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Electrodynamics of relativistic particles through non-standard Lagrangian

Rami Ahmad El-Nabulsi*

College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641112, China

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Abstract. The main aim of this paper is to discuss the electrodynamics of relativistic dynamics of particles bases on the notion of the non-standard Lagrangians which have gained increasing importance in the theory of nonlinear differential equations, dissipative dynamical systems and theoretical physics. The mathematical settings are constructed starting from the modified Euler-Lagrange equation and modified Hamiltons equations. Some illustrative examples are considered and discussed.

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Key words: non-standard Lagrangians, modified relativistic electrodynamics, modified Hamiltonian

1 Introduction

The notion of non-standard Lagrangians (NSL) is not new and it as in fact introduced by Arnold in 1978 [1]. In reality, NSL were not considered seriously in the past for two major reasons. First of all, the physical meaning of NSL is still obscure and besides their Hamiltonian formalism was problematic in particular when it is related to quantization process. However, in the progress of years, it was observed that NSL plays an important role in the theory of nonlinear differential equations [2-4], dissipative dynamical systems [5-15] and in many problems related to theoretical physics [16-18]. Their applications seem wide yet lots of works are required for a better understanding of their physical significances. Some advances to understand the root of NSL based on the theories of inverse variational problem were discussed in [15] yet the problem is still open. It should be emphasized that recent works prove that a number of dynamical systems may be described by two different Lagrangians: one is standard and another non-standard one.

*Corresponding author. *Email address:* `nabulsiahmadrami@yahoo.fr` (R. A. El-Nabulsi)

In other words, the resulting differential equations of motion may be obtained from two different Lagrangians [9-11]. Accordingly, one may argue that a complete description of physical systems require the knowledge of both types of Lagrangians. In this work, we would like to explore the implications of NSL in relativistic electrodynamics. It should be noted that NSL may came in different mathematical forms as discussed in [11], yet through this paper we pick the power-law form. It should be stressed that both the kinetic terms and the potential function in the NSL are non-standard yet physically interesting nonlinear dynamics were obtained. There is one more simple elucidation to add is that in our approach NSL refers to "standard Lagrangian function that modifies the Euler-Lagrange equations and accordingly the Hamilton's equations of motion".

The paper is organized as follows: in Sec. 2, we introduce basic settings mainly the power-law NSL and its corresponding Euler-Lagrange and Hamilton's equations of motions in the presence of electromagnetic forces (EM). In Sec. 3, we discuss the modified dynamics of relativistic particles in the absence and in the presence of the electromagnetic field for different values of ξ. The paper concludes in Sec. 4 with a brief summary of main results and perspectives.

2 Electrodynamics with non-standard Lagrangians and the modified Euler-Lagrange and Hamilton's equations

Through this work we define the power-law NSL by $L_{NSL}=L^{1+\xi}(\dot{q},q,t)$ where $L(\dot{q},q,t)\epsilon C^2([a,b]\times\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R})$ is the standard Lagrangian of the theory $(\dot{q},q,t)\to L(\dot{q},q,t)$ assumed to be a C^2 function with respect to all its arguments and ξ is a free parameter which is different from -1. Here $\dot{q}=dq/dt$ is the time-derivative of the generalized coordinate. The action functional of the theory is defined by:

$$S=\int_a^b \Lambda L^{1+\xi}(\dot{q}(t),q(t),t)dt, \tag{1}$$

and the basic problem is to define the extremum of the functional $S=D\to\mathbb{R}$ where D is the subset of D which is the set of all functions $q:[a+b]\to\mathbb{R}^n$ such that the temporal derivative of $\dot{q}$ exists and is continuous on $[a+b]$. In Eq. (1), the parameter Λ is a free parameter that is introduced for physical arguments (this parameter guarantees the correct physical dimensionalities for all terms). It is an easy exercise to prove that if $q(t)$ is a local minimizer to the action (1) then the following modified Euler-Lagrange equation (MELE) holds [11]:

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)=\xi\frac{1}{L}\frac{\partial L}{\partial \dot{q}}\left(\frac{\partial L}{\partial t}+\dot{q}\frac{\partial L}{\partial q}+\ddot{q}\frac{\partial L}{\partial \dot{q}}\right). \tag{2}$$

