(\tilde{y}^{(\mp)}, \tilde{u}^{(\mp)}, t_0)}{a(t_0)\tilde{y}^{(\mp)} + b(t_0)\tilde{u}^{(\mp)}}\, d\tilde{y} \to \min_{\tilde{u}^{(\mp)}(\tilde{y}^{(\mp)})}. \tag{4.8}$$

The necessary condition for a minimum of the integrand has the form

$$\Big(a(t_0)\tilde{y}^{(\mp)} + b(t_0)\tilde{u}^{(\mp)}\Big) f_u - b(t_0) f(\tilde{y}^{(\mp)}, \tilde{u}^{(\mp)}, t_0) = -b(t_0) f(\varphi_{1,2}(t_0), \alpha_{1,2}(t_0), t_0). \tag{4.9}$$

In view of (3.6), we have that $\tilde{u}^{(\mp)} = h^{(\mp)}(\tilde{y}^{(\mp)}, t_0)$ is the minimum, as it satisfies

$$(a(t_0)\tilde{y} + b(t_0)\tilde{u})^{-2}(a(t_0)\tilde{y} + b(t_0)\tilde{u}^{(\mp)}) f_{\tilde{u}^2} > 0. \tag{4.10}$$

The equations to determine $Q_0^{(\mp)} y$ are given by

$$\frac{dQ_0^{(\mp)} y}{d\tau} = a(t_0)\Big(\varphi_{1,2}(t_0) + Q_0^{(\mp)} y\Big) + b(t_0) h^{(\mp)}(\varphi_{1,2}(t_0) + Q_0^{(\mp)} y, t_0).$$

A_5. Suppose that the following initial problems

$$\begin{cases} \dfrac{dQ_0^{(\mp)} y}{d\tau} = a(t_0)(\varphi_{1,2}(t_0) + Q_0^{(\mp)} y) + b(t_0) h^{(\mp)}(\varphi_{1,2}(t_0) + Q_0^{(\mp)} y, t_0), \\ Q_0^{(\mp)} y(0) = \beta(t_0) - \varphi_{1,2}(t_0), \end{cases} \tag{4.11}$$

have continuously differentiable solutions $Q_0^{(\mp)} y(\tau), \ -\infty \le \tau \le +\infty$.

Substituting $Q_0^{(\mp)} y(\tau)$ into (4.5), it is easy for us to get $Q_0^{(\mp)} u(\tau)$, thus $Q_0^{(\mp)} y(\tau)$ and $Q_0^{(\mp)} u(\tau)$ are determined. From Lemma 3.4 we get

$$a(t_0) + b(t_0) h_y^{(-)}(\varphi_1(t_0), t_0) > 0, \quad a(t_0) + b(t_0) h_y^{(+)}(\varphi_2(t_0), t_0) < 0,$$

which imply that

$$\begin{aligned} |Q_0^{(-)} y(\tau)| &\le C_0^{(-)} e^{\kappa_0 \tau}, &\quad \kappa_0 > 0,\ \tau < 0, \\ |Q_0^{(+)} y(\tau)| &\le C_0^{(+)} e^{-\kappa_1 \tau}, &\quad \kappa_1 > 0,\ \tau > 0, \\ |Q_0^{(-)} u(\tau)| &\le C_1^{(-)} e^{\kappa_0 \tau}, &\quad \kappa_0 > 0,\ \tau < 0, \\ |Q_0^{(+)} u(\tau)| &\le C_1^{(+)} e^{-\kappa_1 \tau}, &\quad \kappa_1 > 0,\ \tau > 0. \end{aligned} \tag{4.12}$$