Two main features of Eq. (2) concern first its RHS which depends on the total derivative of $L(\dot{q},q,t)$ and the form of the momentum conjugate and its time derivative which take respectively the forms $p=\Lambda\partial L^{1+\xi}/\partial\dot{q}$ and $\dot{p}=\Lambda\partial L^{1+\xi}/\partial q$. We will prove that these modified features may have interesting consequences on the relativistic electrodynamics of particles. In order to derive Hamilton's equations, we consider a dynamical system with N -degrees of freedom for which we correlate the following modified Hamiltonian function:

$$H=\sum_{k=1}^{N}(1+\xi)\Lambda\dot{q}_k L^{\xi}\frac{\partial L}{\partial\dot{q}_k}-\Lambda L^{1+\xi}. \tag{3}$$

Let us first look at the change in the Hamiltonian along the actual motion with time. In fact, we can write:

$$\begin{aligned}\frac{dH}{dt}&=\Lambda\sum_{k=1}^{N}\left((1+\xi)\left(\ddot{q}_k L^{\xi}\frac{\partial L}{\partial\dot{q}_k}+\dot{q}_k L^{\xi}\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_k}-\xi\frac{dL}{dt}\dot{q}_k L^{\xi-1}\frac{\partial L}{\partial\dot{q}_k}\right)-(1+\xi)L^{\xi}\frac{dL}{dt}\right),\\&=\Lambda(1+\xi)\sum_{k+1}^{N}\left(\ddot{q}_k L^{\xi}\frac{\partial L}{\partial\dot{q}_k}+\dot{q}_k L^{\xi}\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_k}-\xi\left(\frac{\partial L}{\partial t}+\dot{q}_k\frac{\partial L}{\partial q}+\ddot{q}\frac{\partial L}{\partial\dot{q}_k}\right)\right.\\&\qquad\left.\dot{q}_k L^{\xi-1}\frac{\partial L}{\partial\dot{q}_k}-L^{\xi}\left(\frac{\partial L}{\partial t}+\dot{q}+k\frac{\partial L}{\partial q}+\ddot{q}_k\frac{\partial L}{\partial\dot{q}_k}\right)\right)\end{aligned} \tag{4}$$

If, for instance, the Lagrangian does not depend explicitly on time, Eq. (4) is simplified to:

$$\frac{dH}{dt}=\Lambda(1+\xi)\sum_{k=1}^{N}L^{\xi}\dot{q}_k\left(\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_k}-\xi\frac{1}{L}\left(\dot{q}_k\frac{\partial L}{\partial\dot{q}}+\ddot{q}_k\frac{\partial L}{\partial\dot{q}_k}\right)\frac{\partial L}{\partial\dot{q}_k}-\frac{\partial L}{\partial q_k}\right)=0, \tag{5}$$

and the Hamiltonian is a constant of motion. In order to derive the corresponding Hamilton's equations of motion, we write:

$$\begin{aligned}(1+\xi)\Lambda L^{\xi}dL&=\Lambda\sum_{k=1}^{N}\left(\frac{\partial L^{1+\xi}}{\partial q_k}dq_k+\frac{\partial L^{1+\xi}}{\partial\dot{q}_k}dq_k\right)-(1+\xi)\Lambda L^{\xi}\frac{\partial L}{\partial t}\\&=\sum_{k=1}^{N}(\dot{p}dq_k+pd\dot{q}_k)-(1+\xi)\Lambda L^{\xi}\frac{\partial L}{\partial t}dt,\end{aligned} \tag{6}$$

which gives

$$\begin{aligned}dL&=\frac{1}{(1+\xi)\Lambda L^{\xi}}\sum_{k=1}^{N}\left(\frac{\partial L^{1+\xi}}{\partial q_k}dq_k+\frac{\partial L^{1+\xi}}{\partial\dot{q}_k}dq_k\right)-\frac{\partial L}{\partial t}dt\\&=\sum_{k=1}^{N}(\dot{p}dq_k+pd\dot{q}_k)-\frac{\partial L}{\partial t}dt,\end{aligned} \tag{7}$$

and hence

$$dH=\sum_{k=1}^{N}(\dot{q}_k dp_k-\dot{p}_k dq_k)-\frac{\partial L}{\partial t}dt. \tag{8}$$

One hence conclude that

$$\dot{q}_k=\frac{\partial H}{\partial p_k}, \tag{9}$$

and

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}. \tag{10}$$

In that way, the Hamilton's equations of motion are not modified and take a standard forms. It is noteworthy that the modified Hamiltonian of the system may be also defined by $H/\Lambda L^{\xi} = H = \Sigma_{k=1}^{N}(1+\xi)\dot{q}_k(\partial L/\partial \dot{q}_k) - L$ which is deduced from Eq. (3). This is closely related to the standard Hamiltonian form which is given by $H = \Sigma_{k=1}^{N}\dot{q}_k(\partial L/\partial \dot{q}_k) - L$.

3 Modified electrodynamics of relativistic particles

To incorporate the relativistic theory, we assume that Eq. (2) is valid in every inertial frame, i.e. the action (1) is a Lorentz scalar. Usually for a free particle of rest mass m and moving at a velocity v, the standard Lagrangian given by $L = -mc^2\sqrt{1-\beta^2}$ where $\beta = v/c$ and is the celerity of light [19]. It is easy to check that the corresponding MELE is derived from Eq. (2) and takes the form ($q = x, \dot{q} = v = \dot{x}$):

$$\frac{d}{dt}(\gamma m v) = \xi \gamma^2 m^2 v^2 a, \tag{11}$$

where $a = \ddot{x}$ is the corresponding acceleration of the particles and $\gamma = 1/\sqrt{1-\beta^2}$. The equation of motion is modified accordingly. In order now to incorporate the EM forces, we need to take into account the EM interaction and hence we add to the standard Lagrangian the well-known Lagrangian $l_{int} = q(-\phi + \bar{v}\cdot\bar{A}/c)$ where $\bar{A}$ is a vector potential and $\phi = A^{\circ}$ is a scalar potential. The total Lagrangian of the theory is then given by $L_{total} = -mc^2\sqrt{1-\beta^2} + q(-\phi + \bar{v}\cdot\bar{A}/c)$ and hence we can write Eq. (2) as:

$$\begin{aligned} &\frac{\partial L}{\partial x} + \frac{\partial L_{int}}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) - \frac{d}{dt}\left(\frac{\partial L_{int}}{\partial \vec{v}}\right) \\ &= -\xi \frac{1}{L_{total}}\left(\frac{\partial L}{\partial \vec{v}} + \frac{\partial L_{int}}{\partial \vec{v}}\right)\left(\vec{v}\left(\frac{\partial L}{\partial x} + \frac{\partial L_{int}}{\partial x}\right) + \dot{\vec{v}}\left(\frac{\partial L}{\partial \vec{v}} + \frac{\partial L_{int}}{\partial \vec{v}}\right)\right). \end{aligned} \tag{12}$$

After simple algebraic arrangement, we find:

$$\begin{aligned} &\left(-q\nabla\phi + \frac{q}{c}\nabla(\vec{v}\cdot\vec{A}) - \frac{d}{dt}(\gamma m\vec{v}) - \frac{q}{c}\frac{d\vec{A}}{dt}\right)\left(-mc^2\sqrt{1-\beta^2} + q\left(-\phi + \frac{\vec{v}\cdot\vec{A}}{c}\right)\right) \\ &= -\xi\left(\gamma m\vec{v} + \frac{q}{c}\vec{A}\right)\left(\vec{v}\left(-q\nabla\phi + \frac{q}{c}\nabla(\vec{v}\cdot\vec{A})\right) + \dot{\vec{v}}\left(\gamma m\vec{v} + \frac{q}{c}\vec{A}\right)\right) \end{aligned} \tag{13}$$

Using the fact that:

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A} = \frac{\partial\vec{A}}{\partial t} + \left(\nabla(\vec{v}\cdot\vec{A}) - \vec{v}\times\vec{B}\right), \tag{14}$$

where $\vec{B}$ is the magnetic field, we can rewrite Eq. (13) as:

$$\left(-q\nabla\phi-\frac{q}{c}\left(\frac{\partial\vec{A}}{\partial t}+\left(\nabla(\vec{v}\cdot\vec{A})-\vec{v}\times\vec{B}\right)\right)+\frac{q}{c}\nabla(\vec{v}\cdot\vec{A})-\frac{d}{dt}(\gamma m\vec{v})\right)$$
$$\left(-mc^2\sqrt{1-\beta^2}+q\left(-\phi+\frac{\nu\cdot\vec{A}}{c}\right)\right)$$
$$=-\xi\left(\gamma m\vec{v}+\frac{q}{c}\vec{A}\right)\left(\vec{v}\left(-q\nabla\phi+\frac{q}{c}\nabla(\vec{v}\cdot\vec{A})\right)+\frac{d\vec{v}}{dt}\left(\gamma m\vec{v}+\frac{q}{c}\vec{A}\right)\right). \tag{15}$$

Now using the fact that $\vec{E}=-\nabla\phi-\frac{1}{c}\frac{\partial t}{\partial t}$, $\nabla(\vec{v}\cdot\vec{A})=(\vec{v}\cdot\nabla)\vec{A}+\vec{v}\times\vec{B}$ and $\vec{B}=\nabla\times\vec{A}$ we find:

$$\left(q\left(\vec{E}+\frac{1}{c}\vec{v}\times\vec{B}\right)-\frac{d}{dt}(\gamma m\vec{v})\right)\left(-mc^2\sqrt{1-\beta^2}+q\left(-\phi+\frac{\nu\cdot\vec{A}}{c}\right)\right)$$
$$=-\xi\left(\gamma m\vec{v}+\frac{q}{c}\vec{A}\right)\left(q\vec{v}\left(\vec{E}+\frac{1}{c}\left(\frac{\partial\vec{A}}{\partial t}+\nabla(\vec{v}\cdot\vec{A})\right)\right)+\gamma m\nu\frac{d\vec{v}}{dt}+\frac{q}{c}\vec{A}\frac{d\vec{v}}{dt}\right), \tag{16}$$

and from Eq. (14), we can rewrite Eq. (16) as:

$$\left(q\left(\vec{E}+\frac{1}{c}\vec{v}\times\vec{B}\right)-\frac{d}{dt}(\gamma m\vec{v})\right)\left(-mc^2\sqrt{1-\beta^2}+q\left(-\phi+\frac{\nu\cdot\vec{A}}{c}\right)\right)$$
$$=-\xi\left(\gamma m\vec{v}+\frac{q}{c}\vec{A}\right)\left(q\vec{v}\left(\vec{E}+\frac{1}{c}\vec{v}\times\vec{B}\right)+q\frac{\nu}{c}\frac{d\vec{A}}{dt}+\left(\gamma m\vec{v}+\frac{q}{c}\vec{A}\right)\frac{d\vec{v}}{dt}\right). \tag{17}$$

We can arrange terms in Eq. (17) and write the final form as:

$$q\left(\vec{E}+\frac{1}{c}\vec{v}\times\vec{B}\right)\left(1+\xi\nu\left(\gamma m\vec{v}+\frac{q}{c}\vec{A}\right)\right)\left(-mc^2\sqrt{1-\beta^2}+q\left(-\phi+\frac{\nu\cdot\vec{A}}{c}\right)\right)$$
$$\xi\left(\gamma m\vec{v}+\frac{q}{c}\vec{A}\right)\left(\gamma m\vec{v}\frac{d\vec{v}}{dt}+\frac{q}{c}\frac{d(\vec{v}\cdot\vec{A})}{dt}\right)=m\frac{d(\gamma\vec{v})}{dt}. \tag{18}$$