Below, we give the equations and their conditions for determining $\{L_0y(\tau_0), L_0u(\tau_0)\}$

$$\begin{cases} L_0J = \int_0^\infty \Delta_0 f(\varphi_1(0) + L_0y, \alpha_1(0) + L_0u, 0)\, d\tau_0 \to \min\limits_{L_0u}, \\ \dfrac{d}{d\tau_0} L_0y = a(0)(\varphi_1(0) + L_0y) + b(0)(\alpha_1(0) + L_0u), \\ L_0y(0) = y^0 - \varphi_1(0), \quad L_0y(\infty) = 0, \end{cases} \tag{4.13a}$$

where

$$\Delta_0 f = f(\varphi_1(0) + L_0y, \alpha_1(0) + L_0u, 0) - f(\varphi_1(0), \alpha_1(0), 0), \tag{4.13b}$$

and the problem to determine $\{R_0y(\tau_1), R_0u(\tau_1)\}$ is given by

$$\begin{cases} R_0J = \int_{-\infty}^0 \Delta_0 f(\varphi_2(T) + R_0y, \alpha_2(T) + R_0u, T)\, d\tau_1 \to \min\limits_{R_0u}, \\ \dfrac{d}{d\tau_1} R_0y = a(T)(\varphi_2(T) + R_0y) + b(T)(\alpha_2(T) + R_0u), \\ R_0y(0) = y^T - \varphi_2(T), \quad R_0y(-\infty) = 0, \end{cases} \tag{4.14a}$$

where

$$\Delta_0 f = f(\varphi_2(T) + R_0y, \alpha_2(T) + R_0u, T) - f(\varphi_2(T), \alpha_2(T), T). \tag{4.14b}$$

A_6. Suppose that the boundary data $y^0 - \varphi_1(0)$ and $y^T - \varphi_2(T)$ in the problems L_0J and R_0J belong to certain neighborhoods of the origin that guarantee the existence of these optimal control problems.

Then, we have so far constructed the leading terms

$$\{\bar{y}_0^*(t),\ \bar{u}_0^*(t)\},\ \{L_0y^*(\tau_0),\ L_0u^*(\tau_0)\},\ \{Q_0y^*(\tau),\ Q_0u^*(\tau)\},\ \{R_0y^*(\tau_1),\ R_0u^*(\tau_1)\}$$

of asymptotic series for the problem (4.1) and (4.2). Additionally, we can obtain the minimum values of the corresponding optimal control problems J_0^*, L_0J^*, $Q_0^{(\mp)}J^*$, R_0J^*:

$$J_0^*(\bar{u}_0) = \int_0^T f(\bar{y}_0^*, \bar{u}_0^*, t)\, dt, \tag{4.15a}$$

$$L_0J^* = \int_{y^0}^{\varphi_1(0)} \frac{\Delta_0^{(\mp)} f(\check{y}^*, \check{u}^*, 0)}{a(0)\check{y}^* + b(0)\check{u}^*}\, d\check{y}, \tag{4.15b}$$

$$Q_0^{(\mp)}J^* = \pm \int_{\varphi_{1,2}(t_0)}^{\beta(t_0)} \frac{\Delta_0^{(\mp)} f(\tilde{y}^{(\mp)*}, \tilde{u}^{(\mp)*}, t_0)}{a(t_0)\tilde{y}^{(\mp)*} + b(t_0)\tilde{u}^{(\mp)*}}\, d\tilde{y}, \tag{4.15c}$$

$$R_0J^* = \int_{\varphi_2(T)}^{y^T} \frac{\Delta_0^{(\mp)} f(\hat{y}^*, \hat{u}^*, T)}{a(T)\hat{y}^* + b(T)\hat{u}^*}\, d\hat{y}, \tag{4.15d}$$

where

$$\check{y}^* = \varphi_1(0) + L_0y^*(\tau_0), \quad \check{u}^* = \alpha_1(0) + L_0u^*(\tau_0),$$
$$\hat{y}^* = \varphi_2(T) + R_0y^*(\tau_1), \quad \hat{u}^* = \alpha_2(T) + R_0u^*(\tau_1).$$