Eq. (18) is the modified relativistic EM equation of motion. By defining $\vec{P}=\gamma m\vec{v}+q\vec{A}/c$, it is easy to check that we can write Eq. (19) as:

$$q\left(\vec{E}+\frac{1}{c}\vec{v}\times\vec{B}\right)\left(1+\xi\vec{v}\cdot\vec{P}\right)\left(-mc^2\sqrt{1-\beta^2}+q\left(-\phi+\frac{\nu\cdot\vec{A}}{c}\right)\right)$$
$$+\xi\vec{P}\left(\frac{d(\vec{v}\kappa\cdot\vec{P})}{dt}-m\nu^2\frac{d\gamma}{dt}-\gamma\nu^2\frac{dm}{dt}\right)=m\frac{d(\gamma\nu)}{dt}. \tag{19}$$

In the case of a uniform magnetic field and uniform vector potential ($\vec{B}$ and $\vec{A}$ are assumed constants), we can simplify Eq. (19) to:

$$\begin{aligned}\frac{d\vec{P}}{dt}=&\left(\frac{q}{c}\vec{v}\times\vec{B}\right)\left(1+\xi\nu\cdot\vec{P}\right)\left(-mc^2\sqrt{1-\beta^2}+q\left(-\phi+\frac{\vec{\nu\cdot\vec{A}}}{c}\right)\right)\\&+\xi\vec{P}\left(\frac{d(\vec{v}\kappa\cdot\vec{P})}{dt}-mv^2\frac{d\gamma}{dt}-\gamma v^2\frac{dm}{dt}\right)+\gamma v\frac{dm}{dt},\end{aligned}\tag{20}$$

where $\vec{P}=m\gamma\vec{v}$. However, once can evaluate the time-variation of the energy E:

$$\begin{aligned}\frac{DE}{dt}=&\vec{v}\cdot\frac{d\vec{P}}{dt}=\underbrace{\frac{q}{c}\vec{v}\cdot(\vec{v}\times\vec{B})}_{=0}\left(1+\xi\nu\cdot\vec{P}\right)\left(-mc^2\sqrt{1-\beta^2}+q\left(-\phi+\frac{\vec{\nu\cdot\vec{A}}}{c}\right)\right)\\&+\xi\vec{v}\cdot\vec{P}\left(\frac{d(\vec{v}\kappa\cdot\vec{P})}{dt}-mv^2\frac{d\gamma}{dt}-\gamma v^2\frac{dm}{dt}\right)+\gamma v^2\frac{dm}{dt},\\=&\xi\left(\gamma mv^2+\frac{q}{c}\vec{v}\cdot\vec{A}\right)\left(\frac{d}{dt}\left(\gamma mv^2+\frac{q}{c}\vec{v}\cdot\vec{A}\right)-mv^2\frac{d\gamma}{dt}-\gamma v^2\frac{dm}{dt}\right)+\gamma v^2\frac{dm}{dt},\\=&\xi\vec{v}\cdot\left(\gamma mv+\frac{q}{c}\vec{A}\right)\left(2\gamma mv+\frac{q}{c}\vec{A}\right)\frac{d\vec{v}}{dt}+\gamma v^2\frac{dm}{dt},\end{aligned}\tag{21}$$

and hence the energy is not conserved unless the speed of particles and their masses remain constants. The modified Hamiltonian is derived from Eq. (3):

$$\begin{aligned}H=&(1+\xi)\Lambda\left(-mc^2\sqrt{1-\beta^2}+q\left(-\phi+\frac{\vec{\nu\cdot\vec{A}}}{c}\right)\right)^{\xi}\\&\vec{v}\cdot\left(\gamma mv+\frac{q}{c}\vec{A}\right)-\Lambda\left(-mc^2\sqrt{1-\beta^2}+q\left(-\phi+\frac{\vec{\nu\cdot\vec{A}}}{c}\right)\right)^{1+\xi}.\end{aligned}\tag{22}$$

It is easy to check that when $\xi=0$, i.e. , Eq. (22) is reduced to the standard Hamiltonian form: $H=\Lambda\nu\cdot(\gamma M\vec{v}+q\vec{A})/c-\wedge(-mc^2\sqrt{1-\beta^2}+q(-\phi+\vec{v}\cdot\vec{A}/c))=mc^2\sqrt{1-\beta^2}+q\phi$. However, the canonical formalism supplies us with the canonically modified conjugate momenta which gives:

$$\gamma\vec{v}=-\frac{q}{mc}\vec{A}+\frac{1}{(1+\xi)\Lambda m}\vec{p}L^{-\xi},\tag{23}$$

and hence

$$\gamma^2v^2=\left(-\frac{q}{mc}\vec{A}+\frac{1}{(1+\xi)\Lambda m}\vec{p}L^{-\xi}\right)^2=\frac{1}{\frac{1}{v^2}\frac{1}{c^2}},\tag{24}$$

from which one deduces:

$$\gamma = \frac{1}{mc}\sqrt{m^2c^2 + \left(-\frac{q}{mc}\vec{A} + \frac{1}{(1+\xi)\Lambda}\vec{p}L^{-\xi}\right)^2}. \tag{25}$$

Then we get:

$$\begin{aligned} H = &(1+\xi)\Lambda\left(-\frac{m^2c^2}{\sqrt{m^2c^2+\left(-\frac{q}{c}\vec{A}+\frac{1}{(1+\xi)\Lambda}\vec{p}L^{-\xi}\right)^2}} + q\left(-\phi + \frac{\vec{v}\cdot\vec{A}}{c}\right)\right)^{\xi}\vec{v}\left(\gamma m v + \frac{q}{c}\vec{A}\right) \\ &-\Lambda\left(-\frac{m^2c^2}{\sqrt{m^2c^2+\left(-\frac{q}{c}\vec{A}+\frac{1}{(1+\xi)\Lambda}\vec{p}L^{-\xi}\right)^2}} + q\left(-\phi + \frac{\vec{v}\cdot\vec{A}}{c}\right)\right)^{1+\xi}, \end{aligned} \tag{26}$$

which can be written as:

$$\begin{aligned} H = &(1+\xi)\Lambda\left(-mc^2\sqrt{1-\frac{v^2}{c^2}} + q\left(-\phi + \frac{\vec{v}\cdot\vec{A}}{c}\right)\right)^{\xi}\left(\frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} + q\frac{\vec{v}\cdot\vec{A}}{c}\right) \\ &-\Lambda\left(-mc^2\sqrt{1-\frac{v^2}{c^2}} + q\left(-\phi + \frac{\vec{v}\cdot\vec{A}}{c}\right)\right)^{1+\xi}, \end{aligned} \tag{27}$$

Once more, when $\xi = 0$, Eq. (26) is reduced to the standard form:

$$H = \sqrt{m^2c^4 + c^2\left(p - \frac{\vec{q}}{c}\vec{A}\right)^2} + q\phi \equiv c\sqrt{m^2c^2 + \left(p - \frac{\vec{q}}{c}\vec{A}\right)^2} + q\phi. \tag{28}$$

We discuss the following two independent cases:

A-Absence of the EM field: In that case, Eq. (27) is reduced to the following Hamiltonian form $H = (1+\xi)\Lambda(-\gamma^{-1}mc^2)^{\xi}v^2\gamma\Lambda(-\gamma^{-1}mc^2)^{1+\xi}$. From Eq. (25) we find the velocity in terms of the relativistic momenta as

$$v^2 = \frac{c^2\left(\vec{p}(-\gamma^{-1}mc^2)^{-\xi}\right)^2}{m^2c^2(1+\xi)^2\Lambda^2 + \left(\vec{p}(-\gamma^{-1}mc^2)^{-\xi}\right)^2}, \tag{29}$$

which gives

$$m^2c^2(1+\xi)^2\Lambda^2\gamma^2 + \left(\vec{p}(-mc^2)^{-\xi}\right)^2\gamma^{2\xi-2} = -m^2c^2(1+\xi)^2\Lambda^2. \tag{30}$$