Theorem 4.1. *Suppose that A_1-A_6 hold. Then for sufficiently small $\mu > 0$ there exists a step-like contrast structure solution $y(t, \mu)$ of the problem* (2.1). *Moreover, the following asymptotic expansions hold*

$$y(t, \mu) = \begin{cases} \varphi_1(t) + L_0y(\tau_0) + Q_0^{(-)}y(\tau) + \mathcal{O}(\mu), & 0 \le t < t_0, \\ \varphi_2(t) + R_0y(\tau_1) + Q_0^{(+)}y(\tau) + \mathcal{O}(\mu), & t_0 < t \le T. \end{cases} \tag{4.16a}$$

$$u(t,\mu)=\begin{cases}\alpha_1(t)+L_0u(\tau_0)+Q_0^{(-)}u(\tau)+\mathcal{O}(\mu), & 0\le t<t_0,\\ \alpha_2(t)+R_0u(\tau_1)+Q_0^{(+)}u(\tau)+\mathcal{O}(\mu), & t_0<t\le T.\end{cases} \tag{4.16b}$$

5. An Example

Consider the problem

$$\begin{cases}J[u]=\int_0^{2\pi}\left(\frac{1}{4}y^4-\frac{1}{3}y^3\sin t-y^2+y\sin t+\frac{1}{2}u^2\right)\mathrm{d}t\to\min_u,\\ \mu\frac{\mathrm{d}y}{\mathrm{d}t}=-y+u,\\ y(0,\mu)=0\,,\quad y(2\pi,\mu)=2,\end{cases} \tag{5.1}$$

where

$$f(y,u,t)=\frac{1}{4}y^4-\frac{1}{3}y^3\sin t-y^2+y\sin t+\frac{1}{2}u^2.$$

For each t, we have

$$\bar{y}_0(t)=\begin{cases}-1, & 0\le t<\pi,\\ 1, & \pi<t\le 2\pi.\end{cases} \tag{5.2}$$

$$\min_{\bar{y}}F(\bar{y}_0,t)=\begin{cases}-\frac{1}{4}-\frac{2}{3}\sin t, & 0\le t\le\pi,\\ -\frac{1}{4}+\frac{2}{3}\sin t, & \pi\le t\le 2\pi.\end{cases} \tag{5.3}$$

The transition point $t_0=\pi$ is determined by the equation $\sin t_0=0$.

In this example, different orbits S_{M_1} and S_{M_2}, passing through the saddle points $M_1(\bar{t})$ and $M_2(\bar{t})$, respectively, have the form

$$S_{M_1}:u^{(-)}=y^{(-)}+\frac{\sqrt{2}}{2}(1-y^{(-)2}),\quad S_{M_2}:u^{(+)}=y^{(+)}+\frac{\sqrt{2}}{2}(1-y^{(+)2}). \tag{5.4}$$

The left and right zero-order terms of transition layer are determined by the following problems

$$\frac{\mathrm{d}Q_0^{(\mp)}y}{\mathrm{d}\tau}=-Q_0^{(\mp)}y+Q_0^{(\mp)}u\,,\quad Q_0^{(\mp)}y(0)=\pm1,\quad Q_0^{(\mp)}y(\mp\infty)=0, \tag{5.5}$$

whose solutions are

$$Q_0^{(-)}y=\frac{2e^{\sqrt{2}\tau}}{1+e^{\sqrt{2}\tau}},\quad Q_0^{(-)}u=\frac{(2+2\sqrt{2}+2e^{\sqrt{2}\tau})e^{\sqrt{2}\tau}}{(1+e^{\sqrt{2}\tau})^2}, \tag{5.6a}$$

$$Q_0^{(+)}y=\frac{-2}{1+e^{\sqrt{2}\tau}},\quad Q_0^{(+)}u=\frac{(2\sqrt{2}e^{\sqrt{2}\tau}-2e^{\sqrt{2}\tau}-2)}{(1+e^{\sqrt{2}\tau})^2}. \tag{5.6b}$$