To illustrate we choose $\xi=1$, i.e. $S=\int_a^b \Lambda L^2 dt$. It is notable that for $\xi=1$, the dimension of Λ is [Energie]$^{-2}$. Hence, we may set $\Lambda=1/2mc^2$ and accordingly Eq. (30) gives:

$$v^2=c^2\frac{2m^2c^2+p^2}{m^2c^2+p^2}. \tag{31}$$

We can write the Hamiltonian entirely in terms of the (relativistic) momentum which takes after simple algebra:

$$H=\frac{m^3c^4-2m^3v^2c^2-2mv^2p^2}{2(m^2c^2+p^2)}. \tag{32}$$

We note that when $p=0, v^2=2c^2$ and the Hamiltonian is reduced to $H=-3mc^2/2$ whereas in the standard approach we find $H=mc^2$. This case corresponds for superluminal particles. It is notable that available experimental results neither exclude subluminal nor superluminal propagation [20,21] and several superluminal phenomena are known in literature [22-26]. However, if we choose $\xi=1/2$, i.e. $S=\int_a^b \Lambda L^{3/2} dt$. the dimension of Λ is [Enerige]$^{-1/2}$. Hence, we may set $\Lambda=4/3\sqrt{mc^2}$ and consequently Eq. (30) gives

$$\gamma^{-1}=\frac{1}{2}\left(\frac{p^2}{m^2c^2}\pm\sqrt{\frac{p^4}{m^4c^4}-4}\right). \tag{33}$$

After replacing into the Hamiltonian function, we find

$$\begin{aligned} H &=2imv^2\gamma^{1/2}+\frac{4imc^2}{3}\gamma^{-3/2} \\ &=\frac{2\sqrt{2}imv^2}{\sqrt{\frac{p^2}{m^2c^2}\pm\sqrt{\frac{p^4}{m^4c^4}}-4}}+\frac{4mc^2i}{3}\left(\frac{p^2}{2m^2c^2}\pm\frac{1}{2}\sqrt{\frac{p^4}{m^4c^4}-4}\right)^{\frac{3}{2}}. \end{aligned} \tag{34}$$

In this case, the Hamiltonian is complexified and this special case is not new and was discussed largely in theoretical physics within different aspects [27-33]. Despite the fact that complex energies cannot be produced in the physical world, we expect that they may have some applications in non-local quantum mechanics [34] as stated in [35]. We argue that power-law NSL may be useful to investigate some hidden properties in relativistic quantum mechanics [36]. This problem stimulated in fact a lot of work in the future.

B-Presence of the EM field: For mathematical simplicity, we discuss the case of relativistic particles moving with a relativistic speed, we can approximate Eq. (27) as to:

$$H\approx(1+\xi)\Lambda\left(q\left(-\phi+\frac{\vec{v}\cdot\vec{A}}{c}\right)\right)^{\xi}\left(\frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}}+q\frac{\vec{v}\cdot\vec{A}}{c}\right)-\Lambda\left(q\left(-\phi+\frac{\vec{v}\cdot\vec{A}}{c}\right)\right)^{1+\xi}. \tag{35}$$

If, for instance, we set $\xi=1$, Eq. (35) is reduced to:

$$\begin{aligned}H&\approx2\Lambda q\frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{\vec{v}\cdot\vec{A}}{c}-\phi\right)+\Lambda q^2\left(\left(\frac{\vec{v}\cdot\vec{A}}{c}\right)^2-\phi^2\right)\\&=\Lambda q\left(\frac{\vec{v}\cdot\vec{A}}{c}-\phi\right)\left(\frac{2mv^2}{\sqrt{1-\frac{v^2}{c^2}}}+q\left(\frac{\vec{v}\cdot\vec{A}}{c}+\phi\right)\right).\end{aligned} \tag{36}$$

Eq. (30) is the relativistic single-particle Hamiltonian in the EM field for $\xi=1$. In that case, we have $\gamma\gg1$, the momentum is approximated by $\vec{p}\approx2\gamma m\vec{v}\Lambda q(-\phi+\vec{v}\cdot\vec{A}/c)$ and then we can simplify Eq. (36) to be:

$$(H+2\gamma mv^2\Lambda q\phi)^2-(\vec{p}-v^2\phi)^2=\Lambda^2q^4\phi^4. \tag{37}$$