Similarly, we have

$$L_0y=\frac{2e^{-\sqrt{2}\tau_0}}{1+e^{-\sqrt{2}\tau_0}},\qquad L_0u=\frac{2e^{-\sqrt{2}\tau_0}+2e^{-2\sqrt{2}\tau_0}-2\sqrt{2}e^{-\sqrt{2}\tau_0}}{(1+e^{-\sqrt{2}\tau_0})^2}, \tag{5.7a}$$

$$R_0y=\frac{2}{3e^{-\sqrt{2}\tau_1}-1},\quad R_0u=\frac{6e^{-\sqrt{2}\tau_1}-2+6\sqrt{2}e^{-\sqrt{2}\tau_1}}{(3e^{-\sqrt{2}\tau_1}-1)^2}. \tag{5.7b}$$

Finally, the formal asymptotic solution is

$$y(t,\mu)=\begin{cases}-1+\dfrac{2e^{-\sqrt{2}\tau_0}}{1+e^{-\sqrt{2}\tau_0}}+\dfrac{2e^{\sqrt{2}}\tau}{1+e^{\sqrt{2}\tau}}+\mathcal{O}(\mu), & 0\le t<\pi,\\ 1+\dfrac{2}{3e^{-\sqrt{2}\tau_1}-1}+\dfrac{-2}{1+e^{\sqrt{2}\tau}}+\mathcal{O}(\mu), & \pi<t\le 2\pi.\end{cases}$$

and

$$u(t,\mu)=\begin{cases}-1+\dfrac{2e^{-\sqrt{2}\tau_0}+2e^{-2\sqrt{2}\tau_0}-2\sqrt{2}e^{-\sqrt{2}\tau_0}}{(1+e^{-\sqrt{2}\tau_0})^2}+\dfrac{(2+2\sqrt{2}+2e^{\sqrt{2}\tau})e^{\sqrt{2}\tau}}{(1+e^{\sqrt{2}\tau})^2}+\mathcal{O}(\mu), & t<\pi,\\ 1+\dfrac{6e^{-\sqrt{2}\tau_1}-2+6\sqrt{2}e^{-\sqrt{2}\tau_1}}{(3e^{-\sqrt{2}\tau_1}-1)^2}+\dfrac{(2\sqrt{2}e^{\sqrt{2}\tau}-2e^{\sqrt{2}\tau}-2)}{(1+e^{\sqrt{2}\tau})^2}+\mathcal{O}(\mu), & \pi<t\le 2\pi.\end{cases}$$

Acknowledgments. The authors are grateful to Professor Zhiming Wang of East China Normal University for his comments which improved the presentation of the paper. The research was supported by the National Natural Science Foundation of China (11071075, 30921064, 90820307), Natural Science Foundation of Shanghai (10ZR1409200), E-Institutes of Shanghai Municipal Education Commission (E03004), the Knowledge Innovation Program of the Chinese Academy of Sciences.

References

[1] A.B. Vasil'eva and V.F. Butuzov, Asymptotic methods in the theory of singular perturbations, Vysshaya Shkola, Moscow, 1990 (in Russian).

[2] V.F. Butuzov, A.B. Vasil'eva and N. N. Nefedov, Asymptotic theory of contrast structures(Overview), *Avtom. Telemekh.*, **7** (1997), 4-32.

[3] A.B. Vasil'eva, V.F. Butuzov and N. N. Nefedov, Contrast structures in singularly perturbed problems, *Fundam. Prikl. Mat.*, **4** (1998), 799-851.

[4] M.K. Ni and M.G. Dmitriev, Contrast structures in the simplest vector variational problem and their asymptotics, *Avtom. Telemekh.*, **5** (1998), 41-52.

[5] A.B. Vasil'eva, M.G. Dmitriev and M.K. Ni, On a steplike contrast structure in a problem of variational calculus, *Comput. Math. Math. Phys.*, **44** (2004), 1203-1212.

[6] A.B. Vasil'eva, On contrast steplike structure for a system of singularly perturbed equations, *Zh. Vychisl. Mat.Fiz.*, **34** (1994), 1401-1411.

[7] S.V. Belokopytov and M.G. Dmitriev, Solution of classical optimal control problems with boundary layer, *Avtom. Telemekh.*, **7** (1989), 71-82.