We may set $\Lambda=1/2mc^2$ and reduce Eq. (37) with $v\approx c$:

$$(H+\gamma q\phi)^2-(\vec{p}-c^2\phi)^2=\frac{q^4\phi^4}{4m^2c^2}. \tag{38}$$

In that case the time-like component is $H+\gamma q\phi$, the space-like component is $\vec{p}-c^2\phi$, the modified scalar potential is $\Theta=-\gamma\phi$ and the modified relativistic mass is $m=q^2\phi^2/2mc$. This relation is interesting as it shows that m$m=q^2\phi^2/2c^4=M^2$. The mass $|M|\propto q\phi/c^2$ is too small for relativistic velocities and the associated modified energy is given by $E\approx\pm q\phi$. Eq. (38) is different from the standard relation $(H-q\phi)^2-(\vec{p}-q\vec{A}/c)^2c^2=m^2c^4$ which corresponds for ξ=0. If we choose $\xi=-2$, i.e. $S=\int_a^b\Lambda L^{-2}dt$, then $\vec{p}=-\Lambda(\gamma m\vec{v}+q\vec{A}/c)L^{-2}$ and one may check that the following relation holds:

$$H\approx\frac{m^2c^4\left(\gamma mv^2+q\frac{\vec{v}\cdot\vec{A}}{c}\right)}{q^2\left(-\phi v^2+v^2\frac{\vec{v}\cdot\vec{A}}{c}\right)^2}\approx\frac{\gamma m^3c^6}{q^2\phi^2}\equiv Mc^2, \tag{39}$$

where $M=\gamma m^3c^4/q^2\phi^2\equiv m^3/m^{*2}$ and $|m^*|=q\phi/\gamma c^2$. This mass occurs in the previous case yet in the present case it depends on γ and is hence too small. From what is discussed above we can predict that there are electrified particles accompanied by the modified energy relation $Mc^2 \approx q\phi$. It is notable that the results presented in both cases A and B are inconclusive with respect to their physical consequences and several analysis are in progress in order to support phenomenologically the presented ideas of NSL in relativistic EM theory.

4 Concluding remarks

In this work, we envisaged to study the relevance of NSL in relativistic electrodynamics theory. We have picked the power-law NSL by $L_{NSL} = L^{1+\xi}(\dot{q},q,t)$ and we have derived the corresponding modified Euler-Lagrange equation and the modified Hamilton's equations where we have discussed their implications in relativistic electrodynamics theory. We have observed that for the equations of motion depend on the value of the parameter ξ and which are reduced to their standard forms for $\xi = 0$. We have discussed two different cases: absence and presence of the EM field for different values of . In the absence of the EM field, the theory predicts superluminal particles for $\xi = 1$ and complexified Hamiltonian for $\xi = 1/2$. In the presence of the EM field, it was observed that for $\xi = 1$, the Hamiltonian energy of relativistic moving particles is modified and that the energy is equal to a modified relativistic kinetic energy plus an additional modified energy due to a modified scalar potential where the vector potential $\vec{A}$ plays no role. The modified relativistic mass in the theory is $m = q^2\phi^2/2mc^4$ which indicates that the term $|q\phi/c^2|$ plays the role of a mass. For the case $\xi = -2$, the Hamiltonian energy is modified as well yet the relativistic electromagnetic mass that occurs in that case is $|m^*| = q\phi/\gamma c^2$. Modified expressions for energies are obtained accordingly for different values for ξ and a modified Lorentz force law emerges. More values of ξ may be discussed as well and we expect that more nice properties may be obtained accordingly.

We do not claim that the models as they stand in this paper describe a well-known physical system. In fact, our main aim is to show readers that some concrete uninvestigated possibilities unseen in the mathematics of NSL may be obtained. The good feature of such NSL models is the simplicity of consideration they provide. We conclude that a good number of equations that are expected to have important applications in modern physics can be derived from NSL functions. Specifying the NSL is equivalent to specifying some hidden properties not found in the standard approach where new Lorentz invariance and gauge invariance are obtained. These modifications lead also to derive the modified equations of motion satisfied by the EM field and by particles traveling in them. Using ideas presented in this paper it would be interesting to investigate a fully relativistic system where interactions are taken into account. Some applications of the results obtained here in quantum field theory are under progress.

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